THE DERIVATION OF SUBJECTIVE SCALES FROM JUST NOTICEABLE DIFFERENCES [1]

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The study of cumulated jnd scales began with Fechner; Fechner's law is such a scale. Psychophysicists have been deriving such scales and comparing them with scales derived in other ways, notably by fractionation, ever since, and a lot of controversy has resulted. The controversy is particularly hot at present because Stevens and Galanter (16) and Stevens (14) have assembled a lot of data which indicate that cumulated jnd scales do not agree with magnitude scales derived by other methods for intensity continua such as loudness, brightness, and pain.

Unfortunately, Fechner's procedure for cumulating jnds, which has been widely defended but not widely applied since his day, rests on an assumption which is inconsistent with one of his definitions. This means that cumulated jnd scales developed by his procedure are incorrect, and so comparisons between them and other kinds of scales are meaningless.

This paper begins by showing that Fechner's method contains internal contradictions for all but a few special cases, and that it cannot be rescued by minor changes. It goes on to derive a new and mathematically appropriate method for cumulating jnd's. This method turns out to be the simplest possible one: you can best cumulate jnd's simply by adding them on top of each other, like a stack of plates. Unfortunately, the detailed mathematical equivalent of this very simple operation is often fairly complicated. A simple but sometimes tedious graphic procedure, however, is readily available—and indeed has customarily been used by most scientists when developing cumulated jnd scales. This paper ends by discussing practical applications of this method, the relation it bears to scaling methods based on the law of comparative judgment, and the current controversy about scaling methods in psychophysics.

The model of a sensation scale. The psychophysical model of a sensation scale is a mathematical model; a sensation scale is an intervening variable. The rules by which sensation scales should be constructed are to some degree arbitrary, limited by logic, convenience, intuition, and best fit to data.

The model of a sensation scale goes as follows. Corresponding to many of the major subjective dimensions of change of sensory experience, there are primary physical dimensions of change (e.g., pitch and frequency, loudness and amplitude, etc.). Once parametric conditions for significant variables have been specified, we assume that a single-valued, monotonic, everywhere differentiable (smooth) function exists that relates the subjective dimension to its corresponding physical dimension. From here on, we shall use the words "dimension" and "continuum" interchangeably; we shall usually talk about a stimulus continuum and its corresponding sensory continuum.

That is the model, and it is very easy to state. The big difficulty comes when we try to decide how to fit data to it. All methods for doing this must introduce definitions and assumptions beyond those listed in the previous paragraph. These differ from one method to another.

The oldest sensory scaling method, Fechner's, is based upon a further condition that says that any jnd on a given sensory continuum is subjectively equivalent to any other jnd on that continuum. Whether this added condition is to be interpreted as merely a definition of the scale under consideration or as an assumption is a matter of opinion. Textbooks usually say Fechner "assumed" that all jnd's for a sensory dimension are equal to one another (1). It is not easy to know what he had in mind, but judging by his writings he probably did view it as an assumption having implications beyond scale construction. Since it is not directly observable and since its indirect consequences are highly debatable, others since Fechner have suggested that it might better be viewed as a definition.

It is our view that this is the more sensible position; it certainly is the one for us to take in this paper since our point is a logical one, not a substantive one. If equality of jnd's is taken as a definition, then it cannot be proved or disproved by any kind of empirical evidence. An experiment, for instance, that showed that a tone 20 jnd's loud is not half as loud (according to fractionation judgments) as a tone 40 jnd's loud would not have any relevance to what we may call Fechner's definition; it would only show that the kind of sensation scale implied by his definition does not agree with the kind implied by the definitions used in fractionation experiments. The issue, then, becomes what are the different scales useful for, and what is their relationship one to another. The latter part of this paper touches briefly on this problem.

The main purpose of the paper is to explore the consequences of Fechner's definition. Therefore, we must be certain of its meaning. It clearly does not mean, for instance, that all jnd's for loudness contain the same number of physical units. It is only on the sensory continuum, not on the stimulus continuum, that jnd's are defined as equal to one another. Furthermore, this definition holds only if all stimulus properties except those on the primary stimulus continuum remain constant. So there is no reason, for instance, to expect that the subjective size of a loudness jnd at 1,000 cycles per second (cps) should be the same as that of a loudness jnd at 4,000 cps. Of course, it would be pleasant if they were equal in size, but the model does not require it.

From here on we will talk about two kinds of jnd's. A sensation jnd is the magnitude of a jnd as measured in the units of the appropriate sensa-

tion continuum. By definition, all sensation jnd's for a given sensation continuum are equal to one another, given unchanged values of all stimulus properties except those on the primary stimulus continuum. A stimulus jnd is the magnitude of the change on the primary stimulus continuum, measured in appropriate physical units, which is just sufficient to produce a change of one sensation jnd upward at that point. (A discussion of the essentially statistical nature of jnd's appears later in this paper.) In general, stimulus jnd's will have different sizes at different points on the primary stimulus continuum. The rest of this paper will not be intelligible unless you keep the distinction between these two kinds of jnd's in mind.

We have assumed that jnd's are measured upward on the stimulus continuum. They could also be measured downward, and the possibility exists that the two measurements might not agree. In fact, they are certain not to agree if the distance spanned is more than two jnd's, and if the size of the jnd at the end where measurement starts is used as the unit of measurement, since this means that the size of the measurement unit will be different depending on direction of measurement. However, such discrepancies might exist in the measurement of a single jnd; this, if it happened, would mean that jnd's are not suitable units of measurement unless direction is specified. We have therefore confined ourselves to upward jnd's.

Now we can say exactly what this paper is about. Given a function (obtained from experiment, theory, or both) relating stimulus to sensation jnd's for all points of the primary stimulus continuum, what may we in-

fer about the sensory scale implied by that jnd function?

Fechner's derivation of Fechner's law. On October 22, 1850, Fechner (2) thought up the first (incorrect) answer to the question which ended the previous paragraph. Let us call any function that gives the size of a stimulus jnd at each point of the stimulus continuum a *Weber function* (corresponding to "a function relating stimulus to sensation jnd's" of the previous section), and any one-to-one function based on cumulated jnd's which relates the stimulus continuum to a sensory scale a *Fechner function* (corresponding to "a sensation scale" of the previous section). These definitions do *not* restrict our attention to those two special functions which have come to be known in psychophysics as Weber's law and Fechner's law! Fechner believed that the Fechner function corresponding to any Weber function could be expressed as the solution (integral) of a first order linear differential equation involving that Weber function. He applied this procedure to Weber's aw, which asserts that for a given stimulus continuum the size of the stimulus jnd, Δx, divided by the value of the stimulus at that point, x, is a constant ($\Delta x/x = k$). Let us examine his argument.

If Weber's law is true, then, since all sensation jnd's are equal by definition, there is a constant A such that

$$\frac{\Delta u}{\Delta x} = \frac{A}{x} \qquad [1]$$

where Δu denotes the size of the sensation jnd. The heart of Fechner's solution to his and our basic problem was to "rewrite" Equation 1 as the differential equation

$$\frac{du}{dx} = \frac{A}{x} \qquad [2]$$

How did Fechner make this step from differences (deltas) to differentials? He used what he called a "mathematical auxiliary principle," the essence of which is that what is true for differences as small as jnd's ought also to be true for all smaller differences and so true in the limit as they approach zero (differentials). If this argument were acceptable (which it is not), the rest would be simple. Equation 2, when integrated, yields the familiar logarithmic relationship between sensation and stimulus which is known as Fechner's law.

Fechner thought that his general procedure ought to be applicable to any Weber function, not just to Weber's law. It is not. Except for a few special cases like Weber's law, the definition of sensation jnd's as equal and the "mathematical auxiliary principle" are mutually contradictory. For example, consider the Weber function $\Delta x/x^2 = k$. Then, following Fechner's procedure, we should write:

$$\frac{\Delta u}{\Delta x} = \frac{A}{x^2} \quad \text{and so} \quad \frac{du}{dx} = \frac{A}{x^2}$$

Integrating, we get

$$u = B - \frac{A}{x}$$

Let us now check to see whether this new Fechner function satisfies the definition which says that sensation jnd's are equal to one another. If we are at point x on the stimulus continuum, a stimulus jnd, according to the Weber function used in this example, is kx^2. The sensation increment corresponding to this change, the sensation jnd at this point, is therefore given by:

$$u(x + kx^2) - u(x)$$

$$= B - \frac{A}{(x + kx^2)} - B + \frac{A}{x}$$

$$= \frac{Ak}{1 + kx}$$

which is clearly not a constant for any value of the constant A except zero.

This, although only one example, is typical in the sense that almost any example you could think of would show the same discrepancy. Only for a very few Weber functions—some pathological ones, Weber's law, and its generalization $\Delta x = kx + c$—does the "mathematical auxiliary principle" yield a Fechner function with equal jnd's. We will not take space to prove this formally, but a formal proof is available.

The functional equation solution. We have shown that Fechner's procedure involves a self-contradiction. We shall show later that it leads to wrong results in all important cases except Weber's law. Obviously the "mathematical auxiliary principle" is wrong and must go.

How, then, should we cumulate jnd's? The simplest, most obvious procedure (which has very often been used exactly because it is simplest and most obvious) is simply to add them up one at a time. If the first jnd on a primary stimulus continuum is 5 stimulus units, then two points on our cumulated jnd scale should be 0, 0 and 1, 5, where the first number is the scale value on the y axis and the second number is the corresponding stimulus value on the x axis. If we then find that the size of the stimulus jnd at 5 on the stimulus continuum is is 8, then the third point is 2, 13. If we find that the size of the stimulus jnd at 13 is 10, then the fourth point is 3, 23, and so on.

Fechner and some of his more modern imitators went way out of their way to avoid this simple and sensible procedure; in retrospect it is hard to decide why they did so. At any rate, the next two sections of this paper will develop a formal mathematical solution to Fechner's mathematical problem—a solution which turns

out to be the mathematical equivalent of the simple graphical or arithmetic technique discussed in the previous paragraph. The mathematical problem centers about how to fill in the curve between the discrete points arrived at by the graphical method.

What mathematical tools can we use to replace Fechner's "mathematical auxiliary principle"? Equation 1, and the corresponding ones based on other Weber functions, can be solved directly without any mathematical auxiliary principles or other further assumptions. They are examples of what mathematicians call functional equations. The papers on which most of our discussion is based (5, 6) were published in the 1880s, twenty years after Fechner first published his work.

The kind of functional equation implied by the definition of equality of sensation jnd's is soluble for a very wide class of Weber functions. Unfortunately, there is an infinity of inherently different solutions to each of these equations. However, further consideration of what we mean by a sensation scale will lead us to properties which we usually take for granted and which are enough to narrow the solutions down to just one interval scale, unique except for its zero point and unit of measurement. It is interesting that in the case of the linear generalization of Weber's law, and in that case only, the functional-equation solution is the same as that obtained by Fechner's auxiliary principle; for all other Weber functions the two solutions are different.

First, we will state the general mathematical problem and its solution. Let x, $x \geq 0$, denote a typical value of the stimulus continuum, and let u denote the (unknown) Fechner function. Let g be the (given) Weber function; i.e., a stimulus magnitude

y, $y \geq x$, is detected as larger (in a statistical sense) than x if $y > x + g(x)$, whereas it is not discriminated as different from x if $x \leq y \leq x + g(x)$. We write $x + g(x) = f(x)$. By definition, a sensory jnd at the sensation $u(x)$ is given by the increment[2]

$$u[f(x)] - u(x)$$

(In the usual "delta" notation, $g(x) = \Delta x$ and $u[f(x)] - u(x) = \Delta u$.) The condition that sensation jnd's be equal simply means that all sensation jnd's are a constant, which we may take to be 1 for convenience, since an arbitrary change of unit does not matter. Thus, we have our major mathematical problem:

Find those real-valued differentiable functions u, defined for all $x \geq 0$, such that $u[f(x)] - u(x) = 1$, for all $x \geq 0$.

Note that we have said *those* functions, not *that* function, for there may be more than one such function. This uniqueness question has not traditionally been raised, for so long as the problem was formulated in terms of linear differential equations, the uniqueness theorems of that branch of mathematics insured only one solution. In the realm of functional equations, we have no such assurances.

It is very lucky that the functional equation which has arisen in this problem is one of the more famous in the

[2] Throughout this paper we shall have to use functions of functions. In general, if v and w are two real-valued functions of a real variable x, $v[w(x)]$ denotes the number obtained by calculating $y = w(x)$ and then finding $v(y)$. Clearly, the order of writing v and w is material, for $v[w(x)]$ does not generally equal $w[v(x)]$. Consider, for example, $v(x) = ax$, where $a \neq 1$, and $w(x) = x^2$. Then, $v[w(x)] = v(x^2) = ax^2$, whereas $w[v(x)] = w(ax) = a^2x^2$.

literature; it is called Abel's equation.[3] The principal results we shall need concerning this equation were presented by Koenigs (5, 6) in 1884 and 1885.[4] First, we will present his uniqueness results, which illustrate the method of attack and lead up to the general solution. Suppose that $u_0(x)$ is a solution to Abel's equation, and suppose $p(x)$ is an arbitrary periodic function with period 1—in other words, any function satisfying

$$p(x + 1) = p(x)$$

$K \sin 2\pi x$ is periodic with period 1, and so is an example of a function $p(x)$. It is easly to show that the function $u_p(x) = u_0(x) + p[u_0(x)]$ is also a solution to Abel's equation:

$$\begin{aligned} u_p[f(x)] &= u_0[f(x)] + p\{u_0[f(x)]\} \\ &= 1 + u_0(x) + p[1 + u_0(x)] \\ &= 1 + u_0(x) + p[u_0(x)] \\ &= 1 + u_p(x) \end{aligned}$$

Furthermore, it can be shown that if u and u^* are two solutions to Abel's equation, then there exists a periodic function p with period 1 such that

$$u(x) = u^*(x) + p[u^*(x)]$$

Thus, if we have any solution u_0 to our problem and if we choose p to be a differentiable periodic function with period 1, then $u_p = u_0 + p(u_0)$ is also differentiable and solves the problem.

In the case of Weber's law, we have $f(x) = kx$, $k > 1$, and the differentiable function $u_0(x) = \dfrac{\log x}{\log k}$ is

easily shown to satisfy the condition of equal sensation jnd's. Therefore,

$$\frac{\log x}{\log k} + p\left(\frac{\log x}{\log k}\right)$$

is also a solution if p is differentiable and periodic with period 1.

There is an infinity of such functions p, and so in infinity of different solutions to the problem for any Weber function, including Weber's law. This, of course, is quite unsatisfactory; later on we will show that one of the properties that we usually attribute to jnd's, and which as yet we have not used, enables us to insure a unique solution. However, first it will be useful to present Koenigs's results on the existence of solutions to Abel's equation.

The existence of solutions to Abel's equation. In psychophysical problems, there is always a threshold $R > 0$, such that $g(x)$ is not observable in the range $0 \leq x \leq R$. Thus, it is only a matter of convenience what we assume about the behavior of g near 0; we shall suppose that

$$g(0) = 0 \quad \text{and} \quad 0 < g'(0) < \infty$$

where $g'(x) = \dfrac{dg}{dx}$. It is known also

from experimental work that g is never 0 and that on the whole it will increase with x, except for limited ranges of some stimuli, where it may decrease slowly. With little or no loss of generality, we may suppose it never decreases so rapidly as to have a slope less than -1. In other words, we also assume:

$$g(x) > 0 \quad \text{and} \quad g'(x) > -1 \quad \text{for} \quad x > 0$$

From these assumptions, it follows that $f(x) = x + g(x)$ has these properties:

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[3] Sometimes this equation is spoken of as the Abel-Schroder equation, but more often Abel's name is attached to this equation and Schroder's name to the equation $v[f(x)] = cv(x)$, which arises from Abel's equation through the substitution $v = c^u$.

[4] We are indebted to Richard Bellman of the RAND Corportion for directing us to the literature on the Abel equation.

f if strictly monotonic in x, i.e., if $x < y$, then $f(x) < f(y)$; 0 is the only fixed point of f (x is a fixed point if $f(x) = x$); and $1 < f'(0) < \infty$.

The strict monotonicity of f implies that there exists an inverse function f^{-1}, i.e., a function such that

$$f^{-1}[f(x)] = x = f[f^{-1}(x)]$$

It is easy to show that:

f^{-1} is strictly monotonic increasing, x is a fixed point of f^{-1} if and only if $x = 0$, $0 < f^{-1'}(0) < 1$.

Observe that if we know a solution v to the equation

$$v[f^{-1}(x)] = 1 + v(x) \qquad [3]$$

then $u = -v$ is a solution to

$$u[f(x)] = 1 + u(x) \qquad [4]$$

So it will suffice to deal with f^{-1}. If, in addition to the three properties mentioned, f^{-1} is analytic, i.e., if there exist constants a_i such that

$$f^{-1}(x) = \sum_{i=0}^{\infty} a_i x^i,$$

then Koenigs has shown that a differentiable solution exists to Abel's equation. In applications, analyticity is no real restriction. For simplicity of notation, let us denote f^{-1} by h; then Koenigs' theorem (which is not easy to prove) may be expressed as follows: Let $h^{(n)}$ denote the nth iterate of h (i.e., $h^{(n)}(x)$ is the result of n successive applications of h beginning at the point x), and let

$$\phi(x) = \lim_{n \to \infty} \frac{h^{(n)}(x)}{[h'(0)]^n}$$

then ϕ exists and is differentiable, and

$$v_0(x) = \frac{\log \phi(x)}{\log h'(0)}$$

is a solution to Abel's Equation 3. Therefore, since $h'(0) = 1/f'(0)$,

$$u_0(x) = \frac{\log \phi(x)}{\log f'(0)}$$

is a solution to Equation 4 and so to our problem.

The difficult part of the proof is to show that the limit exists. Assuming that it does, it is easy to show that $u_0(x)$ is a solution. Since $h[f(x)] = x$, $u_0[f(x)]$

$$= \left\{ \log \lim_{n \to \infty} \frac{h^{(n)}[f(x)]}{[h'(0)]^n} \right\} \Big/ \log f'(0)$$

$$= \left\{ \log \frac{1}{h'(0)} \lim_{n \to \infty} \right.$$
$$\left. \times \frac{h^{(n-1)}(x)}{[h'(0)]^{n-1}} \right\} \Big/ \log f'(0)$$

$$= \frac{\log f'(0) + \log \lim_{n \to \infty} \dfrac{h^{(n-1)}(x)}{[h'(0)]^{n-1}}}{\log f'(0)}$$

$$= 1 + u_0(x)$$

The evaluation of the above limit for ϕ is rarely a simple task. Furthermore, the conditions under which it has been shown to exist and to provide a solution to Abel's equation are only sufficient conditions—there are other circumstances in which solutions exist. For example, the function $f(x) = ax^b$, $b \neq 1$, fails to satisfy $1 < f'(0) < \infty$, yet by direct verification one can show that

$$u_0(x) = \frac{\log \log [a^{1/(b-1)}x]}{\log b}$$

satisfies $u_0(ax^b) = 1 + u_0(x)$. The function $f(x) = x + ax^b$ also fails to meet the same condition, but a solution probably exists in this case too. Presumably, other functions can be found which approximate empirical data and which meet the assumed conditions, but it remains to be seen whether the limit ϕ can be evaluated.

The difficulty is, first, in inverting f, and second, in finding a simple expression for $h^{(n)}$. Since this is generally difficult, we doubt that the mathematics of this section will be useful to psychophysicists who want a non-graphic method for cumulating jnd's.

It should be pointed out again that for the empirically important Weber function $g(x) = kx + c$ the solution is known: it is

$$u_0(x) = \frac{\log (kx + c)}{\log (1 + k)}$$

A further definition of the sensation continuum. So far we have examined two formulations of Fechner's problem, both of which are unsatisfactory. The first, that of Fechner, contains an internal contradiction. The second, the functional equation formulation, we have shown can be solved. Unfortunately, we have also shown that it has infinitely many families of different solutions, which is intolerable. In this section we shall propose an addition to the second formulation which amounts to a method of summating jnd's. We shall show that if we demand a particular form of invariance of distances measured in jnd units, then there is a unique (except for zero and unit) sensation scale for each of a wide variety of Weber functions, and for Weber's law this sensation scale is Fechner's law.

The common psychological custom for measuring distances in jnd's between two points is to use the size of the jnd at the lower point as the unit of measurement. Although it is rarely if ever explicitly stated, it is certainly implicitly assumed that if the distances ab and cd are both α stimulus jnd's in length, then they have an equal number, say $K(\alpha)$, of sensation jnd's. As a formal mathematical condition, this states that

$$u[x + \alpha g(x)] - u(x) = K(\alpha)$$

where K is some fixed, but unknown, function of α. It can be shown, first, that if u is a solution to this problem, then it must be the integral $\int \dfrac{dx}{g(x)}$ given by Fechner, but, second, that there are no solutions except when $g(x) = cx$ (Weber's law). We will not present a proof of this result since it is a blind alley, but we believe that it suggests that this customary measurement of distances should be abandoned.

We must now consider how such distances really should be measured. If x and y are more than one jnd apart, we may expect the size of the jnd to change as we go from x to y. That fact should be taken into account in using jnd's as units of measurement; failure to take it into account is what makes Fechner's auxiliary principle and the standard measuring procedure unacceptable. We shall proceed to formulate this more sensible method of using jnd's as measuring units.

Let $f(x) = x + g(x)$; then the point $f(x)$ is one x-jnd larger than x. The point $f[f(x)] = f^{(2)}(x)$ is one $f(x)$-jnd larger than $f(x)$. In general $f^{(n)}(x)$ is one $f^{(n-1)}(x)$-jnd larger than the point $f^{(n-1)}(x)$. Clearly, for $y > x$, we can find some integer n such that

$$f^{(n)}(x) \leq y < f^{(n+1)}(x)$$

and it is reasonable to say that y is between n and $n + 1$ jnd's larger than x. For the moment, let us suppose that y was chosen so that $y = f^{(n)}(x)$, then we can say y is exactly n jnd's larger than x. It seems plausible to require that the same be true of the sensory continuum, i.e.,

$$u[f^{(n)}(x)] - u(x) = n$$

In words, we are saying that if point y is 20 stimulus jnd's higher than

point x on the stimulus continuum, then it must also be 20 sensation jnd's higher than point x on the sensation continuum. If the above condition is met for $n = 1$ (in other words, if all sensation jnd's for a given sensory continuum are equal), then it must also be met for all larger values of n, since

$$u[f^{(n)}(x)] - u(x)$$
$$= u\{f[f^{(n-1)}(x)]\} - u(x)$$
$$= 1 + u[f^{(n-1)}(x)] - u(x)$$
$$\cdots$$
$$= n$$

But this takes care of relatively few points, and does not allow us to say exactly how many jnd's y is from x unless the difference is a whole number of jnd's. We must find a definition which tells us how to subdivide a jnd into fractional parts. How to do this is not obvious, since the definition of distances given above involves iterates of f, and these are apparently defined only for integers. Fortunately, it is possible to generalize the notion of an iterate to arbitrary, rather than integral, indices. This problem is closely related to that of Abel's functional equation which Koenigs examined; we shall be able to use his results.

First, we can set up some properties that a generalized iterate $f^{(t)}(x)$, where t is any non-negative number, should meet. In essence, they amount to stipulating that $f^{(t)}(x)$ should coincide with the usual definition when t is an integer and that the same law of composition should hold. Formally, it is sufficient to require that

$$f^{(0)}(x) = x, \quad f^{(1)}(x) = f(x)$$

and for every s and $t \geq 0$,

$$f^{(s+t)}(x) = f^{(s)}[f^{(t)}(x)]$$

For integers, the generalized iterate coincides with the usual notion, as you can see, by repeatedly applying the last condition to the second one.

We have already presented a result of Koenigs which showed that if f is strictly monotonic and analytic, $1 < f'(0) < \infty$, and 0 is the only fixed point of f, then there exists a function ϕ defined in terms of the iterates of f^{-1} such that

$$u_0(x) = \frac{\log \phi(x)}{\log f'(0)}$$

is a basic solution to Abel's equation. This means that ϕ is itself a solution to what is called Schroder's equation

$$v[f(x)] = f'(0)v(x)$$

which is obtained from Abel's by taking exponentials on both sides. Using this fact and following Koenigs, it is easy to show that ϕ^{-1} exists and that the function

$$f^{(t)}(x) = \phi^{-1}\{[f'(0)]^t\phi(x)\}$$

satisfies the three conditions of a generalized iterate. We show the latter. First,

$$f^{(0)}(x) = \phi^{-1}[\phi(x)] = x$$

Second,

$$f^{(1)}(x) = \phi^{-1}[f'(0)\phi(x)]$$

And so, using the fact that ϕ satisfies Schroder's equation,

$$\phi[f^{(1)}(x)] = f'(0)\phi(x) = \phi[f(x)]$$

Hence,

$$f^{(1)}(x) = f(x)$$

Finally,

$$f^{(s)}[f^{(t)}(x)]$$
$$= \phi^{-1}[[f'(0)]^s\phi(\phi^{-1}\{[f'(0)]^t\phi(x)\})]$$
$$= \phi^{-1}\{[f'(0)]^s[f'(0)]^t\phi(x)\}$$
$$= \phi^{-1}\{[f'(0)]^{s+t}\phi(x)\}$$
$$= f^{(s+t)}(x)$$

So, with this definition of the generalized iterate, we can generalize the above definition of distances in jnd's to prescribe how to deal with fractional jnd's.

We reformulate our major mathematical problem:

> Given a Weber function g which is analytic, $g'(x) > -1$ for all $x > 0$, $g'(0) > 0$, and $g(0) = 0$, to find those functions $u(x)$ such that $u[f^{(t)}(x)] - u(x) = t$, for all $x > 0$, and all $t > 0$, where $f^{(t)}$ is the generalized iterate of $f(x) = x + g(x)$.

Note that by setting $t = 1$, this condition implies the equality of sensation jnd's.

First, we show that $u_0(x) = \dfrac{\log \phi(x)}{\log f'(0)}$

solves the reformulated problem:

$$u_0[f^{(t)}(x)] - u_0(x)$$

$$= \frac{\log \phi[f^{(t)}(x)]}{\log f'(0)} - \frac{\log \phi(x)}{\log f'(0)}$$

$$= \frac{\log \{[f'(0)]^t \phi(x)\} - \log \phi(x)}{\log f'(0)}$$

$$= \frac{t \log f'(0)}{\log f'(0)}$$

$$= t$$

Second, from the results about Abel's equation, we know that if there are any other solutions to this problem, they must be of the form $u_p = u_0 + p(u_0)$, where p is periodic with period 1. For u_p actually to solve the reformulated problem, it is necessary that for every $t \geq 0$,

$$t = u_p[f^{(t)}(x)] - u_p(x)$$

$$= u_0[f^{(t)}(x)] + p\{u_0[f^{(t)}(x)]\}$$

$$\qquad - u_0(x) - p[u_0(x)]$$

$$= t + p[t + u_0(x)] - p[u_0(x)]$$

Thus, for every $t \geq 0$,

$$p[t + u_0(x)] = p[u_0(x)]$$

That is, p must be periodic with every period t, and so p is a constant. Thus, up to an additive constant, u_0 is the unique function which solves our reformulated problem.

In nonmathematical language, introducing the method of measuring fractional jnd's has enabled us to eliminate all solutions to Abel's equation save u_0, thus cutting down the number of acceptable solutions from infinity to one.

We conclude, therefore, that the condition stated in the reformulated problem constitutes an acceptable definition of a psychophysical sensation continuum, in the sense that it yields a unique Fechner function for any reasonable Weber function. We also find that for Weber's law this condition yields Fechner's law. The solution of our reformulated problem may cause unhappiness because it is not the same as the integral "solution" proposed by Fechner, except in the special case of the linear generalization of Weber's law. However, we have already shown that the integral "solution" contradicts the definition of equal sensation jnd's.

It is sad that the integral is not the right solution, for its evaluation is often easy, and we fear that no working psychophysicist will find in our mathematics a tool for determing a summated jnd scale any better or more efficient than the simple graphic procedure of adding jnd's up one at a time.

The statistical nature of jnd's. So far we have sounded as though we were treating jnd's as fixed quantities, although every psychophysicist knows that jnd's are statistical fictions, defined by an arbitrarily chosen cutoff on a cumulative frequency curve.

However, we now show that our method of reducing the infinity of solutions to Abel's equation to one is equivalent to treating jnd's as just such statistical fictions.

We start with the old, famous psychological rule of thumb: equally often noticed differences are equal, unless always or never noticed. We define $P(y,x)$ as the probability that y is discriminated as larger than x. Now, this rule of thumb simply means that on the sensation continuum the function $P(y,x)$ is transformed in such a way that it no longer depends on x and y separately, but only on the difference of their transformed values. Put another way, the subjective continuum u is a strictly monotonic transformation of the stimulus continuum such that the probability that a change of δ units on the sensation scale will be detected depends only upon δ, and not on the place at which δ begins or ends.

Formally, if we are at a point x of the stimulus continuum, and therefore at $u(x)$ on the sensation scale, and if a stimulus y is presented such that $u(y) = u(x) + \delta$, then the chance that y will be detected depends upon δ, but not on x. If we note that

$$y = u^{-1}[u(x) + \delta]$$

then the condition is that

$$P\{u^{-1}[u(x) + \delta], x\} = P(\delta)$$

Our problem is to decide under what conditions this problem has a solution and what that solution is. To this end, we make the assumption that for each x, $P(y,x)$ is a strictly monotonic increasing function of y.

We show the following: If the above problem has a solution, then there exists a function $f(x)$ such that $P[f^{(t)}(x), x]$ is independent of x, where $f^{(t)}(x)$ is the tth iterate of $f(x)$ previously defined. The function

$f(x) - x$ is a Weber function naturally defined in terms of P. If there is a solution, it is unique and it is the solution u_0 to Abel's equation $u[f(x)] - u(x) = 1$. In other words, if there is any solution to the problem of equally often noticed differences being equal, then it is unique and it is the solution to our proposed reformulation of Fechner's problem.

The proof is comparatively simple and runs as follows. Suppose there exists a solution u to the condition that $P\{u^{-1}[u(x) + \delta], x\}$ is independent of x for all $\delta \geq 0$. Since P is strictly montonic in y for all x, there is a unique solution to $P(y,x) = k$ for each k, $0 < k < 1$; call it $y = f_k(x)$. For any δ, let $k = P(\delta)$, and so by our assumption u must satisfy

$$u^{-1}[u(x) + \delta] = f_\delta(x)$$

where we have written f_δ for $f_{P(\delta)}$. Applying u to this, we have

$$u[f_\delta(x)] - u(x) = \delta$$

Let $f = f_1$. We observe that if $\delta = 0$, then $f_0(x) = x$. Suppose we choose any $\delta, \epsilon \geq 0$ and let $y = f_\epsilon(x)$; then

$$\delta = u[f_\delta(y)] - u(y)$$
$$= u\{f_\delta[f_\epsilon(x)]\} - u[f_\epsilon(x)]$$
$$= u\{f_\delta[f_\epsilon(x)]\} - u(x) - \epsilon$$

Thus,

$$u\{f_\delta[f_\epsilon(x)]\} - u(x) = \delta + \epsilon$$

But, from above,

$$u[f_{\delta+\epsilon}(x)] - u(x) = \delta + \epsilon$$

so

$$u[f_{\delta+\epsilon}(x)] = u\{f_\delta[f_\epsilon(x)]\}$$

whence

$$f_{\delta+\epsilon}(x) = f_\delta[f_\epsilon(x)]$$

Thus, we have shown that f_δ must satisfy the three conditions of a gen-

eralized iterate of f, i.e., $f_\delta = f^{(\delta)}$ for all δ, so a necessary condition for a solution is that

$$P[f^{(\delta)}(x), x]$$

shall be independent of x. From the fact that $u[f_\delta(x)] - u(x) = \delta = u[f^{(\delta)}(x)] - u(x)$, it follows that the solution is uniqe and that it is the same as that given for our reformulation of Fechner's problem.

It probably is not obvious, but the point of this section extends beyond sensory psychophysics into the scaling procedures based on Thurstone's law of comparative judgment. Case V of that law is based on the assumption that equally often noticed differences are equal unless always or never noticed. This fact has two interesting implications. The first and more obvious one is that these two apparently different branches of psychological measurements are actually doing the same thing (namely, using a measure of confusion as a unit of measurement by assuming that confusion is equal at all places on the subjective scale). The second, less obvious implication is that perhaps sensory psychophysics can profit by considering, as Thurstone and his followers have, scaling methods with less rigid assumptions which nevertheless are based on confusability data. One of us (Luce) will pursue this possibility further in a forthcoming book (7).

Graphic methods for cumulating jnd's. Psychophysical data do not come in mathematical form. In order to apply our method for cumulating jnd's (or Fechner's, for that matter), it is necessary either to put the Weber function into equation form, or else to develop a graphic equivalent of the appropriate mathematical operations. The graphical equivalent of Fechner's technique is well known, although rarely used (see, e.g., **15**, pp. 94 and
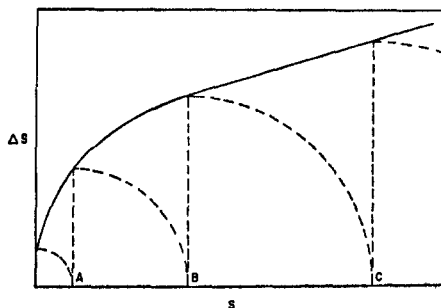

FIG. 1. How to cumulate jnd's. The size of the jnd at the origin is marked off on the x axis to find point A, the size of the jnd at A is marked off to find point B, and so on. The stimulus values $A, B, C, \ldots$ correspond to the points 1, 2, 3, . . . on the cumulated jnd scale.

147–148). It is, of course, wrong, since Fechner's technique is wrong. If our technique is to be of greatest applicability, we should provide a graphic equivalent also. Unfortunately, it seems difficult to find a truly convenient one. The only method we know of is to go back to the basic idea of adding up jnd's—the idea that one jnd plus one jnd is two jnd's. The method of applying this basic idea is given in Figure 1, and was discussed earlier in the paper. Its error characteristics are abour the same as those of the graphic techniques of integration which have been used in the past. Unfortunately, the method is tedious; if there are 170 jnd's between absolute threshold and the upper limit of discrimination, then 170 separate operations are required to determine the cumulated-jnd scale. The errors in these successive operations do not multiply, however.

Practial effects of the new procedure. No doubt it is important to understand Fechner's logical error and to know how to avoid it, but the burning question for working psychophysicists is: What, if anything, does this do to the currently accepted conclusions

about the uselessness of adding up jnd's?

First, it is easy to show that under some circumstances the difference between integration and the functional-equation solution is substantial. Consider the class of Weber functions $g(x) = ax^{1+e}$: if e is greater than zero, the asymptotic error of the integral solution as x approaches infinity is infinite; while if e is less than zero, the asymptotic error is zero. Of course, if e equals zero (Weber's law), the two procedures give identical results. The order of magnitude of the error for small numbers of jnd's depends on the constants in the equation; it can be of significant size even if e is less than zero. One way of looking at it is that the integral solution is the approximation given by the first two terms of a Taylor series expansion of the functional equation; all square and higher power terms of the expansion are omitted:

$$u[x + g(x)] - u(x) = 1$$
$$= u(x) + u'(x)g(x)$$
$$\quad + u''(x)\frac{g(x)^2}{2!} + \ldots - u(x)$$
$$= u'(x)g(x)$$
$$\quad + \left\{ u''(x)\frac{g(x)^2}{2!} + \ldots \right\}$$

A number of experimental determinations of jnd's, particularly for intensive continua, produce a curve of $g(x)/x$ that first falls and then is flat —a function often well approximated by $g(x) = kx + c$. However, for some continua the picture is less simple. There are some (pitch, for example) where the curve appears to rise again at the high end. The falling section of these curves corresponds to the case $e < 0$; the flat section corresponds to the case $e = 0$; the rising section corresponds to the case $e > 0$.

However, the x-axis of such graphs is usually plotted logarithmically. This means that the rising section may cover most of the range within which the stimulus can be varied—a fact which the logarithmic x-axis tends to conceal. So it is quite possible that the error in using the integration technique is substantial for many sense modalities and for large ranges within each.

But the possibility of error is irrelevant unless someone has actually made the error. Has anyone? Extensive examination of the literature suggests that the answer is that not very many such errors have occurred. Some authors are quite unclear about how they added up jnd's, but many of them have preferred the step-by-step method which corresponds to the functional-equation solution because it was very simple to do. How simple it is, of course, depends on the number of jnd's to be added; we doubt very much if the jnd's for pitch will ever be added this way, since there are several thousand of them. We have found only one clear instance (15) in which the graphic equivalent of integration has been used (to cumulate pitch jnd's, as it happens), though it has been vigorously recommended. The general avoidance of the graphic equivalent of integration may be caused by shrewd intuition that something is wrong with Fechner's mathematical auxiliary principle. Or it may simply be a rare instance in which the fear of mathematical complexity has benefited science.

Do cumulated jnd's agree with other scales? The results of cumulating jnd's have often been compared with the results of other psychophysical procedures (4). The most common finding has been that the cumulated jnd scales do not agree with scales de-

termined by fractionation or direct magnitude estimation, at least for such continua as loudness. A review of this literature might seem appropriate here, but it is quite unnecessary, since the relation between scales based on confusion data (like cumulated jnd scales) and those based on fractionation or magnitude estimation has been extensively and excellently discussed in recent studies by Stevens (14), Stevens and Galanter (16), and Piéron (8, 9, 10).

The controversy over the relation between cumulated jnd scales and scales determined by other methods is embedded in a larger, sometimes acrimonious controversy about the relationships among various methods of sensory scaling. To some extent we shall have to enter the fray.

The first and most important question is this: Do the different scaling procedures, if properly used, lead to different scales? Unless we reject a great many experiments as improperly performed, we must answer "Yes." But the issue is not as simple or unambiguous as that answer. For example, Garner (3) has developed a loudness scale based on both fractionation and multisection judgments that fits a large number of experimental results in auditory psychophysics better than does the old sone scale (his paper was written prior to the development of the new sone scale [13]). Figure 2 shows the relationship between that scale and a cumulated jnd scale for loudness prepared by us from Riesz's data (11, 12). The two scales seem to be roughly linearly related—but does it mean anything for the controversy? Riesz's procedure has often been criticized, and his data are almost 30 years old. The form of Garner's scale (which is all that matters for this argument) is based primarily on his multisection
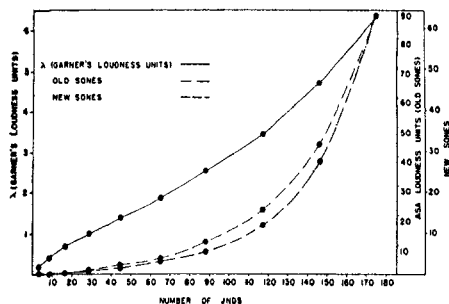

FIG. 2. The relation between Garner's loudness scale and Riesz's cumulated jnd scale. The old sone scale (ASA loudness scale) and Stevens's recent revision of it are included for comparison.

rather than his fractionation data. Scales based upon multisection data usually agree with those constructed by confusability methods; the explanation proposed by critics of these methods is that the adjustment of five or six stimuli in a multisection experiment may produce confusion among the tones being adjusted. If this argument is correct, and if the form of Garner's scale is based upon multisection data, it is not surprising that the two agree. Our reason for so extensive a discussion of Garner's scale is that loudness is the central battleground of this controversy. If the verdict of psychophysical history is that confusability and multisection scales give results different from fractionation results for loudness, then psychologists will amost certainly assume that the two procedures yield different results in other intensive (or, as Stevens calls them, Class I or prothetic) continua. Unfortunately, even in psychophysics, not enough universally accepted data are available to settle the argument.

If confusability scales and scales based upon fractionation or direct magnitude estimation agree, no problem arises. If not (and we suspect

they will not), psychophysicists must still evaluate each kind of procedure and its resulting scale. Some psychophysicists feel that fractionation and magnitude estimation have great face validity, and that confusability scales are distortions of the scales obtained by these procedures. They say that fractionation and estimation scales correspond to what *S*s say they feel, they are obtained by straightforward procedures rather than indirect ones, and, after all, what logic is there in basing a measure of magnitude on variance or "noise."

Other psychophysicists feel that confusability scaling is the better method. They say that fractionation and estimation data are unreliable, variable, and, as a rule, at least fractionation data cannot be turned into scales unless obtained from a "good," which means extensively trained, subject. The estimation techniques have not been used enough times in enough places to indicate clearly what effect, if any, training may have on the results. Confusability scales can be obtained from untrained *S*s who have no idea what form of scale is wanted from them; they can even be obtained from animals.

Each group asserts that its preferred scales are more nearly consistent with the bulk of psychophysical data than the other kind of scales; each group can produce impressive arguments to buttress its claim.

Still another position is possible: perhaps two different kinds of sensory processes are being tapped by these two different kinds of procedures. If so, both kinds of scales are useful, but for different purposes. This could well be the eventual end-point of the argument.

Yet another source of confusion in the argument is the treatment of individual differences. The custom has been to take means or medians, and recently a number of psychophysicists have raised vigorous questions about the appropriateness of doing so. W. J. McGill[5] is currently attempting to find a better way of respecting individual differences while still obtaining a "universal" scale. It will be interesting to see what light serious attempts to do justice to individual differences sheds on the differences between the two classes of scales.

The status of cumulated jnd's has been controversial for more than a hundred years, and this paper is not intended as an attempt to settle the controversy. Our main point is that Fechner's problem has been improperly formulated and that the integral usually offered as a solution is not in fact a solution when the Weber function differs from $g(x) = kx + c$. We have also developed what appears to be the correct solution, only to find that in computational work it has usually been used in spite of its disagreement with the integral solution. This means that our clarification of the logical issues underlying Fechner's formulation does little to change the status of the present, primarily empirical, controversy about scaling methods. However, one of us (Luce [7]) has recently developed a way of dealing with confusability data based on a simple axiom which, if it works out successfully, may resolve the difficulty by changing our ideas about the meaning of confusability scales; this development will be described in another publication.

Summary

Fechner's method for adding up just noticeable differences (jnd's) to obtain sensory scales is based on a

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[5] W. J. McGill, Personal communication, 1957.

mathematical error: he used a differential equation approximation to a functional equation instead of the functional equation itself. The functional equation can, however, be solved directly. The solution coincides with the differential equation solution only in the special case in which the linear generalization of Weber's law holds exactly. The mathematical properties of the formal solution are such that it probably will not be very useful for practical computation, but the extremely simple graphical procedure of adding up jnd's one at a time is the graphical equivalent of the mathematically correct solution. The amount of difference between the two procedures can be calculated for some special cases; its size depends on the form of the function relating size of jnd's to stimulus magnitude.

This error does not seem to have any significant impact upon the controversy over the relation between cumulated jnd scales and scales based on fractionation and direct estimation data because most psychophysicists have, in fact, ignored the recommended (incorrect) procedure and have stubbornly summated jnd's in the obvious and correct way.

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