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THE DENSITY OF SOLENOIDAL FUNCTIONS AND  
THE CONVERGENCE OF A DUAL FINITE ELEMENT  
METHOD

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Some a priori error estimates have been given in [1] for the dual variational method, using piecewise linear equilibrium finite element model of Veubeke-Hogge ([2]) and assuming that the exact solution is sufficiently regular. It is the aim of the present paper to prove the convergence of the procedure without any regularity hypothesis. The crucial point of the proof is the following density theorem: the infinitely differentiable solenoidal vector-functions are dense in the space of admissible functions, which are solenoidal in the distribution sense.

We present the proof of the density theorems separately for the Dirichlet (Section 1), Neumann (Section 2) and the mixed boundary value problem (Section 3). Whereas arbitrary bounded plane domains with Lipschitz boundary are considered in the case of the Dirichlet problem, the remaining cases are restricted to bounded polygonal domains. Finally, the convergence proof of the dual finite element procedure is given in Section 4.

1. THE DIRICHLET PROBLEM

Let  $\Omega \subset R^2$  be a bounded domain with Lipschitz boundary. We shall denote by  $H^k(\Omega)$  the Sobolev spaces  $W^{k,2}(\Omega)$ , with the norm  $\|\cdot\|_{k,\Omega}$ . The same notation will be used for norms of vector-functions in  $[H^k(\Omega)]^2$ .  $H_0^1(\Omega)$  is the subspace of  $H^1(\Omega)$  of functions, the traces of which vanish on the boundary  $\partial\Omega$ . Define

$$Q_0(\Omega) = \left\{ \mathbf{q} \in [L_2(\Omega)]^2 \mid \int_{\Omega} \mathbf{q} \cdot \text{grad } v \, dx = 0 \quad \forall v \in H_0^1(\Omega) \right\},$$

(i.e.,  $\text{div } \mathbf{q} = 0$  in the sense of distributions).

**Theorem 1.** *The set*

$$Q_0(\Omega) \cap [C^\infty(\bar{\Omega})]^2$$

*is dense in  $Q_0(\Omega)$  (with respect to  $[L_2(\Omega)]^2$  - norm).*

Proof. Let  $\Omega^*$  be a bounded domain with Lipschitz boundary such that  $\Omega^* \supset \bar{\Omega}$ . In case that  $\Omega$  is a domain of connectivity  $m$ , we choose  $\Omega^*$  of the same connectivity. Then

$$\Omega^* \supset \Omega = \bigcup_{j=1}^m G_j,$$

where  $G_j$  are doubly-connected domains.

Let a  $\mathbf{q} \in Q_0(\Omega)$  be given. We construct an extension  $E\mathbf{q} \in Q_0(\Omega^*)$ ,  $E\mathbf{q}|_{\Omega} = \mathbf{q}$  as follows.

In every  $G_j$  let us consider the following auxiliary problem: to find

$$u \in V(G_j) = \{v \in H^1(G_j) \mid v = 0 \text{ on } \partial G_j \supset \partial \Omega\}$$

such that

$$(1.1) \quad \int_{G_j} \text{grad } u \cdot \text{grad } v \, dx = - \int_{\Omega} \mathbf{q} \cdot \text{grad } Pv \, dx \quad \forall v \in V(G_j),$$

where  $Pv$  is any extension of  $v \in V(G_j)$  such that

$$\begin{aligned} Pv \in V_j(\Omega) &= \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial \Omega \supset \partial G_j\}, \\ Pv &= v \text{ on } \partial \Omega \cap \partial G_j. \end{aligned}$$

The right-hand side of (1.1) is independent of the kind of extension from  $V(G_j)$  into  $V_j(\Omega)$ . In fact, since  $\tilde{P}v - Pv = 0$  on  $\partial \Omega$ ,  $\tilde{P}v - Pv \in H_0^1(\Omega)$  and

$$\int_{\Omega} \mathbf{q} \cdot \text{grad} (\tilde{P}v - Pv) \, dx = 0$$

follows from the definition of  $Q_0(\Omega)$ .

There exists a linear mapping  $P$  of  $H^{1/2}(\partial \Omega \cap \partial G_j)$  into  $V_j(\Omega)$  (see e.g. [3]) such that

$$\|Pv\|_{1,\Omega} \leq C \|v\|_{1/2,\partial \Omega \cap \partial G_j} \leq C_1 \|v\|_{1,G_j}.$$

Consequently, we have

$$\left| \int_{\Omega} \mathbf{q} \cdot \text{grad } Pv \, dx \right| \leq \|\mathbf{q}\|_{0,\Omega} \|Pv\|_{1,\Omega} \leq C_1 \|\mathbf{q}\|_{0,\Omega} \|v\|_{1,G_j},$$

and the right-hand side of (1.1) is a linear bounded functional on  $H^1(G_j)$ . Since the left-hand side is  $V(G_j)$ -elliptic, i.e.

$$\|w\|_{1,G_j}^2 = \int_{G_j} |\text{grad } w|^2 \, dx \geq C \|w\|_{1,G_j}^2 \quad \forall w \in V(G_j),$$

there exists a unique solution  $u$  of the problem (1.1).

Setting  $E\mathbf{q} = \text{grad } u$  in  $G_j$  for all  $j = 1, \dots, m$ ,  $E\mathbf{q} = \mathbf{q}$  in  $\Omega$ , we shall prove that  $E\mathbf{q} \in Q_0(\Omega^*)$ . In fact, let  $v \in H_0^1(\Omega^*)$ . Then

$$\int_{\Omega^*} E\mathbf{q} \cdot \text{grad } v \, dx = \int_{\Omega} \mathbf{q} \cdot \text{grad } v \, dx + \sum_{j=1}^m \int_{G_j} \text{grad } u \cdot \text{grad } v \, dx.$$

We have  $v|_{G_j} \in V(G_j) \forall j$  and

$$v \in H^1(\Omega) \Rightarrow v = \sum_{j=1}^m w_j, \quad w_j \in V_j(\Omega), \quad w_j = v \quad \text{on} \quad \partial G_j \cap \partial \Omega$$

(see e.g. [3] — chapt. 2, the proof of Th. 5.7).

Thus we may write

$$\int_{\Omega^*} E \mathbf{q} \cdot \text{grad } v \, dx = \sum_{j=1}^m \left[ \int_{\Omega} \mathbf{q} \cdot \text{grad } w_j \, dx + \int_{G_j} \text{grad } u \cdot \text{grad } v \, dx \right] = 0,$$

because  $w_j$  can be identified with an extension  $Pv$  of  $v$  and  $u$  is the solution of (1.1).

Let us regularize  $E \mathbf{q}$  by means of a kernel  $\omega_\kappa(x - y)$ , where  $A = \text{const}$ .

$$A \omega_\kappa(z) = \begin{cases} \exp(|z|^2/(|z|^2 - \kappa^2)) & \text{for } |z| < \kappa, \\ 0 & \text{for } |z| \geq \kappa, \end{cases}$$

$\kappa < \text{dist}(\partial \Omega^*, \partial \Omega)$ . We obtain  $q_i^\kappa \in C^\infty(\bar{\Omega})$ ,

$$q_i^\kappa(x) = \int_{\Omega^*} \omega_\kappa(x - y) E q_i(y) \, dy, \quad i = 1, 2,$$

$$\text{div } \mathbf{q}^\kappa(x) = \int_{\Omega^*} - \sum_{i=1}^2 \frac{\partial}{\partial y_i} \omega_\kappa(x - y) E q_i(y) \, dy = 0 \quad \forall x \in \Omega,$$

because  $\omega_\kappa \in C_0^\infty(\Omega^*)$  for  $\kappa < \text{dist}(\partial \Omega^*, \partial \Omega)$  and  $E \mathbf{q} \in Q_0(\Omega^*)$ . Consequently,  $\mathbf{q}^\kappa \in Q_0(\Omega)$ . Moreover, we have

$$\|\mathbf{q}^\kappa - E \mathbf{q}\|_{0,\Omega} = \|\mathbf{q}^\kappa - \mathbf{q}\|_{0,\Omega} \rightarrow 0 \quad \text{for } \kappa \rightarrow 0.$$

Q. E. D.

## 2. THE NEUMANN PROBLEM

From the dual variational formulation ([1], [2]) it follows that in case of the Neumann boundary value problem the admissible vector-functions have to be not only solenoidal but also their fluxes on the boundary have to vanish. Therefore we define

$$Q_0(\Omega) = \left\{ \mathbf{q} \in [L_2(\Omega)]^2 \mid \int_{\Omega} \mathbf{q} \cdot \text{grad } v \, dx = 0 \quad \forall v \in H^1(\Omega) \right\},$$

(i.e.,  $\text{div } \mathbf{q} = 0$  in the sense of distributions and  $\mathbf{q} \cdot \mathbf{n} = 0$  as a functional from  $H^{-1/2}(\partial \Omega)$ ).

**Theorem 2.** *Let  $\Omega \subset R^2$  be a bounded polygonal domain, (the boundary consists of a finite number of non-intersecting closed polygons). Then the set*

$$Q_0(\Omega) \cap [C^\infty(\bar{\Omega})]^2$$

*is dense in  $Q_0(\Omega)$ . \**

\*) The density is true even for any bounded domain with Lipschitz boundary (see [4] — Chap. I, § 2). We present the proof of Theorem 2., because its idea and some parts will be used in Section 3.

The proof is based on several lemmas.

**Lemma 2.1.** Let  $\Omega_h \subset \Omega$  be a polygonal domain inscribed into the domain  $\Omega'_h$  (see Fig. 1), where

$$\Omega'_h = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > h\}.$$

Denote  $G_h = \Omega \setminus \Omega_h$ .

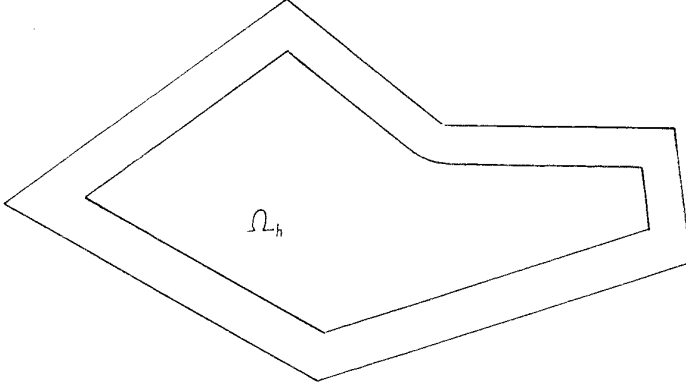


Fig. 1.

To every  $\mathbf{q} \in Q_0(\Omega)$  a function  $\mathbf{q}^h \in Q_0(\Omega)$  exists such that  $\mathbf{q}^h = 0$  in the boundary strip  $\Omega \setminus \Omega_{h/2}$  and

$$(2.1) \quad \|\mathbf{q} - \mathbf{q}^h\|_{0,\Omega} < C \|\mathbf{q}\|_{0,G_h}$$

where  $C$  is independent of  $h$ .

*Proof.* Let us define

$$(2.2) \quad \begin{aligned} \mathbf{q}^h &= \mathbf{q} \quad \text{in } \Omega_h, \\ \mathbf{q}^h &= 0 \quad \text{in } \Omega \setminus \Omega_{h/2}, \\ \mathbf{q}^h &= \text{grad } u \quad \text{in } \Omega_{h/2} \setminus \Omega_h = G_h^0. \end{aligned}$$

where  $u \in H^1(G_h^0)$  is a solution of the following problem:

$$(2.3) \quad \int_{G_h^0} \text{grad } u \cdot \text{grad } v \, dx = \int_{G_h} \mathbf{q} \cdot \text{grad } Pv \, dx \quad \forall v \in H^1(G_h^0).$$

and  $Pv \in H^1(G_h)$  is an arbitrary extension of  $v \in H^1(G_h^0)$ .

The right-hand side of (2.3) is a functional  $f(v)$ , independent of the kind of the extension  $P$ . In fact,  $\tilde{P}v - Pv = 0$  on  $\partial\Omega_h$  for any two extensions  $\tilde{P}$  and  $P$ . If we prolong  $\tilde{P}v - Pv$  by zero onto  $\Omega_h$ ,

$$\int_{G_h} \mathbf{q} \cdot \text{grad}(\tilde{P}v - Pv) \, dx = \int_{\Omega} \mathbf{q} \cdot \text{grad}(\tilde{P}v - Pv) \, dx = 0$$

by virtue of the definition of  $Q_0(\Omega)$ .

**Lemma 2.2** For sufficiently small  $h$  there exists an extension  $P : H^1(G_h^0) \rightarrow H^1(G_h)$  such that

$$(2.4) \quad |Pu|_{1,G_h} \leq C|u|_{1,G_h^0},$$

with  $C$  independent of  $h$ .

*Proof.* We construct the extension  $P$  as follows. We divide the boundary strips  $G_h$  into “straight segments” by means of the axes of the angles at the vertices of  $\partial\Omega$ . Let  $V_r^h$  be a straight segment,  $V_r^{h0} = V_r^h \cap G_h^0$ . Denote

$$K_r^h = \left\{ y \in \mathbb{R}^2 \mid a_r < y_1 < b_r, -\frac{h}{2} < y_2 < \frac{h}{2} \right\}$$

a rectangle in the  $(y_1, y_2)$ -plane.

Let us put the origin of a local cartesian system  $(x_1^r, x_2^r)$  into the point of intersection of the axes which limit the segment  $V_r^h$  and let the  $x_2^r$ -axis be orthogonal to the segment. Then (omitting the superscript  $r$  by  $x^r$ ):

$$V_r^h = \left\{ x \in \mathbb{R}^2 \mid d_r - \frac{h}{2} < x_2 < d_r + \frac{h}{2}, a_r < \frac{x_1}{x_2} < b_r \right\}$$

and the mapping

$$x = T_r^h y \equiv \begin{cases} x_1 = y_1(d_r + y_2), \\ x_2 = y_2 + d_r, \end{cases}$$

maps the rectangle  $K_r^h$  onto  $V_r^h$ , being one-to-one.

In case of parallel limiting axes, we put the origin into the center of  $V_r^h$ , consequently

$$V_r^h = \left\{ x \in \mathbb{R}^2 \mid -\frac{h}{2} < x_2 < \frac{h}{2}, -a_r < x_1 - k_r x_2 < a_r \right\},$$

$$x^r = T_r^h y \equiv \begin{cases} x_1 = y_1 + k_r y_2, \\ x_2 = y_2. \end{cases}$$

Let a  $u \in H^1(G_h^0)$  be given. Then  $u|_{V_r^{h0}} \in H^1(V_r^{h0})$ . Defining for  $u(x)$  the function

$$\hat{u}(y) = u(T_r^h y),$$

then  $\hat{u} \in H^1(K_r^{h0})$ , where  $K_r^{h0}$  denotes the corresponding lower (upper) half of the rectangle  $K_r^h$ . In fact,  $T_r^h$  and  $(T_r^h)^{-1}$  are Lipschitz mappings on  $\bar{K}_r^h$  and  $\bar{V}_r^h$ , respectively.

Let us extend  $\hat{u}$  onto the whole rectangle  $K_r^h$  symmetrically, defining for the extension  $\hat{P}_r \hat{u}$

$$\hat{P}_r \hat{u}(y_1, -y_2) = \hat{P}_r \hat{u}(y_1, y_2).$$

Then obviously  $\hat{P}_r \hat{u} \in H^1(K_r^h)$ .

Defining the extension of  $u$  from  $V_r^{h0}$  onto  $V_r^h$  as

$$P_r u(x) = \hat{P}_r \hat{u}((T_r^h)^{-1} x),$$

we obtain  $P_r u \in H^1(V_r^h)$ .

Finally, let  $Pu$  be the function such that

$$Pu|_{V_r^h} = P_r u \quad \forall r = 1, \dots, m.$$

To prove that  $Pu \in H^1(G_h)$ , it suffices to verify that the traces of  $P_r u$  and  $P_{r+1} u$  coincide on the intersegment boundary  $\bar{V}_r^h \cap \bar{V}_{r+1}^h \equiv I_r^h$  for all  $r$ . Since it holds

$$\hat{P}_r \hat{u}(b_r, y_2) = \hat{P}_r \hat{u}(b_r, -y_2) \quad \forall y_2 \in \left(-\frac{h}{2}, \frac{h}{2}\right),$$

the traces of  $P_r u$  are symmetric with respect to the straight line  $x_2^r = d_r$ . The same is true for  $P_{r+1} u$  and the traces coincided on  $G_h^0 \cap I_r^h$ . Consequently, they coincide on the whole  $I_r^h$ .

For sufficiently small  $h$  it holds

$$(2.5) \quad |P_r u|_{1, V_r^h}^2 \leq C |u|_{1, V_r^{h_0}}^2,$$

with  $C$  independent of  $h, r$ . In fact, constants  $h_0 > 0, M > 0$  exist such that for  $h < h_0$

$$(2.6) \quad \max_r \max_{x \in V_r^h} (|x_2^r|, |x_1^r/x_2^r|, |1/x_2^r|) < M.$$

Consequently, the entries of Jacobians of  $T_r^h, (T_r^h)^{-1}$  are uniformly bounded for all  $r$  and  $h < h_0$ . Moreover, we have

$$(2.7) \quad |\hat{P}_r \hat{u}|_{1, K_r^h}^2 = 2 |\hat{u}|_{1, K_r^{h_0}}^2 \quad \forall h < h_0, \forall r.$$

Then (2.5) follows from (2.7) and (2.6).

Altogether we may write

$$|Pu|_{1, G_h}^2 = \sum_{r=1}^m |P_r u|_{1, V_r^h}^2 \leq C \sum_{r=1}^m |u|_{1, V_r^{h_0}}^2 = C |u|_{1, G_h^0}^2.$$

and the proof of (2.4) is complete.

Next we are able to finish the *proof of Lemma 2.1*. The functional  $f(v)$  in the right-hand of (2.3) is linear and continuous on  $H^1(G_h^0)$ . In fact, using (2.4), we may write

$$(2.8) \quad |f(v)| \leq \|q\|_{0, G_h} \cdot |Pv|_{1, G_h} \leq C \|q\|_{0, G_h} \cdot |v|_{1, G_h^0}.$$

The bilinear form in the left-hand side of (2.3) generates a scalar product in the Hilbert space  $\tilde{H} = H^1(G_h^0)/P_0$ , where  $P_0$  is the subspace of constant functions. Since  $f(1) = 0$ ,  $f$  is linear and continuous over  $\tilde{H}$ . Consequently, there exists a unique class of equivalence  $\tilde{u} \in \tilde{H}$  such that

$$\int_{G_h^0} \text{grad } \tilde{u} \cdot \text{grad } \tilde{v} \, dx = f(\tilde{v}) \quad \forall \tilde{v} \in \tilde{H}.$$

Let  $u \in \tilde{u}$  and choose  $v = u$  in (2.3). Thus we obtain

$$|u|_{1, G_h^0}^2 \leq C \|q\|_{0, G_h} \cdot |u|_{1, G_h^0}$$

using also (2.8). Setting  $q^h = \text{grad } u$ , it holds

$$(2.9) \quad |u|_{1, G_h^0} = \|q^h\|_{0, G_h^0} \leq C \|q\|_{0, G_h}.$$

By virtue of (2.2) and (2.9) we may write

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}^h\|_{0,\Omega}^2 &= \|\mathbf{q} - \mathbf{q}^h\|_{0,G_{h^0}}^2 + \|\mathbf{q}\|_{0,G_h - G_{h^0}}^2 \leq \\ &\leq 2\|\mathbf{q}\|_{0,G_{h^0}}^2 + 2C^2\|\mathbf{q}\|_{0,G_h}^2 + \|\mathbf{q}\|_{0,G_h - G_{h^0}}^2 \leq (2 + 2C^2)\|\mathbf{q}\|_{0,G_h}^2. \end{aligned}$$

Hence (2.1) follows.

It remains to verify that  $\mathbf{q}^h \in Q_0(\Omega)$ . For any  $v \in H^1(\Omega)$  we have

$$(2.10) \quad \int_{\Omega} \mathbf{q}^h \cdot \text{grad } v \, dx = \int_{G_{h^0}} \text{grad } u \cdot \text{grad } v \, dx + \int_{\Omega_h} \mathbf{q} \cdot \text{grad } v \, dx,$$

$$(2.11) \quad \mathbf{q} \in Q_0(\Omega) \Rightarrow \int_{\Omega_h} \mathbf{q} \cdot \text{grad } v \, dx + \int_{G_h} \mathbf{q} \cdot \text{grad } v \, dx = 0.$$

Inserting (2.11) into (2.10), realizing that  $v|_{G_h} = P(v|_{G_{h^0}})$  and utilizing (2.3), we are led to the conclusion that the integral in the left-hand side of (2.10) vanishes.

Q. E. D.

**Proof of Theorem 2.** Let  $\mathbf{q} \in Q_0(\Omega)$ . Consider the function  $\mathbf{q}^h$  from Lemma 1, extend it by zero outside  $\Omega$  and regularize. Thus we obtain  $R_x q_i^h \in C^\infty(\bar{\Omega})$ ,

$$R_x q_i^h(x) = \int_{\Omega} \omega_x(x-y) q_i^h(y) \, dy, \quad i = 1, 2.$$

Obviously,  $R_x q_i^h = 0$  on  $\partial\Omega$  for  $x < h/2$ . Moreover,

$$\text{div } R_x q_i^h(x) = - \int_{\Omega} \sum_{i=1}^2 \frac{\partial}{\partial y_i} \omega_x(x-y) q_i^h(y) \, dy = 0 \quad \forall x \in \Omega,$$

because the function  $\omega_x \in C^\infty(\bar{\Omega}) \subset H^1(\Omega)$ . Finally, using (2.1), we obtain

$$\|R_x \mathbf{q}^h - \mathbf{q}\|_{0,\Omega} \leq \|\mathbf{q} - \mathbf{q}^h\|_{0,\Omega} + \|\mathbf{q}^h - R_x \mathbf{q}^h\|_{0,\Omega} \rightarrow 0$$

for  $h \rightarrow 0, x < \frac{1}{2}h$ .

### 3. THE MIXED BOUNDARY VALUE PROBLEM

Let us consider a bounded polygonal domain  $\Omega \subset R^2$ , the boundary  $\Gamma$  of which consists of two parts,

$$\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_g, \quad \Gamma_u \cap \Gamma_g = \emptyset.$$

Let the boundary conditions for  $u$  be

$$\begin{aligned} u &= \bar{u} \quad \text{on } \Gamma_u, \\ \frac{\partial u}{\partial n_A} &= g \quad \text{on } \Gamma_g, \end{aligned}$$



where  $\partial u / \partial n_A$  denotes the derivative with respect to the conormal. It is well-known (see [1], [2]), that the admissible functions of the dual variational formulation have to satisfy

$$\begin{aligned} \operatorname{div} \mathbf{q} &= 0 \quad \text{in } \Omega, \\ \mathbf{q} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_g, \end{aligned}$$

in the following weak sense:

$$\mathbf{q} \in Q_0(\Omega) = \left\{ \mathbf{q} \in [L_2(\Omega)]^2 \mid \int_{\Omega} \mathbf{q} \cdot \operatorname{grad} v \, dx = 0 \quad \forall v \in V \right\},$$

where

$$V = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on } \Gamma_u\}.$$

**Theorem 3.** Let  $\Gamma_g$  consist of a finite number of connected parts of the polygonal boundary  $\Gamma$ .

Then the set

$$Q_0(\Omega) \cap [C^\infty(\bar{\Omega})]^2$$

is dense in  $Q_0(\Omega)$ .

The main idea of the proof is similar to that of Theorem 2. First we establish an auxiliary

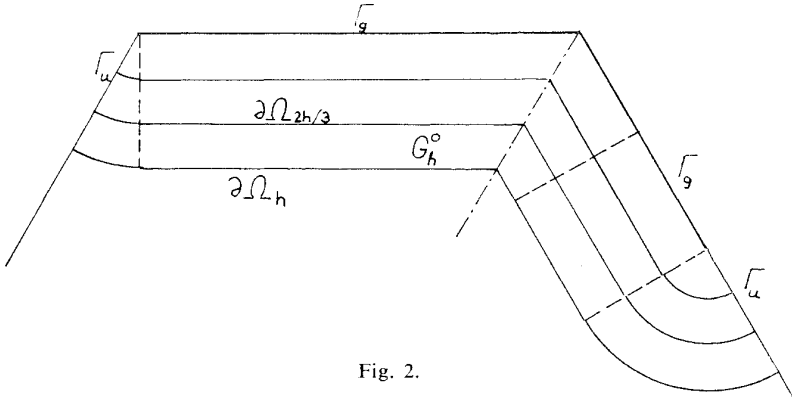


Fig. 2.

**Lemma 3.1.** Let us define (see Fig. 2)

$$\Omega_h = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma_g) > h\},$$

(replacing the circular arcs at concave angles of  $\Gamma_g$  by tangents, see Fig. 1)

$$G_h = \Omega \setminus \Omega_h, \quad G_h^0 = G_h \cap \Omega_{2h/3}.$$

Then to every  $\mathbf{q} \in Q_0(\Omega)$  a function  $\mathbf{q}^h \in Q_0(\Omega)$  exists such that  $\mathbf{q}^h = 0$  in the boundary strip  $\Omega \setminus \Omega_{2h/3}$  and

$$(3.1) \quad \|\mathbf{q} - \mathbf{q}^h\|_{0,\Omega} < C \|\mathbf{q}\|_{0,G_h}$$

with  $C$  independent of  $h$ .

Proof. Let us define

$$(3.2) \quad \begin{aligned} \mathbf{q}^h &= \mathbf{q} && \text{in } \Omega_h, \\ \mathbf{q}^h &= 0 && \text{in } \Omega \setminus \Omega_{2h/3}, \\ \mathbf{q}^h &= \text{grad } u && \text{in } G_h^0, \end{aligned}$$

where

$$u \in V(G_h^0) = \{v \in H^1(G_h^0) \mid v = 0 \text{ on } \partial G_h^0 \cap \Gamma_u\},$$

$$(3.3) \quad \int_{G_h^0} \text{grad } u \cdot \text{grad } v \, dx = \int_{G_h} \mathbf{q} \cdot \text{grad } Pv \, dx \quad \forall v \in V(G_h^0),$$

and

$$Pv \in V(G_h) = \{v \in H^1(G_h) \mid v = 0 \text{ on } \Gamma_u \cap \partial G_h\}$$

is an arbitrary extension of  $v \in V(G_h^0)$ .

The right-hand side of (3.3) is a functional  $f(v)$ , independent of the kind of the extension  $P$ . In fact,  $\tilde{P}v - Pv = 0$  on  $\partial\Omega_h$  for any extensions  $\tilde{P}, P$ . If we prolong  $\tilde{P}v - Pv$  by zero into  $\Omega_h$ , we obtain a function, belonging to  $V$ . Consequently,

$$\int_{G_h} \mathbf{q} \cdot \text{grad} (\tilde{P}v - Pv) \, dx = \int_{\Omega} \mathbf{q} \cdot \text{grad} (\tilde{P}v - Pv) \, dx = 0$$

holds by virtue of the definition of  $Q_0(\Omega)$ .

**Lemma 3.2.** *For sufficiently small  $h$  there exists an extension  $P : V(G_h^0) \rightarrow V(G_h)$  such that*

$$(3.4) \quad |Pv|_{1, G_h} \leq C|v|_{1, G_h^0}$$

with  $C$  independent of  $h$ .

Proof. Throughout the proof  $C$  will denote generic constants, independent of  $h$ . We construct the extension  $P$  as follows. We divide the boundary strips  $G_h$  into “straight segments” (cf. the proof of Lemma 2.2) and sectors of circles, to which adjacent squares are joined (see Fig. 2).

1° Let  $V_r^h$  be an arbitrary “straight segment”,  $V_r^{h0} = V_r^h \cap G_h^0$ .

Let

$$K_r^h = \left\{ y \in \mathbb{R}^2 \mid a_r < y_1 < b_r, -\frac{h}{3} < y_2 < \frac{2}{3}h \right\}$$

be a rectangle in the  $(y_1, y_2)$ -plane. We use the local Cartesian coordinate system  $(x_1^r, x_2^r)$  and a mapping similar to that of Section 2.

Let us consider a convex part of  $\Gamma_g$ , i.e.,  $x_2^r$ -axis is oriented as an outward normal. We extend the function  $v \in H^1(V_r^{h0})$  into  $Pv \in H^1(V_r^h)$  by means of successive symmetric extensions in  $K_r^h$ :

- (i) with respect to the straight line  $y_2 = 0$ ,
- (ii) with respect to  $y_2 = \frac{1}{3}h$ .

(In case that the  $x_2$ -axis is oriented inward, we also extend symmetrically in  $K_r^h$  but applying the inverse order.)

Thus we obtain a function  $Pv \in H^1(V_r^h)$ . By an argument parallel to that of Section 2, we deduce the estimate

$$(3.5) \quad |Pv|_{1, V_r^h}^2 \leq C|v|_{1, V_r^{h_0}}^2 \quad \forall r, \forall h < h_0.$$

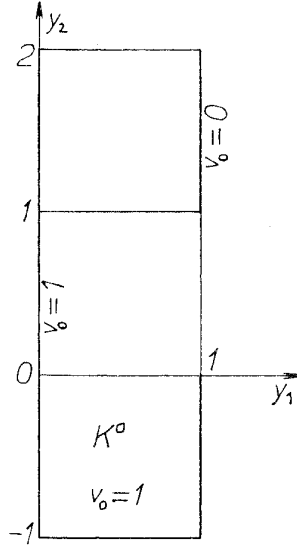


Fig. 3.

2° Let  $V_s$  be a square (with the side  $h$ ), adjacent to the point  $\bar{\Gamma}_g \cap \bar{\Gamma}_u$ . Define the rectangle (see Fig. 3)

$$K = \{y \in R^2 \mid 0 < y_1 < 1, -1 < y_2 < 2\}.$$

The mapping

$$x = Ty \equiv \begin{cases} x_1 = hy_1 \\ x_2 = \frac{1}{3}hy_2 \end{cases}$$

maps the rectangle  $K$  onto the square

$$V_s = \left\{ x^r \in R^2 \mid 0 < x_1^r < h, -\frac{h}{3} < x_2^r < \frac{2}{3}h \right\}.$$

Let  $V_s$  be adjacent to a "convex part" of  $\Gamma_g$ .

Consider a function  $v_0 \in C^{(1)}(\bar{K})$  such that  $0 \leq v_0 \leq 1$ ,  $v_0 = 1$  on the segment  $\{y_1 = 0, 0 \leq y_2 \leq 2\}$  and on the rectangle  $\{0 \leq y_1 \leq 1, -1 \leq y_2 \leq 0\}$ ,  $v_0 = \varphi$  on the segment  $\{y_1 = 1, 0 \leq y_2 \leq 2\}$ , where  $\varphi(y_2) = 0$  for  $1 \leq y_2 \leq 2$ .

Let  $v \in H^1(V_s^0)$  be given, where  $V_s^0 = V_s \cap G_h^0$ . Defining

$$\hat{v}(y) = v(T(y)),$$

$\hat{v} \in H^1(K^0)$ , where  $K^0 = \{y \mid 0 < y_1 < 1, -1 < y_2 < 0\}$ .

Let us extend  $\hat{v}$  into  $K$  in the same way as in the previous part 1°, denoting the extension by  $\hat{P}\hat{v} \in H^1(K)$ . Then obviously  $\|\hat{P}\hat{v}\|_{1,K}^2 = 3\|\hat{v}\|_{1,K^0}^2$ . If we define

$$\hat{w} = v_0 \hat{P}\hat{v},$$

then  $\hat{w} \in H^1(K)$ ,  $\hat{w} = \hat{P}\hat{v} = \hat{v}$  on  $K^0$ ,

$$\begin{aligned} \hat{w} &= \hat{P}\hat{v} \quad \text{on } y_1 = 0, \\ \hat{w} &= \varphi(y_2) \hat{P}\hat{v} \quad \text{on } y_1 = 1. \end{aligned}$$

Defining

$$Pv(x) = \hat{w}(T^{-1}x),$$

then  $Pv \in H^1(V_s)$  and we may write

$$\begin{aligned} (3.6) \quad |Pv|_{1,V_s}^2 &= \int_{V_s} \sum_{i=1}^2 \left| \frac{\partial Pv}{\partial x_i} \right|^2 dx = \int_{V_s} \sum_{i=1}^2 \left| \sum_{j=1}^2 \frac{\partial \hat{w}}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right|^2 dx = \\ &= \int_K \left( \frac{1}{h^2} \left| \frac{\partial \hat{w}}{\partial y_1} \right|^2 + \frac{9}{h^2} \left| \frac{\partial \hat{w}}{\partial y_2} \right|^2 \right) \frac{h^2}{3} dy \leq 3|\hat{w}|_{1,K}^2, \\ \left| \frac{\partial \hat{w}}{\partial y_i} \right|^2 &= \left| \frac{\partial v_0}{\partial y_i} \hat{P}\hat{v} + v_0 \frac{\partial \hat{P}\hat{v}}{\partial y_i} \right|^2 \leq 2\|v_0\|_{C^1(K)}^2 \left( |\hat{P}\hat{v}|^2 + \left| \frac{\partial \hat{P}\hat{v}}{\partial y_i} \right|^2 \right). \end{aligned}$$

Since  $\|v_0\|_{C^1(\bar{K})}$  is a fixed constant, we have

$$|\hat{w}|_{1,K}^2 \leq C_0 \|P\hat{v}\|_{1,K}^2 = 3C_0 \|\hat{v}\|_{1,K^0}^2.$$

From (3.6) we obtain

$$(3.7) \quad |Pv|_{1,V_s}^2 \leq 9C_0 \|\hat{v}\|_{1,K^0}^2.$$

3° Let  $V_0$  be a sector of a circle, adjacent to the point  $\bar{\Gamma}_u \cap \bar{\Gamma}_g$ . The mapping

$$x = T(\vartheta, \varrho) = \begin{cases} x_1 = \frac{1}{3}h\varrho \cos \vartheta, \\ x_2 = \frac{1}{3}h\varrho \sin \vartheta \end{cases}$$

maps the rectangle (in  $(\vartheta, \varrho)$ -plane)

$$K = \{(\vartheta, \varrho) \mid \vartheta_0 < \vartheta < \vartheta_1, 1 < \varrho < 3\}$$

onto the sector of the annulus  $V_c^1 \subset V_c$ ,

$$V_c^1 = \{x \equiv (\vartheta, r) \mid \vartheta_0 < \vartheta < \vartheta_1, \frac{1}{3}h < r < h\}.$$

Denote

$$V_c^0 = \{x \equiv (\vartheta, r) \mid \vartheta_0 < \vartheta < \vartheta_1, \frac{2}{3}h < r < h\} = V_c \cap G_h^0.$$

Let a function  $v \in H^1(V_c^0)$ ,  $v = 0$  for  $\vartheta = \vartheta_1$  be given. (Suppose that the straight line segment  $\vartheta = \vartheta_1$  belongs to  $\Gamma_u$ ). Defining

$$\hat{v}(\vartheta, \varrho) = v(T(\vartheta, \varrho)),$$

then  $\hat{v} \in H^1(K_c^0)$ , where

$$K_c^0 = \{(\vartheta, \varrho) \mid \vartheta_0 < \vartheta < \vartheta_1, 2 < \varrho < 3\},$$

Let us extend  $\hat{v}$  symmetrically across the line  $\varrho = 2$ , i.e., let

$$(3.8) \quad \hat{P}\hat{v}(\vartheta, \varrho) = \hat{P}\hat{v}(\vartheta, 4 - \varrho) \quad \forall \vartheta, \forall 1 < \varrho < 3.$$

Moreover, we define

$$(3.9) \quad \hat{w}(\vartheta, \varrho) = \varphi(2 - \varrho) \hat{P}\hat{v}(\vartheta, \varrho),$$

where  $\varphi$  is the function used in the previous part 2°, and extend  $\hat{w}$  by zero for  $\vartheta_0 < \vartheta < \vartheta_1, 0 < \varrho < 1$ . Obviously, we have  $\hat{w} \in H^1(K)$ ,  $\hat{w} = \hat{v}$  on  $K^0$ .

Defining

$$Pv = \begin{cases} 0 & \text{for } V_c \setminus V_c^1, \\ \hat{w}(T^{-1}x) & \text{for } x \in V_c^1, \end{cases}$$

we may write

$$(3.10) \quad \begin{aligned} |Pv|_{1, V_c}^2 &= \int_{V_c^1} |\text{grad } Pv|^2 dx = \\ &= \frac{9}{h^2} \int_1^3 d\varrho \int_{\vartheta_0}^{\vartheta_1} \left( \left| \frac{\partial \hat{w}}{\partial \varrho} \right|^2 + \frac{1}{\varrho^2} \left| \frac{\partial \hat{w}}{\partial \vartheta} \right|^2 \right) \frac{h^2}{9} \varrho d\vartheta \leq \\ &\leq 3 \int_1^3 d\varrho \int_{\vartheta_0}^{\vartheta_1} \left( \left| \frac{\partial \hat{w}}{\partial \varrho} \right|^2 + \left| \frac{\partial \hat{w}}{\partial \vartheta} \right|^2 \right) d\vartheta. \end{aligned}$$

Using (3.9), we deduce easily

$$\begin{aligned} \left| \frac{\partial \hat{w}}{\partial \varrho} \right|^2 &\leq (|\varphi'|^2 + |\varphi|^2) |\hat{P}\hat{v}|^2 + \left| \frac{\partial \hat{P}\hat{v}}{\partial \varrho} \right|^2, \\ \left| \frac{\partial \hat{w}}{\partial \vartheta} \right|^2 &\leq \left| \frac{\partial \hat{P}\hat{v}}{\partial \vartheta} \right|^2 \end{aligned}$$

and from (3.10) we obtain

$$(3.11) \quad |Pv|_{1, V_c}^2 \leq C_\varphi \int_1^3 d\varrho \int_{\vartheta_0}^{\vartheta_1} \left( |\hat{P}\hat{v}|^2 + \left| \frac{\partial \hat{P}\hat{v}}{\partial \varrho} \right|^2 + \left| \frac{\partial \hat{P}\hat{v}}{\partial \vartheta} \right|^2 \right) d\vartheta$$

where  $C_\varphi$  depends on the function  $\varphi$  only.

By virtue of the symmetric extension (3.8), the last integral equals to the double of the integral

$$I \equiv \int_2^3 d\varrho \int_{\vartheta_0}^{\vartheta_1} \left( |\hat{v}|^2 + \left| \frac{\partial \hat{v}}{\partial \varrho} \right|^2 + \left| \frac{\partial \hat{v}}{\partial \vartheta} \right|^2 \right) d\vartheta.$$

By an analogous argument as in (3.10) we deduce

$$\|\hat{\delta}\|_{1,K_c^0}^2 = \int_2^3 d\varrho \int_{\vartheta_0}^{\vartheta_1} \left( |\hat{\delta}|^2 + \left| \frac{\partial \hat{\delta}}{\partial \varrho} \right|^2 + \frac{1}{\varrho^2} \left| \frac{\partial \hat{\delta}}{\partial \vartheta} \right|^2 \right) \varrho \, d\vartheta \geq \frac{2}{9} I.$$

Inserting this into (3.11), we obtain

$$(3.12) \quad |Pv|_{1,V_c}^2 \leq 9C_\varphi \|\hat{\delta}\|_{1,K_c^0}^2.$$

4° Let us consider the square  $V_s$  together with the sector  $V_c$  of a circle. In the points 2° and 3° we have deduced the estimates (3.7) and (3.12), from which

$$(3.13) \quad |Pv|_{1,V_s}^2 + |Pv|_{1,V_c}^2 \leq C(\|\hat{\delta}\|_{1,K^0}^2 + \|\hat{\delta}\|_{1,K_c^0}^2)$$

follows. Since  $\hat{\delta}|_{K^0}$  and  $\hat{\delta}|_{K_c^0}$  have the same traces on the intersection  $\partial K^0 \cap \partial K_c^0$ ,  $\hat{\delta} \in H^1(K^0 \cup K_c^0)$  and

$$(3.14) \quad \|\hat{\delta}\|_{1,K^0}^2 + \|\hat{\delta}\|_{1,K_c^0}^2 = \|\hat{\delta}\|_{1,K^0 \cup K_c^0}^2 \leq C_1 |\hat{\delta}|_{1,K^0 \cup K_c^0}^2$$

holds by virtue of the generalized Friedrichs inequality (note that  $\hat{\delta} = 0$  for  $\vartheta = \vartheta_1$ ).

It is easy to deduce

$$(3.15) \quad \begin{aligned} |\hat{\delta}|_{1,K^0}^2 &\leq 3|v|_{1,V_s^0}^2, \\ |\hat{\delta}|_{1,K_c^0}^2 &= |v|_{1,V_c^0}^2. \end{aligned}$$

Consequently, (3.13), (3.14) and (3.15) yield the estimate

$$(3.16) \quad |Pv|_{1,V_s}^2 + |Pv|_{1,V_c}^2 \leq C|v|_{1,V_s^0 \cup V_c^0}^2.$$

Remark 3.1 In case that the point  $\bar{\Gamma}_g \cap \bar{\Gamma}_u$  coincides with a vertex of  $\Gamma$ , the internal angle of which is not greater than  $\pi/2$ , it suffices to construct the extension according to the point 1°, thus obtaining the estimate (3.5).

5° Let a function  $v \in V(G_h^0)$  be given. We define the extension  $Pv$  separately in the straight segments, squares and sectors of circles according to the points 1°, 2° and 3°. Then  $Pv \in H^1(G_h)$ . In fact, the traces on the common boundaries  $\bar{V}_r^h \cap \bar{V}_{r+1}^h$ ,  $\bar{V}_r \cap \bar{V}_s$  and  $\bar{V}_s \cap \bar{V}_c$  coincide. According to the point 3°,  $Pv = 0$  on  $\vartheta = \vartheta_1$ , consequently  $Pv \in V(G_h)$ . Using the estimates (3.5) for all  $r$ , (3.16) for the squares and sectors, we obtain the estimate (3.4).

Now we are able to finish the *proof of Lemma 3.1*. The right-hand side of (3.3) is a linear bounded functional  $f(v)$  on  $V(G_h^0)$ . In fact, using (3.4), we have

$$|f(v)| \leq \|\mathbf{q}\|_{0,G_h} \cdot |Pv|_{1,G_h} \leq C \|\mathbf{q}\|_{0,G_h} \cdot |v|_{1,G_h^0}.$$

Hence the problem (3.3) has a unique solution  $u$ . Inserting  $v = u$  in (3.3), we obtain

$$|u|_{1,G_h^0} \leq C \|\mathbf{q}\|_{0,G_h}.$$

From the definition (3.2) it follows

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}^h\|_{0,\Omega}^2 &= \|\mathbf{q}\text{-grad } u\|_{0,G_h^0}^2 + \|\mathbf{q}\|_{0,G_h - G_h^0}^2 \leq \\ &\leq 2\|\mathbf{q}\|_{0,G_h^0}^2 + 2|u|_{1,G_h^0}^2 + \|\mathbf{q}\|_{0,G_h - G_h^0}^2 \leq (2 + 2C^2) \|\mathbf{q}\|_{0,G_h}^2. \end{aligned}$$

Hence the assertion (3.1) follows.

It remains to verify that  $\mathbf{q}^h \in Q_0(\Omega)$ . Let  $v \in V$  be given. We may write on the basis of (3.2)

$$(3.17) \quad \int_{\Omega} \mathbf{q}^h \cdot \text{grad } v \, dx = \int_{G_h^o} \text{grad } u \cdot \text{grad } v \, dx + \int_{\Omega_h} \mathbf{q} \cdot \text{grad } v \, dx$$

and the definition of  $Q_0(\Omega)$  yields that

$$\int_{\Omega_h} \mathbf{q} \cdot \text{grad } v \, dx + \int_{G_h} \mathbf{q} \cdot \text{grad } v \, dx = 0.$$

Inserting into (3.17), we obtain

$$\int_{\Omega} \mathbf{q}^h \cdot \text{grad } v \, dx = \int_{G_h^o} \text{grad } u \cdot \text{grad } v \, dx - \int_{G_h} \mathbf{q} \cdot \text{grad } v \, dx = 0,$$

if we realize that the restriction  $v|_{G_h}$  is an extension of  $v|_{G_h^o}$  into  $V(G_h)$ , and  $u$  is a solution of (3.3). Q. E. D.

**Proof of Theorem 3.** Let a  $\mathbf{q} \in Q_0(\Omega)$  be given. Consider the corresponding function  $\mathbf{q}^h$  from Lemma 3.1 and extend it out of  $\Omega$  as follows.

Let  $\Omega^* \supset \bar{\Omega}$ ,  $\Omega^*$  be bounded with Lipschitz boundary. Let  $h < \text{dist}(\partial\Omega, \partial\Omega^*)$  and denote

$$G_h^* = \{x \notin \Omega \mid \text{dist}(x, \Gamma_g) < \frac{2}{3}h\}$$

the ‘‘external boundary strip’’,

$$\Omega_1 = \Omega \cup G_h^*,$$

$$V^* = \{v \in H^1(\Omega^* \setminus \Omega_1) \mid v = 0 \text{ on } \partial\Omega^*\}.$$

Consider the auxiliary problem to find  $u \in V^*$  such that

$$(3.18) \quad \int_{\Omega^* \setminus \Omega_1} \text{grad } u \cdot \text{grad } v \, dx = - \int_{\Omega_{2h/3}} \mathbf{q}^h \cdot \text{grad } Pv \, dx \quad \forall v \in V^*$$

where  $Pv$  is (a restriction of) an arbitrary extension of the function  $v \in V^*$  into  $H_0^1(\Omega^*)$ .

The right-hand side of (3.18) does not depend on the kind of the extension  $P$ . In fact,  $\tilde{P}v - Pv = 0$  on  $\partial\Omega_1$  for any two extensions  $\tilde{P}, P$ . Consequently, we may extend  $\tilde{P}v - Pv$  by zero into  $G_h^*$  and then

$$\tilde{P}v - Pv = 0 \text{ on } \Gamma_u \Rightarrow \tilde{P}v - Pv \in V,$$

$$\int_{\Omega_{2h/3}} \mathbf{q}^h \cdot \text{grad}(\tilde{P}v - Pv) \, dx = \int_{\Omega} \mathbf{q}^h \cdot \text{grad}(\tilde{P}v - Pv) \, dx = 0.$$

A linear extension  $P : V^* \rightarrow H_0^1(\Omega^*)$  exists (see e.g. [3] – Chapt. 2, Th. 3.9), such that

$$\|Pv\|_{1, \Omega^*} \leq C \|v\|_{1, \Omega^* \setminus \Omega_1} \quad \forall v \in V^*.$$

Consequently, the problem (3.18) has a unique solution  $u$ .

Defining the extension  $E\mathbf{q}^h$  as follows

$$E\mathbf{q}^h = \begin{cases} 0 & \text{in } G_h^*, \\ \text{grad } u & \text{in } \Omega^* \div \Omega_1, \\ \mathbf{q}^h & \text{in } \Omega, \end{cases}$$

then

$$(3.19) \quad \int_{\Omega^*} E\mathbf{q}^h \cdot \text{grad } v \, dx = \int_{\Omega_{2h/3}} \mathbf{q}^h \cdot \text{grad } v \, dx + \int_{\Omega^* \div \Omega_1} \text{grad } u \cdot \text{grad } v \, dx = 0 \quad \forall v \in H_0^1(\Omega^*)$$

holds by virtue of (3.18).

Let us regularize  $E\mathbf{q}^h$  by means of a kernel  $\omega_\varkappa$ , where  $\varkappa < 2h/3$ . Then obviously

$$R_\varkappa E\mathbf{q}^h \in [C^\infty(\bar{\Omega})]^2, \quad R_\varkappa E\mathbf{q}^h = 0 \quad \text{on } \Gamma_g,$$

$$\text{div } R_\varkappa E\mathbf{q}^h(x) = - \int_{\Omega^*} \sum_{i=1}^2 \frac{\partial}{\partial y_i} \omega_\varkappa(x-y) E q_i^h \, dy = 0 \quad \forall x \in \Omega,$$

because  $\omega_\varkappa \in H_0^1(\Omega^*)$  can be inserted for  $v$  in (3.19). Hence for  $v \in V$

$$\int_{\Omega} R_\varkappa E\mathbf{q}^h \cdot \text{grad } v \, dx = - \int_{\Omega} \text{div } R_\varkappa E\mathbf{q}^h v \, dx + \int_{\Gamma_g} v \mathbf{n} \cdot R_\varkappa E\mathbf{q}^h \, ds = 0$$

and consequently  $R_\varkappa E\mathbf{q}^h \in Q_0(\Omega)$ .

Finally, from Lemma 3.1 it follows

$$\|R_\varkappa E\mathbf{q}^h - \mathbf{q}\|_{0,\Omega} \leq \|R_\varkappa E\mathbf{q}^h - \mathbf{q}^h\|_{0,\Omega} + \|\mathbf{q}^h - \mathbf{q}\|_{0,\Omega} \rightarrow 0$$

for  $h \rightarrow 0, \varkappa < \frac{2}{3}h$ .

Q. E. D.

#### 4. CONVERGENCE OF THE DUAL FINITE ELEMENT PROCEDURE

In the present Section we apply the previous results to the proof of convergence of the dual finite element procedure without any regularity assumption. Let us consider a polygonal bounded domain  $\Omega \subset R^2$  and the following problem:

$$(4.1) \quad - \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) = f \quad \text{in } \Omega,$$

$$u = \bar{u} \quad \text{on } \Gamma_u,$$

$$a_{ij} \frac{\partial u}{\partial x_j} n_j = g \quad \text{on } \Gamma_g,$$

where  $\partial\Omega = \bar{\Gamma}_g \cup \bar{\Gamma}_u$ ,  $\Gamma_u \cap \Gamma_g = \emptyset$ , a repeated index implies summation over the



range  $\{1, 2\}$ ,  $a_{ij}$  are bounded measurable functions in  $\Omega$ ,  $a_{ij} = a_{ji}$ , and there exists  $\alpha_0 > 0$  such that

$$a_{ij}(x) \zeta_i \zeta_j \geq \alpha_0 \zeta_i \zeta_i \quad \forall \zeta \in \mathbb{R}^2,$$

holds almost everywhere in  $\Omega$ . The functions  $f \in L_2(\Omega)$ ,  $\bar{u} \in H^1(\Omega)$  and  $g \in L_2(\Gamma_g)$  are given.

Let  $\Gamma_g$  be either empty or consist of a finite number of connected parts.

In the paper [1] the dual variational formulation of the problem (4.1) was presented, as follows.

Define  $V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_u\}$ ,

$$A_{f,g} = \left\{ \mathbf{q} \in [L_2(\Omega)]^2 \mid \int_{\Omega} \mathbf{q} \cdot \text{grad } v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_g} g v \, ds \quad \forall v \in V \right\}$$

and assume that we have an element  $\bar{\mathbf{q}} \in A_{f,g}$ .

Let us define the functional

$$\Phi(\boldsymbol{\chi}) = \frac{1}{2} \int_{\Omega} b_{ij} \chi_i \chi_j \, dx + \int_{\Omega} (b_{ij} \bar{\mathbf{q}}_i \chi_j - \boldsymbol{\chi} \cdot \text{grad } \bar{u}) \, dx,$$

where  $\boldsymbol{\chi} \in A_{0,0}$  and  $[b]$  is the matrix inverse to  $[a]$ .

The dual variational problem is: to find  $\boldsymbol{\chi}^0 \in A_{0,0}$  such that

$$(4.2) \quad \Phi(\boldsymbol{\chi}^0) \leq \Phi(\boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in A_{0,0}.$$

In [1] we studied the subspaces  $\mathcal{N}_h$  of triangular elements, proposed by Veubeke and Hoggie [2].  $\mathcal{N}_h$  consists of piecewise linear vector-functions on a given triangulation  $\mathcal{T}_h$  of  $\Omega$ . In every triangle they satisfy the condition  $\text{div } \boldsymbol{\chi} = 0$  and their fluxes  $\boldsymbol{\chi} \cdot \mathbf{n}$  are continuous when crossing any interelement boundary.

Denoting

$$V_h = \mathcal{N}_h \cap A_{0,0} = \{\boldsymbol{\chi} \in \mathcal{N}_h \mid \boldsymbol{\chi} \cdot \mathbf{n} = 0 \text{ on } \Gamma_g\},$$

we define the dual finite element approximation:

$$(4.3) \quad \boldsymbol{\chi}^h \in V_h, \quad \Phi(\boldsymbol{\chi}^h) \leq \Phi(\boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in V_h.$$

**Theorem 4.1** *The dual finite element approximations converge to the solution  $\boldsymbol{\chi}^0$  of the problem (4.2), i.e.,*

$$(4.4) \quad \|\boldsymbol{\chi}^h - \boldsymbol{\chi}^0\|_{0,\Omega} \rightarrow 0 \quad \text{for } h \rightarrow 0.$$

*Proof.* Note that  $A_{0,0} = Q_0(\Omega)$  in Section 1–3. From the density theorems it follows that to every  $\varepsilon_1 > 0$  there exists a function  $\mathbf{q}^0 \in [C^\infty(\bar{\Omega})]^2 \cap Q_0(\Omega)$  such that

$$(4.5) \quad \|\boldsymbol{\chi}^0 - \mathbf{q}^0\|_{0,\Omega} < \varepsilon_1.$$

In [1] (see the proof of Theorem 3.1 there) it was shown that an element  $r_h \mathbf{q}^0 \in V_h$  exists such that

$$(4.6) \quad \|\mathbf{q}^0 - r_h \mathbf{q}^0\|_{0,\Omega} \leq Ch^2 \|\mathbf{q}^0\|_{[C^2(\bar{\Omega})]^2}.$$

Moreover

$$(4.7) \quad \|\chi^0 - \chi^h\|_{0,\Omega} \leq C \inf_{\chi \in V_h} \|\chi^0 - \chi\|_{0,\Omega}$$

follows from [1] – Theorem 1.2. Consequently, for any  $\varepsilon > 0$

$$\|\chi^0 - \chi^h\|_{0,\Omega} \leq C(\|\chi^0 - \mathbf{q}^0\|_{0,\Omega} + \|\mathbf{q}^0 - r_h \mathbf{q}^0\|_{0,\Omega}) < \varepsilon$$

holds for sufficiently small  $h$ .

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#### Souhrn

### HUSTOTA SOLENOIDÁLNÍCH FUNKCÍ A KONVERGENCE DUÁLNÍ METODY KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

V práci [1] byly podány a priori odhady chyb pro duální variační metodu, která užívá po částech lineární rovnovážný model konečných prvků podle Veubeke a Hoggea [2], a to za předpokladu, že přesné řešení je dostatečně regulární. V tomto článku se dokazuje konvergence bez předpokladu regularity. Základním bodem důkazu je věta o hustotě: nekonečně hladké solenoidální funkce jsou husté v prostoru přípustných funkcí, které jsou solenoidální jen ve smyslu distribucí.

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