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The density of the extinction probability of a time homogeneous linear birth and death process under the influence of randomly occurring disasters

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Abstract

Under the influence of randomly occurring disasters, the eventual extinction probability, q, of a birth and death process, Z, is a random variable. In this paper, we obtain an integral expression for the probability density function $g(x)$ of q under the assumption that the population process Z is a time homogeneous linear birth and death process and the disasters occur according to an arbitrary renewal process so that its interarrival times have a density. An example is provided to demonstrate how to evaluate the integral numerically. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider a birth and death process $Z = \{Z(t); t \geqslant 0\}$, with $Z(0) = 1$. If the evolution of Z is under the influence of a sequence of randomly occuring disasters, then the probability of extinction of Z is a random variable and not a constant as is the case in the absence of outside influence.

In this paper we consider a time homogeneous linear birth and death process Z, under the influence of randomly occuring disasters whose timing is governed by an arbitrary renewal process. Many natural phenomena can be modeled by this process. For example, a population of boars in the Central Range of Taiwan are living under the risk of forest fires. We may assume that the forest fires occur according to a Poisson process, with each forest fire killing a binomial

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proportion of the boar population. Between fires, the size of boar population is modeled by, e.g., a time-homogeneous Markov branching process. A second example is a population of insects on a farm. To protect the crops, farmers spray pesticide repeatedly over time. If we assume that the waiting time between sprays are i.i.d. random variables with a common probability density function $f(t)$, then the randomly occurring disasters (sprays) form a renewal process. Each spray causes a binomial proportion of death of the insects. Between two sprays, the population of insects follows a linear birth and death process.

Still another example is in pharmacology. Suppose a certain anticancer drug is administered to patients. The drug stays in the body for a period of time whose duration (the residence time) depends on the rate of metabolism of each individual patient. Here, the drug may be viewed as a continuous stream of 'disasters', and each disaster kills a binomial proportion of cancer cells. Between two disasters, cancer cells grow and die following a linear birth and death process.

The model of a branching process with random environments has been discussed by many authors like Athreya and Karlin [1], Kaplan, Sudbury and Nilsen [2]. Some excellent summaries can be found in [3,4]. The topic was further studied by Brockwell et al. [5]; Pakes [6]; Zeiman [7]; Grey and Lu [8].

The main results we obtained are the integral form of the density of the asymptotic distribution of the probability of population extinction and its numerical solution if the random disasters are Poissonian. We believe that these results have not been discussed before in the literature.

The paper is organized as follows. In Section 2 we systematically utilize a regenerative property of the Z process and an embedded Galton–Watson process with random environments (GWRE process) to derive the extinction probability after each disaster. It has a very simple form, see (2.7). In Section 3 we discuss the distribution $G(x)$ of the eventual extinction probability (which is a random variable). The distribution possesses a density $g(x)$ which is the functional solution to (3.5). An example is given to illustrate how to obtain the solution numerically.

2. Random disasters

Let $Z = \{Z(t); t \geqslant 0\}$, with $Z(0) = 1$, be the population process under consideration where $Z(t)$ is the number of individuals that are alive at time t. At the start, Z grows as a time homogeneous linear birth and death process with birth and death rates λ and μ until a disaster strikes. Suppose the population is subject to disasters which occur at random times $\tau_1 < \tau_2 < \tau_3, \ldots$ When a disaster happens each individual in the population has the probability δ of surviving the disaster. Alternatively we say the killing rate is $\varepsilon = 1 - \delta$. It is assumed that the survival event of any individual is independent of that of the rest in the population. Then, the number of individuals alive prior to the ith disaster is $Z(\tau_i^-)$ and the number of individuals that survive the ith disaster, $Z(\tau_i^+)$, is a binomial random variable with parameters δ and $Z(\tau_i^-)$.

Let $X = \{X(t); t \geqslant 0\}$ be a natural birth and death process, i.e., it is the Z process free of disasters. Let $M(s,t) = E(s^{X(t)})$ be the probability generating function of the 'natural X'. To proceed, it is convenient to work with the interarrival times of the disasters, namely, $t_i = \tau_i - \tau_{i-1}$ for $i = 1, 2, \ldots$ and $t_0 = 0$, so that $\tau_1 = t_1$. We assume that $t_1, t_2, t_3, \ldots$ are i.i.d. random variables with density function $f(t)$.

By a standard renewal approach (see, e.g., [9]) it is easy to see that the probability generating function $E(s^{Z(t_1^+)})$ of the population size right after the first disaster is $M((1-\varepsilon)s+\varepsilon, t_1)$.

Now consider the embedded process $Z_n = Z((\sum_{i=1}^n t_i)^+)$ which is the number of individuals that survive the nth disaster. Define $T_n = (t_1, t_2, \dots, t_n)$ $\forall n \geqslant 1$ and $S(T_n) = (t_2, t_3, \dots, t_n)$ $\forall n \geqslant 2$, and let $F_n(s, T_n) = E(s^{Z_n})$ be the probability generating function of Z_n. Then by the i.i.d. property of $t_1, t_2, \dots,$ we obtain

$$F_1(s, T_1) = F_1(s, t_1) = M((1-\varepsilon)s+\varepsilon, t_1),$$

and for $n \geqslant 2$,

$$\begin{aligned} F_n(s, T_n) &= E(s^{Z_n}) = E(E(s^{Z_n} \mid Z_1)) = E(F_{n-1}(s, S(T_n)))^{Z_1} \\ &= F_1(F_{n-1}(s, S(T_n)), t_1) \\ &= M((1-\varepsilon)F_{n-1}(s, S(T_n)) + \varepsilon, t_1). \end{aligned} \tag{2.1}$$

Define $v = v(t) = (\mu - \lambda)t$, $\beta = \beta(t) = (\mu/(\mu-\lambda))(\exp(\mu-\lambda)t - 1)$. Then (see e.g., [10] or [11])

$$M(s,t) = E(s^{X(t)}) = \frac{\beta - (\beta - e^v)s}{1 + \beta - (1 + \beta - e^v)s}. \tag{2.2}$$

Let $\beta_i = \beta(t_i)$, $v_i = v(t_i)$, $S_n = \sum_{i=1}^n v(t_i)$ and define

$$\begin{aligned} A(t_i) &= -(1-\varepsilon)(\beta_i - e^{v_i}), \\ B(t_i) &= \beta_i - \varepsilon(\beta_i - e^{v_i}), \\ C(t_i) &= -(1-\varepsilon)(1 + \beta_i - e^{v_i}), \\ D(t_i) &= 1 + \beta_i - \varepsilon(1 + \beta_i - e^{v_i}). \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} F_1(s, t_1) = M((1-\varepsilon)s+\varepsilon, t_1) &= \frac{\beta_1 - (\beta_1 - e^{v_1})((1-\varepsilon)s+\varepsilon)}{1 + \beta_1 - (1 + \beta_1 - e^{v_1})((1-\varepsilon)s+\varepsilon)} \\ &= \frac{A_1 s + B_1}{C_1 s + D_1}, \end{aligned} \tag{2.4}$$

where

$$A_1 = A(t_1), \quad B_1 = B(t_1), \quad C_1 = C(t_1), \quad D_1 = D(t_1).$$

Similarly, use (2.1),

$$F_2(s, T_2) = F_1(F_1(s, t_2), t_1) = \frac{(A_2 s + B_2}{(C_2 s + D_2)},$$

where

$$\begin{aligned} A_2 &= A_1 A(t_2) + B_1 C(t_2) = (1-\varepsilon)e^{S_2} - (1-\varepsilon)(\beta_2 + \varepsilon)e^{S_1} - (1-\varepsilon)^2 \beta_1, \\ B_2 &= A_1 B(t_2) + B_1 D(t_2) = \varepsilon e^{S_2} + (1-\varepsilon)(\beta_2 + \varepsilon)e^{S_1} + (1-\varepsilon)^2 \beta_1, \\ C_2 &= C_1 A(t_2) + D_1 C(t_2) = (1-\varepsilon)e^{S_2} - (1-\varepsilon)(\beta_2 + \varepsilon)e^{S_1} - (1-\varepsilon)^2 \beta_1 - (1-\varepsilon)^2, \\ D_2 &= C_1 B(t_2) + D_1 D(t_2) = \varepsilon e^{S_2} + (1-\varepsilon)(\beta_2 + \varepsilon)e^{S_1} + (1-\varepsilon)^2 \beta_1 + (1-\varepsilon)^2. \end{aligned} \tag{2.5}$$

By induction, it can be shown that $F_n(s, T_n) = (A_n s + B_n)/(C_n s + D_n)$, where

$$
\begin{aligned}
A_n &= (1-\varepsilon)\mathrm{e}^{S_n} - \sum_{i=1}^{n-1}(1-\varepsilon)^i(\beta_{n-i+1}+\varepsilon)\mathrm{e}^{S_{n-i}} - (1-\varepsilon)^n\beta_1,\\
B_n &= \varepsilon\mathrm{e}^{S_n} + \sum_{i=1}^{n-1}(1-\varepsilon)^i(\beta_{n-i+1}+\varepsilon)\mathrm{e}^{S_{n-i}} + (1-\varepsilon)^n\beta_1,\\
C_n &= (1-\varepsilon)\mathrm{e}^{S_n} - \sum_{i=1}^{n-1}(1-\varepsilon)^i(\beta_{n-i+1}+\varepsilon)\mathrm{e}^{S_{n-i}} - (1-\varepsilon)^n\beta_1 - (1-\varepsilon)^n,\\
D_n &= \varepsilon\mathrm{e}^{S_n} + \sum_{i=1}^{n-1}(1-\varepsilon)^i(\beta_{n-i+1}+\varepsilon)\mathrm{e}^{S_{n-i}} + (1-\varepsilon)^n\beta_1 + (1-\varepsilon)^n.
\end{aligned}
\tag{2.6}
$$

And the extinction probability right after the nth disaster is

$$
q_n = F_n(s, T_n)|_{s=0} = \frac{B_n}{D_n}. \tag{2.7}
$$

When $\varepsilon = 0$ (i.e., disasters do not cause death) and $\lambda > \mu$, then B_n and D_n become telescoping series that $B_n = (\mu/(\mu-\lambda))\exp(\mu-\lambda)\sum_{i=1}^{n} t_i + \mu/(\lambda-\mu) \to \mu/(\lambda-\mu)$ and $D_n = B_n + 1$. Therefore $q_n \to \mu/\lambda$ a.e. If $\varepsilon = 0$ and $\lambda < \mu$, then $q_n \to 1$ a.e. The special case $\lambda = \mu$ also implies $q_n \to 1$.

3. Probability of extinction under random disasters

To study the eventual extinction probability of the process Z we utilize the embedded process $\{Z_n\}_{n=1}^{\infty}$ in which Z_n is the population size immediately after the nth disaster. This is a Galton–Watson process in a random environment. We shall consider the supercritical case, $\lambda > \mu$, in which the extinction is not certain. By (2.7), the extinction probability of Z_n, q_n, right after the nth disaster is a random variable which depends on the timing of all previous disasters, i.e.,

$$
q_n = q_n(t_1, t_2, \ldots, t_n).
$$

Let T_n and $S(T_n)$ be defined as in Section 2, then q_n satisfies the following iterative property:

$$
\begin{aligned}
q_{n+1}(T_{n+1}) &= q_{n+1}(t_1, t_2, \ldots, t_{n+1})\\
&= P\left\{Z\left(\sum_{i=1}^{n+1} t_i\right)^+ = 0\right\}\\
&= \sum_{m=0}^{\infty} P\{Z(t_1^+) = m\}(q_n(t_2, \ldots, t_{n+1}))^m\\
&= F_1(q_n(t_2, \ldots, t_{n+1}), t_1)\\
&= F_1(q_n(S(T_{n+1})), t_1).
\end{aligned}
\tag{3.1}
$$

Passing to the limit, the probability of extinction, $q = \lim_{n\to\infty} q_n$, of the process $\{Z(t); t \geqslant 0\}$ is easily seen to satisfy the equation

$$q(T) = \lim_{n\to\infty} q_n(T_n) = F_1(q(S(T)), t_1), \tag{3.2}$$

where $T = (t_1, t_2, \ldots)$ and $S(T) = (t_2, t_3, \ldots)$

It is worth noting that the almost sure convergence limit, q, of q_n, is a random variable which depends only on the interarrival times and the killing rate.

Based on this equation we shall derive the distribution function of q, $G(x) = P\{q \leqslant x\}$, and show that the density $g(x)$ of q exists under very mild conditions. The proofs are long and will be presented in Sections 3.1 and 3.2. They are followed by an example in Sections 3.3 and 3.4.

3.1. Differentiability of G

Theorem 1. *Under the assumptions stated in Section* 2. *Let* $F_1(s,t) = M((1-\varepsilon)s + \varepsilon, t)$, *where* $M(s,t)$ *is as defined in* (2.2). *Let* F_1^* *be the inverse of* F_1 *with respect to its first argument*, *and* F_1^{**} *be the inverse of* F_1^* *with respect to its second argument* (*see* (3.8) *and* (3.10) *below*). *Assume that the density* $f(t)$ *of the interarrival times of disasters is such that* $\partial/\partial x[f(F_1^{**}(x,y))(\partial/\partial y)F_1^{**}(x,y)]$ *exists and is integrable on* $[a,1)$, $a = \max(0,(r-\varepsilon)/(1-\varepsilon))$, *r is the natural extinction probability*, *i.e.* $r = \mu/\lambda$. *Then* $G(x)$ *is differentiable on* (0,1) *with the only exception* $x = r$.

Proof. By (3.2), conditioning on t_1, we obtain

$$\begin{aligned} G(x) &= P[q(T) \leqslant x] \\ &= E\{P[q(T) \leqslant x | t_1]\} \\ &= \int_0^\infty f(t) P\{q(T) \leqslant x | t_1 = t\}\, \mathrm{d}t \\ &= \int_0^\infty f(t) P\{F_1(q(S(T)), t) \leqslant x\}\, \mathrm{d}t \\ &= \int_0^\infty f(t) P\{q(S(T)) \leqslant F_1^*(x,t)\}\, \mathrm{d}t \\ &= \int_0^\infty f(t) G(F_1^*(x,t))\, \mathrm{d}t \end{aligned} \tag{3.3}$$

since $q(T)$ and $q(S(T))$ have the same probability distribution. To simplify (3.3), we notice that when x is fixed, $F_1^*(x,t)$ is a monotone function of t, therefore if we change variable (fix x) by $y = F_1^*(x,t)$, then $t = F_1^{**}(x,y)$ for some function F_1^{**}. Also note that the integration limit $t = 0$ in (3.3) corresponds to $y = (x-\varepsilon)/(1-\varepsilon)$ (for $M((1-\varepsilon)s+\varepsilon, 0) = (1-\varepsilon)s + \varepsilon = x$ implies $s = (x-\varepsilon)/(1-\varepsilon)$) and $t = \infty$ corresponds to $y = 1$, therefore by change of variable the last integral can be written as below and we have

$$G(x) = \int_{\frac{x-\varepsilon}{1-\varepsilon}}^{1} f(F_1^{**}(x,y))G(y)\frac{\partial F_1^{**}(x,y)}{\partial y}\,\mathrm{d}y. \tag{3.4}$$

It is obvious that adding disasters only increases extinction probability; thus $0 < r \leqslant q \leqslant 1$. Therefore $G(x) = 0$ for all $0 \leqslant x < r$, and $G(x)$ is differentiable (in fact, $G'(x) = 0$) at least for $0 \leqslant x < r$. Let us write $G(x) = \int_{(x-\varepsilon)/(1-\varepsilon)}^{1} H(x,y)G(y)\,\mathrm{d}y$, where $H(x,y) = f(F_1^{**}(x,y))\partial F_1^{**}(x,y)/\partial y$. First we show that $G(x)$ is continuous on [0,1). We already know that $G(x) = 0$ for all x in $[0,r)$. It suffices to consider $r < x_0 < x < 1$. We have

$$\begin{aligned} G(x) - G(x_0) &= \int_{\frac{x-\varepsilon}{1-\varepsilon}}^{1} H(x,y)G(y)\,\mathrm{d}y - \int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{1} H(x_0,y)G(y)\,\mathrm{d}y \\ &= \int_{\frac{x-\varepsilon}{1-\varepsilon}}^{1} [H(x,y) - H(x_0,y)]G(y)\,\mathrm{d}y - \int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{\frac{x-\varepsilon}{1-\varepsilon}} H(x_0,y)G(y)\,\mathrm{d}y. \end{aligned}$$

The continuity of $H(x,y)$ in x and boundedness of $G(y)$ imply that the first integral of the last equality tends to zero when x tends to x_0 from the right. The integrability of $H(x_0,y)$ with respect to $G(y)$ ensures that the last integral also tends to zero as x tends to x_0 (see e.g. [12]). Hence $G(x)$ is right continuous. The proof for the left continuity is similar. Next we show that $G(x)$ is differentiable on (0,1). Consider for $r < x_0 < x < 1$,

$$\begin{aligned} \lim_{x\to x_0} \frac{G(x) - G(x_0)}{x - x_0} &= \lim_{x\to x_0} \int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{1} \left[\frac{H(x,y) - H(x_0,y)}{x - x_0}\right] G(y)\,\mathrm{d}y \\ &\quad - \lim_{x\to x_0} \int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{\frac{x-\varepsilon}{1-\varepsilon}} \left[\frac{H(x,y) - H(x_0,y)}{x - x_0}\right] G(y)\,\mathrm{d}y - \lim_{x\to x_0} \int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{\frac{x-\varepsilon}{1-\varepsilon}} \frac{H(x_0,y)}{x - x_0} G(y)\,\mathrm{d}y. \end{aligned}$$

The first integral tends to

$$\int_{\frac{x_0-\varepsilon}{1-\varepsilon}}^{1} \left[\left.\frac{\partial H(x,y)}{\partial x}\right|_{x=x_0}\right] G(y)\,\mathrm{d}y$$

by the Lebesgue Dominate Convergence Theorem. This integral is finite by the assumption of $H(x,y)$. The second one tends to zero and the third one tends to a finite value both by the Mean–Value Theorem for integrals, similarly for the case when x tends to x_0 from the left. This proves Theorem 1. □

Remark. Given an arbitrary $f(t)$, it is not difficult to check the integrability of the function

$$\partial/\partial x\left[f(F_1^{**}(x,y))\frac{\partial}{\partial y}F_1^{**}(x,y)\right]$$

since $(\partial/\partial y)F_1^{**}(x,y)]$ is a function of y alone (see 3.11).

3.2. Density of q

Differentiating $G(x)$ in (3.3) to obtain the density of q, $g(x) = G'(x)$, then

$$g(x) = \int_0^\infty f(t) g(F_1^*(x,t)) \frac{\partial F_1^*(x,t)}{\partial x} \, \mathrm{d}t,$$

or equivalently (replace t by $F_1^{**}(x,y)$)

$$g(x) = \int_{\frac{x-\varepsilon}{1-\varepsilon}}^1 f(F_1^{**}(x,y)) g(y) \left(\frac{\partial F_1^*(x,t)}{\partial x} \bigg|_{t=F_1^{**}(x,y)} \right) \frac{\partial F_1^{**}(x,y)}{\partial y} \, \mathrm{d}y. \tag{3.5}$$

Unfortunately there is no close form solution of $g(x)$ for the general case in (3.5); however for the special case that the occurrence of disasters follows a Poisson process, we develop a simple method that can evaluate the numerical solution easily.

3.3. An example

To illustrate how $g(x)$ can be evaluated, we assume that the occurrence of disasters follows a Poisson process with intensity $\alpha > 0$, then the interarrival density is $f(t) = \alpha \mathrm{e}^{-\alpha t}$, $t > 0$.

By (2.2) and (2.3),

$$M(s,t) = \frac{(\mu - \mu \mathrm{e}^{(\lambda-\mu)t}) - (\lambda - \mu \mathrm{e}^{(\lambda-\mu)t}) s}{(\mu - \lambda \mathrm{e}^{(\lambda-\mu)t}) - (\lambda - \lambda \mathrm{e}^{(\lambda-\mu)t}) s}. \tag{3.6}$$

$$F_1(s,t) = M((1-\varepsilon)s + \varepsilon, t) = \frac{(\mu - \mu \mathrm{e}^{(\lambda-\mu)t}) - (\lambda - \mu \mathrm{e}^{(\lambda-\mu)t})((1-\varepsilon)s + \varepsilon)}{(\mu - \lambda \mathrm{e}^{(\lambda-\mu)t}) - (\lambda - \lambda \mathrm{e}^{(\lambda-\mu)t})((1-\varepsilon)s + \varepsilon)}. \tag{3.7}$$

To invert $F_1(s,t)$ in (3.7) for fixed t, set $x = F_1(s,t)$, we obtain

$$s = F_1^*(x,t) = \frac{1}{1-\varepsilon} \left\{ \frac{(\mu - \lambda \mathrm{e}^{(\lambda-\mu)t}) x - (\mu - \mu \mathrm{e}^{(\lambda-\mu)t})}{(\lambda - \lambda \mathrm{e}^{(\lambda-\mu)t}) x - (\lambda - \mu \mathrm{e}^{(\lambda-\mu)t})} - \varepsilon \right\}, \tag{3.8}$$

so that

$$\frac{\partial}{\partial x} F_1^*(x,t) = \frac{1}{1-\varepsilon} \frac{(\lambda - \mu)^2 \mathrm{e}^{(\lambda-\mu)t}}{[(\lambda - \lambda \mathrm{e}^{(\lambda-\mu)t}) x - (\lambda - \mu \mathrm{e}^{(\lambda-\mu)t})]^2}. \tag{3.9}$$

Next we invert $F_1^*(x,t)$ with x fixed. Put $y = F_1^*(x,t)$, then

$$t = F_1^{**}(x,y) = \frac{1}{\lambda - \mu} \log \left\{ \frac{(\lambda((1-\varepsilon)y + \varepsilon) - \mu)(1-x)}{(\lambda x - \mu)(1 - ((1-\varepsilon)y + \varepsilon))} \right\}, \tag{3.10}$$

hence

$$\frac{\partial}{\partial y} F_1^{**}(x,y) = \frac{1-\varepsilon}{[\lambda((1-\varepsilon)y + \varepsilon) - \mu][1 - ((1-\varepsilon)y + \varepsilon)]}. \tag{3.11}$$

If we write $\gamma = \alpha/(\lambda - \mu)$, then (3.5) becomes

$$g(x) = \frac{\alpha(\lambda x - \mu)^{\gamma - 1}}{(1-x)^{\gamma+1}} \int_{\frac{x-\varepsilon}{1-\varepsilon}}^{1} \left[\frac{1 - ((1-\varepsilon)y + \varepsilon)}{\lambda((1-\varepsilon)y + \varepsilon) - \mu}\right]^{\gamma} g(y)\,\mathrm{d}y, \tag{3.12}$$

for $\mu/\lambda \leqslant x < 1$ and $g(x) = 0$ for all x in $[0, \mu/\lambda)$.

Put

$$c = \int_{\frac{\mu}{\lambda}}^{1} \left[\frac{1 - ((1-\varepsilon)y + \varepsilon)}{\lambda((1-\varepsilon)y + \varepsilon) - \mu}\right]^{\gamma} g(y)\,\mathrm{d}y \tag{3.13}$$

then obviously

$$g(x) = \frac{\alpha(\lambda x - \mu)^{\gamma - 1}}{(1-x)^{\gamma+1}} .c$$

for all x in the interval $[\mu/\lambda, (\mu/\lambda) + (1 - \mu/\lambda)\varepsilon]$. To determine the value of $g(x)$ for x in the interval $((\mu/\lambda) + (1 - \mu/\lambda)\varepsilon,\ 1)$, we observe that

$$\int_{\frac{x-\varepsilon}{1-\varepsilon}}^{1} \left[\frac{1 - ((1-\varepsilon)y + \varepsilon)}{\lambda((1-\varepsilon)y + \varepsilon) - \mu}\right]^{\gamma} g(y)\,\mathrm{d}y = c - \int_{\frac{\mu}{\lambda}}^{\frac{x-\varepsilon}{1-\varepsilon}} \left[\frac{1 - ((1-\varepsilon)y + \varepsilon)}{\lambda((1-\varepsilon)y + \varepsilon) - \mu}\right]^{\gamma} g(y)\,\mathrm{d}y.$$

The integral can be estimated subinterval by subinterval that $g(x)$ is a certain multiple of c by Simpson's rule. This will be demonstrated in Section 3.4.

3.4. Numerical results

Suppose $\lambda = 6$, $\mu = 2$, $\varepsilon = 0.4$, $\alpha = 4$. Then (3.12) becomes

$$g(x) = (4/(1-x)^2) \int_{(x-0.4)/0.6}^{1} h(y)g(y)\,\mathrm{d}y, \quad 1/3 \leqslant x < 1,$$

where $h(y) = [1 - (0.6y + 0.4)]/[6(0.6y + 0.4) - 2]$. Let

$$c = \int_{1/3}^{1} h(y)g(y)\,\mathrm{d}y,$$

then for all x in $[1/3, 0.6]$, $g(x)$ is a multiple of c. For example,

$$g(1/3) = 9\int_{1/3}^{1} h(y)g(y)\,\mathrm{d}y = 9c, \quad g(0.5) = 16c, \quad g(0.6) = 25c, \ldots$$

The values of $g(x)$, $x \in (0.6, 0.76]$, can be estimated from the values of $g(x)$ for x in $[1/3, 0.6]$. For example,

$$g(0.7) = (4/0.09)\int_{0.5}^{1} h(y)g(y)\,\mathrm{d}y = (4/0.09)\left(c - \int_{1/3}^{0.5} h(y)g(y)\,\mathrm{d}y\right).$$

Table 1
The density of the extinction probability, $g(x)$, given by (3.12) ($\lambda = 6$, $\mu = 2$, $\varepsilon = 0.4$, $\alpha = 4$)

x	$g(x)$ as multiple of c	Numerical values of $g(x)$
0.333	9.0000c	0.72809
0.367	9.9723c	0.80675
0.400	11.1111c	0.89888
0.433	12.4567c	1.00774
0.467	14.0625c	1.13764
0.500	16.0000c	1.29438
0.533	18.3673c	1.48590
0.567	21.3018c	1.72329
0.600	25.0000c	2.02247
0.633	26.0962c	2.11116
0.667	27.2419c	2.20385
0.700	28.2807c	2.28788
0.733	28.8807c	2.33642
0.767	28.3461c	2.29317
0.800	28.8609c	2.17302
0.833	24.4717c	1.97974
0.867	20.8776c	1.68898
0.900	16.2419c	1.31395
0.933	10.6727c	0.86341
0.967	4.6793c	0.37855

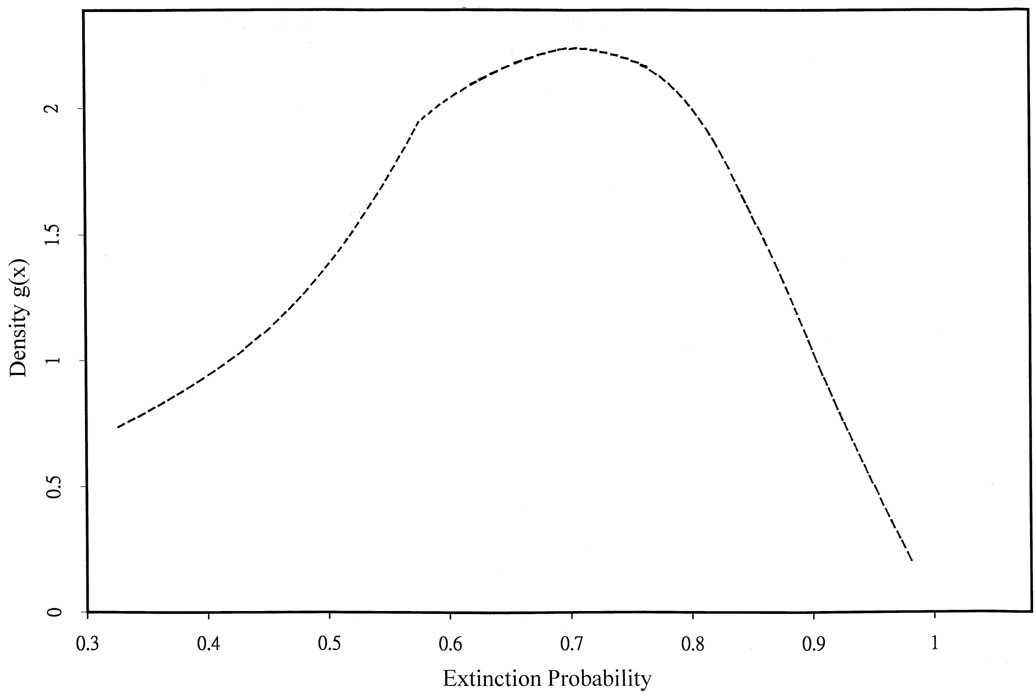

Fig. 1.

The integration limit $[1/3, 0.5]$ is contained in the interval $[1/3, 0.6]$ therefore the integral $\int_{1/3}^{0.5} h(y)g(y)\,\mathrm{d}y$ can be estimated by Simpson's rule. (For example, divide $[1/3, 0.5]$ into

$$\{8/24, 9/24, 10/24, 11/24, 12/24\}, \text{ then } \int_{1/3}^{0.5} h(y)g(y)\,\mathrm{d}y \approx 0.3636843c$$

and therefore $g(0.7) \approx 28.2807c$). Continue in this way and using the fact that $\int_{1/3}^{1} g(x)\,\mathrm{d}x = 1$, the constant c can be determined as $c = 0.080899$. Table 1 lists 20 values of $g(x)$ and the graph of $g(x)$ is sketched in Fig. 1.

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