

- [16] K. G. Merryfield, *On the area integral, Carleson measure and H^p in the polydisc*, Indiana Univ. Math. J. 34 (1985), 663–685.
- [17] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
- [18] B. Muckenhoupt and R. L. Wheeden, *Norm inequalities for the Littlewood–Paley function g_λ^** , *ibid.* 191 (1974), 95–111.
- [19] —, —, *Weighted bounded mean oscillation and the Hilbert transform*, Studia Math. 54 (1976), 221–237.
- [20] —, —, *On the dual of weighted H^1 of the half-space*, *ibid.* 63 (1978), 57–79.
- [21] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.
- [22] A. Torchinsky, *Weighted norm inequalities for the Littlewood–Paley function g_λ^** , in: Proc. Sympos. Pure Math. 35, Part 1, Amer. Math. Soc., Providence, R.I., 1979, 125–139.
- [23] R. L. Wheeden, *A boundary value characterization of weighted H^1* , Enseign. Math. 22 (1976), 121–134.

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The Denjoy extension of the Bochner, Pettis, and Dunford integrals

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Abstract. In this paper the Denjoy–Dunford, Denjoy–Pettis, and Denjoy–Bochner integrals of functions mapping an interval $[a, b]$ into a Banach space X are defined and studied. Necessary and sufficient conditions for the existence of the Denjoy–Dunford integral are determined. It is shown that a Denjoy–Dunford (Denjoy–Bochner) integrable function on $[a, b]$ is a Dunford (Bochner) integrable on some subinterval of $[a, b]$ and that for spaces that do not contain a copy of c_0 , a Denjoy–Pettis integrable function on $[a, b]$ is Pettis integrable on some subinterval of $[a, b]$. For measurable functions, the Denjoy–Dunford and Denjoy–Pettis integrals are equivalent if and only if X is weakly sequentially complete. Several examples of functions that are integrable in one sense but not another are included.

The Denjoy integral of a real-valued function is, in the descriptive sense (that is, specifying the properties of the primitive), a natural extension of the Lebesgue integral of a real-valued function. The Bochner, Pettis, and Dunford integrals are generalizations of the Lebesgue integral to Banach-valued functions. In this paper we will study the Denjoy extension of the Bochner, Pettis, and Dunford integrals.

Before embarking on this study a firm foundation must be laid. The reader may wish to begin with Definition 25 and refer to the introductory material as the need arises. We begin with the notions of bounded variation and absolute continuity on a set. Throughout this paper X will denote a real Banach space and X^* its dual.

DEFINITION 1. Let $F: [a, b] \rightarrow X$ and let E be a subset of $[a, b]$.

(a) The function F is BV on E if $\sup \sum_i \|F(d_i) - F(c_i)\|$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of nonoverlapping intervals that have endpoints in E .

(b) The function F is AC on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i) - F(c_i)\| < \varepsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \delta$.

(c) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.

(d) The function F is ACG on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

The proofs of the next two theorems are tedious but not difficult and are left to the reader.

THEOREM 2. Let $F: [a, b] \rightarrow X$, let E be a subset of $[a, b]$, and suppose that F is continuous on a set H that contains \bar{E} . If F is BV (AC) on E , then F is BV (AC) on \bar{E} .

THEOREM 3. Let $F: [a, b] \rightarrow X$ and let E be a closed subset of $[a, b]$ with bounds c and d . Let $G: [c, d] \rightarrow X$ be the function that equals F on E and is linear on the intervals contiguous to E . If F is BV (AC) on E , then G is BV (AC) on $[c, d]$.

The following theorem and lemma are useful in proving results in the theory of the Denjoy integral. The proof of the first is quite similar to a proof in Saks [6, p. 233] and the proof of the second can be found in Romanovski [5]. Recall that a portion of a set $E \subset \mathbb{R}$ is a nonempty set P of the form $P = E \cap I$ where I is an open interval.

THEOREM 4. Let E be a closed subset of $[a, b]$ and let $F: [a, b] \rightarrow X$ be continuous on E . Then F is BVG (ACG) on E if and only if every perfect set in E contains a portion on which F is BV (AC).

LEMMA 5. Let \mathcal{F} be a family of open intervals in (a, b) and suppose that \mathcal{F} has the following properties:

- (1) If (α, β) and (β, γ) belong to \mathcal{F} , then (α, γ) belongs to \mathcal{F} .
- (2) If (α, β) belongs to \mathcal{F} , then every open interval in (α, β) belongs to \mathcal{F} .
- (3) If (α, β) belongs to \mathcal{F} for every interval $[\alpha, \beta] \subset (c, d)$, then (c, d) belongs to \mathcal{F} .
- (4) If all of the intervals contiguous to the perfect set $E \subset [a, b]$ belong to \mathcal{F} , then there exists an interval I in \mathcal{F} such that $I \cap E \neq \emptyset$.

Then \mathcal{F} contains the interval (a, b) .

The function $F: E \rightarrow \mathbb{R}$ satisfies condition (N) on E if $\mu^*(F(A)) = 0$ for every set $A \subset E$ of measure zero. Here $\mu^*(A)$ represents the Lebesgue outer measure of the set A . The next two theorems reveal the importance of this concept. See Saks [6] for the proofs of these results.

THEOREM 6. If $F: E \rightarrow \mathbb{R}$ is ACG on E , then F satisfies condition (N) on E .

THEOREM 7. Let E be a bounded, closed set with bounds a and b and let $F: [a, b] \rightarrow \mathbb{R}$ be continuous on E . Then F is AC on E if and only if F is BV on E and satisfies condition (N) on E .

Next, we define approximate derivatives and state two theorems that illustrate the usefulness of this type of derivative. The proofs can be found in Saks [6].

DEFINITION 8. Let $F: [a, b] \rightarrow X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that

$$\lim_{\substack{s \rightarrow t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z.$$

We will write $F'_{\text{ap}}(t) = z$.

THEOREM 9. Let $F: [a, b] \rightarrow \mathbb{R}$ be measurable and let E be a subset of $[a, b]$. If F is BVG on E , then F is approximately differentiable almost everywhere on E .

THEOREM 10. Let $F: [a, b] \rightarrow \mathbb{R}$ be ACG on $[a, b]$. If $F'_{\text{ap}} = 0$ almost everywhere on $[a, b]$, then F is constant on $[a, b]$.

We can now define the Denjoy integral of a real-valued function. Recall that a function $f: [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable on $[a, b]$ if there exists an AC function $F: [a, b] \rightarrow \mathbb{R}$ such that $F' = f$ almost everywhere on $[a, b]$.

DEFINITION 11. The function $f: [a, b] \rightarrow \mathbb{R}$ is Denjoy integrable on $[a, b]$ if there exists an ACG function $F: [a, b] \rightarrow \mathbb{R}$ such that $F'_{\text{ap}} = f$ almost everywhere on $[a, b]$. The function f is Denjoy integrable on the set $E \subset [a, b]$ if $f\chi_E$ is Denjoy integrable on $[a, b]$. When it is necessary to distinguish between the Denjoy and Lebesgue integrals we will use the prefixes (D) and (L).

THEOREM 12. Let $f: [a, b] \rightarrow \mathbb{R}$.

- (a) If f is Denjoy integrable on $[a, b]$, then f is measurable.
- (b) If f is nonnegative and Denjoy integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.
- (c) If f is Denjoy integrable on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which f is Lebesgue integrable.

The proofs of these facts can be found in Saks [6]. Note that (c) implies that a Denjoy integrable function on $[a, b]$ is Lebesgue integrable on some subinterval of $[a, b]$. Another useful property of the Denjoy integral is given by the next theorem (see Saks [6, p. 257]). Let $\omega(F, [\alpha, \beta]) = \sup\{|F(t) - F(s)|: \alpha \leq s < t \leq \beta\}$ denote the oscillation of F on $[\alpha, \beta]$.

THEOREM 13. Let E be a bounded, closed set with bounds a and b and let $\{(a_k, b_k)\}$ be the sequence of intervals in $[a, b]$ contiguous to E . Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is Denjoy integrable on E and on each $[a_k, b_k]$. If

$$\sum_k \int_{a_k}^{b_k} |f| < \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \omega\left(\int_{a_k}^t f, [a_k, b_k]\right) = 0,$$

then f is Denjoy integrable on $[a, b]$. In addition,

$$\int_a^t f = \int_a^t f \chi_E + \sum_{b_k \leq t} \int_{a_k}^{b_k} f \quad \text{for } t \in E,$$

$$\int_a^t f = \int_a^t f \chi_E + \sum_{b_k \leq t} \int_{a_k}^{b_k} f + \int_{a_n}^t f \quad \text{for } t \in (a_n, b_n).$$

The next group of definitions and theorems extends the familiar concept of uniform integrability into the context of Denjoy integrals.

DEFINITION 14. Let $\{f_\alpha\}$ be a family of Denjoy integrable functions defined on $[a, b]$. The family $\{f_\alpha\}$ is *uniformly Denjoy integrable* on $[a, b]$ if for each perfect set $E \subset [a, b]$ there exists an interval $[c, d] \subset [a, b]$ with $c, d \in E$ and $E \cap (c, d) \neq \emptyset$ such that every f_α is Lebesgue integrable on $E \cap [c, d]$ and for every α the series $\sum_n |\int_{c_n}^{d_n} f_\alpha|$ is convergent where $[c, d] - E = \bigcup_n (c_n, d_n)$.

DEFINITION 15. Let $\{F_\alpha\}$ be a family of functions defined on $[a, b]$ and let E be a subset of $[a, b]$. The family $\{F_\alpha\}$ is *uniformly BVG (ACG)* on E if each F_α is BVG (ACG) on E and if each perfect set in E contains a portion on which every F_α is BV (AC).

The proof of Theorem 4 can be adapted to prove the following theorem.

THEOREM 16. Let $\{F_\alpha\}$ be a family of functions defined on $[a, b]$. Suppose that E is a closed subset of $[a, b]$ and that each F_α is continuous on E . Then the family $\{F_\alpha\}$ is *uniformly BVG (ACG)* on E if and only if $E = \bigcup_n E_n$ where every F_α is BV (AC) on each E_n .

The next theorem ties together the concepts of uniform Denjoy integrability and uniformly ACG.

THEOREM 17. Let $\{f_\alpha\}$ be a family of Denjoy integrable functions defined on $[a, b]$ and for each α let $F_\alpha(t) = \int_a^t f_\alpha$. Then the family $\{f_\alpha\}$ is *uniformly Denjoy integrable* on $[a, b]$ if and only if the family $\{F_\alpha\}$ is *uniformly ACG* on $[a, b]$.

Proof. Suppose first that $\{f_\alpha\}$ is uniformly Denjoy integrable on $[a, b]$ and let E be a perfect set in $[a, b]$. Then there exists an interval $[c, d]$ with $c, d \in E$ and $E \cap (c, d) \neq \emptyset$ such that every f_α is Lebesgue integrable on $E \cap [c, d]$ and for every α the series

$$\sum_n \int_{c_n}^{d_n} f_\alpha = \sum_n (F_\alpha(d_n) - F_\alpha(c_n))$$

is absolutely convergent where $[c, d] - E = \bigcup_n (c_n, d_n)$. By Theorem 13 we

have

$$F_\alpha(t) - F_\alpha(c) = (L) \int_c^t f_\alpha \chi_E + \sum_{d_n \leq t} \int_{c_n}^{d_n} f_\alpha$$

for all t in $E \cap [c, d]$ and for all α . We will show that each F_α is AC on $E \cap [c, d]$.

To this end fix α and let $\varepsilon > 0$. Choose a positive integer N such that $\sum_{n=N+1}^\infty |F_\alpha(d_n) - F_\alpha(c_n)| < \varepsilon/2$ and choose a positive number $\delta < \min\{d_n - c_n : 1 \leq n \leq N\}$ such that $\int_H |f_\alpha| \chi_E < \varepsilon/2$ whenever $\mu(H) < \delta$ and $H \subset [c, d]$. Now let $\{[u_i, v_i]\}$ be a finite collection of nonoverlapping intervals that have endpoints in $E \cap [c, d]$ and satisfy $\sum_i (v_i - u_i) < \delta$. Let $H = \bigcup_i [u_i, v_i]$ and for each i let $\pi_i = \{n : (c_n, d_n) \subset (u_i, v_i)\}$. Since the π_i 's are disjoint and contain no integers less than or equal to N we have

$$\begin{aligned} \sum_i |F_\alpha(v_i) - F_\alpha(u_i)| &= \sum_i \left| \int_{u_i}^{v_i} f_\alpha \chi_E + \sum_{n \in \pi_i} \int_{c_n}^{d_n} f_\alpha \right| \\ &\leq \sum_i \int_{u_i}^{v_i} |f_\alpha| \chi_E + \sum_i \sum_{n \in \pi_i} \left| \int_{c_n}^{d_n} f_\alpha \right| \\ &\leq \int_H |f_\alpha| \chi_E + \sum_{n=N+1}^\infty |F_\alpha(d_n) - F_\alpha(c_n)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, the function F_α is AC on $E \cap [c, d]$.

Now suppose that the family $\{F_\alpha\}$ is uniformly ACG on $[a, b]$ and let E be a perfect set in $[a, b]$. Then there exists an interval $[c, d]$ such that $c, d \in E$, $E \cap (c, d) \neq \emptyset$, and each F_α is AC on $E \cap [c, d]$. Fix α , let $F = F_\alpha$, and let $G: [c, d] \rightarrow \mathbf{R}$ be the function that equals F on E and is linear on the intervals contiguous to E . By Theorem 3 the function G is AC on $[c, d]$ and hence G' exists almost everywhere on $[c, d]$ and is Lebesgue integrable on $[c, d]$. Since $G' = F'_{np} = f_\alpha$ almost everywhere on $E \cap [c, d]$, the function f_α is Lebesgue integrable on $E \cap [c, d]$. Furthermore, since F is BV on $E \cap [c, d]$, the series $\sum_n |\int_{c_n}^{d_n} f_\alpha|$ converges where $[c, d] - E = \bigcup_n (c_n, d_n)$. Since this is valid for each α it follows that the family $\{f_\alpha\}$ is uniformly Denjoy integrable on $[a, b]$.

DEFINITION 18. Let $\{f_\alpha\}$ be a family of Denjoy integrable functions defined on $[a, b]$. The family $\{f_\alpha\}$ is *uniformly integrable in a generalized sense* on $[a, b]$ if for each perfect set E in $[a, b]$ there is a portion $E \cap I$ of E such that the family $\{f_\alpha \chi_E\}$ is uniformly integrable (Lebesgue sense) on I .

We are now ready to look at the integrals of vector-valued functions.

We first state and prove a theorem that guarantees the existence of the Dunford integral. Our proof of this theorem is not the standard closed graph argument (see Diestel and Uhl [2, p. 52]) but it is in the spirit of several proofs that appear later in this paper.

THEOREM 19. *Let $f: [a, b] \rightarrow X$. If x^*f is Lebesgue integrable on $[a, b]$ for each x^* in X^* , then for each measurable set E in $[a, b]$ there exists a vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^*f$ for all x^* in X^* .*

Proof. Let $B = \{x^* \in X^*: \|x^*\| \leq 1\}$ and for each positive integer n let $V_n = \{x^* \in B: \int_a^b |x^*f| \leq n\}$. Then $B = \bigcup_n V_n$ and we show next that each V_n is closed. Let y^* be a limit point of V_n and let $\{x_k^*\}$ be a sequence in V_n that converges to y^* . The sequence $\{|x_k^*f|\}$ converges pointwise to $|y^*f|$ and Fatou's Lemma yields

$$\int_a^b |y^*f| \leq \liminf_{k \rightarrow \infty} \int_a^b |x_k^*f| \leq n.$$

This shows that $y^* \in V_n$ and we conclude that V_n is closed.

By the Baire Category Theorem there exist an integer N , a real number $\varrho > 0$, and a vector x_0^* in B such that $\{x^*: \|x^* - x_0^*\| \leq \varrho\} \subset V_N$. Take $x^* \in B$ and compute

$$\int_a^b |x^*f| = \varrho^{-1} \int_a^b |\varrho x^*f + x_0^*f - x_0^*f| \leq \varrho^{-1} \int_a^b (|\varrho x^*f + x_0^*f| + |x_0^*f|) \leq 2N/\varrho.$$

Now let E be a measurable subset of $[a, b]$ and let l_E be the linear functional defined on X^* by $l_E(x^*) = \int_E x^*f$. Since

$$\sup_B \left| \int_E x^*f \right| \leq \sup_B \int_a^b |x^*f| \leq \sup_B \int_a^b |x^*f| \leq 2N/\varrho,$$

the linear functional l_E is bounded and hence defines an element of X^{**} . This completes the proof.

DEFINITION 20. (a) The function $f: [a, b] \rightarrow X$ is *Dunford integrable* on $[a, b]$ if x^*f is Lebesgue integrable on $[a, b]$ for each x^* in X^* . The *Dunford integral* of f on the measurable set $E \subset [a, b]$ is the vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_a^b x^*f$ for all x^* in X^* .

(b) The function $f: [a, b] \rightarrow X$ is *Pettis integrable* on $[a, b]$ if f is Dunford integrable on $[a, b]$ and $x_E^{**} \in X$ for every measurable set E in $[a, b]$.

(c) The function $f: [a, b] \rightarrow X$ is *Bochner integrable* on $[a, b]$ if there exists an AC function $F: [a, b] \rightarrow X$ such that F is differentiable almost everywhere on $[a, b]$ and $F' = f$ almost everywhere on $[a, b]$.

The function f is Dunford, Pettis, or Bochner integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is Dunford, Pettis, or Bochner integrable on $[a, b]$. The next three theorems list properties of the Dunford and Pettis integrals. The last two follow from the Bessaga-Pelczyński characterization of Banach spaces that do not contain a copy of c_0 (see Diestel and Uhl [2, p. 22]).

THEOREM 21. *If $f: [a, b] \rightarrow X$ is Pettis integrable on $[a, b]$, then the family $\{x^*f: \|x^*\| \leq 1\}$ is uniformly integrable on $[a, b]$.*

THEOREM 22. *Suppose that X contains no copy of c_0 and let $f: [a, b] \rightarrow X$ be Dunford integrable on $[a, b]$. If f is measurable, then f is Pettis integrable on $[a, b]$.*

THEOREM 23. *Suppose that X contains no copy of c_0 and let $f: [a, b] \rightarrow X$ be Dunford integrable on $[a, b]$. If $\int_I f \in X$ for every interval I in $[a, b]$, then f is Pettis integrable on $[a, b]$.*

The foundation to study the Denjoy extension of Banach-valued integrals is now in place. The next theorem guarantees the uniqueness of the Denjoy-Bochner integral. We will use θ to represent the zero of a Banach space.

THEOREM 24. *Let $F: [a, b] \rightarrow X$ be ACG on $[a, b]$ and suppose that F is approximately differentiable almost everywhere on $[a, b]$. If $F'_{ap} = \theta$ almost everywhere on $[a, b]$, then F is constant on $[a, b]$.*

Proof. Suppose that F is not constant on $[a, b]$. Then there exist points t_1 and t_2 in $[a, b]$ such that $F(t_1) \neq F(t_2)$. Choose x^* in X^* such that $x^*F(t_1) \neq x^*F(t_2)$. Since x^*F is ACG on $[a, b]$ and since $(x^*F)_{ap} = \theta$ almost everywhere on $[a, b]$, the function x^*F is constant on $[a, b]$ by Theorem 10. This contradiction completes the proof.

DEFINITION 25. (a) The function $f: [a, b] \rightarrow X$ is *Denjoy-Dunford integrable* on $[a, b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a, b]$ and if for every interval I in $[a, b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^* .

(b) The function $f: [a, b] \rightarrow X$ is *Denjoy-Pettis integrable* on $[a, b]$ if f is Denjoy-Dunford integrable on $[a, b]$ and if $x_I^{**} \in X$ for every interval I in $[a, b]$.

(c) The function $f: [a, b] \rightarrow X$ is *Denjoy-Bochner integrable* on $[a, b]$ if there exists an ACG function $F: [a, b] \rightarrow X$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$.

The function f is integrable in one of the above senses on the set $E \subset [a, b]$ if the function $f\chi_E$ is integrable in that sense on $[a, b]$. We first

examine the properties of the Denjoy–Bochner integral. As with the Bochner integral, a Denjoy–Bochner integrable function must be measurable.

THEOREM 26. *If $f: [a, b] \rightarrow X$ is Denjoy–Bochner integrable on $[a, b]$, then f is measurable.*

Proof. Since it is clear that each x^*f is Denjoy integrable on $[a, b]$, each x^*f is measurable by Theorem 12(a). Let $F(t) = \int_a^t f$. Since F is continuous, the set $\{F(t): t \in [a, b]\}$ is compact and hence separable. Let Y be the closed linear subspace spanned by $\{F(t): t \in [a, b]\}$. Then Y is separable and Y contains the set $\{f(t): F'_{\text{ap}}(t) = f(t)\}$. Hence, the function f is essentially separably valued. It follows from the Pettis Measurability Theorem that f is measurable.

The next two theorems were proved by Alexiewicz [1]. The first, which we state without proof, is important in the theory of the differentiation of vector-valued functions (see also Gordon [3]). The proof of the second is included for completeness.

THEOREM 27. *Let $F: [a, b] \rightarrow X$ be AC on $[a, b]$. If F is approximately differentiable almost everywhere on $[a, b]$, then F is differentiable almost everywhere on $[a, b]$.*

THEOREM 28. *If $f: [a, b] \rightarrow X$ is Denjoy–Bochner integrable on $[a, b]$, then each perfect set in $[a, b]$ contains a portion on which f is Bochner integrable.*

Proof. Let E be a perfect set in $[a, b]$. Since the function $F(t) = \int_a^t f$ is AC on $[a, b]$, we find, using Theorem 4, that there exists an interval $[c, d]$ in $[a, b]$ such that $c, d \in E$, $(c, d) \cap E \neq \emptyset$, and F is AC on $[c, d] \cap E$. Let $G: [c, d] \rightarrow X$ be the function that equals F on $[c, d] \cap E$ and is linear on the intervals contiguous to $[c, d] \cap E$. Then G is AC on $[c, d]$ by Theorem 3 and it is not difficult to show that G is approximately differentiable almost everywhere on $[c, d]$. By Theorem 27 the function G is differentiable almost everywhere on $[c, d]$ and it follows that G' is Bochner integrable on $[c, d]$. Since $G' = F'_{\text{ap}} = f$ almost everywhere on $E \cap [c, d]$ the function f is Bochner integrable on $E \cap (c, d)$.

We next consider the Denjoy–Dunford integral. The Lebesgue integrability of each x^*f is sufficient to guarantee the existence of the Dunford integral, that is, the linear functional $l_f: X^* \rightarrow \mathbb{R}$ defined by $l_f(x^*) = \int_l x^*f$ is continuous. We have been unable to prove this result (or find a counterexample) when each x^*f is Denjoy integrable. However, we have obtained a necessary and sufficient condition for a function to be Denjoy–Dunford integrable and the next few results lay the base for this condition.

LEMMA 29. *Let $F: [a, b] \rightarrow X^{**}$. If for each x^* in X^* the function Fx^* is continuous and BVG on $[a, b]$, then the family $\{Fx^*: x^* \in X^*\}$ is uniformly BVG on $[a, b]$.*

Proof. Let E be a perfect set in $[a, b]$ and let $\{I_n\}$ be the sequence of all open intervals in (a, b) that intersect E and have rational endpoints. For each pair of positive integers m and n let $A_m^n = \{x^* \in X^*: V(Fx^*, E \cap I_n) \leq m\}$ where $V(f, A)$ is the variation of f on A . By Theorem 4 we see that $X^* = \bigcup_m \bigcup_n A_m^n$. The next step is to show that each of the sets A_m^n is closed.

Let x^* be a limit point of A_m^n and let $\{x_k^*\}$ be a sequence in A_m^n that converges to x^* . Let $\{[c_i, d_i]\}$ be a finite collection of nonoverlapping intervals that have endpoints in $E \cap I_n$ and compute

$$\begin{aligned} \sum_i |F(d_i)x^* - F(c_i)x^*| &= \liminf_{k \rightarrow \infty} \left\{ \sum_i |F(d_i)x_k^* - F(c_i)x_k^*| \right\} \\ &\leq \liminf_{k \rightarrow \infty} \{V(Fx_k^*, E \cap I_n)\} \leq m. \end{aligned}$$

Hence, we have $V(Fx^*, E \cap I_n) \leq m$ and this shows that the set A_m^n is closed.

By the Baire Category Theorem there exist M, N, x_0^* , and $\varrho > 0$ such that $\{x^*: \|x^* - x_0^*\| \leq \varrho\} \subset A_M^N$. For each x^* in X^* with $\|x^*\| \neq 0$ we find that

$$\begin{aligned} V(Fx^*, E \cap I_N) &= \frac{\|x^*\|}{\varrho} V\left(\frac{\varrho}{\|x^*\|} Fx^* + Fx_0^* - Fx_0^*, E \cap I_N\right) \\ &\leq \frac{\|x^*\|}{\varrho} \left\{ V\left(F\left(\frac{\varrho}{\|x^*\|} x^* + x_0^*\right), E \cap I_N\right) \right. \\ &\quad \left. + V(Fx_0^*, E \cap I_N) \right\} \leq \frac{2M}{\varrho} \|x^*\|. \end{aligned}$$

Hence, the function Fx^* is BV on $E \cap I_N$ for each x^* in X^* and it follows that the family $\{Fx^*: x^* \in X^*\}$ is uniformly BVG on $[a, b]$.

LEMMA 30. *Let $F: [a, b] \rightarrow X^{**}$. If the function Fx^* is ACG on $[a, b]$ for each x^* in X^* , then the family $\{Fx^*: x^* \in X^*\}$ is uniformly ACG on $[a, b]$.*

Proof. Let E be a perfect set in $[a, b]$. By the previous lemma there exists an interval $[c, d]$ in $[a, b]$ such that $E \cap (c, d) \neq \emptyset$ and such that the function Fx^* is BV on $E \cap [c, d]$ for each x^* in X^* . By Theorem 6 the function Fx^* satisfies condition (N) on $E \cap [c, d]$ for each x^* in X^* and by Theorem 7 the function Fx^* is AC on $E \cap [c, d]$ for each x^* in X^* . Hence, the family $\{Fx^*: x^* \in X^*\}$ is uniformly ACG on $[a, b]$.

The next theorem is the analogue of Theorem 19. However, to guarantee the continuity of the linear functional we need to assume more than just the Denjoy integrability of each x^*f .

THEOREM 31. *Let $f: [a, b] \rightarrow X$ and suppose that x^*f is Denjoy integrable on $[a, b]$ for each $x^* \in X^*$. Then f is Denjoy–Dunford integrable on $[a, b]$ if and only if the family $\{x^*f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$.*

Proof. Suppose first that f is Denjoy–Dunford integrable on $[a, b]$ and let $F(t) = \int_a^t f$. Then $F(t)x^* = \int_a^t x^*f$ for each x^* in X^* and it follows that the function Fx^* is ACG on $[a, b]$ for each x^* in X^* . By Lemma 30 the family $\{Fx^*: x^* \in X^*\}$ is uniformly ACG on $[a, b]$ and therefore the family $\{x^*f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$ by Theorem 17.

Now suppose that the family $\{x^*f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. For each open interval I in (a, b) let l_I be the linear functional on X^* defined by $l_I(x^*) = \int_I x^*f$. Let $\mathcal{F} = \{I \subset (a, b): l_I \in X^{**}\}$ for every open interval $K \subset I$. We must show that the interval (a, b) belongs to \mathcal{F} . To this end we will verify that \mathcal{F} satisfies Romanovski's four conditions and apply Lemma 5.

Conditions (1) and (2) are trivially verified. Suppose that $(\alpha, \beta) \in \mathcal{F}$ for every interval $[\alpha, \beta] \subset (c, d)$. For those positive integers n for which $I_n = (c + 1/n, d - 1/n)$ is nonempty define $x_n^{**} = l_{I_n}$. For each x^* in X^* we have

$$l_{(c,d)}(x^*) = \int_c^d x^*f = \lim_{n \rightarrow \infty} \int_{c+1/n}^{d-1/n} x^*f = \lim_{n \rightarrow \infty} x_n^{**}(x^*).$$

By the Banach–Steinhaus Theorem the functional $l_{(c,d)}$ is an element of X^{**} . It follows easily that (c, d) belongs to \mathcal{F} . Thus, condition (3) is satisfied.

Let E be a perfect set in $[a, b]$ such that each interval contiguous to E in (a, b) belongs to \mathcal{F} . Since the family $\{x^*f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$ there exists an interval $[c, d]$ in $[a, b]$ with $c, d \in E$ and $E \cap (c, d) \neq \emptyset$ such that each x^*f is Lebesgue integrable on $E \cap [c, d]$ and for each x^* in X^* we have $\sum_n \left| \int_{c_n}^{d_n} x^*f \right| < \infty$ and (see Theorem 13)

$$\int_c^d x^*f = (L) \int_c^d x^*f \chi_E + \sum_n \int_{c_n}^{d_n} x^*f$$

where $[c, d] - E = \bigcup_n (c_n, d_n)$. It follows that f is Dunford integrable on $E \cap [c, d]$ and that the series $\sum_n x_n^{**}$, where $x_n^{**} = l_{(c_n, d_n)}$, is weak* Cauchy. Let x_E^{**} be the Dunford integral of f on $E \cap [c, d]$ and let x_0^{**} be the weak* limit of the series $\sum_n x_n^{**}$. Then for each x^* in X^* we have

$$l_{(c,d)}(x^*) = \int_c^d x^*f = (L) \int_c^d x^*f \chi_E + \sum_n x_n^{**}(x^*) = (x_E^{**} + x_0^{**})(x^*).$$

Hence, the functional $l_{(c,d)}$ is an element of X^{**} . Using a similar argument for the subintervals of (c, d) we find that (c, d) belongs to \mathcal{F} . Therefore, condition (4) is satisfied and the proof is complete.

COROLLARY 32. *If $f: [a, b] \rightarrow X$ is Denjoy–Dunford integrable on $[a, b]$, then each perfect set in $[a, b]$ contains a portion on which f is Dunford integrable. In particular, the function f is Dunford integrable on some subinterval of $[a, b]$.*

Proof. Let E be a perfect set in $[a, b]$. Since f is Denjoy–Dunford integrable on $[a, b]$ the family $\{x^*f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. Hence, there is a portion $E \cap I$ of E on which each x^*f is Lebesgue integrable and it follows that f is Dunford integrable on $E \cap I$.

The next theorem gives a result that is formally stronger than that of the above corollary.

THEOREM 33. *Let $f: [a, b] \rightarrow X$. If the function x^*f is Denjoy integrable on $[a, b]$ for every $x^* \in X^*$, then each perfect set in $[a, b]$ contains a portion on which f is Dunford integrable.*

Proof. Let E be a perfect set in $[a, b]$ and let $\{I_n\}$ be the sequence of all open intervals in (a, b) that intersect E and have rational endpoints. For each n let $E_n = E \cap I_n$. For each pair of positive integers m and n let $A_m^n = \{x^* \in X^*: \int_{E_n} |x^*f| \leq m\}$. By Theorem 12(c) we find that $X^* = \bigcup_n \bigcup_m A_m^n$. We claim that each of the sets A_m^n is closed.

Let x^* be a limit point of A_m^n and let $\{x_k^*\}$ be a sequence in A_m^n that converges to x^* . Then the sequence $\{|\int_{E_n} x_k^*f|\}$ converges pointwise on $[a, b]$ to the function $|\int_{E_n} x^*f|$ and by Fatou's Lemma we have

$$\int_{E_n} |x^*f| \leq \liminf_{k \rightarrow \infty} \left\{ \int_{E_n} |x_k^*f| \right\} \leq m.$$

This shows that $x^* \in A_m^n$ and we conclude that the set A_m^n is closed.

By the Baire Category Theorem there exist M, N, x_0^* , and $\varrho > 0$ such that $\{x^*: \|x^* - x_0^*\| \leq \varrho\} \subset A_M^N$. For each x^* in X^* with $\|x^*\| \neq 0$ we find that

$$\int_{E_N} |x^*f| \leq \frac{\|x^*\|}{\varrho} \left\{ \int_{E_N} \left| \frac{\varrho}{\|x^*\|} x^*f + x_0^*f \right| + \int_{E_N} |x_0^*f| \right\} \leq \frac{2M}{\varrho} \|x^*\|.$$

Hence, for each x^* in X^* the function x^*f is Lebesgue integrable on $E \cap I_N$. This shows that f is Dunford integrable on $E \cap I_N$.

Let $f: [a, b] \rightarrow X$ and suppose that each x^*f is Denjoy integrable on $[a, b]$. The proof of Theorem 31 shows that f is Denjoy–Dunford integrable on $[a, b]$ if for each perfect set E in $[a, b]$ there exists a portion $E \cap (c, d)$ such that for each x^* in X^* the function x^*f is Lebesgue integrable on

$E \cap (c, d)$ and the series $\sum_n \int_{c_n}^{d_n} |x^* f|$ converges where $(c, d) - E = \bigcup_n (c_n, d_n)$. The proof of Theorem 33 shows that there is a portion on which each $x^* f$ is Lebesgue integrable. However, this argument fails to provide a portion on which the series converges for every x^* in X^* because the corresponding sets A_n are not necessarily closed. Hence, this approach does not establish the existence of the Denjoy–Dunford integral.

The next two theorems provide conditions for the existence of the Denjoy–Dunford integral. The theorems give no new information since each condition implies that the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable. However, these conditions may be easier to verify in a particular case. We first need a definition.

DEFINITION 34. Let $f: [a, b] \rightarrow X$ and suppose that each $x^* f$ is Denjoy integrable on $[a, b]$. A point t in $[a, b]$ is called a *Denjoy point* of f if for each interval I in $[a, b]$ that contains t there exists a vector x_t^* in X^* such that the function $x_t^* f$ is not Lebesgue integrable on I .

It is clear that the set of Denjoy points of a function is closed. Furthermore, if the interval I contains no Denjoy points of f , then there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = (D) \int_I x^* f$ for all x^* in X^* . Here x_I^{**} is the Dunford integral of f on I .

THEOREM 35. Let $f: [a, b] \rightarrow X$ and suppose that $x^* f$ is Denjoy integrable on $[a, b]$ for each x^* in X^* . If the set of Denjoy points of f is countable, then f is Denjoy–Dunford integrable on $[a, b]$.

Proof. By Theorem 31 it is sufficient to prove that the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. Let H be the set of Denjoy points of f and let E be a perfect set in $[a, b]$. Since the set E is uncountable and since the set H is closed and countable there exists an interval $[c, d]$ in $[a, b]$ such that $[c, d] \cap H = \emptyset$ and $(c, d) \cap E \neq \emptyset$. Now each $x^* f$ is Lebesgue integrable on $[c, d]$ and, in particular, Lebesgue integrable on $E \cap [c, d]$. Let $(c, d) - E = \bigcup_n (c_n, d_n)$ and note that

$$\sum_n \int_{c_n}^{d_n} |x^* f| \leq \sum_n \int_{c_n}^{d_n} |x^* f| \leq \int_c^d |x^* f| < \infty$$

for all x^* in X^* . Hence, the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$.

THEOREM 36. Let $f: [a, b] \rightarrow X$ and suppose that $x^* f$ is Denjoy integrable on $[a, b]$ for each x^* in X^* . Let H be the set of Denjoy points of f , let $(a, b) - H = \bigcup_n (a_n, b_n)$, and let x_n^{**} be the Denjoy–Dunford integral of f on (a_n, b_n) . If f is Dunford integrable on H and if the series $\sum_{n=1}^{\infty} |x_n^{**}(x^*)|$ converges for each x^* in X^* , then f is Denjoy–Dunford integrable on $[a, b]$.

Proof. By Theorem 31 it is sufficient to prove that the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. Let E be a perfect set in $[a, b]$. If E is not a subset of H , then as in the previous proof there is a portion of E with the desired properties. Suppose that E is a subset of H and let $(a, b) - E = \bigcup_i (c_i, d_i)$. Since each $x^* f$ is Lebesgue integrable on $E \cap [a, b]$ we need only prove that $\sum_{i=1}^{\infty} \int_{c_i}^{d_i} |x^* f| < \infty$ for each x^* in X^* .

For each i let $\pi_i = \{n: (a_n, b_n) \subset (c_i, d_i)\}$ and note that the π_i 's form a partition of the positive integers since $E \subset H$. Using Theorem 13 we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \int_{c_i}^{d_i} |x^* f| &\leq \sum_{i=1}^{\infty} \left(\int_{c_i}^{d_i} |x^* f \chi_H| + \sum_{n \in \pi_i} \int_{a_n}^{b_n} |x^* f| \right) \\ &\leq \int_H |x^* f| + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} |x^* f| = \int_H |x^* f| + \sum_{n=1}^{\infty} |x_n^{**}(x^*)| < \infty \end{aligned}$$

for each x^* in X^* . This completes the proof.

The next theorem gives necessary and sufficient conditions for a function to be Denjoy–Pettis integrable.

THEOREM 37. A function $f: [a, b] \rightarrow X$ is Denjoy–Pettis integrable on $[a, b]$ if and only if there exists a function $F: [a, b] \rightarrow X$ such that the family $\{x^* F: x^* \in X^*\}$ is uniformly ACG on $[a, b]$ and $(x^* F)_{ap} = x^* f$ almost everywhere on $[a, b]$ for each x^* in X^* . (The exceptional set may vary with each x^* .)

Proof. Suppose first that f is Denjoy–Pettis integrable on $[a, b]$ and let $F(t) = \int_a^t f$. By Theorem 31 the family $\{x^* F: x^* \in X^*\}$ is uniformly ACG on $[a, b]$. Since $x^* F(t) = \int_a^t x^* f$ we see that $(x^* F)_{ap} = x^* f$ almost everywhere on $[a, b]$.

Now suppose that a function F with the given properties exists and fix $x^* \in X^*$. The function $x^* F$ is ACG on $[a, b]$ and $(x^* F)_{ap} = x^* f$ almost everywhere on $[a, b]$. Hence, the function $x^* f$ is Denjoy integrable on $[a, b]$ and for each interval $[c, d] \subset [a, b]$ we have $x^*(F(d) - F(c)) = \int_c^d x^* f$. Since this is valid for all x^* in X^* the function f is Denjoy–Pettis integrable on $[a, b]$.

It should be noted that the above theorem remains valid if we only require $x^* F$ to be ACG for each x^* in X^* .

We have shown that a Denjoy–Bochner (Denjoy–Dunford) integrable function is Bochner (Dunford) integrable on a portion of every perfect set. We have been unable to prove that a Denjoy–Pettis integrable function is Pettis integrable on a portion of every perfect set. However, we do have the following result.

THEOREM 38. Suppose that X contains no copy of c_0 and let $f: [a, b] \rightarrow X$. If f is Denjoy–Pettis integrable on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which f is Pettis integrable.

Proof. Let E be a perfect set in $[a, b]$. By Theorem 31 the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. Consequently, there exists an interval $[c, d]$ in $[a, b]$ with $c, d \in E$ and $E \cap (c, d) \neq \emptyset$ such that each $x^* f$ is Lebesgue integrable on $E \cap [c, d]$ and $\sum_n \int_{c_n}^{d_n} x^* f < \infty$ for each $x^* \in X^*$ where $(c, d) - E = \bigcup_n (c_n, d_n)$. We will show that f is Pettis integrable on $E \cap [c, d]$. Since the function f is Dunford integrable on $E \cap [c, d]$ it is sufficient to prove that the Dunford integral of $f \chi_E$ is X -valued for every interval in $[c, d]$ and apply Theorem 23.

Let t be a point in (c, d) and suppose that $t \notin E$. Choose an integer N such that $t \in (c_N, d_N)$. Since $\sum_{d_n \leq t} \int_{c_n}^{d_n} x^* f < \infty$ for each $x^* \in X^*$, the series $\sum_{d_n \leq t} \int_{c_n}^{d_n} f$ is unconditionally convergent by the Bessaga-Pelczyński Theorem. Let x_t be the sum of the series. Fix $x^* \in X^*$ and use Theorem 13 to compute

$$(D) \int_c^t x^* f = (L) \int_c^t x^* f \chi_E + \sum_{d_n \leq t} (D) \int_{c_n}^{d_n} x^* f + (D) \int_{c_N}^t x^* f.$$

We then have

$$(L) \int_c^t x^* f \chi_E = x^* \left(\int_c^t f \right) - x^*(x_t) - x^* \left(\int_{c_N}^t f \right).$$

Since this is valid for all $x^* \in X^*$ the Dunford integral of $f \chi_E$ on $[c, t]$ is X -valued. The case for $t \in E$ is similar and it follows easily that the Dunford integral of $f \chi_E$ is X -valued for every interval in $[c, d]$.

If $f: [a, b] \rightarrow X$ is Pettis integrable on $[a, b]$, then the family $\{x^* f: \|x^*\| \leq 1\}$ is uniformly integrable on $[a, b]$. Theorem 38 yields the following extension of this result. (See Definition 18.)

COROLLARY 39. *Suppose that X contains no copy of c_0 and let $f: [a, b] \rightarrow X$. If f is Denjoy-Pettis integrable on $[a, b]$, then the family $\{x^* f: \|x^*\| \leq 1\}$ is uniformly integrable in a generalized sense on $[a, b]$.*

Suppose that X contains no copy of c_0 . In Theorem 22 we have stated that every measurable, Dunford integrable function is Pettis integrable in such a space. Morrison [4] states and proves that every measurable, Denjoy-Dunford integrable function is Denjoy-Pettis integrable in such a space. However, he uses Solomon's [7] definitions of these integrals and it is difficult to relate Solomon's integrals to those defined here. Using the definitions of the Denjoy-Dunford and Denjoy-Pettis integrals developed in this paper we prove that the result is only valid in weakly sequentially complete spaces.

THEOREM 40. *Let X be weakly sequentially complete and let $f: [a, b] \rightarrow X$ be Denjoy-Dunford integrable on $[a, b]$. If f is measurable, then f is Denjoy-Pettis integrable on $[a, b]$.*

Proof. For each interval I in $[a, b]$ let x_I^{**} be the Denjoy-Dunford integral of f on I . Let \mathcal{F} be the collection of all open intervals I in (a, b) such that $x_K^{**} \in X$ for every open interval $K \subset I$. We must show that \mathcal{F} contains (a, b) and by Lemma 5 it is sufficient to verify that \mathcal{F} satisfies Romanovskii's four conditions.

Conditions (1) and (2) are easily verified. Suppose that (α, β) belongs to \mathcal{F} for every interval $[\alpha, \beta]$ in (c, d) . For each positive integer $n > 2/(d-c)$ define $I_n = (c+1/n, d-1/n)$ and let $x_n = x_{I_n}^{**}$. Then we have

$$x_{(c,d)}^{**}(x^*) = \int_c^d x^* f = \lim_{n \rightarrow \infty} \int_{I_n} x^* f = \lim_{n \rightarrow \infty} x^*(x_n)$$

for each $x^* \in X^*$. Since X is weakly sequentially complete the sequence $\{x_n\}$ converges weakly to an element x_0 of X and we must have $x_{(c,d)}^{**} = x_0$. It follows easily that (c, d) belongs to \mathcal{F} and this verifies condition (3).

Now let E be a perfect set in $[a, b]$ such that each of the intervals in (a, b) contiguous to E belongs to \mathcal{F} . Since f is Denjoy-Dunford integrable on $[a, b]$ the family $\{x^* f: x^* \in X^*\}$ is uniformly Denjoy integrable on $[a, b]$. Hence, there exists an interval $[u, v]$ with $u, v \in E$ and $E \cap (u, v) \neq \emptyset$ such that each $x^* f$ is Lebesgue integrable on $E \cap [u, v]$ and the series $\sum_i \int_{u_i}^{v_i} x^* f$ converges for each $x^* \in X^*$ where $(u, v) - E = \bigcup_i (u_i, v_i)$.

Let (c, d) be a subset of (u, v) , let $(c, d) - E = \bigcup_k (c_k, d_k)$, and for each k let $x_k = x_{(c_k, d_k)}^{**}$. Since $\sum_k |x^*(x_k)| < \infty$ for each $x^* \in X^*$ the series $\sum_k x_k$ is unconditionally convergent by the Bessaga-Pelczyński Theorem. Let $z = \sum_k x_k$. Since $f \chi_E$ is Dunford integrable on $[c, d]$ and since X contains no copy of c_0 , the function $f \chi_E$ is Pettis integrable on $[c, d]$ by Theorem 22. Let y be the Pettis integral of $f \chi_E$ on $[c, d]$. For each $x^* \in X^*$ we have

$$x_{(c,d)}^{**}(x^*) = (L) \int_c^d x^* f \chi_E + \sum_k (D) \int_{c_k}^{d_k} x^* f = x^*(y) + x^*(z)$$

and this shows that $(c, d) \in \mathcal{F}$. Therefore, the interval (u, v) belongs to \mathcal{F} and \mathcal{F} satisfies condition (4). This completes the proof.

EXAMPLE 41. Suppose that X is not weakly sequentially complete. There exists a series $\sum_n x_n$ in X such that the series $\sum_n x^*(x_n)$ converges for each $x^* \in X^*$ but the series $\sum_n x_n$ converges weak* to $x_0^{**} \in X^{**} - X$. For each positive integer n let $I_n = (1/(n+1), 1/n)$ and define $f: [0, 1] \rightarrow X$ by

$$f(t) = \sum_n n(n+1) x_n \chi_{I_n}(t).$$

The function f is clearly measurable. Fix $x^* \in X^*$ and note that $x^* f$ is Lebesgue integrable on the interval $[\alpha, 1]$ for each $\alpha \in (0, 1)$. For $\alpha \in (1/(N+1), 1/N]$ we find that

$$\int_\alpha^1 x^* f = \int_\alpha^{1/N} x^* f + \int_{1/N}^1 x^* f = N(N+1)(1/N - \alpha) x^*(x_N) + \sum_{n=1}^{N-1} x^*(x_n).$$

It is not difficult to see that

$$\lim_{\alpha \rightarrow 0^+} \int_{\alpha}^1 x^* f = \sum_n x^*(x_n).$$

Hence, the function $x^* f$ is Denjoy integrable on $[0, 1]$ and

$$\int_0^1 x^* f = \sum_n x^*(x_n) = x_0^{**}(x^*).$$

It follows easily that f is Denjoy–Dunford integrable on $[0, 1]$ but not Denjoy–Pettis integrable on $[0, 1]$.

We conclude this paper with three more examples. Example 43 is due to Alexiewicz [1].

EXAMPLE 42: A measurable, Pettis integrable function that is not Denjoy–Bochner integrable.

Let $\{r_k\}$ be a listing of the rational numbers in $[0, 1)$ and for each pair of positive integers n and k let

$$I_n^k = \left(r_k + \frac{1}{n+1}, r_k + \frac{1}{n} \right).$$

For each k define $f_k: [0, 1] \rightarrow l_2$ by

$$f_k(t) = \{(n+1)\chi_{I_n^k}(t)\}.$$

We claim that the series $\sum_k 4^{-k} f_k$ is l_2 -valued almost everywhere on $[0, 1]$.

For each positive integer j let $A_j = \cup_k \{t \in [0, 1]: |t - r_k| < 2^{-j-k}\}$ and let $A = \cap_j A_j$. Then $\mu(A) = 0$ and $\{r_k\} \subset A$. If $t \notin A$, then $t \notin A_{j_0}$ for some j_0 and it follows that $\|f_k(t)\| \leq 2^{j_0+k}$ for all k . Hence we have $\sum_k \|4^{-k} f_k(t)\| \leq 2^{j_0}$ and this shows that the series $\sum_k 4^{-k} f_k(t)$ converges in l_2 . This establishes the claim.

Define $g: [0, 1] \rightarrow l_2$ by $g(t) = \sum_k 4^{-k} f_k(t)$ for t in $[0, 1] - A$ and $g(t) = \theta$ for t in A . We first show that g is Pettis integrable on $[0, 1]$.

Since l_2 is reflexive we need only prove that the function $x^* g$ is Lebesgue integrable on $[0, 1]$ for each x^* in $l_2^* = l_2$. For each positive integer n define $g_n: [0, 1] \rightarrow \mathbf{R}$ by $g_n(t) = \sum_k 4^{-k} (n+1)\chi_{I_n^k}(t)$ for t in $[0, 1] - A$ and $g_n(t) = 0$ for t in A and note that $g(t) = \{g_n(t)\}$ for all t in $[0, 1]$. Since

$$\sum_k \int_0^1 4^{-k} (n+1)\chi_{I_n^k} \leq \sum_k 4^{-k} \cdot \frac{1}{n} = \frac{1}{3n},$$

the function g_n is Lebesgue integrable on $[0, 1]$ for each n by the Beppo Levi

Theorem. Let $x^* = \{\alpha_n\} \in l_2$. Then $x^* g = \sum_n \alpha_n g_n$ and we have

$$\sum_n \int_0^1 |\alpha_n g_n| \leq \sum_n |\alpha_n| \cdot \frac{1}{3n} < \infty.$$

Using the Beppo Levi Theorem once again we find that the function $x^* g$ is Lebesgue integrable on $[0, 1]$. This proves that g is Pettis integrable on $[0, 1]$.

Now we will show that g is not Denjoy–Bochner integrable on $[0, 1]$. By Theorem 28 it is sufficient to prove that f is not Bochner integrable on any subinterval of $[0, 1]$. Let $[a, b] \subset [0, 1]$ and choose a rational number r_k and a positive integer N such that $(r_k, r_k + 1/N) \subset (a, b)$. We then have

$$\int_a^b \|g\| \geq \int_{r_k}^{r_k + 1/N} \|g\| \geq \int_{r_k}^{r_k + 1/N} \|4^{-k} f_k\| = 4^{-k} \sum_{n=N}^{\infty} 1/n = \infty.$$

Hence, the function g is not Bochner integrable on $[a, b]$.

EXAMPLE 43: A Denjoy–Bochner integrable function that is Pettis integrable but not Bochner integrable.

Let X be an infinite-dimensional Banach space. By the Dvoretzky–Rogers Theorem there exists a series $\sum_n x_n$ in X that converges unconditionally but not absolutely. For each positive integer n let $I_n = (1/(n+1), 1/n)$ and define $f: [0, 1] \rightarrow X$ by $f(t) = (1/\mu(I_n))x_n$ for t in I_n and $f(t) = \theta$ for all other values of t . The function f is measurable since it is countably valued, but f is not Bochner integrable on $[0, 1]$ since

$$\int_0^1 \|f\| = \sum_n \int_{I_n} \frac{1}{\mu(I_n)} \|x_n\| = \sum_n \|x_n\| = \infty.$$

We will show that f is both Denjoy–Bochner and Pettis integrable on $[0, 1]$.

We first show that f is Denjoy–Bochner integrable on $[0, 1]$. Define $F: [0, 1] \rightarrow X$ by

$$F(t) = \frac{t - \frac{1}{n+1}}{\mu(I_n)} x_n + \sum_{k=n+1}^{\infty} x_k \quad \text{for } t \text{ in } \left(\frac{1}{n+1}, \frac{1}{n} \right]$$

and $F(0) = \theta$. Then F is continuous on $[0, 1]$ and $F' = f$ almost everywhere on $[0, 1]$. Furthermore, the function F is ACG on $[0, 1]$ since F is AC on $\{0\}$ and on each of the intervals $[1/(n+1), 1/n]$. Hence, the function f is Denjoy–Bochner integrable on $[0, 1]$.

Now we will show that f is Pettis integrable on $[0, 1]$. For each x^* in X^* we have

$$\int_0^1 |x^* f| = \sum_n \int_{I_n} \frac{1}{\mu(I_n)} |x^*(x_n)| = \sum_n |x^*(x_n)| < \infty$$

since the series $\sum_n x_n$ is unconditionally convergent. Hence, the function f is Dunford integrable on $[0, 1]$. Let E be a measurable set in $[0, 1]$ and let x_E be the sum of the unconditionally convergent series

$$\sum_n \frac{\mu(E \cap I_n)}{\mu(I_n)} x_n.$$

Then

$$\int_E x^* f = \sum_n \int_{E \cap I_n} x^* f = \sum_n \frac{\mu(E \cap I_n)}{\mu(I_n)} x^*(x_n) = x^*(x_E)$$

for each x^* in X^* . This shows that f is Pettis integrable on $[0, 1]$.

EXAMPLE 44: A Denjoy-Pettis integrable function that is Dunford integrable but not Pettis integrable.

For each positive integer n let

$$I'_n = \left(\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)} \right), \quad I''_n = \left(\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n} \right)$$

and define $f_n: [0, 1] \rightarrow \mathbf{R}$ by $f_n(t) = 2n(n+1)(\chi_{I'_n}(t) - \chi_{I''_n}(t))$. Then the sequence $\{f_n\}$ converges to 0 pointwise and it is not difficult to show that $\{\int_I f_n\}$ converges to 0 for each interval $I \subset [0, 1]$. Define $f: [0, 1] \rightarrow c_0$ by $f(t) = \{f_n(t)\}$.

Let $x^* = \{\alpha_n\} \in l_1$; then $x^* f = \sum_n \alpha_n f_n$. Since

$$\sum_n \int_0^1 |\alpha_n f_n| = \sum_n 2|\alpha_n| < \infty$$

the Beppo Levi Theorem applies to show that $x^* f$ is Lebesgue integrable on $[0, 1]$ and

$$\int_E x^* f = \sum_n \int_E \alpha_n f_n = \{\alpha_n\} \cdot \left\{ \int_E f_n \right\}.$$

Hence, the function f is Dunford integrable on $[0, 1]$ and $\int_E f = \{\int_E f_n\}$ for every measurable set $E \subset [0, 1]$. For each interval $I \subset [0, 1]$ we have $\int_I f \in c_0$ by the choice of $\{f_n\}$ and it follows that f is Denjoy-Pettis integrable on $[0, 1]$. But f is not Pettis integrable on $[0, 1]$ since for the set $E = \bigcup_n I'_n$ we have $\{\int_E f_n\} = \{1\} \in l_\infty - c_0$.

References

- [1] A. Alexiewicz, *On Denjoy integrals of abstract functions*, C. R. Soc. Sci. Lett. Cl. III Sci. Math. Phys. 41 (1948), 97-129.
- [2] J. Diestel and J. J. Uhl, *Vector Measures*, Amer. Math. Soc., Providence 1977.

[3] R. A. Gordon, *Differentiation in Banach spaces*, to appear.

[4] T. J. Morrison, *A note on the Denjoy integrability of abstractly-valued functions*, Proc. Amer. Math. Soc. 61 (1976), 385-386.

[5] P. Romanovski, *Essai d'une exposition de l'intégrale de Denjoy sans nombres transfinis*, Fund. Math. 19 (1932), 38-44.

[6] S. Saks, *Theory of the Integral*, 2nd revised ed., Hafner, New York 1937.

[7] D. W. Solomon, *Denjoy integration in abstract spaces*, Mem. Amer. Math. Soc. 85 (1969).

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