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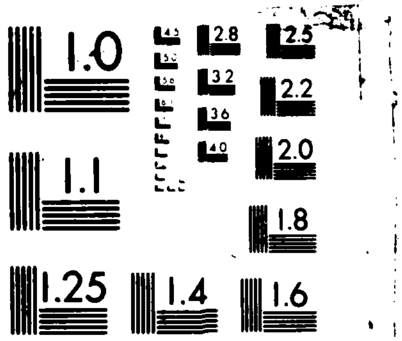

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by

Kenneth J. Arrow

Technical Report No. 494

November 1986

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
FOURTH FLOOR, ENCINA HALL
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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California
94305

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THE DEMAND FOR INFORMATION AND THE DISTRIBUTION OF INCOME*

by

Kenneth J. Arrow

1. Remarks on the Distribution of Income

The income of an individual is the sum of income derived from the sale of labor and income derived from the return on wealth. The increment of wealth in any period is the excess of total income over consumption. Therefore, the inequality of income among the members of a population depends at a moment of time on the inequality in labor income, the inequality in property income, and the covariance between them. However, the inequality in the holdings of wealth is to some extent derived from the inequality in past labor income.

In this paper, I take the inequality in labor income as a fact. However, it is well known that the inequality in property income is considerably greater proportionately. It would be natural to assume that saving (excess of income over consumption) is proportional to income. In that case, if it is assumed that labor income comes first, the inequality in wealth should be equal to the inequality in income. Wealth and income are closely related, in that those with high incomes tend to be those with large accumulations of property. But the relation

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is far from linear. It is well known that the proportion of wealth held by the high income recipients is much higher than the proportion of income that they receive.

One can think of a number of possible explanations for this nonlinearity. Among these is one that does not seem to have been much explored, namely, that those with high incomes receive higher rates of return on their investments. The hypothesis that different individuals receive different rates of return has been suggested by Becker [1967], where it is one component of a theory of income distribution. Becker does not, however, offer any explanation why individuals should have differing rates of return. In particular, he does not suggest that they will increase with income.

However, a recent paper of Yitzhaki [1986] has shown that the rates of return on investments are systematically higher for those with higher incomes. It is to be noted that his evidence makes clear tht there is a line of causation from wealth to higher return in that it is those who are wealthier at the beginning of a period who get higher returns. The correlation is not spurious.

There are, it must be admitted, several conceivable explanations of this correlation. I am going to develop just one. One alternative is that individuals who have the ability to secure systematically higher rates of return become wealthier; this is pretty much Becker's position. Actually, the model which follows is consistent with this hypothesis also, in that individuals with lower costs of information acquisition will buy more information and therefore get higher rates of return.

Another alternative hypothesis, which is suggested by Yitzhaki, is that richer individuals have lower relative risk aversion than poorer ones. The rich will therefore have a higher proportion of high-risk investments. In equilibrium, high-risk investments must have a higher expected rate of return, otherwise they will not be held. Hence, the expected rate of return will be an increasing function of wealth.

The hypothesis that relative risk aversion is decreasing itself has other difficulties. On general grounds, it may be argued that the utility function for risk-bearing is bounded both above and below (see Arrow [1971], pages 63-65, using an argument orginally due to Menger [1934]; this is a generalized version of the St. Peterburg paradox). But then it can be shown that relative risk aversion must exceed 1 for arbitrarily large values of wealth and be less than 1 for arbitrarily small values (Arrow [1971], pages 110-111], which contradicts the hypothesis that relative risk aversion is decreasing as wealth increases.

I would like to advance another hypothesis, the more interesting because it started with a question that had nothing to do with income distribution. Assume not only that there is uncertainty about the rates of return on alternative investments but also that is is possible to learn something about these distributions at a cost. What is being learned is information about <u>rates</u>. Hence, the value of the information depends on the amount to be invested. The cost, on the other hand, depends only on the distributions being studied and is therefore independent of the amount invested. Assume that one can invest more or less in information; increased expenditures buy more accurate knowledge of the rates of return. Then the optimal amount of information purchased by an investor is an increasing function of initial wealth (amount to be invested). Under this argument, it is not at all surprising that the rate of return will be higher, on the average, the higher is initial wealth. It follows that the distribution of final wealth (the return on the investment) will be more unequal than that of initial wealth.

It is necessary to show that these hypotheses can be embodied in a consistent story. I will generalize a model that I have studied earlier (Arrow [1971], chapter 12). In section 2, I consider a situation with given uncertainty and no possibility of acquiring information. An individual has resources to invest in a portfolio of different securities. The allocation among securities is to be optimal according to the expected utility of wealth achieved by the investment. This case is analyzed as a preliminary to the case where information can be acquired, which is studied in section 3. There it is shown that indeed the amount of information (interpreted as increased accuracy in forecasting) is an increasing function of initial wealth. From this, it is shown that the rate of return does indeed increase with wealth, and therefore terminal wealth (at the end of the investment) is more unequal than initial wealth.

2. Portofolio Choice under Uncertainty

As usual, uncertainty is represented by unknown states of nature. I simplify by assuming that the securities among which choices are to be made are elementary. By this is meant that each security is a possible bet, at fixed odds, on one state of nature.1/ Let X be the random variable, state of nature, x a representative value of X, and $p(x)$ the probability that $X = x$. (For the general theoretical presentation, I assume that X is a discrete variable. But the specific example will assume that X is normally distributed. I make the obvious adaptations without comment.) The security x can be bought in any amount; one unit of security x costs one monetary unit and pays one monetary unit if the realized state X is equal to x and 0 otherwise. (The case where security x pays r monetary units if $X = x$, the same for all x, is only trivially different. If the payment for a winning bet depends on the value of x, there are complications in presentation which obscure the point being made here, though they do not obviate it.)

The investor has an initial amount of wealth A. This is to be devoted to the purchase of elementary securities. Let $a(x)$ be the amount invested in elementary security x. This must satisfy the budget constraint,

$$\sum_x a(x) = A. \tag{1}$$

A-1

As a result of the investment, there is a payoff when the value of X becomes known. Call this payoff, W, the terminal wealth. In this model,

$$W = a(X). \tag{2}$$

The choice of the investment allocations, $a(x)$, thus determine a random variable, W. Among all feasible allocations, as defined by (1), we wish to choose the one which yields the best distribution of W. In accordance with standard theory, we assume that random variables ae ranked by the expected value of the utility obtained. Let $U(w)$ be the utility of a value, w, of terminal wealth. Then, the measure of merit for a random variable, W, is, $E[U(W)]$, and this quantity is to be maximized. In view of (2), the optimal portfolio is defined as that which maximizes,

$$E\{U[a(X)]\} = \sum_{x} p(x)\, U[a(x)], \tag{3}$$

subject to (1) (and the implicit condition that $a(x) \geq 0$ for all x).

To get definite results, it will be assumed that the utility function belongs to a specific class, that of power functions. These have been the most used in practice and are at least not strongly contradicted by evidence. Since risk aversion is certainly assumed, the power functions must be concave. Specifically, it is assumed that,

$$U(w) = w^{1-\alpha}/(1 - \alpha),\ \alpha > 0. \tag{4}$$

Since utility functions are defined only up to positive linear transformations, the constant can be chosen to insure that U is strictly increasing. The case, $\alpha = 1$ is not covered by (4), but if the constant, $-1/(1-\alpha)$, is subtracted from the right-hand side of (4) and then α is made to approach 1, the limit is the function, $\ell n \quad \alpha$, which is therefore one of the class of power functions used.

In the terminology of Pratt [1964] and Arrow [1965, Chapter 2; 1971, Chapter 3], the functions (4) are those with constant <u>proportionate</u> or <u>relative</u> risk aversion.

If we substitute (4) into (3) and then maximize with respect to the allocations, $a(x)$, subject to (1), we find

$$a(x) = \{[p(x)]^{1/\alpha} / \sum_y [p(y)]^{1/\alpha}\} A. \tag{5}$$

The proportions of amounts invested in different securities is independent of initial wealth. More is bet on more probable states of nature, but the extent to which this is true depends on the coefficient of relative risk aversion, α. If α tends to 0, the absence of risk aversion, then the bets are more and more concentrated on the most probable outcome. If $\alpha = 1$, then amounts invested are proportional to probabilities. If α tends to ∞, then amounts invested in all securities tend to the same value; the extreme of risk aversion is concern for only the worst possible outcome.

We will be interested here and in the following section in some magnitudes derived from the optimal portfolio, (5). One is the maximum expected utility which is achieved, that, the expected utility obtained if the policy (5) is followed. Closely related is the maximum certainty-equivalent terminal wealth, that, the value of W which, if received with certainty would have the same utility as that derived from the optimal policy (5). Finally, we can ask a different question, one more related to the statistics of income distribution: what is the expected terminal wealth or, equivalently, what is the expected return on initial wealth (the expected ratio of terminal to initial wealth)?

The maximum expected utility is obtained simply by substituting (5) into (4) and that into (3). The algebra is straightforward.

$$\max E\left[U(W)\right] = \left\{\sum_x \left[p(x)\right]^{1/\alpha}\right\}^{\alpha} A^{1-\alpha}/(1-\alpha). \tag{6}$$

The certainty-equivalent terminal wealth, W_c is defined by the equation,

$$U\left(W_c\right) = \max E\left[U(W)\right]. \tag{7}$$

From (6) and (4),

$$W_c = \left\{\sum_x \left[p(x)\right]^{1/\alpha}\right\}^{[\alpha/(1-\alpha)]} A. \tag{8}$$

This result shows already that, in one sense, the effect of pure uncertainty does not change the distribution of wealth. The certainty-equivalent terminal wealth is simply proportional to initial wealth.

This measure represents, so to speak, the *ex ante* perception of terminal wealth. An alternative would look at the actual outcomes, at least on the average.

$$E(W) = E\,[a(X)] = \sum_x p(x)\,a(x)$$

$$= \Big\{ \sum_x [p(x)]^{[1+\alpha)/\alpha]} / \sum_y [p(y)]^{1/\alpha} \Big\}\, A, \qquad (9)$$

again proportional to A.

Theorem 1: In the absence of possibility of information-gathering, the distribution of terminal wealth is the same as the distribution of initial wealth, when the porfolio has been selected optimally for a risk-aversion represented by a power function. This is true whether the distribution of terminal wealth is measured by certainty-equivalents or expected value.

The effect of risk aversion on these results may be easily presented. If $\alpha = 0$ (no risk aversion), then both W_c and $E(W)$ are equal to $\left[\max_x p(x)\right] A$. If $\alpha = 1$ (logarithmic utility), then,

$$W_c = e^{-I} A,$$

where,

$$I = -\sum_x p(x)\,\ell n\, p(x),$$

the Shannon measure of information, while,

$$E(W) = \sum_x [p(x)]^2\, A.$$

Finally, as $\alpha \to +\infty$, W_c tends to $e^{-n}A$, $E(W)$ to A/n, where n is the number of distinct values of x for wich $p(x) > 0$.

To prepare for the case with information and to get more particular results, I assume specifically that X is normally distributed. Though there are some interpretive questions, I will simply substitute integrals for sums as needed. It will be useful to characterize the normal distribution by its mean, as usual, and its precision, which is the reciprocal of the variance, more usually employed.

X is normally distributed with mean μ and precision h. Then the density is,

$$p(x) = (h/2\pi)^{1/2}e^{-h(x-\mu)^2/2}. \tag{10}$$

From the simple fact that

$$\int e^{-h(x-\mu)^2/2}\,dx = (h/2\pi)^{-1/2},$$

it can be seen that

$$\begin{aligned}\int [p(x)]^r\,dx &= (h/2\pi)^{r/2}\int e^{-hr(x-\mu)^2/2}dx \\ &= (h/2\pi)^{r/2}(hr/2\pi)^{-1/2} \\ &= r^{-1/2}(h/2\pi)^{(r-1)/2}. \end{aligned} \tag{11}$$

Substitute into the general formulas (6), (8), and (9), with sums replaced by integrals, with $r = 1/\alpha$ in (6) and the denominator of (9) and $r = (1 + \alpha)/\alpha$ into the numerator of (9). After straightforward simplification, we find,

$$\max E[U(W)] = \alpha^{\alpha/2}(h/2\pi)^{(1-\alpha)/2}A^{1-\alpha}/(1-\alpha), \tag{12}$$

$$W_c = \alpha^{\alpha/2(1-\alpha)}(h/2\pi)^{1/2} A, \tag{13}$$

$$E(W) = (1+\alpha)^{-1/2}(h/2\pi)^{1/2} A. \tag{14}$$

That W_c and $E(W)$ are proportional to A is not surprising; we already had this result in the general case. It more interesting to see that, for given A (and given risk aversion α), both W_c and $E(W)$ are proportional to $h^{1/2}$. The more precise the knowledge of the state of nature, X, the higher is terminal wealth, however measured. This suggests, as will be confirmed in the next section, that if the opportunity to increase h exists, it will be undertaken even for some decline in assets, A.

The effects of risk aversion may be quickly noted. For $\alpha = 0$, the constant factors independent of both h and A in (13) and (14) are both equal to 1, while for $\alpha = +\infty$, both constant factors are 0. For $\alpha = 1$, the constant in (13) is $e^{-1/2}$, in (14) $2^{-1/2}$.

3. The Demand for Information

Suppose the economic agent can observe some random variable, say S, before choosing the portfolio. To be useful, of course, it should not be independent of X. Suppose that the purchase of the signal costs C, to be paid for out of initial wealth. Clearly, once S is observed, the optimal portfolio is selected according to the same principles as before, except that the probabilities used are the conditional probabilities of the different values of X, given the observed value of S. Let $a(x,s)$ be the amount invested in securities of type x given that $S = s$, $p(x|s)$ = probability that $X = x$ given that $S = s$. Then, since the amount available for investment is now $A - C$, the optimal portfolio conditional on S is,

$$a(x,s) = \{[p(x|s)]^{1/\alpha} / \sum_y [p(y|s)]^{1/\alpha}\} (A - C), \tag{15}$$

by adaptation of (5) to the conditional case. For a given value of S, the parallel to (6) is,

$$\max E[U(W)|S] = \{ \sum_x [p(x|S)^{1/\alpha}\}^{\alpha} (A - C)^{1-\alpha}/(1 - \alpha). \tag{16}$$

This last expression is a function of the random variable, S, and hence is still a random variable. To get a measurement of the value of the signal, we need its expected value with respect to S, since the realization of S is unknown when the signal is chosen.

$$\max E[U(W)] = E_S\{ \sum_x [p(x|S)]^{1/\alpha}\}^{\alpha} (A - C)^{1-\alpha}/(1 - \alpha)$$

$$= [(A-C)^{1-\alpha}/(1-\alpha)] \sum_s p(s) \{ \sum_x [p(x|s)]^{1/\alpha}\}^{\alpha}. \tag{17}$$

If the choice were simply whether or not to observe the signal S, one would simply compare (17) with (6). The second factor can easily be shown (by Jensen's inequality) to be larger than the corresponding factor in (6), while the first factor is obviously less.

More generally, however, we want to admit many possible signals, of greater or lesser reliability and correspondingly of greater or lesser cost. To keep matters tractable, I now assume that X and S have a joint normal distribution. This will be described by specifying the distribution of X as in (10) and the conditional distribution of S given X, which will have a mean which is linear in X and a precision, to be denoted for reasons of convenience by $H-h$, which is independent of X.

If S is not independent of X, $E(S|X)$ is a non-trivial linear function of X. By making a simple linear transformation on S, we can assume without loss of generality that,

$$E(S|X) = X. \tag{18}$$

It will now be assumed that <u>the portfolio-chooser can select the conditional precision of S</u>, i.e., $H-h$. After choosing $H-h$, he/she observes S and then chooses $a(x,s)$ for all x in accordance with (15).

It remains to consider the cost associated with a choice of precision of the signal. Think of S as a random sample from a population with unknown mean X. Then the cost may be thought of as proportional to the size of the sample. But elementary statistical theory tells us that the precision of S (given X) is proportional to the sample size (the <u>variance</u> of a sample mean is inversely proportional to sample size). It will be assumed that <u>the cost of a signal S with conditional precision $H-h$ is $c(H-h)$</u>, for some constant c. We substitute this for C in (17).

Under the assumption of normality, the posterior of X given S, $p(x|s)$, can be computed by Bayes' Theorem along well-known lines.

$$p(x|s) = p(s|x)\ p(x)/p(s).$$

With the notations introduced above,

$$p(s|x) = \left[(H-h)/2\pi\right]^{1/2} e^{-(H-h)(s-x)^2/2},$$

and $p(x)$ is given by (10). Multiply, collect the terms in the exponent, and complete the square. Then,

$$p(x|s) = Ke^{-H(x-m)^2/2}, \tag{19}$$

where K is independent of x but might depend on s, and,

$$m = \left[(H-h)s + h\mu\right]/H. \tag{20}$$

Since (19) is a distribution and its integral with respect to x is 1,

$$K = (H/2\pi)^{1/2},$$

independent of s. Then, as is well known, the posterior distribution of X given S is normal with mean m and precision H (the latter independent of S).

If we substitute from (19) and the assumption, $C = c(H - h)$, into (16), we have the same result as (12), with h replaced by H and A by $A - C = A - c(H - h) = (A + ch) - cH = c(A' - H)$, where

$$A' = (A/c) + h, \tag{21}$$

the initial level of information or precision plus the ability of initial assets to buy information. The analogue of (12) is,

$$E[U(W)|S] = \alpha^{\alpha/2}(H/2\pi)^{(1-\alpha)/2}c^{1-\alpha}(A'-H)^{1-\alpha}/(1-\alpha). \qquad (22)$$

The right-hand side is independent of S. Hence, if we take expectations over S, we get the same expression for $E[U(W)]$ when the optimal portfolio for a signal with precision $H-h$ is used. The result takes a slightly simpler form when restated in terms of the certainty-equivalent terminal wealth as defined in (7).

$$W_c = \alpha^{\alpha/2(1-\alpha)}c(A'-H)H^{1/2}(2\pi)^{-1/2}. \qquad (23)$$

Since W_c is a simple monotone transformation of $E[U(W)]$, the same value of H will be optimal for both. To maximize (23) with respect to H, it suffices to maximize,

$$(A'-H)H^{1/2},$$

and clearly this is done by setting,

$$H = A'/3. \qquad (24)$$

Notice that this is independent of the degree of risk-aversion, α. It is also independent of the initial information, h, subject to one proviso. The optimization should have had the inequality constraint, $H \geq h$, since initial information cannot be sold, or, otherwise put, the precision of the signal, $H-h$, cannot be non-negative. Strictly speaking (24) holds only if H, so defined, satisfies this constraint;

otherwise, $H = h$. I will assume, to avoid minor expository problems, that the constraint is not binding.

If we substitute (24) into (23), we find,

$$\max_{H} W_c = \text{constant } (A')^{3/2}. \tag{25}$$

Here we see confirmation of the hypothesis presented here. The magnitude, A', is a measure of initial wealth, where initial information is included. Then terminal wealth (as measured by certainty equivalents) is no longer proportional to initial wealth but increases more than proportionately. The possibility of costly information acquisitions acts to increase the concentration of wealth beyond the initial.

It may be worth presenting the exact formula for the optimal portfolio when H has been chosen optimally, as given by (24). We need only substitute (19) and (24) into (15).

$$a(x,s) = (2/3)\ (H/2\pi\alpha)^{1/2} e^{-H(x-m)^2/2\alpha} cA', \tag{26}$$

where H is understood to be given by (24).

As in section 2, we also want to look at $E(W)$, the actual (rather than certainty-equivalent) average outcomes. First, note that the conditional expected terminal wealth, given the signal S, $(E(W|S)$, is the same as that for $E(W)$, in formula (14), with $p(x)$ replaced by $p(x|s)$. This means replacing h by H and A by cA'.

$$E(W|S) = (2/3)(1 + \alpha)^{-1/2}(H/2\pi)^{1/2} cA'.$$

But since the expectation is in fact independent of S, we can write,

$$E(W) = (2/3)[(1 + \alpha)2\pi]^{-1/2}c\ H^{1/2}A', \tag{27}$$

and, since H is proportional to A' by (24),

$$E(W) = \text{constant } (A')^{3/2}. \tag{28}$$

<u>Theorem 2</u>: If the state of nature and a signal are jointly normally distributed, if the precision of the signal (conditional on the state of nature) can be increased at constant cost, and if the utility of terminal wealth is a power function, then the amount of total information (initial and purchased) is 1/3 of the initial wealth measured in information units (including initial information) independent of the amount of risk aversion, and the expected terminal wealth (in natural units or certainty-equivalent) is proportional to the 3/2 power of initial wealth (or, to put it differently, the expected rate of return is proportional to the square root of initial wealth).

<u>Remark</u>: It must be agreed that the prediction of the theorem is for an unrealistically strong relation between initial wealth and rate of return. It is not credible that someone with 100 times to invest receives 10 times the return, nor do Yitzhaki's data support such a view. One possible explanation in terms of this model is that wealthier individuals have better paid alternative uses for their time, so that the cost, c, of information gathering goes up with A, and therefore A' rises more slowly. Another is that only part of the initial uncertainty can be decreased by information.

4. Remark on Effect of Cost of Information on Rate of Return

Becker's hypothesis that individuals have differing rates of return can be restated in a more fundamental way here as differing costs of acquiring information. Naturally, an individual who is more efficient at information-gathering, a lower c in this paper's notation, will buy more information. The precise relation is seen by substituting (21) into (24). Thus, for a given initial wealth, the rate of return should be higher the lower is c. This is confirmed by using, for example, (27), where expected terminal wealth is proportional to,

$$c\,H^{1/2}A' = c\,(A')^{3/2}/3^{1/2} = (Ac^{-1/3} + hc^{2/3})^{3/2}3^{1/2}.$$

This is decreasing as c increases if and only if the expression,

$$Ac^{-1/3} + hc^{2/3},$$

does the same. This will be true for $A \geq 2hc$. But for the individual to purchase more precision it is necessary and sufficient that $H \geq h$, i.e., $A'/3 \geq h$, or $A \geq 2hc$, so that the rate of return increases with decreasing c provided, of course, that A is sufficiently large so that some information will be purchased.

FOOTNOTES

1/ It may be noted that every actual security can be regarded as a function specifying the payoff to a unit security in each state of nature and therefore as a vector combination of elementary securities. Conversely, if there is a linearly independent set of securities equal in number to the number of states of nature, then each individual can in effect construct elementary securities by suitable combinations of the existing securities.

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