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The design and evaluation of three competitive bidding models for application in the construction industry.

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THE DESIGN AND EVALUATION OF THREE COMPETITIVE BIDDING MODELS FOR APPLICATION IN THE CONSTRUCTION INDUSTRY

A Dissertation Presented

By

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ABSTRACT

The Design and Evaluation of Three Competitive Bidding Models for Application in the Construction Industry

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This dissertation deals with modeling the recurring bidding decisions made by construction contractors. Construction firms specializing in roadwork construction obtain the majority of their work contracts through open competitive bidding. Since under competitive bidding contracts are awarded to the lowest bidder, participating construction firms must decide upon and submit their bids under the uncertainty of their competitors' similar actions.

Several decision models have been developed which capture the probabilistic nature of the bidding process. The principal approach has been to assign a specific probability distribution to competitor bids and to use this probability distribution in selecting the bid which maximizes the expected value of the contract. The effectiveness of such a probabilistic model in a competitive bidding problem is dependent upon the decision maker's ability to choose the appropriate tractable probability distribution and to solve the necessary optimization problem within the limitations of his or her decision making resources.

The development of the models in this dissertation includes the selection of an appropriate tractable probability distribution, the formulation of an expected value expression, and the computation of an optimal bid, where an optimal bid is the one which maximizes the expected monetary value of the contract. The models discussed were developed under the consideration of the limitations of the decision maker in applying quantitative models. Typically these limitations include the lack of computer facilities and the limited analytical training of the decision maker. In consideration of these constraints, a numerical approximation technique is employed in each modeling approach in determining the optimal bid and bid tables are designed to assist in the required computations.

Three decision models designed for application in the construction industry are developed. For the first model a probabiity distribution of the ratio of the lowest competitor bid to the decision maker's cost estimate is used in computing the expected value of the profit to be received from the contract. Assuming a normal probability distribution, the optimal bid is approximated using the Newton-Raphson approximation method. In the second model, a probability distribution of the ratio of competitor bid to the decision maker's cost estimate is assumed to exist for each competitor. Assuming normal probability distributions and assuming independence among these competitor distributions, an expected value expression is derived. The Newton-Raphson approximation method is employed in approximating the bid which maximizes the expected value expression. The bid to decision maker's cost estimate ratio for each compeitor is assumed to be generated by a normal regression process in the third model. The output of each regression model is used to construct a joint probability distribution which is applied in approximating the optimal bid as in the second model. Tables are constructed for terms contained in the analytical optimization expressions of the three models.

The validity of the assumptions under which the models are developed are tested with empirical bidding data. Tests for goodness of fit and for independence are conducted. Actual bidding results, in terms of contracts won and resulting profits, are compared to the results which would have been obtained by applying the bidding models.

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PREVIOUS WORK IN COMPETITIVE BIDDING MODELS

This dissertation deals with modeling the recurring bidding decisions made by construction contractors. Construction firms specializing in roadwork construction obtain the majority of their work contracts through open competitive bidding. Since under competitive bidding contracts are awarded to the lowest bidder, participating construction firms must decide upon and submit their bids under the uncertainty of their competitors' similar actions.

Several decision models have been developed which capture the probabilistic nature of the bidding process. The principal approach has been to assign a specific probability distribution to competitor bids and to use this probability distribution in selecting the bid which maximizes the expected value of the contract. The effectiveness of such a probabilistic model in a competitive bidding problem is dependent upon the decision maker's ability to choose the appropriate tractable probability distribution and to solve the necessary optimization problem within the limits of his resources.

The development of the models in this dissertation includes the selection of an appropriate tractable probability distribution, the formula of an expected value expression, and the computation of an optimal bid, where an optimal is the one which maximizes the expected monetary value of the contract. The models to be discussed were developed under the consideration of the limitations of the decision maker in applying quantitative models. Typically these limitations include the lack of computer facilities and the limited analytical training of the decision maker. In consideration of these constraints, a numerical approximation technique is employed in each modeling approach in determining the optimal bid and bid tables are designed to assist in the required computations.

A summary of a sample of the research work which has been published in the area of competitive bidding models is presented in Chapter 2. The work discussed covers a broad spectrum of competitive bidding decisions in business, including: corporate securities, oil leases, timber purchases and construction work.

Three decision models designed for application in the construction industry are developed in Chapter 3. For the first model a probability distribution of the ratio of the lowest competitor bid to the decision maker's cost estimate is used in computing the expected value of the profit to be received from the contract. Assuming a normal probability distribution, the optimal bid is approximated, for a given cost estimate, using the Newton-Raphson approximation method. In the second model, a probability distribution of the ratio of competitor bid to the decision maker's cost estimte is assumed to exist for each competitor. Assuming a normal probability distribution for each competitor and assuming independence among these competitor distributions, an expected value expression is derived. The Newton-Raphson approximation method is employed in approximating the bid which maximizes the expected value expression. The bid to decision maker's cost estimate ratio for each competitor is assumed to be generated by a normal regression process in the third model. The output of each regression model is used to construct a joint probability distribution which is applied in approximating the optimal bid as in model two.

In Chapter 4 tables are constructed for terms contained in the analytical optimization expressions of the three models developed in Chapter 3. These tables permit the decision maker to compute an approximation of the optimal bid with the parameters of the respective distributions and a cost estimate, with a few simple hand calculations. Examples of the application of the tables and the approximation technique are presented. The precision of the approximation method is demonstrated by comparing the results of the second modeling approach to the optimal bids obtained by computer simulation.

The validity of the assumptions under which the models are developed are tested with empirical data in Chapter 5. Data from sixty-eight sample contracts are used to test the hypothesis of the normality of the distribution of the lowest competitor bid to cost ratios. From the same sample data, sample bid to cost ratios are extracted for eleven individual contractors in order to test the hypotheses of the normality of each individual distribution. Regression models are derived from this same data for the eleven competitors using four independent variables. The significance of the regression coefficients for the four independent variables are tested for each of the eleven models. The independence among the distributions is tested by extracting paired observations for six pairs of competitors and using the computed correlation coefficients to test the hypotheses that the coefficients equal zero. The actual bidding results, in terms of contracts won and resulting profits, for the sixty-eight sample bids are compared to the results which would have been obtained by applying models one and two to the same sixty-eight bids.

CHAPTER 2

PREVIOUS WORK IN COMPETITIVE BIDDING MODELS

Competitive bidding under conditions of uncertainty has been discussed in quantitative methods literature as it is applied to several business environments, ranging from bidding on construction contracts to corporate bond issues. The approaches can be classified into two general areas; decision and game theoretic. The application of game theory to most bidding decisions encountered in business is limited by the number of participants, which generally exceeds two. The two person game in competitive bidding provides an interesting framework for a theoretic solution, but the situation is rarely encountered in many business applications where competitive bidding is encountered. The decision theoretic approaches vary in degrees of complexity and applicability to actual bidding problems. A review of the work done in developing quantitative models designed to be applied to bidding problems will provide a background and framework for the models to be developed in this research.

The first appearance in the literature of the application of operations research models to the competitive bidding decision was in 1955 in an article by Lawrence Friedman.[1] Much of the subsequent work in the area has been built upon

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[1] Lawrence Friedman, "A Competitive Bidding Strategy," Operations Research, 4 (1956), 104-112.

his initial ideas. While mentioning several objectives which the bidder may have in bidding, the model presented is based upon the objective of maximization of the expected profit resulting from the bid on each individual contract. The general expression for this expected profit for a bid of x is:

$$E(x)=p(x)(x-c')$$

where c' is the estimated cost of completing the contract, x is the bidder's bid and p(x) is the probability of winning with a bid of x. Recognizing the uncertainty of the true cost, a probability distribution, f(s), of the ratio of true cost to estimated cost, s, is used to compute the expected value of the estimated cost, where:

$$c'=c\int_0^\infty sf(s)ds$$

This expected value expression is independent of the bid. The bidder's objective is then to select the bid which maximizes the expected value, E(x), given an expected cost of g'. If the bidder is bidding against n competitors, the probability of winning is the product of the marginal probabilities of winning against each competitor. Friedman extends his model to the case where the bidder is bidding against an unknown number of bidders. In this case a density function, f(r), for an average bidder's bid to cost ratio is used. This bid to cost ratio is the ratio of the competitor's actual bid to the decision maker's estimated cost. The probability of winning when bidding against one average bidder is therefore:

$$p(x) = \int_{x/c'}^{\infty} f(r)dr,$$

which is the probability that the ratio of the average bidder's bid to the decision maker's cost is greater than the ratio of the decision maker's bid to his cost. The probability of winning against k independent average bidders is then:

$$p(x) = \left(\int_{x/c'}^{\infty} f(r)dr\right)^k.$$

When the number of bidders, k, is unknown, it is assumed that a probability density function, g(k), can be determined. The probability of winning when bidding against an unknown number of average bidders can then be expressed as:

$$p(x) = \sum_{k=0}^{\infty} g(k)\left(\int_{x/c'}^{\infty} f(r)dr\right)^k.$$

Friedman suggested that f(r) could be approximated with a gamma distribution and g(k) with a poisson distribution. Substituting these probability functions, the expected value expression becomes:

$$E(x) = (x-c')\exp(-\lambda(1-\sum_{i=1}^{b} (1/1!)(ax/c)^i e^{-ax/c})),$$

where: $f(r) = (a^{b+1}/b!)r^b e^{-ar}$

and: $g(k) = \lambda^k e^{-\lambda}/k!$.

Friedman suggests obtaining the optimal bid graphically and notes that a solution for the optimal bid is not available in closed form.

Edelman discusses the value of a quantitative approach to competitive bidding in a non-mathematical presentation.[2] The model described was incorporated and tested at the Radio Corporation of America. Using a case study as a vehicle, Edelman analyzes the trade-off between the marginal profit if the contract is won and the marginal loss if the contract is lost. Probabilities for winning at various price levels are determined subjectively from management judgment and an optimal trade-off price is selected as the one which maximizes the expected marginal profit contribution.

It is assumed that the contract is not necessarily won by the lowest bidder. Edelman graphs the probability of winning a bid against the percent that protagonist's bid is above or below his competitor's bid. These probabilities are subjectively assigned. The decision maker assigns a likelihood to each of a series of competitor bids, over a particular relevant range. A range of possible protagonist and competitor bids are used to construct a matrix of award probabilities. An example of this matrix is shown in figure 1. The A_{ij} entries in the matrix are obtained from a subjective probability graph as described above. Each pair of competitor

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[2]Franz Edelman, "Art and Science of Competitive Bidding," Harvard Business Review, 43 (August, 1965), 53-66.

FIGURE 1

Computation of winning probabilities for a competitive bidding model by Franz Edelman.

		Competitor Bid $C_1 \; C_2 \; . \; . \; C_j \; . \; . \; . \; C_n$	Expected Probability of winning
Protagonist Bid	B_1	$A_{11} A_{12} \; . \; . \; A_{1j} \; . \; . \; . \; A_{1n}$	P_1
	B_2	$A_{21} A_{22} \; . \; . \; A_{2j} \; . \; . \; . \; A_{2n}$	P_2
	.	. . . .	.
	.	. . . .	.
	B_i	$A_{i1} \; . \; . \; . \; A_{ij} \; . \; . \; . \; A_{in}$	P_i
	.	. . . .	.
	.	. . . .	.
	B_m	$A_{m1} \; . \; . \; . \; A_{mj} \; . \; . \; . \; A_{mn}$	P_m
Likelihood		$L_{C_1} \; . \; . \; . \; L_{c_j} \; . \; . \; . \; L_{C_n}$	

C_j - competitor's bid

B_i - protagonist's bid

A_{ij} - award possibilities if competitor bids C_j and protagonist bids B_i

L_{C_j} - likelihood that competitor will bid C_j

P_i - discounted probability that protagonist will win with bid B_i

and protagonist bids yields a ratio with which a probability of winning can be determined from the graph. Each column entry in the matrix is discounted by the likelihood of the bid associated with the respective column and the resulting discounted probabilities are summed across the rows, yielding the expected probability of winning, P_i, given a bid of B_i. These probabilities are then used to compute the expected marginal profit contribution for each possible bid in order to determine the bid which maximizes this expected value.

A decision model for the competitive bidding situation as encountered in timber purchasing was presented by Taylor.[3] In bidding on timber in government sponsored sealed auctions, each competitor must submit a bid in excess of the United States Forest Service's appraised value for the timber (stumpage). This appraised value is published by the Forest Service prior to the invitation for bids. The firm submitting the highest bid wins the purchase rights to the timber on a specified government owned parcel of land, at the bid price. The profit to the winning firm, P, is:

$$P=R-V-S$$

where R is the market value of the processed timber, V is the cost of processing the timber and S is the bid price.

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[3] Norman Taylor, "A Bidding Model for Timber Purchasing," Research Program in Marketing, Graduate School of Business Administration, University of California at Berkley, special publication of the Institute of Business and Economic Research (1963), 28-44.

For each competitor, Taylor suggests deriving from past bidding behavior a cumulative probability function for the ratio of the bid price to the appraised value. From the respective cumulative probability distribution, the bidder can assess the probability of winning against each competitor. Assuming independence among competitor bids, the probability of winning a contract is equal to the product of the probabilities of winning against each individual competitor. With these probability assessments, expected profit values are enumerated for a range of possible bids and the bid which results in the highest expected profit is chosen. In cases where the bidder is not aware of the identity of his competitors prior to submission of a bid, Taylor suggests using a cumulative probability function for the average bidder.

A Bayesian decision theoretic modeling approach to the modeling decision was presented by Christenson.[4] The application of his work was in the investment banking field, where competitive bidding is encountred in the pricing of corporate securities. The basic structure of the approach was based upon the initial work by Friedman. The value to the bidder of winning a bid, net of all costs except the bid, is normalized to a value of one. The return to the bidder if the bid is won is therefore, $(1-b_o)$, where b_o is the normalized

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[4]Charles Christenson, Strategic Aspects of Competitive Bidding for Corporate Securities (Boston: Division of Research, Graduate School of Business Administration, Harvard University, 1965), 72-89.

value of the bid. Defining $Q(b_o)$ as the probability that each competitor bid is less than b_o, the expected monetary value resulting from a bid of b_o is expressed as:

$$M(b_o)=(1-b_o)Q(b_o).$$

The first order condition for a maximum is obtained by differentiating the above expression with respect to b_o, which yields:

$$M'(b_o)=(1-b_o)q(b_o)-Q(b_o)=0$$

$$1/(1-b_o)=q(b_o)/Q(b_o).$$

Noting the difficulty in assessing the joint probability distribution, $Q(b_o)$, for each possible subset of competitors and the large number of these potential subsets (2^n for n competitors), Christenson suggests deriving a conditional marginal probability distribution for each competitor, which can be assumed to be independent. Defining a vector of characteristics for issued to be bid on, a marginal distribution function for each competitor can be assessed, conditional on this vector of characteristics. It is reasoned that any dependence among the competitor bids is a consequence of their common dependence upon the characteristics of the issue being bid on. Under the assumption of independence, based upon this reasoning, the probability that an issue will be won with a bid of b_o is expressed as:

$$Q(b_o|x_i)=\sum_{j=1}^{n} F_j(b_o|x_i),$$

where x_i represents the characteristics vector of the i^{th} issue and $F_j(b_o|x_i)$ represents the j^{th} competitor's conditional probability distribution for the i^{th} issue.

Christenson develops a procedure for assessing these conditional distributions for each competitor based upon a normal regression process. The theory upon which this approach was based was developed by Raiffa and Schlaifer.[5]

Lavalle also viewed the bidding decision from a Bayesian decision theoretic viewpoint.[6] It is assumed in his work that there are two bidders and that the protagonist is uncertain of both value value of the object being bid on, W, and his opponent's bid, M. The value of the bid to the protagonist, v, is expressed as:

$$v(a,M)= \begin{cases} W-a & \text{if } a \geq M \\ 0 & \text{if } a<M \end{cases}$$

where a is the value of the protagonist's bid. The expected gain to the bidder for a bid of a is:

$$V(a)=E[v(a,M)]=E_W E_{M|W} v(a,M)=E_W(W-a)F_{M|W}(a).$$

Lavalle suggested that an optimal bid a* can be derived from the above expression by a search procedure. If M and

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[5]Howard Raiffa and Robert Schlaifer, Applied Statistical Decision Theory (Boston, 1961), 290-309.

[6]Irving H. Lavalle, "A Bayesian Approach to an Individual Player's Choice of Bid in Competitive Sealed Auctions," Management Science, 13 (March, 1967), 584-597.

W are assumed to be independent, the above expression becomes:

$$V(a)=(E_W W-a)F_M(a),$$

where $E_W W$ is a certainty equivalent. Setting the first derivative of the expected value expression, V(a), equal to zero to satisfy the first order condition for the root a*:

$$v'(a^*)=-F_M(a^*)+E_W W-a^*)f_M(a^*)$$

$$F_M(a^*)/f_M(a^*)=E_W W-a^*.$$

With this result, Lavalle discusses the effects of acquired perfect information on M and W by the protagonist.

Capen, Clapp and Campbell discuss a bidding model which they developed and implemented at the Atlantic Richfield Corporation.[7] Development of the model, which applies to bidding on oil leases, resulted from investigations of the bidding process by Atlantic Richfield's own team of analysts. The paper represents one of the few public discussions of a working bidding model by a source within industry. As Friedman noted in his work, details of successful applications of operations research to the development of bidding strategies ate not ordinarily made public for reasons of industrial security.[8] This inside view of the work being done within industry provides a motivation for external efforts.

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[7] E. C. Capen, R. V. Clapp, and W. M. Campbell, "Competitive Bidding in High Risk Situations," Journal of Petroleum Technology, (June, 1971), 641-653.

[8] Lawrence Friedman, op. cit., p. 106.

As a motivation for their work, the authors cite compiled data from the results of the 1969 Alaska North Slope sale, in which the major oil companies engated in competitive bidding for oil leases. The sum of the winning bids from the sale was $900 million, while the sum of the second highest bids was $370 million, in other words on average the second highest bidder was willing to bid only 41 percent as much as the winner. In addition, in 26 percent of the instances if the second highest bidders had increased their bids by 400 percent, they still would have lost.

The model developed by Capen et al. utilizes maximization of the expected monetary value of the bid as the criterion for bid selection. The value of the bid is the present value of the tract being bid on, discounted at the firm's internal rate of return, net of the amount of the bid. A bid on an oil lease is ordinarily a fraction of the estimated value of the oil reserves recoverable from the tract. This estimated tract value is regarded as a random variable for both the bidder and his competitors. Various values of the bid level, the fraction of the estimated value which is bid, are assumed for the bidder and his competitors in simulating the model. Defining $f_i(\cdot)$ as the probability density function of the i^{th} competitor's bid, and $g(\cdot)$ as the probability density function of the bidder's bid, the probability density function of the bidder's winning bid, x, becomes:

$$h(x)=k_n\left(\prod_{i=1}^{n} F_i(x)\right)g(x),$$

where k_n is a constrant which makes the integral of h(x) over all possible values of x equal to one, and $F_i(x)$ is the probability that the i^{th} competitor bids less than x. The expected value of the winning bid, x_w, is expressed as:

$$E(x_w)=\int_{-\infty}^{\infty} x k_n\left(\prod_{i=1}^{n} F_i(x)\right)g(x)dx.$$

The objective is then to select the value of x_w which maximizes this expected value. It is suggested that the probability distributions for the value estimates of the bidder and the competitors can be approximated with a log-normal probability distribution, although no empirical evidence was presented to justify the selection of this distribution. The optimum bids were selected by computer simulation of the model.

Dougherty and Nozaki also discuss a modeling approach to a competitive bidding situation in the oil industry.[9] As in the work of Capen et al. the values of the tract are estimated by the bidder and his competitors are treated as random variables.[10] The modeling approach assumes that the

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[9] E. L. Dougherty and M. Nozaki, "Determining Optimum Bid Fraction," Journal of Petroleum Technology, (March, 1975), 349-356.

[10] Capen, Clapp, and Campbell, op. cit., p. 646.

the bidder's objective is to select the bid which maximizes the expected value of the gain from the bid. The expected value of the bidder's gain from a tract is given by:

$$EV = \binom{\text{true value of}}{\text{the tract-bid}} \binom{\text{Prob. bidder}}{\text{bids } x} \binom{\text{Prob. Competitors}}{\text{bid less than } x}$$

The bidder's bid x is the product of the bidder's estimate of the value of the tract, v_o, and the bidder's bid fraction, c_o. The objective of the model is to select the optimum bid fraction, the value of c_o corresponding to the maximum value of EV. Assuming that the value estimats of the bidder, v_o, and v_i for the i^{th} competitor are gamma distributed with parameters (λ_o,Γ_o) and (λ_i,Γ_i), respectfully, the probability that the bidder's bid will be between x and x+dx is given by:

$$g_{\Gamma_o}(x)dx=(\Gamma_o/c_o(\Gamma_o x/c_o)^{\Gamma_o -1} e^{-\Gamma_o x/c_o})/(\Gamma_o-1)!dx$$

In this experssion, it is assumed that the mean of the standardized value estimate is one, from which follows:

$$\mu=\Gamma/\lambda=1 \quad \text{therefore} \quad \lambda=\Gamma$$

The variance of the distribution is therefoer:

$$\sigma_{\Gamma,\lambda}{}^2=\Gamma/\lambda^2=1/\Gamma$$

The probability that competitor i will bid less than x is given by:

$$F_i(x)=\int_0^x f_{\Gamma_i}(x)dx=\int_0^x (\Gamma_i/c_i(\Gamma_i x/c_i)e^{-\Gamma_i x/c_i})/(\Gamma_i-1)!dx$$

The expected value of the bid resulting from a bid of x can be expressed as:

$$EV=\int_0^{\infty}(1-x)((\Gamma_o/c_o(\Gamma_o x/c_o)^{\Gamma_o-1}e^{-\Gamma_o x/c_o})/(\Gamma_o-1!).$$
$$(\int_0^{x}(\Gamma_i/c_i(\Gamma_i x/c_i)e^{-\Gamma_i/c_i})/(\Gamma_i-1)!dx)^n dx$$

where n represents the number of competitors. Assuming that $\Gamma_i=1$ for all n competitors, integration of the above expression yields:

$$EV=\sum_{k=0}^{n}((-1)^k\binom{n}{k}/((kc_o)/(\Gamma_o c_i)+1)^{\Gamma_o}(1-c_o/((kc_o)/(\Gamma_o c_i)+1))$$

For assumed values of Γ_o, Γ_i, c_i, and n, a Fibonacci search procedure is used to locate the value of c_o for which the expected value is greatest. Various relationships between the number of competitors and the optimum bid fraction can be examined graphically after simulating the above expression for the expected value.

The case of bidding on a series of contracts when the bid total for the contracts is limited is discussed by Stark and Mayer.[11] Expressing the expected value of contract i as:

$$(b_i-c_i)P(b_i,k_i),$$

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[11]Robert Stark and Robert H. Mayer, "Some Multi-Contract Decision Teeoretic Competitive Bidding Models," Operations Research, 19 (March-April, 1971), 469-483.

where b represents the amount of the bid, c the associated cost of performing the contract, and $P(b_i, k_i)$ the probability of winning the contract with a bid of b_i and a contract size k_i, the expected value of a series of n contracts can be expressed as:

$$E = \sum_{i=1}^{n} (b_i - c_i) P_i (b_i, k_i).$$

If the total amount to be bid is constrained by the bid total B, that is $b_1 + b_2 + \cdot \cdot \cdot + b_n \leq B$, the optimal bid mix, in the case where the unconstrained bid total exceeds B, can be determined by the method of LaGrangian multiplier. The LaGrangian formulation is:

$$L = \sum_{i=1}^{n} (b_i - c_i) P_i (b_i, k_i) + \lambda \left(\sum_{i=1}^{n} b_i - B \right).$$

The constrained optimum bid mix can be determined by solving the simultaneous equations; $\delta L/\delta b_i = \delta L/\delta b_2 = \cdot \cdot \cdot = \delta L/\delta \lambda = 0$. This approach is limited in that it is necessary to assume that bids must be submitted on all n contracts.

Another approach, which was discussed, was to use dynamic programming and consider each bid selection as a stage in the program formulation. Letting $\Pi_i(b_i, s)$ represent the expected profit resulting from an allocation of dollars among the last i contracts, the optimal bid selection for the i^{th} contract would be:

$$b_i{}^*(s)=\mathrm{Max}_{0\leq b_i\leq s}(\Pi_i(b_i,s))$$

$$=\mathrm{Max}_{0\leq b_i\leq s}((b_i-c_i)P_i(b_i,k_i)(1-\delta(b_i))+\Pi_{i-1}{}^*(s-b_i)),$$

where $\delta(u)$ is the Kronecker Delta, that is $\delta(u)=1$ when $u=0$ and zero otherwise.

A third approach, discussed in the work, is to formulate the problem as a zero-one integer programming problem. The range of feasible bids on each contract is divided into a number of intervals, s. The problem of selecting the bid which maximizes the expected value for an individual contract is equivalent to selecting the appropriate interval. The problem would be formulated as:

$$\text{Maximize} \quad z=\sum_{j=0}^{s} x_j(b_j-c)P(b_j,k)$$

$$\text{Subject to} \quad \sum_{j=0}^{s} x_j\leq 1 \quad \text{and } x_j=0 \text{ or } 1,$$

where b_j represents the bid level at the upper extreme of the j^{th} interval. For a series of n contracts, the problem formulation would become:

$$\text{Maximize} \quad z=\sum_{i=1}^{n}\sum_{j=0}^{s} x_{ij}(b_{ij}-c_i)P_i(b_{ij},k_i)$$

$$\text{Subject to} \quad \sum_{j=0}^{s}(x_{ij})\leq 1 \quad \text{for } i=1 \text{ to } n$$

$$\sum_{i=1}^{n}\sum_{j=0}^{s} b_{ij}z_{ij}\leq B$$

$$x_{ij}=0 \text{ or } 1,$$

where B is the total amount to be bid on the n contracts.

Bidding models ordinarily view the selection of an optimal bid from the perspective of one of the n bidders. This approach does not consider the implications of the adoption of similar optimization models by other competitors. In other words, what if each bidder were to bid to maximize his expected value? Rothkopf explores this issue and proves the existence of an equilibrium set of strategies for n bidders.[12] Rothkopf assumes that each bidder is unaware of the true cost of performing the contract, c, and selects a bid based upon a cost estimate, c'. It is further assumed that the ratios of the cost estimates to the actual cost, for all competitors, are independent with known probability distributions. The bidding strategy of competitor i is a function of his cost estimate c_i'. Assuming a multiplicative strategy for the i^{th} competitor, the bid can be expressed as:

$$x_i=h(c_i')=p_ic_i'$$

where p_i is the markup multiplier of the i^{th} competitor and x_i is the bid of competitor i. The cumulative probability distribution of the i^{th} competitor's bid, x_i, is given by $F_i(x_i)$ and the density of x_i by $f_i(x_i)$. Rothkopf assumes a two parameter Weibull distribution for these functions. The

[12]Michael H. Rothkopf, "A Model of Rational Competitive Bidding," Management Science, 15 (March, 1969), 362-373.

expected profit of the i^{th} competitor can be expressed as:

$$E_i=\int_0^{\infty}(x_i-c_i)f_i(x_i)\prod_{j\neq 1}(1-F_j(x_i))dx_i$$

Rothkopf describes a rational bidder as one who will bid to maximize the expected profit of the bid, in other words a rational bidder will select the markup multiplier, p, which maximizes the expected value expression. This maximization can be achieved by setting the partial derivative of the expected value expression, with respect to p, equal to zero. In order to insure a maximum expected value, the second partial derivative of the expected value, with respect to p, must be less than zero. The conditions for an optimal strategy for competitor i are:

$$\delta E_i/\delta p_i=0 \qquad \text{and} \qquad \delta^2 E_i/\delta p_i^2<0.$$

An equilibrium set of strategies for n competitors, $(p_i^*, p_2^* \cdot \cdot \cdot p_n^*)$ exists if the above conditions hold for all n competitors. Under the assumptionof the appropriateness of a two parameter Weibull distribution, Rothkopf solves for the equilibrium set of strategies analytically in the cases where there are n bidders with equal costs and two bidders with unequal costs. No analytical solution was offered for the case of more than two bidders with unequal costs. A table of numerically obtained strategies was presented for three and five bidders. This approach provides a means of selecting an equilibrium point strategy, but under the limitation of the assumption that each competitor behaves in the same rational manner. A single spiteful or ignorant competitor can make this modeling approach useless.

The independence among competitor bids is assumed in most quantitative modeling approaches to the competitive bidding problem. The assumption of independence allows one to express the probability of winning against n competitors as the product of the probabilities of winning agsint each individual competitor. This assumption of independence was questioned, in the case of bidding in the construction industry, by Gates.[13] The alternative proposed by Gates was to determine the probability of winning against n competitors from the equation:

$$P=1/((1-p(A))/p(A)+(1-p(B))/p(B)+ \cdot \cdot \cdot +(1-p(N))/p(N)+1)$$

where p(A) represents the probability of winning against competitor A. This representation was not derived or defended in the article. In response to Gates' article, Stark, while concurring in the general notion that bids may in fact be dependent, questioned Gates' representation of the probability of winning.[14]

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[13] Martin Gates, "Bidding Strategies and Probabilities," Journal of the Construction Division: Proceedings of the American Society of Civil Engineers, (March, 1967), 75-107.

[14] Robert Stark, "Bidding Strategies and Probabilities, Discussion," Journal of the Construction Division: Proceedings of the American Society of Civil Engineers, (January, 1968), 109-112.

Gates, Baumgarten, and Benjamin present a derivation of the equation in a later work.[15] It is reasoned that if a bidder were bidding against two competitors, A and B, the probability of winning over both competitors would be:

$$\frac{P(A \cap B)}{P(A \cup B)} = \frac{P(A)P(B)}{P(A)+P(B)-P(A)P(B)}$$

$$= \frac{P(A)P(B)}{P(A)+P(B)-P(A)P(B)-P(A)P(B)+P(A)P(B)}$$

$$= \frac{P(A)P(B)}{P(A)(1-P(B))+P(B)(1-P(A))+P(A)P(B)}$$

$$= \frac{1}{\frac{1-P(A)}{P(A)} + \frac{1-P(B)}{P(B)} + 1} ,$$

which is the two competitor case of the previously stated general equation. This expression is not the joint probability of winning against both competitors, but rather the conditional probability of winning against both competitors given the bidder wins against at least one of them:

$$P((A \cap B) \mid (A \cup B)) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(A \cap B)}{P(A \cup B)} .$$

It is also interesting to note that in the derivation of an expression designed to present an alternative to the assumption of independence, the relation $P(A \cap B)=P(A)P(B)$ is used repeatedly. This relationship is true only if the events A and B are in fact independent.

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[15] Ralph Baumgarten, Neal Benjamin, and Marvin Gates, "OPBID: Competitive Bidding Strategy Model," Journal of the Construction Division: Proceedings of the American Society of Civil Engineers, (June, 1970), 88-91.

In the absence of a valid alternative, it would appear reasonable to assume independence when the events can be reasoned to be independent. Given the quantity of information which is shared by competitors in bidding, it is reasonable to assume that conditional on this common information the distribution of their bids would be independent.

C H A P T E R 3

THE DESIGN OF THREE COMPETITIVE BIDDING MODELS

Prior to submitting a bid on a contract, a bidder is typically unaware of the bids to be submited by his competitors. It will be assumed in this analysis that the bidder possesses only publicly available information on past competitor bidding behavior and characteristics of the contract being bid on. This assumption is necessary in order to exclude the possibility of collusion and other unfair bidding practices. With this available information on past bidding behavior and contract characteristics, the bidder must select a bid. This bid can be expressed in relation to the bidder's estimated cost of completing the contract, as a ratio of the bid to the cost estimate. A bid to cost ratio of one would mean that the bidder selects a bid equal to his estimated cost. For purposes of clarification in this analysis, the bidding decision will be viewed from the perspective of one bidder, who will be referred to as 'the bidder.' In this analysis, all bids will be expressed in terms of a bid to cost ratio and in all cases the cost used in computing this ratio, will be the estimated cost of the bidder.

The bidder can utilize various criteria in selecting his bid to cost ratio, however the decision is typically based upon intuitive judgment. The criterion upon which this analysis will be based, is the maximization of the expected monetary value of each individual contract.

The probability that a contract is won with a particular bid to cost ratio, is equal to the probability that the bidder's bid to cost ratio is lower than all competitor bid to cost ratios, where each competitor bid to cost ratio is based upon the bidder's estimated cost. Two approaches will be employed in assessing this probability.

In the first approach, it will be assumed that the lowest competitor bid to cost ratio behaves as a random variable. The parameters of the assumed probability distribution can be estimated from past competitor bidding behavior. The probability that the bidder wins a contract with a particular bid ratio can be computed from the estimated distribution by computing the area to the right of the bidder's bid ratio, which is equal to the probability that the lowest competitor bid ratio is greater than the bidder's.

The second approach to estimating the probability of winning with a particular bid ratio, is to view each individual competitor bid to cost ratio as a random variable possessing its own probability distribution. The parameters of each of these assumed distributions can be estimated from historical bidding behavior. The probability that the bidder wins a contract is therefore equal to the probability that each individual competitor bid to cost ratio exceeds the bidder's bid to cost ratio. Assuming independence among the competitor bid to cost ratios, a joint probability distribution can be derived as the product of the marginal distributions of each participating competitor.

Three models, which are designed to approximate the optimal bid under the decision criterion of the maximization of the expected monetary value of each contract, will be presented. The Newton-Raphson technique of approximating the root of an equation will be employed in each of the models to approximate this optimal bid. In the first model it will be assumed that the probability distribution of the lowest competitor bid to cost ratio is a normal distribution whose parameters can be estimated from historical bidding data. In the second model it will be assumed that the distribution of individual competitor bid to cost ratios for each competitor is normal. In the third model it will be assumed that each competitor bid to cost ratio is generated by a normal regression process.

Model I

The difference between the bid to be submitted and the estimated cost of the contract, provides an estimate of the profit to be received from each contract. Let this profit be designated P. Letting the bidder's cost estimate b C and the bidder's bid be B_o, the estimated profit from the contract would be; $P=(B_o-C)$. The expected value of this profit for a bid of B_o, would be estimated profit, if the contract is won, times the probability of winning the bid with a bid of B_o, plus the probability of losing the contract with a bid of B_o times the profit if the contract is lost, which is zero.

The probability distribution of the lowest competitor bid to cost ratio can be estimated, in terms of its parameters, from historical bidding data. Assuming that this distribution is approximately normal, the mean and standard deviation of the past lowest competitor bid ratios will provide unbiased estimates of the required parameters. Let M and S be the estimates of the mean and standard deviation of this distribution of B/C, where B is the lowest competitor bid and C is the bidder's cost estimate, as previously defined. The probability of winning a contract with a bid of B_o is therefore equal to the area under this probability distribution to the right of the bidder's bid ratio of B_o/C. Define:

$$G(B_o/C) = \int_{B_o/C}^{\infty} (2\pi S^2)^{-\frac{1}{2}} \exp -\tfrac{1}{2}(((B/C)-M)/S)^2 d(B/C)$$

This right tail integral represents the probability that the lowest competitor bid ratio will be greater than the bidder's bid ratio of B_o/C, which is therefore the probability that the bidder wins the contract with a bid of B_o.

The profit from a contract, resulting from a bid of B_o, can be expressed as:

$$P = \begin{cases} (B_o - C) & \text{if } B_o/C \text{ is less than } B/C \\ 0 & \text{otherwise} \end{cases}$$

The expected monetary value of the profit from a contract for a bid of B_o can be expressed as:

$$E(P)=(B_o-C)G(B_o/C)+(0)(1-G(B_o/C))$$

$$E(P)=(B_o-C)G(B_o/C). \tag{1}$$

Under the criterion of maximization of the expected monetary value, the bidder would desire to select the bid, B_o*, which maximizes this expected value. This value of B_o which maximizes the expected value will be considered to be the optimal bid and can be computed by setting the first derivative of the expected value (equation 1), with respect to B_o, equal to zero and solving for a root, B_o*. If the second derivative of this expected value equation is negative, this root would yield a maximum value of the expected monetary value and would therefore be the optimal bid. The first derivative of the expected value expression would be:

$$\frac{dE(P)}{dB_o}=(B_o-C)G'(B_o/C)+G(B_o/C), \tag{2}$$

where:

$$G'(B_o/C)=dG(B_o/C)/dB_o=-1/C(2\Pi S^2)^{-\frac{1}{2}}\exp-\tfrac{1}{2}(((B_o/C)-M)/S)^2$$

$$G'(B_o/C)=-\frac{1}{c}g(B_o/C).$$

Setting equation (2) equal to zero:

$$E'(P)=G(B_o{*}/C)-((B_o{*}/C)-1)g(B_o{*}/C)=0 \tag{3}$$

The root which satisfies equation (3), B_o*, is the value of B_o which results in an extreme value of equation (1).

Equation (3) can be expressed in terms of the normal density function as follows:

$$E'(P)=\int_{B_o^*/C}^{\infty}(2\pi S^2)^{-\frac{1}{2}}\exp-\tfrac{1}{2}(((B/C)-M)/S)^2 d(B/C) - ((B_o^*/C)-1)(2\pi S^2)^{-\frac{1}{2}}\exp-\tfrac{1}{2}(((B_o^*/C)-M)/S)^2 \tag{4}$$

Let $z=((B/C)-M)/S$ and $q=((B_o^*/C)-M)/S$ in equation (4). Substituting z and q into equation 4:

$$E'(q)\ \int_{q}^{\infty}(2\pi)^{-\frac{1}{2}}\exp-\tfrac{1}{2}z^2 dz-((qS+M-1)/S)(2\pi)^{-\frac{1}{2}}\exp-\tfrac{1}{2}q^2 \tag{5}$$

Equation (5) can be rewritten in the following form:

$$E'(q)=(1-\Phi(q))-((qS+M-1)/S)\phi(q) \tag{6}$$

where:

$$\Phi(q)=\int_{q}^{\infty}(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}z^2)$$

$$\phi(q)=(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}q^2)$$

Equation (6) can therefore be re-expressed as a function of the variable q:

$$f(q)=(1-\Phi(q))/\ (q)-(qS+M-1)/S=0 \tag{7}$$

The Newton-Raphson method of approximating a root of an equation can be used to approximate a value of q which satisfies equation (7). Since q is the number of standard deviations which the bidder's bid to cost ratio deviates from the mean of the lowest competitor bid to cost ratios, the value of q which satisfies equation (7) yields a value of B_o which satisfies equation (3), since M and S are constants for any contract. In order to employ the Newton-Raphson technique, an initial approximation of the root of the equation must be chosen in order to perform an iteration which will yield a closer approximation. Given the magnitude and the range of the variable q, an initial selection of a value of one for q would always be a reasonably close initial approximation. Each subsequent iteration will yield a closer approximation. The fundamental formula in the Newton-Raphson method is:

$$x_1=x_o-(f(x_o)/f'(x_o))$$

where x_o is an initial approximation of a root of an equation, which is a function of x, and x_1 is a closer approximation than x_o. In order to apply the Newton-Raphson method in approximating a root of equation (7), the first derivative of the function with respect to q must be computed:

$$f'(q)=(\phi(q)(-\phi(q))-(1-\Phi(q))\Phi'(q)/\phi(q)^2-1)$$

$$f'(q)=(-2-((1-\Phi(q))\phi'(q))/\phi(q)^2) \tag{8}$$

Combining equations (7) and (8) in the fundamental formula for the Newton-Raphson method:

$$q_1-q_o-\frac{1-\Phi(q_o))/\phi(q_o)-(q_oS+M+1)/S}{(-2-((1-\Phi(q_o))\phi'(q_o))/\phi(q_o)^2} \tag{9}$$

Equation (8) is the first derivative, with respect to q, of equation (7) and it is therefore also the second derivative, with respect to q, of equation (1), since q is a function of of B_o. Therefore second order conditions for a maximum can be checked by observing the sign of equation (8). If the second derivative is negative, the root obtained by utilizing this approximation will maximize equation (1).

Model II

In this approach it will be assumed that the probability distribution of each competitor's bid to cost ratio is normal with parameters which can be estimated from historic bidding data. It will be assumed that the bid to cost ratios among the competitors are independent. Let (B_i/C) be the bid to cost ratio of the i^{th} competitor, where:

$$(B_i/C) \sim N(\mu_i, \sigma_i^2)$$

The bidder's bid to cost ratio will be denoted by B_o/C. The probability that the bidder wins a contract with a bid of B_o/C is equal to the probability that each competitor's bid to cost ratio is greater than (B_o/C). Assuming independence among the competitor bid to cost ratios, the probability that the bidder wins a contract with a bid ratio of (B_o/C) is equal to the probability that each competitor bid ratio is above (B_o/C), which is equal to the product of the probabilities that each competitor bid ratio is above (B_o/C). Define the following cumulative probability function for each competitor:

$$G_{B_i/C}(x)=\int_x^{\infty}(2\pi S_i^{\,2})^{-\frac{1}{2}}\exp-\tfrac{1}{2}(((B_i/C)-M_i)/S_i)^2 d(B_i/C),$$

where S_i and M_i are unbiased estimators of σ_i and μ_i. The density function for each competitor will be defined as:

$$g_{B_i/C}(x)=(2\pi S_i^{\,2})^{-\frac{1}{2}}\exp-\tfrac{1}{2}((x-M_i)/S_i)^2.$$

The probability that the i^{th} competitor's bid to cost ratio is greater than the bidder's is therefore equal to:

$$G_{B_i/C}(B_o/C)=\int_{B_o/C}^{\infty}(2\pi S_i^{\,2})^{-\frac{1}{2}}\exp-\tfrac{1}{2}(((B_i/C)-M_i)/S_i)^2 d(B_i/C).$$

Under the assumption of independence, the probability that the bidder wins a contract with a bid ratio of (B_o/C), when competing against n competitors, is equal to the product of the n cumulative probability functions:

$$\begin{aligned}\text{the probability that a contract is won with a ratio of } (B_o/C) &= G_{B_1/C}(B_o/C)\cdot G_{B_2/C}(B_o/C)\ldots G_{B_n/C}(B_o/C)\\ &= \prod_{i=1}^{n} G_{B_i/C}(B_o/C).\end{aligned}$$

The profit which the bidder will receive from a contract with a cost of C for which a bid to cost ratio of (B_o/C) is selected can be expressed as:

$$P=\begin{cases}(B_o-C) & \text{if } B_o/C<B_1/C, B_2/C, \cdots B_n/C\\ 0 & \text{otherwise.}\end{cases}$$

The expected monetary value of the profit from a contract for a bid of B_o is therefore equal to:

$$E(P)=(B_o-C)\prod_{i=1}^{n} G_{B_i/C}(B_o/C). \tag{10}$$

Taking the logarithms of both sides of equation (10):

$$\text{Log}(E(P))=\text{Log}(B_o-C)+\sum_{i=1}^{n} \text{Log}(G_{B_i/C}(B_o/C)). \tag{11}$$

A value of B_o which maximizes equation (11) will also yield a maximum value of equation (10). Taking the first derivative of equation (11), with respect to B_o, yields:

$$\frac{d(\text{Log}(E(P)))}{d(B_o)}=1/(B_o-C)-(1/C)\sum_{i=1}^{n}(g_{B_i/C}(B_o/C)/G_{B_i/C}(B_o/C)). \tag{12}$$

Setting equation (12) equal to zero and solving for a root, B_o*, will yield a value of B_o which results in an extreme value of equation (10). Setting equation (12) equal to zero:

$$0=1/(B_o-C)-(1/C)\sum_{i=1}^{n}(g_{B_i/C}(B_o/C)/G_{B_i/C}(B_o/C)).$$

Dividing the numerator and the denominator of the first term in equation (13) by C:

$$0=(1/C)/((B_o/C)-1)-(1/C)\sum_{i=1}^{n}(g_{B_i/C}(B_o/C)/G_{B_i/C}(B_o/C))$$

$$0=1/((B_o/C)-1)-\sum_{i=1}^{n}(g_{B_i/C}(B_o/C)/G_{B_i/C}(B_o/C)). \tag{14}$$

Letting y equal the bidder's bid to cost ratio of B_o/C and substituting y into equation (14):

$$0=1/(y-1)-\sum_{i=1}^{n}(g_{B_i/C}(y)/G_{B_i/C}(y)). \tag{15}$$

Equation (15) can be expressed as a function of the variable y as shown:

$$f(y)=1/(y-1)-\sum_{i=1}^{n}(g_{B_i/C}(y)/G_{B_i/C}(y)). \tag{16}$$

The ratio of the ordinate to the right tail area, as contained in the summation in equation (16) for each competitor, is referred to as the hazard function, where:

$$H(x)=f(x)/(1-F(x))$$

Equation (16) can therefore be re-expressed in the form:

$$f(y)=1/(y-1)-\sum_{i=1}^{n}H_{B_i/C}(y). \tag{17}$$

The value of y for which f(y) is zero yields an extreme value of equation (10). The first derivative of equation (17), with respect to y, would indicate the curvature of equation (10). If the first derivative of equation (17) is negative in the region of the curve around the optimal bid, the curve defined by equation (10) is concave in this same region and the root is therefore a maximum.

The Newton-Raphson method can be employed in approximating a root of equation (17), for which f(y) equals zero.

The first derivative of equation (17), with respect to y, is:

$$f'(y)=(-1/(y-1)^2)-\sum_{i=1}^{n} H'_{B_i/C}(y).$$

Assuming an initial approximation of the root of y_o, the first iteration would yield a second approximation of:

$$y_1=y_o+(1/(y_o-1)-\sum_{i=1}^{n} H_{B_i/C}(y_o))/(1/(y_o-1)^2+\sum_{i=1}^{n} H'_{B_i/C}(y_o)) \quad (18)$$

where:

$$H_{B_i/C}(y)=g_{B_i/C}(y)/G_{B_i/C}(y)$$

$$H'_{B_i/C}(y)=(G_{B_i/C}(y)g'_{B_i/C}(y)+g_{B_i/c}(y)^2)/G_{B_i/C}(y)^2.$$

The approximation of a root of equation (17) would yield a root of equation (10), B_o*, if second order conditions are satisfied. The second order conditions for a maximum would require that the second derivative of equation (10), with respect to B_o, be negative. This condition would be satisfied if the first derivative of equation (17), which is contained in the denominator of the second term in equation (18), is negative.

Model III

In this model it will be assumed that the bid to cost ratio of each competitor is generated by a normal regression process with unknown parameters. The bid to cost ratio of the i^{th} competitor on the j^{th} contract will be defined by y_{ij}. Assume that y_{ij} is generated by the following regression model:

$$y_{ij}=B_{i0}+B_{i1}x_{1j}+B_{i2}x_{2j}+\cdot\cdot\cdot+B_{ik}x_{kj}+e_{ij}$$

Where: Y_{ij} is a typical value of Y_{ij}, the bid to cost ratio of the i^{th} competitor on the j^{th} contract, the dependent variable, $i=1,2,\cdot\cdot\cdot,n$ and $j=1,2,\cdot\cdot\cdot,m$.

$B_{i0},B_{i1},\cdot\cdot\cdot,B_{ik}$ are the population partial regression coefficients of the i^{th} competitor;

$x_{1j},x_{2j},\cdot\cdot\cdot,x_{kj}$ are the observed values characteristics, of the k independent variables for the j^{th} contract.

The following assumptions will be made:

1. The $(x_{1j},x_{2j},\cdot\cdot\cdot,x_{kj})$ terms are fixed variables associated with the j^{th} contract, whose values are known to the bidder prior to submitting a bid.
2. For each combination of the $(x_{1j},x_{2j},\cdot\cdot\cdot,x_{kj})$ terms, there exists a normally distributed subpopulation of Y_{ij} values for the i^{th} competitor on the j^{th} contract.
3. The variances of the subpopulations of Y_{ij} are equal for all combinations of i and j.

4. The Y_{ij} values are independent for each combination of i and j.

5. The e_{ij} values are normally distribued independent random variables with mean zero and variance σ^2_i for the i^{th} competitor.

The least squares estimate of $B_{i0}, B_{i1}, \cdot\cdot\cdot, B_{ik})$, for the i^{th} competitor, can be obtained by minimizing the sum of the squared error terms for a sample of m historic bids with respect to $(B_{i0}, B_{i1}, \cdot\cdot\cdot, B_{ik})$. The sum of the squared error terms can be expressed as:

$$\sum_{j=1}^{m} e_{ij}^{2} = \sum_{j=1}^{m} (y_{ij} - B_{i0} - B_{i1}x_{1j} - B_{i2}x_{2j} - \cdot\cdot\cdot - B_{ij}x_{kj})^2.$$

The solution to this minimization leads to the following sample regression equation for the i^{th} competitor on the j^{th} contract:

$$y_c = b_{io} + b_{i1}x_{1j} + b_{i2}x_{2j} + \cdot\cdot\cdot + b_{ik}x_{kj}.$$

The y_{cij} term is an unbiased point estimator of the mean bid to cost ratio of the i^{th} competitor on the j^{th} contract. The variance of the distribution of bid to cost ratios for the i^{th} competitor on the j^{th} contract can be estimated by the estimate of the variance of the subpopulation of new y_{ij} values for a given set of values of the independent variables from the least squares regression line. The estimate of the variance of the new y_{ij} values, $S^2(y_{ij(new)})$, will be written as $S^2(y_{ijn})$. The distribution of the statistic $(y_{ij}-y_{cij})/s(y_{ijn})$ can be approximated by a student's t distribution with (n-k-1) degrees of freedom. A normal approximation of the distribution can be used for (n-k-1)>30.

The profit to be received from contract j, when m known competitors are submitting bids, can be expressed as:

$$P_j = \begin{cases} (B_{oj}-c_j) & \text{if } y_{oj}<y_{1j}, y_{2j}, \cdots, y_{mj} \\ 0 & \text{otherwise.} \end{cases}$$

Where: $y_{oj}=B_{oj}/C_j$ (the bidder's bid to cost ratio for the j^{th} contract).

The expected value of contract j is therefore equal to:

$$E(P_j)=(B_{oj}-c_j)G_{y_{1j}}(y_{oj})G_{y_{2j}}(y_{oj}) \cdots G_{ymj}(y_{oj})$$

$$E(P_j)=(B_{oj}-C_j)\prod_{i=1}^{m} G_{y_{ij}}(y_{oj}). \tag{19}$$

Where: $G_{y_{ij}}(y_{o_j}) = \int_{y_{oj}}^{\infty} (2\pi S(y_{ijn}))^{-\frac{1}{2}} \exp -\tfrac{1}{2}((y_{ij}-y_{cij})/S(y_{ijn}))^2 dy_{ij}$

Equation (19) is equivalent to equation (10) contained in the discussion of Model II. The value of B_{oj} which maximizes equation (19) can be approximated by application of the Newton-Raphson method, as developed in Model II. Taking the logarithm of both sides of equation (19), and setting the first derivative with respect to B_{oj}, equal to zero yields:

$$\frac{d(\mathrm{Log}(E(P_j))}{d(B_{oj})} = 1/(B_{oj}-C_j)-(1/C_j)\sum_{i=1}^{m} g_{y_{ij}}(y_{oj})/G_{y_{ij}}(y_{oj})=0. \tag{20}$$

Dividing the numerator and denominator of the first term by C_j and multiplying the equation by C_j yields:

$$0=1/(y_{oj}-1)-\sum_{i=1}^{m} g_{y_{ij}}(y_{oj})/G_{y_{ij}}(y_{oj}). \tag{21}$$

Re-expressing equation (21) in terms of the hazard function and writing equation (22) as a function of y_{oj}:

$$0=1/(y_{oj}-1)-\sum_{i=1}^{m} H_{y_{ij}}(y_{oj})=f(y_{oj}). \tag{22}$$

The first derivative of equation (22) equals:

$$f'(y_{oj})=(-1/(y_{oj}-1)^2)-\sum_{i=1}^{m} H'_{y_{ij}}(y_{oj}).$$

Where: $H'_{j_{ij}}(y_{oj})=(G_{y_{ij}}(y_{oj})g'_{y_{ij}}(y_{oj})+g_{y_{ij}}(y_{oj})^2)/G_{y_{ij}}(y_{oj})^2.$

The first iteration of the Newton-Raphson method would yield an approximation of:

$$y_1=y_0+(1/(y_{oj}-1)-\sum_{i=1}^{m} H_{y_{ij}}(y_{oj}))/(1/(y_{oj}-1)^2+\sum_{i=1}^{m} H'_{y_{ij}}(y_{oj})). \tag{23}$$

Each subsequent iteration will yield a value of B_{oj} closer to the optimal bid, $B_{oj}{}^*$, for which equation (19) reaches an actual extreme value. The curvature of equation (19) in the neighborhood of the extreme value can be checked by observing the sign of the derivative of equation (22), which is contained in the denominator of the second term in equation (23).

In each of these models the underlying approach has been to express the optimal bid ratio in an equation whose roots can be approximated by numerican methods. The computation of the iterative formulae for approximating this optimal bid ratio, equations (9), (18) and (23), can be simplified by the tablization of several of the terms contained therein. This tablization for ease of computation, will be presented in the next chapter.

CHAPTER 4

THE DESIGN OF BIDDING TABLES FOR CONSTRUCTION CONTRACT BIDDING

Computation of an approximation of the optimal bid ratio by use of the models presented in the previous chapter, can be facilitated by the combination and tablization of several of the terms contained in equations (9), (18), and (23). The use of such tables will permit rapid computation of the first iteration and therefore a quick approximation of the optimal bid ratio.

For model one, the first iteration would yield an approximation which is based upon an initial guess, expressed as a number of standard deviations from the mean of the lowest competitor bid to cost ratios, and a function of the initial guess. Restating equation (9):

$$q_1 = q_o - \frac{(1-\Phi(q_o))/\phi(q_o) - (q_oS+M-1)/S}{(-2-((1-\Phi(q_o))\phi'(q_o))/\phi(q_o)^2)}$$

In this expression, q_o denotes an initial guess at the number of standard deviations from the mean, and q_1 represents a second approximation, which is closer than q_o to the value of q which maximizes the expected value of the profit from the contract. The terms:

$$(1-\Phi(q_o))/\phi(q_o) \quad \text{and} \quad (-2-((1-\Phi(q_o))\phi'(q_o))/\phi(q_o)^2)$$

are functions of the initial guess, q_o. Expressing these terms as functions u and v, respectively, of q_o, equation

(9) can be rewritten as follows:

$$q_1 = q_o - \frac{u(q_o) - (q_o S + M - 1)/S}{v(q_o)} \quad .$$

The functions u and v have been computed for q_o values ranging from -2.39 to 2.39 and are contained in Appendix I.

As an example of the application of model one, consider the case in which a contractor must decide on a dollar value of a bid to be submitted on a contract which has an estimated cost of $50,000. Assume that historic bidding data indicates that the distribution of lowest competitor bid to cost ratios is approximately normal with a mean of 1.1 and a standard deviation of .20. By computing the probability of winning and the expected value of the profit for a range of possible bids, the optimal bid can be approximated. For this example, the results of such an enumeration process are shown in Table 1. This enumeration indicates that the bid which maximizes the expected value of the profit is $59,000, for an expected profit of $3,101. The value of q which equates equation (7) to zero is the value of q which will maximize the expected value of the profit. This value of q which equates equation (7) to zero can be determined graphically, as shown in Figure (1), where f(g) equals zero for q equal to .42, which would be a bid of $59,200 and an expected profit of $3,102. Each of these methods provides a means to obtain a close approximation of the optimal bid, but with considerable computational effort.

TABLE 1. Computation of the expected value for a bid range of $50,000-$70,000. (one competitor)

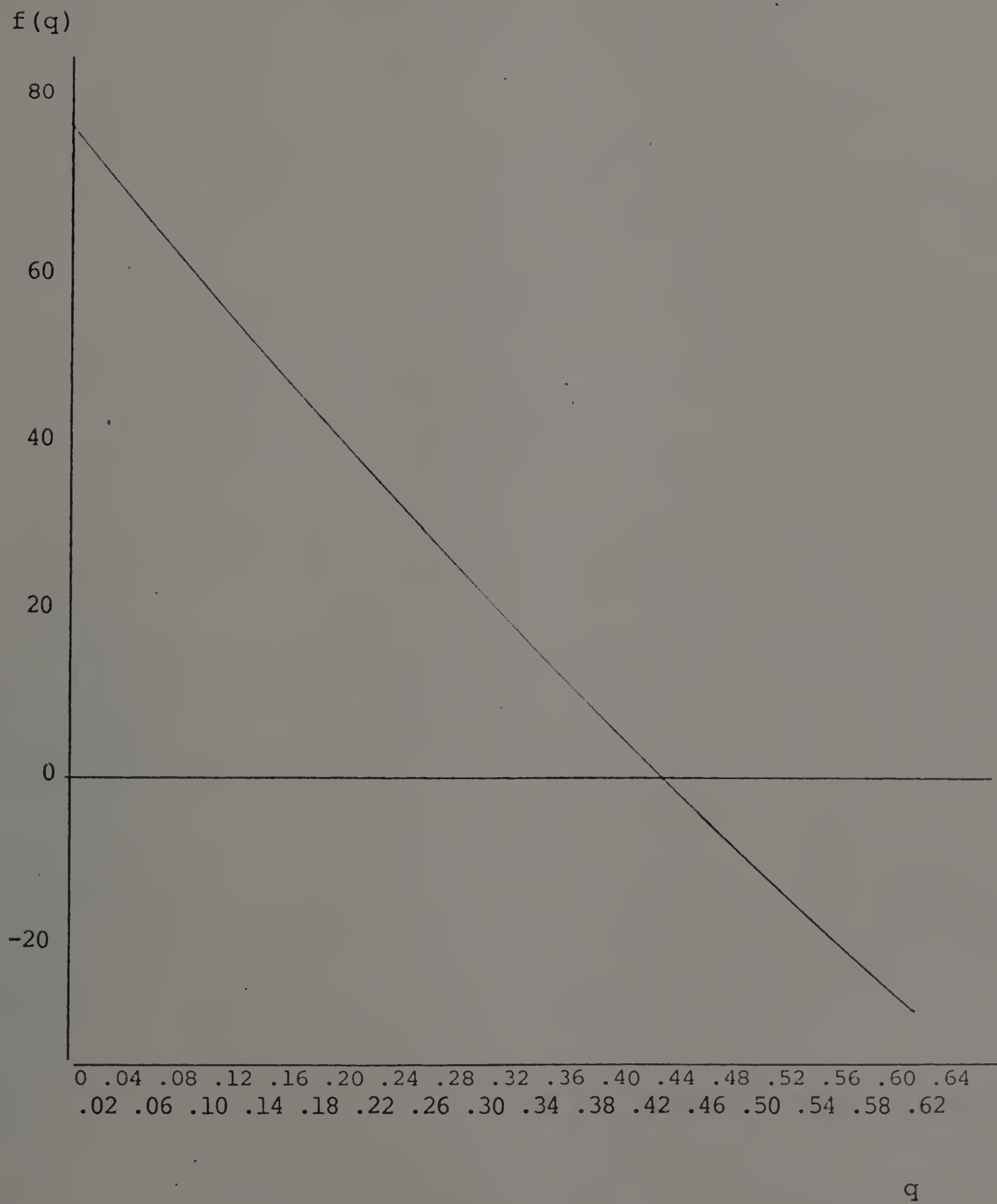
Bid	Bid to cost ratio	No of Std Dev fm Mean	Prob of winning	Profit if contract won	Expected Monetary Value
$50,000	1.00	-.50	.6915	$ 0	$ 0
50,500	1.01	-.45	.6736	500	337
51,000	1.02	-.40	.6554	1,000	665
51,500	1.03	-.35	.6368	1,500	955
52,000	1.04	-.30	.6179	2,000	1,236
52,500	1.05	-.25	.5987	2,500	1,497
53,000	1.06	-.20	.5793	3,000	1.738
53,500	1.07	-.15	.5596	3,500	1,959
54,000	1.08	-.10	.5398	4,000	2,159
54,500	1.09	-.05	.5199	4,500	2,340
55,000	1.10	.00	.5000	5,000	2,500
55,500	1.11	.05	.4801	5,500	2,641
56,000	1.12	.10	.4602	6,000	2,761
56,500	1.13	.15	.4404	6,500	2,863
57,000	1.14	.20	.4207	7,000	2,945
57,500	1.15	.25	.4013	7,500	3,010
58,000	1.16	.30	.3821	8,000	3,057
58,500	1.17	.35	.3632	8,500	3,087
59,000	1.18	.40	.3446	9,000	3,101
59,500	1.19	.45	.3264	9,500	3,101

TABLE 1 (continued). Computation of the expected value for a bid range of $50,000-$70,000. (one competitor)

Bid	Bid to cost ratio	No of Std Dev fm Mean	Prob. of winning	Profit if contract won	Expected Monetary Value
$60,000	1.20	.50	.3085	$10,000	$3,085
60,500	1.21	.55	.2912	10,500	3,058
61,000	1.22	.60	.2743	11,000	3,017
61,500	1.23	.65	.2578	11,500	2,965
62,000	1.24	.70	.2420	12,000	2,904
62,500	1.25	.75	.2266	12,500	2,833
63,000	1.26	.80	.2119	13,000	2,755
63,500	1.27	.85	.1977	13,500	2,669
64,000	1.28	.90	.1841	14,000	2,577
64,500	1.29	.95	.1711	14,500	2,481
65,000	1.30	1.00	.1587	15,000	2,381
65,500	1.31	1.05	.1469	15,500	2,277
66,000	1.32	1.10	.1357	16,000	2,171
66,500	1.33	1.15	.1251	16,500	2,064
67,000	1.34	1.20	.1151	17,000	1,957
67,500	1.35	1.25	.1056	17,500	1,848
68,000	1.36	1.30	.0968	18,000	1,742
68,500	1.37	1.35	.0885	18,500	1,637
69,000	1.38	1.40	.0808	19,000	1,535
69,500	1.39	1.45	.0735	19,500	1,433
70,000	1.40	1.50	.0668	20,000	1,336

FIGURE I

Graph of equation (7)



In applying model one, equation (24) can be used to approximate the optimal bid to cost ratio, and therefore the optimal bid. It is necessary to initialize the model with a crude approximation of the optimal bid to cost ratio. For this example, assume that the first approximation was 1.1, or \$55,000, this bid to cost ratio would be equal to the mean of the lowest competitor bid to cost ratios, therefore q_o equals zero.

Applying model one, equation (24), the first iteration would be:

$$q_1 = 0 - \frac{1.25345 - .50}{-2.0} = .3767.$$

The dollar value of a bid .3767 standard deviations above the mean of the lowest competitor bid to cost ratios equals:

$$(1.1 + .3767(.2))\$50,000 = \$58,767.$$

This bid yields an expected value of the profit of:

$$(\$58,767 - \$50,000).3520 = \$3,085.98.$$

Iterating a second time:

$$q_2 = .38 - \frac{.94828 - .88}{-1.6398} = .38 + .0416 = .4216$$

The dollar value of the bid and the expected profit for a q value of .4216 would be:

$$(1.1 + .4216(.2))\$50,000 = \$59,216$$

$$(\$59,216 - \$50,000).3372 = \$3,107.63.$$

A third iteration indicates that this value is close to the actual maximum value of the expected profit:

$$q_3 = .42 - \frac{.92308-.92}{-1.61237} = .42+.0019=.4219.$$

The dollar value of the bid and the expected profit for a value of q of .4219:

(1.1+.4219(.2))\$50,000=\$59,219

(\$59,219-\$50,000).3372=\$3,108.65.

The first iteration yielded an approximation which was within \$300 of the optimal bid as computed graphically, by enumeration, and by three iterations.

For models two and three, the optimal bid to cost ratio will be the value of the bid to cost ratio, y, which equates equations (15) and the equivalent equations (22) to zero. Approximations of this optimal bid to cost ratio can be obtained by iterating equation (18) or the equivalent equation (23). Restating equation (18):

$$y_1 = y_o + (1/(y_o-1) \sum_{i=1}^{n} H_{B_i/C}(y_o))/(1/(y_o-1)^2 + \sum_{i=1}^{n} H'_{B_i/C}(y_o)).$$

In this expression y_o is an initial approximation of the optimal bid ratio, the value of y which satisfies equations (15) and (22), and y_1 is a closer approximation. The term $H_{B_i/C}(y_o)$ is the hazard function of competitor i and $H'_{B_i/C}(y_o)$ is the first derivative of the i^{th} competitor's hazard function with respect to y_o. The hazard function, H(y), is a function of the density function and the cumulative density:

$$H(y)=g(y)/G(y).$$

The derivative of the hazard function H'(y), is a function of the density function, the cumulative density, and the first derivative of the density:

$$H'(y)=(G(y)g'(y)+g(y)^2)/G(y)^2.$$

Values of H(y) and H'(y), for a particular density function, are therefore functions of the parameters of the density function and the value y. In the case of a normal density, values of H(y) and H'(y) will be defined for values of the mean, standard deviation, and y. The value y_o contained in equation (18) is the initial approximation of the optimal bid to cost ratio. If the same initial value, y_o, is assumed each time that the model is used, values of H(y) and H'(y) can be tablized for combinations of values for the mean and standard deviation. Appendix II contains values of H(y) and H'(y) for combinations of the mean and standard deviation ranging from 1.00 to 1.30 and .01 to .40, respectively, for a y_o value of 1.1.

This initial value of y_o of 1.1 was chosen because empirical evidence has indicated that the lowest competitor bids are on average approximately ten percent above the bidder's cost estimate. The mean of a sample of thirty-six lowest competitor bid to cost ratios was 1.092. Therefore an initial approximation of 1.1 would be expected to be close to the optimal bid ratio. The range of the mean includes all feasible values for the mean bid to cost ratio for any competitor. A competitor would not be expected, on average, to bid below the bidder's cost, nor on average more than twice the bidder's cost. In the selection of values for the range of standard deviations, it was reasoned that a standard deviation of less than .01 would indicate incredibly consistent bidding behavior, which would be unlikely, and a value greater than .3 would extend the ninety-five percent confidence limits beyond the range of feasible bids for any value of the mean.

As an example of the application of the tables contained in Appendix II, consider the case of bidding against three competitors with the following parameters of their respective bid to cost ratio distributions:

Competitor	Mean	Standard Deviation
1	1.05	.16
2	1.10	.14
3	1.15	.12

It will be assumed that the distributions are normal. The bidder's objective would be to select the bid which maximizes the expected value of the profit resulting from the contract, given the above competitor parameters. This expected value for a bid of B_i can be expressed as:

$$E(P)=(B_i-C)\prod_{k=1}^{3} G_{B_k/C}(B_i/C).$$

where $G_{B_k/C}(B_i/C)$ represents the probability that the bid to cost ratio of competitor k exceeds B_i/C.

Assume that the bidder is bidding against the above competitors on a contract with an estimated cost of \$80.000. The expected value of the profit resulting from the contract for a bid range of \$83,840 to \$87,120 is presented in Table 2. This enumeration of a selected range of bids indicates that the optimal bid is approximately \$85,700 with an expected value of the profit of \$1,103.

An approximation of this optimal bid can be computed from equation (18):

$$y_1=y_o+(1/(y_o-1)-\sum_{i=1}^{n} H_{B_i/C}(y_o))/(1/(y_o-1)^2+\sum_{i=1}^{n} H'_{B_i/C}(y_o).$$

Assuming an initial value of y_o of 1.1, equation (18) becomes:

$$y_1=1.1+(10-\sum_{i=1}^{n} H_{B_i/C}(1.1))/(100+\sum_{i=1}^{n} H'_{B_i/C}(1.1)).$$

Values of $H_{B_i/C}(1.1)$ and $H'_{B_i/C}(1.1)$ are tablized in Appendix II for selected values of the mean and standard deviation of competitors' probability distributions of their respective bid to cost ratios.

The appropriate values of $H_{B_i/C}(1.1)$ and $H'_{B_i/C}(1.1)$ for the three competitors, in this sample, obtained from Appendix II are shown in Table 3. As shown in Table 3, the iteration described in equation (18) would yield a second approximation of the optimal bid to cost ratio of 1.067. A bid to cost ratio of 1.067 for a cost estimate of \$80,000 would mean a bid of:

Bid=(1.067)\$80,000=\$85,360.

TABLE 2. Computation of the expected value for a bid range of \$83,840-\$87,120. (three competitors)

Bid(B_i)	Bid to cost ratio	Profit if contract won	Prob Comp 1 Bid Bi	Prob Comp 2 Bid Bi	Prob Comp 3 Bid Bi	Prob of win with Bid of Bi	Expected Monetary Value
$83,840	1.048	$3,840	.50499	.64484	.80234	.26127	$1,003
83,920	1.049	3,920	.50249	.64218	.80001	.25815	1,012
84,000	1.050	4,000	.50000	.63951	.79767	.25506	1,020
84,080	1.051	4,080	.49751	.63683	.79531	.25198	1,028
84,160	1.052	4,160	.49501	.63415	.79294	.24891	1,035
84,240	1.053	4,240	.49252	.63146	.79055	.24586	1,042
84,320	1.054	4,320	.49003	.62876	.78814	.24283	1,049
84,400	1.055	4,400	.48753	.62606	.78572	.23982	1,055
84,480	1.056	4,480	.48504	.62335	.78328	.23683	1,061
84,560	1.057	4,560	.48255	.62063	.78083	.23385	1,066
84,640	1.058	4,640	.48006	.61791	.77836	.23089	1,071
84,720	1.059	4,720	.47757	.61519	.77588	.22795	1,076
84,800	1.060	4,800	.47508	.61245	.77337	.22505	1,080
84,880	1.061	4,880	.47259	.60971	.77085	.22212	1,084
84,960	1.062	4,960	.47011	.60697	.76832	.21923	1,087
85,040	1.063	5,040	.46762	.60422	.76577	.21637	1,090
85,120	1.064	5,120	.46514	.60147	.76321	.21352	1,093
85,200	1.065	5,200	.46265	.59871	.76063	.21069	1,096
85,280	1.066	5,280	.46017	.59594	.75804	.20788	1,098
85,360	1.067	5,360	.45769	.59317	.75543	.20509	1,099

TABLE 2 (continued). Computation of the expected value for a bid range of $83,840-$87,120. (three competitors)

Bid(B_i)	Bid to cost ratio	Profit if contract won	Prob Comp 1 Bid Bi	Prob Comp 2 Bid Bi	Prob Comp 3 Bid Bi	Prob of win with Bid of Bi	Expected Monetary Value
$85,440	1.068	$5,440	.45521	.59040	.75280	.20232	$1,101
85,520	1.069	5,520	.45274	.58762	.75016	.19957	1,102
85,600	1.070	5,600	.45026	.58484	.74751	.19684	1,102
85,680	1.071	5,680	.44779	.58205	.74484	.19413	1,103
85,760	1.072	5,760	.44532	.57926	.74215	.19144	1,103
85,840	1.073	5,840	.44285	.57646	.73945	.18877	1,102
85,920	1.074	5,920	.44038	.57367	.73674	.18612	1,102
86,000	1.075	6,000	.43792	.57086	.73401	.18350	1,101
86,080	1.076	6,080	.43546	.56806	.73127	.18089	1,100
86,160	1.077	6,160	.43300	.56525	.72852	.17830	1,098
86,240	1.078	6,240	.43054	.56243	.72575	.17574	1,097
86,320	1.079	6,320	.42808	.55962	.72296	.17320	1,095
86,400	1.080	6,400	.42563	.55680	.72016	.17067	1,092
86,480	1.081	6,480	.42319	.55398	.71735	.16817	1,090
86,560	1.082	6,560	.42074	.55115	.71453	.16569	1,087
86,640	1.083	6,640	.41830	.54832	.71169	.16324	1,084
86,720	1.084	6,720	.41586	.54449	.70884	.16080	1,081
86,800	1.085	6,800	.41342	.54266	.70598	.15839	1,077
86,880	1.086	6,880	.41099	.53983	.70310	.15599	1,073
86,960	1.087	6,960	.40856	.53699	.70021	.15362	1,060
87,040	1.088	7,040	.40613	.53415	.69731	.15127	1,065
87,120	1.089	7,120	.40371	.51313	.69439	.14894	1,060

TABLE 3. Computation of an approximation of the optimal bid to cost ratio when bidding against three competitors, using model two and Appendix II.

Competitor	Mean	Standard Deviation	H(1.1)	H'(1.1)
1	1.05	.16	6.29181	27.29817
2	1.10	.14	5.69803	32.46756
3	1.15	.12	4.60666	37.21677

$$\sum_{i=1}^{3} H(1.1)=16.5965 \qquad \sum_{i=1}^{3} H'(1.1)=96.9825$$

$$y_1-1.1+\frac{10-16.5965}{100-96.9825} = 1.067$$

Referring to Table 2, a bid of $85,360 would have an expected profit of $1,099 or within $4 of the expected value resulting from a bid of $85,700, obtained by means of the enumeration method. One iteration using this approximation technique, therefore, results, in a close approximation of the optimal bid to cost ratio. It should be noted also that the degree of difficulty in applying the approximation technique does not increase with the number of bidders, as does the enumeration method.

The accuracy of the approximation method used in models one and two was demonstrated by comparing the bids computed by application of the bidding tables with computer simulated optimal bids. Fifty simulated bidding problems were generated by randomly choosing five pairs of means and standard deviations and computing the bid which maximizes the expected value of the bid for a cost estimate of $80,000. The means and standard deviations for each simulated set of five competitors were used to compute an approximation of the optimal bid by using model two and the tables contained in Appendix II. A summary of the results is displayed in Table 4. For the contract with an estimated cost of $80,000, the average absolute value of the difference between the model bid and the bid which actually maximizes the expected value was $33.46. On average the approximated optimal bid deviated less than four one hundredths of one percent from the actual optimal bid.

Table 4. Optimal bids and model two approximations for fifty computer simulated bids against five competitors.

Sim	Comp #1		Comp #2		Comp #3		Comp #4		Comp #5		Actual	Model	Abs
Run	M	S	M	S	M	S	M	S	M	S	Optimal	Apprx	Diff
1	1.24	0.17	1.30	0.13	1.19	0.24	1.22	0.16	1.19	0.16	$87600	$87577	23
2	1.19	0.23	1.17	0.16	1.24	0.17	1.18	0.24	1.31	0.09	87680	87637	43
3	1.19	0.09	1.17	0.23	1.22	0.24	1.17	0.09	1.24	0.09	87880	86831	49
4	1.28	0.09	1.28	0.09	1.19	0.09	1.17	0.19	1.19	0.09	87760	87773	13
5	1.30	0.09	1.17	0.16	1.31	0.09	1.24	0.16	1.17	0.24	88640	88610	30
6	1.31	0.23	1.24	0.09	1.31	0.13	1.31	0.09	1.31	0.13	90640	90498	142
7	1.24	0.16	1.22	0.09	1.31	0.11	1.17	0.23	1.19	0.19	87920	87943	23
8	1.17	0.16	1.22	0.19	1.24	0.09	1.22	0.13	1.17	0.09	86800	86734	66
9	1.17	0.11	1.24	0.11	1.30	0.09	1.22	0.19	1.28	0.24	87920	87939	19
10	1.22	0.23	1.18	0.24	1.17	0.11	1.28	0.09	1.24	0.23	87440	87444	4
11	1.31	0.19	1.19	0.17	1.22	0.23	1.24	0.16	1.18	0.24	87520	87515	5
12	1.30	0.13	1.18	0.09	1.31	0.23	1.19	0.23	1.22	0.13	87600	87619	19
13	1.22	0.16	1.28	0.19	1.18	0.11	1.28	0.16	1.22	0.16	87360	87330	30
14	1.24	0.23	1.22	0.09	1.24	0.16	1.28	0.13	1.28	0.16	88480	88475	5
15	1.28	0.16	1.18	0.23	1.30	0.17	1.18	0.17	1.31	0.13	88320	88299	21
16	1.17	0.09	1.22	0.09	1.24	0.19	1.17	0.23	1.17	0.23	86720	86691	29
17	1.19	0.16	1.24	0.19	1.17	0.19	1.22	0.19	1.31	0.23	87200	87121	79
18	1.30	0.09	1.18	0.17	1.24	0.24	1.19	0.13	1.31	0.13	88240	88235	5
19	1.30	0.17	1.22	0.09	1.22	0.17	1.28	0.16	1.22	0.23	88240	88248	8
20	1.30	0.09	1.22	0.09	1.19	0.17	1.28	0.11	1.28	0.13	89040	89010	30
21	1.22	0.09	1.22	0.19	1.30	0.09	1.22	0.23	1.22	0.23	88400	88432	32
22	1.22	0.11	1.17	0.16	1.18	0.09	1.24	0.17	1.17	0.09	86400	86303	97
23	1.22	0.19	1.22	0.09	1.28	0.09	1.30	0.24	1.22	0.13	88480	88474	6
24	1.17	0.23	1.28	0.13	1.18	0.24	1.28	0.23	1.28	0.23	88160	88164	4
25	1.28	0.09	1.22	0.24	1.31	0.09	1.24	0.09	1.31	0.19	90160	90043	17

Table 4 (continued). Optimal bids and model two approximations for fifty computer simulated bids against five competitors.

Sim Run	Comp #1 M	Comp #1 S	Comp #2 M	Comp #2 S	Comp #3 M	Comp #3 S	Comp #4 M	Comp #4 S	Comp #5 M	Comp #5 S	Actual Optimal	Model Apprx	Abs Diff
26	1.22	0.17	1.17	0.09	1.18	0.17	1.22	0.23	1.24	0.09	$86880	$86836	44
27	1.24	0.11	1.28	0.11	1.24	0.23	1.22	0.17	1.22	0.09	88320	88328	8
28	1.31	0.24	1.31	0.23	1.22	0.09	1.31	0.17	1.17	0.16	88240	88262	22
29	1.19	0.23	1.22	0.11	1.30	0.13	1.28	0.17	1.19	0.13	87760	87772	12
30	1.24	0.17	1.28	0.23	1.22	0.17	1.22	0.24	1.28	0.13	88240	88224	16
31	1.30	0.16	1.28	0.24	1.18	0.23	1.22	0.19	1.17	0.09	87200	87191	9
32	1.30	0.09	1.28	0.09	1.19	0.13	1.18	0.09	1.24	0.24	88080	88073	7
33	1.22	0.11	1.30	0.17	1.31	0.09	1.31	0.09	1.19	0.11	89120	89049	71
34	1.31	0.17	1.22	0.09	1.24	0.09	1.22	0.17	1.17	0.23	88080	88115	35
35	1.31	0.19	1.31	0.19	1.24	0.23	1.24	0.16	1.24	0.16	88640	88595	45
36	1.28	0.11	1.22	0.17	1.31	0.24	1.19	0.09	1.31	0.11	88560	88571	11
37	1.19	0.17	1.19	0.09	1.31	0.09	1.30	0.23	1.19	0.11	87600	87574	26
38	1.28	0.24	1.19	0.16	1.30	0.13	1.24	0.23	1.30	0.11	88880	88840	40
39	1.17	0.16	1.18	0.23	1.18	0.09	1.17	0.09	1.17	0.19	86000	85803	197
40	1.19	0.19	1.31	0.17	1.30	0.23	1.22	0.19	1.22	0.11	87920	87952	32
41	1.24	0.24	1.22	0.19	1.31	0.17	1.28	0.13	1.30	0.17	89040	88999	41
42	1.17	0.09	1.18	0.19	1.22	0.13	1.31	0.09	1.28	0.24	87440	87444	4
43	1.22	0.09	1.30	0.11	1.30	0.16	1.28	0.09	1.28	0.09	90080	90055	25
44	1.31	0.11	1.22	0.16	1.22	0.09	1.19	0.09	1.28	0.24	88080	88053	27
45	1.31	0.19	1.24	0.11	1.22	0.24	1.19	0.19	1.28	0.23	88160	88187	27
46	1.30	0.23	1.30	0.11	1.31	0.19	1.19	0.16	1.22	0.09	88640	88659	19
47	1.30	0.17	1.19	0.09	1.19	0.24	1.24	0.19	1.31	0.19	87840	87863	23
48	1.17	0.23	1.30	0.24	1.30	0.16	1.22	0.09	1.31	0.24	88560	88551	9
49	1.31	0.16	1.18	0.11	1.24	0.16	1.28	0.19	1.22	0.11	87680	87672	8
50	1.22	0.13	1.22	0.17	1.30	0.24	1.30	0.17	1.22	0.09	88160	88144	16

CHAPTER 5

EMPIRICAL TESTS OF THE ASSUMPTIONS IN THE MODELING APPROACHES

The models described in Chapter 3 will be applicable in selecting an optimal bid provided that the assumptions under which the models were developed are valid. In applying model one, it is assumed that the distribution of lowest competitor bid to cost ratios is normal. For model two it is assumed that the probability distribution of bid to cost ratios is normal for each competitor. An assumption of the independence of competitor bid to cost ratios is necessary in applying models two and three.

In order to test these assumptions, and hence the underlying validity of the modeling approach, actual bidding data was collected on past bids submitted by a heavy construction contractor in the state of Rhode Island. The company is involved in heavy roadwork construction and is an active bidder on state and municipal road and sewer contracts. The manager of the firm, who has the sole responsibility for bidding on contracts, agreed beforehand to cooperate in sharing his recollections and records on past contracts. Data on sixty-eight contracts, on which the company had submitted bids, were collected from the public records of the state department of transportation and from the minutes of the meetings of contract review boards of four cities within the state. The data consisted of the identify and the value of the bid of each contractor participating in the bid. In all cases the contract was awarded to the lowest bidder.

The company keeps a file on each contract on which it submits a bid. These files contain copies of the contract specifications, labor and material estimates, and rough scratch work which was used in computing the bid. With this recorded data and the personal recollection of the manager, a cost estimate was determined for each of the sixty-eight contracts. The price bid by the company and its competitors and this cost estimate for each contract are contained in Appendix III. Each contractor was assigned a letter to preserve their anonymity and still allow for further classification and analysis. The cooperating company was assigned the letter A.

Based upon contractor A's estimated cost, a ratio of their respective bid to this estimated cost, was computed for each contractor. Values of these ratios also appearin Appendix III. The values of the lowest competitor bid to cost ratios for each of the sixty-eight contracts, and the sample mean and standard deviation are presented in Appendix IV. For model one, it is assumed that the probability distribution of these lowest competitor bid to cost ratios is normal. A test of the null hypothesis that this distribution is normal was conducted, based upon the sample data contained in Appendix IV.

A modified chi-square test for goodness of fit to a normal distribution was performed on the data presented in Appendix IV. The data contained in Appendix IV were grouped into eight equiprobable class intervals, which are shown in Table 5. Since the formation of the intervals was based upon estimates of the mean and variance of the parent distribution which were obtained from the sample, a modified chi-square value was used in testing the hypotheses. The use of the modified test statistic is appropriate when the class bounds are random. The asymptotic distribution of this modified statistic, X_R^2, is described in an article by Dahiya and Gurland.[16] Computation of the X_R^2 statistic for the data contained in Appendix IV is shown in Table 5. The critical value of the statistic, for an alpha level of .05 and eight classes, is 11.543. Since the test statistic computed from the sample, X_R^2(5.4117) is less than the critical value, $d_{i,.95}$(11.543), the null hypothesis of normality was not rejected. It can therefore be concluded that the sample evidence does not indicate that the distribution of the lowest competitor bid to cost ratios is not normal.

In applying model two, it is assumed that the probability distribution of each competitor's bid to cost ratios is normal. The sample bidding data on the sixty-eight contracts contained in Appendix III includes 311 competitor bid to cost

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[16] Ram Dahiya and John Gurland, "Pearson Chi-Squared Test of Fit with Random Intervals," Biometrika, 59 (1972), 147-153.

TABLE 5. Modified chi-square test for goodness of fit for sixty-eight sample lowest competitor bid yo cost ratios to a normal probability distribution.

	Theoretical Probability $p_i(\theta)$	Theoretical Frequency $np_i(\theta)$	Observed Frequency m_i	$\frac{(m_i - np_i(\hat{\theta}))^2}{np_i(\hat{\theta})}$
below .93149	.125	8.5	8	.0294
.93149 to .97856	.125	8.5	7	.2647
.97856 to 1.03177	.125	8.5	9	.0294
1.01377 to 1.04529	.125	8.5	10	.2647
1.04529 to 1.07681	.125	8.5	6	.7653
1.07681 to 1.11202	.125	8.5	10	.2647
1.11202 to 1.5909	.125	8.5	13	2.3823
above 1.15909	.125	8.5	5	1.4412

$$X_R^2 = \sum_{i=1}^{k} \frac{(m_i - np_i(\hat{\theta}))^2}{np_i(\hat{\theta})} = 5.417$$

$$d_{8,.95} = 11.543$$

ratios for 54 contractors. The range of the number of bid to cost ratios for individual competitors is from one to twenty-five. There were eleven competitors against which competitor A bid more than ten times in the sixty-eight sample bids. The bidding data for these eleven competitors were chosen to test the assumption of normality of the individual bid to cost ratio distributions. Appendix V contains a frequency distribution, a histogram, sample mean and sample standard deviation for each of the eleven competitors for which there were more than ten sample bids.

An analysis of variance test for normality was conducted on each of the eleven samples contained in Appendix V. The test is based upon a statistical procedure discussed in an article by Shapiro and Wilk.[17] Derivation of the test statistic, W, and percentage points of its null distribution are contained in the article. The denominator of the test statistic is the sum of the squared deviations of the order statistics from the sample mean, S^2, where:

$$S^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

The numerator of the test statistic, b^2, is the square of the weighted sum of the differences between decreasing extreme values of the sample:

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[17] S.S. Shapiro and M.B. Wilk, "An Analysis of Variance Test for Normality," Biometrika, 52 (1975), 591.

$$b = \sum_{i=1}^{k} a_{n-i+1}(y_{n-i+1} - y_i)$$

where the values of the weights, a_{n-i+1}, are provided in a table contained in the article describing the test.[18]

Computation of the test statistic, W, for each of the eleven samples is presented in Table 6. Low values of the test statistic are indicative of the non-normality of the parent distribution. Percentage points from the ull distribution for an alpha level of .05 are displayed with the computed value of the test statistic in Table 6. In six of the eleven cases the sample data did not provide sufficient evidence to reject the hypothesis of normality. Although the evidence was not strongly supportive, it would appear that the assumption of the normality of the individual bid to cost ratio distributions is tenable.

In the sixty-eight recorded bids in Appendix III, competitor A bid less than ten times against 43 competitors. Of these, 32 were non-union contractors, 15 of which competitor A bid against once in the sixty-eight sample bids. Since competition against these competitors, on an individual basis, is infrequent, for the purpose of applying model two, these competitors can be grouped together and it can be assumed that they bid individually as an average non-union or average union competitor infrequently encountered. The bidding data,

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[18]Ibid.

TABLE 6. Analysis of variance test for normality of the distribution of individual competitor bid to cost ratios for eleven competitors.

Competitor LL
(n=13)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.5359(1.620- .986) = .33976
2	.3325(1.271- .995) = .09177
3	.2412(1.214-1.038) = .04245
4	.1707(1.126-1.0940 = .00546
5	.1099(1.121-1.103) = .00198
6	.0539(1.120-1.117) = .00016

$b = .48158$

$b^2 = .23192$ $\qquad$ $s^2 = .31276$

$W = .74153$ $\qquad$ $W_{.05} = .866$

Competitor E
(n=18)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4886(1.740- .980) = .37134
2	.3253(1.683-1.072) = .19876
3	.2553(1.342-1.076) = .06791
4	.2027(1.310-1.099) = .04277
5	.1587(1.273-1.104) = .02682
6	.1197(1.205-1.112) = .01113
7	.0837(1.193-1.136) = .00477
8	.0496(1.170-1.154) = .00079
9	.0163(1.159-1.157) = .00003

$b = .72432$

$b^2 = .52464$ $\qquad$ $s^2 = .67312$

$W = .77942$ $\qquad$ $W_{.05} = .897$

TABLE 6 (continued). Analysis of variance test for normality of the distributions of individual competitor bid to cost ratios for eleven competitors.

Competitor C
(n=25)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4450(1.522-1.087) = .19358
2	.3069(1.485-1.097) = .11908
3	.2543(1.464-1.101) = .09231
4	.2148(1.395-1.104) = .06251
5	.1822(1.386-1.108) = .05065
6	.1539(1.379-1.120) = .03986
7	.1283(1.361-1.121) = .03079
8	.1046(1.338-1.139) = .02082
9	.0823(1.333-1.190) = .01177
10	.0610(1.332-1.233) = .00604
11	.0403(1.322-1.240) = .00330
12	.0200(1.292-1.252) = .00080

$b = .63151$

$b^2 = .39880$ $\qquad s^2 = .42868$

$W = .93030$ $\qquad W_{.05} = .918$

Competitor DD
(n=14)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.5251(1.253- .986) = .14020
2	.3318(1.185-1.040) = .04811
3	.2460(1.183-1.052) = .03223
4	.1802(1.167-1.064) = .01856
5	.1240(1.130-1.078) = .00645
6	.0727(1.126-1.082) = .00320
7	.0240(1.122-1.119) = .00007

$b = .24882$

$b^2 = .06191$ $\qquad s^2 = .06294$

$W = .98364$ $\qquad W_{.05} = .874$

TABLE 6 (continued). Analysis of variance test for normality of the distributions of individual competitor bid to cost ratios for eleven competitors.

Competitor M
(n=22)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4590(1.750- .972) = .31120
2	.3156(1.593- .983) = .19252
3	.2571(1.470-1.043) = .10978
4	.2131(1.404-1.046) = .07629
5	.1764(1.330-1.073) = .04533
6	.1443(1.180-1.076) = .01501
7	.1150(1.168-1.102) = .00759
8	.9878(1.164-1.104) = .00527
9	.0618(1.151-1.118) = .00204
10	.0368(1.148-1.125) = .00088
11	.0122(1.145-1.137) = .00010

$b = .76601$

$b^2 = .58677$ $s^2 = .71114$

$W = .82511$ $W_{.05} = .911$

Competitor D
(n=17)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4968(1.614- .888) = .36068
2	.3273(1.573- .959) = .20096
3	.2540(1.549- .961) = .14935
4	.1988(1.521- .999) = .10377
5	.1524(1.308-1.035) = .04161
6	.1109(1.288-1.068) = .02440
7	.0725(1.240-1.071) = .01225
8	.0359(1.176-1.118) = .00208

$b = .89510$

$b^2 = .80120$ $s^2 = .8784$

$W = .91176$ $W_{.05} = .892$

TABLE 6 (continued). Analysis of variance test for normality of the distributions of individual competitor bid to cost ratios for eleven competitors.

Competitor K
(n=17)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4968(1.609-1.036) = .28467
2	.3273(1.407-1.044) = .11881
3	.2540(1.392-1.079) = .07950
4	.1988(1.320-1.095) = .04473
5	.1524(1.212-1.118) = .01433
6	.1109(1.205-1.119) = .00954
7	.0725(1.162-1.123) = .00283
8	.0359(1.155-1.127) = .00101

$b = .55542$

$b^2 = .30849$ $\qquad s^2 = .37111$

$W = .83126$ $\qquad W_{.05} = .892$

Competitor I
(n=14)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.5251(1.364-1.037) = .17171
2	.3318(1.333-1.058) = .09125
3	.2460(1.222-1.072) = .03690
4	.1802(1.211-1.101) = .01982
5	.1240(1.182-1.115) = .00831
6	.0727(1.136-1.118) = .00131
7	.0240(1.130-1.130) = .00000

$b = .32930$

$b^2 = .10844$ $\qquad s^2 = .12133$

$W = .89376$ $\qquad W_{.05} = .874$

TABLE 6 (continued). Analysis of variance test for normality of the distributions of individual competitor bid to cost ratios for eleven competitors.

Competitor B
(n=21)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4643(1.670- .999) = .31155
2	.3185(1.539-1.107) = .13759
3	.2578(1.495-1.120) = .09668
4	.2119(1.464-1.139) = .06887
5	.1736(1.434-1.166) = .04652
6	.1399(1.386-1.195) = .02672
7	.1092(1.317-1.207) = .01201
8	.0804(1.296-1.240) = .00450
9	.0530(1.287-1.243) = .00233
10	.0263(1.267-1.251) = .00042

$b = .70719$

$b^2 = .50012$ $\qquad$ $s^2=.51878$

$W = .96403$ $\qquad$ $W_{.05} = .908$

Competitor J
(n=17)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.4968(1.488-1.033) = .22604
2	.3273(1.327-1.097) = .07528
3	.2540(1.211-1.101) = .02794
4	.1988(1.204-1.110) = .01869
5	.1524(1.199-1.115) = .01280
6	.1109(1.189-1.129) = .00665
7	.0725(1.169-1.145) = .00174
8	.0359(1.168-1.149) = .00068

$b = .36982$

$b^2 = .13677$ $\qquad$ $s^2=.16793$

$W = .81445$ $\qquad$ $W_{.05} = .892$

TABLE 6 (continued). Analysis of variance test for normality of the distributions of individual competitor bid to cost ratios for eleven competitors.

Competitor F
(n=13)

i	$a_{n-i+1}(y_{n-i+1}-y_i)$
1	.5359(1.353- .962) = .20954
2	.3325(1.242-1.039) = .06750
3	.2412(1.230-1.069) = .03883
4	.1707(1.216-1.104) = .01912
5	.1099(1.146-1.113) = .00363
6	.0539(1.135-1.121) = .00075

$b = .33937$

$b^2 = .11517$ $\qquad s^2=.11908$

$W = .96716$ $\qquad W_{.05} = .866$

from Appendix III, on these 43 competitors were grouped together for union and non-union contractors and appear in Appendices VI and VII in a frequency distribution and Histogram with the associated sample mean and standard deviation. In order to utilize these parameters of an average union or non-union competitor in applying model two, it is necessary to assume that the distribution of the respective population is normal.

Modified chi-square tests for goodness of fit to a normal distribution were performed on the data presented in Appendices VI and VII. The computation of the test statistic, X_R^2, and the corresponding critical value for an alpha level of .05, are presented in Tables 7 and 8. The null hypothesis of the normality of the distribution of bid to cost ratios for infrequently encountered non-union competitors was rejected at the .05 level of significance. The normality hypothesis of the distribution of bid to cost ratios for infrequently encountered union contractors was not rejected at the same level of significance.

TABLE 7. Modified chi-square testo for goodness of fit of the bid to cost ratios of non-union contractors, which competitor A bid against less than ten times in sixty-eight sample bids, to a normal probability distribution.

	Theoretical Probability $p_i(\hat{\theta})$	Theoretical Frequency $np_i(\hat{\theta})$	Observed Frequency m_i	$\frac{(m_i - np_i(\hat{\theta}))^2}{np_i(\hat{\theta})}$
below .97361	.125	8.875	10	.1426
.97361 to 1.06325	.125	8.875	5	1.6919
1.06325 to 1.3030	.125	8.875	15	4.2271
1.13030 to 1.19032	.125	8.875	13	1.9173
1.19032 to 1.25034	.125	8.875	8	.0863
1.25034 to 1.31739	.125	8.875	6	.9313
1.31739 to 1.40703	.125	8.875	4	2.6778
above 1.40703	.125	8.875	10	.1426

$$X_R^2 = \sum_{i=1}^{k} \frac{(m_i - np_i(\hat{\theta}))^2}{np_i(\hat{\theta})} = 11.8169$$

$$d_{8,.95} = 11.543$$

TABLE 8. Modified chi-square test for goodness of fit of the bid to cost ratios of union contractors, which competitor A bid against less than ten times in sixty-eight sample bids, to a normal probability distribution.

	Theoretical Probability $p_i(\hat{\theta})$	Theoretical Frequency $np_i(\hat{\theta})$	Observed Frequency m_i	$\frac{(m_i - np_i(\hat{\theta}))^2}{np_i(\hat{\theta})}$
below .96510	.125	5.75	4	.5326
.96510 to 1.06504	.125	5.75	6	.0109
1.06504 to 1.13980	.125	5.75	10	3.1413
1.13980 to 1.20672	.125	5.75	8	.8804
1.20672 to 1.27364	.125	5.75	6	.0109
1.27364 to 1.34840	.125	5.75	1	3.9239
1.34840 to 1.44834	.125	5.75	4	.5326
above 1.44834	.125	5.75	7	.2717

$$X_R^2 = \sum_{i=1}^{k} \frac{(m_i - p_i(\hat{\theta}))^2}{np_i(\hat{\theta})} = 9.3043$$

$$d_{8,.95} = 11.543$$

An assumption necessary for the application of model three is that the bid to cost ratios of each individual competitor are generated by a normal regression process. Under this assumption, the predicted dependent variables in each regression model will be normally distributed random variables. In the basic regression model for each competitor:

$$Y_{ij} = B_{i0} + B_{i1}x_{1j} + B_{i2}x_{2j} + \ . \ . \ . \ B_{ik}x_{kj} + e_{ij}$$

the error term, e_{ij}, is for the i^{th} competitor on the j^{th} contract. The assumption of the normality of these error terms for each competitor, and hence the normality of the predicted bid to cost ratios, follows from the fact that the error terms represent the effects of many factors omitted from the model. If these omitted factors are mutually independent, the sum of these effects would approach a normal distribution as the number of factors becomes large, in accordance with the Central Limit Theorem.[19]

The output of each of these regression models will consist of a predicted bid to cost ratio and a standard error of this predicted value. These values represent estimates of the mean and standard deviation of the subpopulation of bid to cost ratios for a particular set of independent variables. These two values will provide the input required for the computation of an approximation of the optimal bid to

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[19] John Neter and William Wasserman, Applied Linear Statistical Models (Homewood, Illinois, 1974), 47.

cost ratio in a manner identical to that for model two described in Chapter IV.

The data on the eleven competitors which were used to test the normality assumption required for model two were also used as the recorded values of the dependent variable in examining the appropriateness of the Regression approach of model three. Selection of the independent variables was constrained by the availability of historic data on the sixty-eight sample bids contained in Appendix III. The independent variables selected were; size of the contract expressed in thousands of dollars, the number of bidders participating in the bid, the number of material suppliers participating and the number of non-union contractors participating. Each of these variables, which will be described subsequently, were considered as factors in selecting a bid by the manager of firm A.

The size of the contract, as an independent variable, could reveal the underlying preference of individual contractors for large or small work contracts. The capital outlay required for material, equipment and labor is directly proportional to the bid price of the contract. For financial considerations, therefore, it would be expected that smaller firms would bid more competitively on smaller contracts while participating in bidding on larger contracts with correspondingly higher bid to cost ratios.

The number of bidders participating in the bid was introduced as an independent variable because of the suspected increase in the degree of competitiveness associated with a large number of competitors. A general theory of bidding behavior is that individual competitors lower their bids as the number of competitors increases. During recessionary periods in the construction trade, what work does become available is highly sought after and the number of contractors participating in bidding on individual contracts increases. Conversely, when numerous contracts are available and fewer contractors are bidding on individual contracts, the bid to cost ratios would tend to be higher.

In many of the roadwork construction contracts, as those contained in Appendix III, the contractor is required to include materials such as concrete, asphalt, gravel, sand or crushed stone in the bid price. Among the contractors bidding on the sixty-eight contracts with firm A, are six suppliers of such material. Whether this cost advantage is reflected in the bid to cost ratios of these suppliers or whether other bidders lower their bids in response to such competition, could be revealed by using the number of such suppliers participating in the bidding as an independent variable in predicting individual bid to cost ratios.

Contractors participating in the bids contained in Appendix III were either union or non-union contractors. Union contractors hire only union members and pay union wages, while non-union contractors are under no wage restrictions. Non-union contractors are indicated in Appendix III by double

TABLE 9. Regression data for eleven contractors with bid to cost ratios as the dependent variable.

Independent variables
1 bid size in thousands of dollars
2 number of bidders participating
3 number of suppliers participating
4 number of non-union contractors participating

Variable	Coefficient	Standard Error	t-Ratio
	Competitor I		
1	-1.34×10^{-7}	1.48×10^{-4}	-9.06×10^{-4}
2	.0125	.0377	.3304
3	- .0298	.0514	- .5805
4	- .0454	.0518	- .8772
	R^2 = .1630	Critical t for 9 d.f. = 2.262	
	Competitor DD		
1	-8.53×10^{-5}	.0002	- .3661
2	.0093	.0247	.3783
3	- .0027	.0366	- .0742
4	- .0162	.0256	- .6316
	R^2 = .0659	Critical t for 10 d.f. = 2.228	
	Competitor J		
1	-1.54×10^{-4}	3.24×10^{-4}	- .4766
2	- .0345	.0295	-1.169
3	.1031	.0401	2.574
4	.0315	.0354	.8889
	R^2 = .5017	Critical t for 12 d.f. = 2.179	

TABLE 9 (continued). Regression data for eleven contractors with bid to cost ratios as the dependent variable.

Variable	Coefficient	Standard Error	t-Ratio
	Competitor D		
1	$-1.88\text{X}10^{-4}$	$1.51\text{X}10^{-4}$	-1.241
2	.0294	.0463	.6347
3	- .0964	.0501	-1.925
4	- .0892	.0615	-1.451
	R^2 = .3379	Critical t for 12 d.f. = 2.179	
	Competitor E		
1	$4.14\text{X}10^{-5}$	$1.82\text{X}10^{-4}$	.2267
2	.0095	.0324	.2944
3	- .0949	.0525	-1.806
4	- .0806	.0588	-1.370
	R^2 = .2388	Critical t for 13 d.f. = 2.160	
	Competitor M		
1	$-1.51\text{X}10^{-4}$	$2.60\text{X}10^{-4}$	- .5784
2	.0128	.0379	.3379
3	- .0526	.0515	-1.021
4	- .0107	.0420	- .2542
	R^2 = .1197	Critical t for 17 d.f. = 2.110	
	Competitor C		
1	$1.69\text{X}10^{4}$	$1.25\text{X}10^{-4}$	1.352
2	.0118	.0250	.4737
3	- .0267	.0291	- .9171
4	- .0016	.0285	- .0544
	R^2 = .1372	Critical t for 20 d.f. = 2.086	

TABLE 9 (continued). Regression data for eleven contractors with bid to cost ratios as the dependent variable.

Variable	Coefficient	Standard Error	t-Ratio
	Competitor B		
1	$-4.20\text{X}10^{-5}$	$1.44\text{X}10^{-4}$	- .2915
2	.0171	.0733	.2327
3	.0053	.0407	.1298
4	- .0102	.0926	- .1099
	R^2 = .0354	Critical t for 16 d.f. = 2.120	
	Competitor F		
1	$-1.69\text{X}10^{4}$	$1.04\text{X}10^{-4}$	-1.627
2	.0499	.0274	1.825
3	$-7.73\text{X}10^{-4}$	.0378	-0.020
4	- .0494	.0307	-1.609
	R^2 = .5462	Critical t for 8 d.f. = 2.306	
	Competitor K		
1	$5.74\text{X}10^{-4}$	$1.57\text{X}10^{-4}$	3.649
2	- .0034	.0245	- .1381
3	- .0692	.0684	-1.011
4	.0388	.0315	1.232
	R^2 = .6829	Critical t for 12 d.f. = 2.179	
	Competitor LL		
1	.0083	.0026	3.254
2	- .1201	.0811	-1.482
3	.1569	.1437	1.092
4	.1721	.0692	2.488
	R^2 = .7325	Critical t for 8 d.f. = 2.306	

letters. The number of non-union contractors being bid against by Firm A was a consideration in selecting a bid. The number of non-union contractors participating in each contract was therefore used as an independent variable in the regression formulation.

The results of the eleven computed regression models are contained in Table 9. The proportion of the total variation of the dependent variable explained by the regression model is indicated by the coefficient of determination, R^2. A reduction in the total variation of the bid to cost ratios for each competitor, would cause a corresponding reduction in the standard error of the predicted bid to cost ratios. In model three, it is this standard error of the predicted bid to cost ratio which is used as an estimate of the standard deviation of the distribution of bid to cost ratios, which is incorporated into the approximation technique as in model two. Any reduction in these standard error terms would improve the accuracy of the estimation of the probability of winning a contract with a particular bid.

In each of the eleven cases, the four coefficients computed in the regression model were tested to determine if they differ significantly from zero. The computed t-ratio and the critical value of the t statistic, for a significance level of 95% and the appropriate number of degrees of freedom, are shown in Table 10. The null hypothesis in each case was that population regression coefficient is equal to zero.

The null hypotheses of the equality of the true regression coefficients to zero was not rejected in all but four instances. This result indicates an apparent lack of predictability of these selected independent variables. Of the eleven regression models, four accounted for more than 50% of the variance of the dependent variable.

In computing the probability of winning with a particular bid in each of the three models, it is assumed that the probabilities of winning against each individual competitor are independent. If it is assumed that the joint probability distribution of the bid to cost ratios of any two competitors is a bivariate normal distribution, then the bid to cost ratios of the two competitors are independent if and only if the correlation coefficient, ρ, between the two variables is equal to zero. For any joint distribution, independence implies that the correlation coefficient is equal to zero. For the bivariate normal distribution, the equality of the correlation coefficient to zero implies and is implied by the statistical independence of the two variables.

The sample correlation coefficient, r, can be used to estimate ρ. Under the assumption of bivariate normality, the equality of the correlation coefficient to zero is equivalent to the independence of the two variables. Testing the hypothesis of independence is therefore equivalent to testing the null hypothesis that the correlation coefficient, ρ, is equal to zero, against the alternative that it is not

TABLE 10. Sample correlation coefficients for six pairs of competitors who bid more than six times against each other in sixty-eight sample bids.

Competitor pair	Correlation Coefficient	Degrees of freedom (n-2)	Computed t Statistic	Critical t value (α=.05)
C - E	.3253	10	1.0879	$\pm$ 2.228
C - M	.1316	6	0.3252	$\pm$ 2.447
K - LL	.5321	10	1.9870	$\pm$ 2.228
B - E	-.0777	9	-0.2400	$\pm$ 2.262
B - C	.1929	10	0.6217	$\pm$ 2.228
B - M	.1028	6	0.2531	$\pm$ 2.447

equal to zero. The test statistic $(r(n-2)^{\frac{1}{2}})/(1-r^2)^{\frac{1}{2}}$ has a t distribution with (n-2) degrees of freedom. This statistic can be used to test the hypothesis of independence.

In order to utilize the date in Appendix III to test the assumption of independence, pairs of competitors were selected who bid against each other more often than others. There were six pairs of contractors who bid against each other more than six times in the sixty-eight sample bids. From the sample bidding data, a set of paired observations was recorded for each of the six pairs of competitors and appear in Appendix XI.

A sample correlation coefficient and an associated t value were computed for each pair of competitors. These values of the correlation coefficient and t value are shown in Table 10, with the corresponding critical value of t for a level of significance of 95% and the respective number of degrees of freedom. In each of the six cases, the sample data did not provide sufficient evidence to reject the null hypothesis of the equality of the correlation coefficient to zero. This result, while not proof of independence, would tend to substantiate the assumption of independence which was necessary in the application of each of the models presented previously.

The fact that five of the six sample correlation coefficients were positive could be attributed to the general behavior of all competitors on each contract. For example,

TABLE 11. Sample correlation coefficients for standardized bid to cost ratios for six pairs of competitors who bid more than six times against each other in sixty-eight sample bids.

Competitor pair	Correlation Coefficient	Degrees of freedom (n-2)	Computed t Statistic	Critical t value (α=.05)
C - E	.2282	10	0.7032	$\pm$ 2.228
C - M	-.3322	6	-1.1138	$\pm$ 2.447
K - LL	.4783	10	1.7223	$\pm$ 2.228
B - E	-.6880	6	-2.3222	$\pm$ 2.262
B - C	-.2282	10	-0.7032	$\pm$ 2.228
B - M	-.8013	6	-3.2808	$\pm$ 2.447

if all contractors were bidding high on a particular contract, all bid ratios would be high. This would contribute to the positive correlation in bid ratios of the sample pairs of contractors. In order to remove the possible effect of the magnitude of the bid ratios of all competitors on a particular contract from the correlation between the six sample pairs, the bid ratios were expressed as percentages of the average bid ratio on the respective contract. The paired values of these standardized bid to cost ratios for the six sample pairs of contractors appear in Appendix XII. The sample correlation coefficients and corresponding sample t values for these standardized bid ratios are shown in Table 11 with the critical t value for an alpha of .05. The two-tailed tests of the hypotheses that the population correlation coefficient is equal to zero was not rejected in all but two cases.

A comparison of the application of models one and two with competitor A's current method of bid selection was made by applying models one and two to the sixty-eight sample bids contained in Appendix III. A summary of the results of this comparison are shown in Table 12. Of the sixty-eight contracts bid on, competitor A won fourteen. Applying models one and two to compute a bid for each contract, resulted in twenty-seven and twenty-four winning bids respectfully. The total profit received under the current bidding method, based upon estimated costs, was $220,807. The total profit resulting from the application of model one was $532,662 and $398,346 for model two.

TABLE 12. A comparison of the application of models one and two to sample bidding data with the existing method of bid selection.

Contract Number	Estimated Cost	Actual Bid	Model I Bid	Model II Bid	Actual Profit	Model I Profit	Model II Profit
2	$ 52,900	$ 59,092	$ 57,777	$ 56,680	$ 6,192	$ 4,877	$ 3,780
4	274,200	303,361	299,478	296,297	29,161	25,278	22,097
6	298,000		325,473			27,473	
9	99,000		108,127	106,956		9,127	7,956
10	50,300	55,344	54,937	57,556	5,044	4,637	7,256
13	85,800	95,421	93,710	92,182	9,621	7,910	6,382
15	48,500		52,971	53,725		4,471	5,225
16	8,750		9,557			807	
17	47,800		52,207	52,266		4,407	4,466
21	220,800	235,084	241,156	246,691	14,284	20,356	25,891
23	176,100	191,321	192,335	190,859	15,221	16,235	14,759
26	28,300		30,909	31,332		2,609	3,032
27	935,000		1,021,198			86,198	
30	150,200	161,060	164,047	162,307	10,860	13,847	12,107
32	650,600		710,579	722,129		59,979	71,529
34	185,700		202,820	203,077		17,120	17,377
40	10,200	11,957	11,140	11,107	1,757	940	907
47	180,000	184,870	196,594	195,311	4,870	16,594	15,311
48	143,000	159,618	156,183	152,256	16,618	13,183	9,226
50	111,500	125,172	121,779	118,054	13,672	10,279	6,554
52	723,200	791,630	789,872	775,539	68,430	66,672	52,339
53	531,304	495,059	580,285	586,382	36,245	48,981	55,078
54	46,000		50,241	48,391		4,241	2,391
57	99,500	110,668	108,673	107,381	11,168	9,173	7,881
60	339,900		371,235	359,778		31,335	19,878
66	250,100		273,157	273,511		23,057	23,411
67	31,200		34,076	34,713		2,876	3,513
Total Contracts Won		14	27	24			
Total Profit					220,807	532,662	398,346

CHAPTER 6

POSSIBLE EXTENSIONS OF THE BIDDING MODELS

The three models presented present a basis for quantifying the bidding decision and providing the decision maker with additional information in making a bid decision. The basic modeling approach and the approximation technique can be extended to include other applications and alternative assumptions. Many of the possible extensions would call for new research endeavors which would be beyond the scope of this present work.

Model one was developed under the assumption that the distribution of lowest competitor bid to cost ratios is normal. Model two similarly was developed under the assumption that the distribution of each individual competitor's bid to cost ratios is normal. The tables contained in Appendices I and II were based upon these assumptions of normality. The numerical approximation techniques, upon which the tables are based, is not limited to the normal distribution. Similar tables could be generated for any assumed tractable distribution. Although the data collected for this study did not cast doubt upon the assumptions of normality, the distributions of other bidding data could possibly be more closely approximated with a gamma or log-normal probability distribution. Both the gamma and log-normal distributions would allow for skewness in the distribution and a minimum value of zero, characteristics which could be appropriate for the distribution of a random variable which is the ratio of two positive numbers.

In each of the modeling approaches, it was assumed that the parameters of the respective distributions were fixed but unknown values. Sample data provided unbiased point estimators of these parameters. Without the basis of sample information, estimates of these parameters would be unavailable, unless the idea of an average bidder were employed. In practice such a situation would arise each time the bidder encountered a competitor which the bidder had not previously bid against. The classical approach to estimating the parameters would not provide a means for incorporating the models in such instances. An additional limitation of the classical approach to estimating the relevant parameters would be that additional data, drawn from the individual populations of competitor bid to cost ratios, would not alter the decision maker's knowledge or degree of belief about the parameters.

A Bayesian approach to the estimation of the necessary parameters would allow for the incorporation of new competitors into the modeling approach and for the use of all available data in the model. Formulation of prior distributions on the parameters to be estimated, would enable the decision maker to update these distributions upon the receipt of additional bidding data. In the bidding process, the decision maker is in receipt of a continual inflow of free information on the bidding behavior of his competitors. Bayesian natural conjugate theory would provide the decision maker with the mechanism for combining sample data with a prior distribution to form a posterior distribution which contains all the available information about the relevant parameters.

If it is assumed that the distribution of each competitor's bid to cost ratios is normal, an assumption which the data in this work have supported, then natural conjugate theory can be applied in estimating the distributions of these parameters. For purposes of exposition, assume that the distribution of an individual competitor's bid to cost ratios is normal with a mean M and a precision R. By natural conjugate theory, if M and R have a normal-gamma joint prior density, the posterior marginal density of M is a student's t distribution and the posterior marginal density of R is a gamma distribution. Specifically, if the conditional prior distribution of M, when $R=r$, is a normal distribution with mean μ and precision τr and the marginal distribution of R is a gamma distribution with parameters α and β, then the posterior joint distribution of M and R is a normal-gamma, where the posterior conditional distribution of M when $R=r$ is a normal distribution with mean μ' ("'" indicates a posterior parameter) and precision $(\tau+n)r$, where:

$$u'=(\tau\mu+n(\overline{B/C}))/(\tau+n)$$

and the marginal distribution of R is a gamma distribution with parameters α' and β', where:

$$\alpha'=\alpha+(n/2)$$

$$\beta'=\beta+\tfrac{1}{2}\sum_{i=1}^{n}((B_i/C)-(\overline{B/C}))^2+\frac{\tau n((\overline{B/C})-\mu)^2}{2(\tau+n)}$$

The posterior marginal distribution of M is a Student's t distribution with $2\alpha'$ degrees of freedom, location parameter μ' and precision $\alpha'\tau'/\beta'$. The prior marginal distribution of M is equivalent using prior parameters α, β, τ, and μ.[20]

The prior mean and variance of M and R can be estimated subjectively or from historical data. For example, if previous lowest bid to cost ratios are grouped by quarters of the year in which they occurred for n years into the past, there would be 4(n) individual groups of lowest competitor bid to cost ratios for each competitor. The mean bid to cost ratio for each group, $(\overline{B/C})$, and precision, $(1/s^2)$, could be computed and the means of these means and precisions could be used as prior estimates of E(m) and E(R). The variances of the means and precisions about E(M) and E(R), respectively, could be computed and used as prior estimates of Var(M) and Var(r). In the case of a new or not previously encountered competitor, subjective estimates of E(M), Var(M), E(R), and Var(R) could be used. These estimates could be based upon the updated distributions of competitors against which the bidder has had previous bidding experience, which have similarities to the new competitor. The posterior distributions

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[20]Morris H. DeGroot, Optimal Statistical Decisions (New York, 1970), 168-171.

would summarize all the available information which the bidder possesses about each competitor.

One aspect of the bidding decision for a construction contractor, which is not captured in the modeling approaches discussed in this work, is the contractor's "degree of hunger" for each pending contract. The degree of hunger is a term used to describe the strength of the firm's desire to win a particular contract. This measure is commonly related to the firm's current or projected workload, in terms of its personnel and equipment utilization. When personnel and equipment are idle, the winning of a contract, which would utilize these resources, would be more important to the firm than if these resources were engated in other work. This slack in resource utilization is often taken up by bidding low on smaller contracts with close starting dates and short completion times. The lost profit on these contracts is compensated for by the utilization of the resources, which enables the firm to cover its fixed costs. The payoffs on such contracts, in terms of the profit as defined for the models in this work, do not reflect the true worth to the firm.

This change in attitude toward the profit to be received from a contract when resources are underutilized, can be thought of as a movement along the firm's utility function. The firm's utility function can be found empirically by personally interviewing the decision maker. The utility function can be described graphically by the firm's preference curve, drawn over the relevant range of potential asset positions. Choosing the appropriate preference curve would enable the decision maker to assign an approximation of the true worth to the firm of winning or losing a bid on a particular contract.

Utilizing an appropriate utility function, utility values could be derived for values of the potential asset positions resulting from winning losing a bid on a contract. The expected utility of the contract would be expressed as:

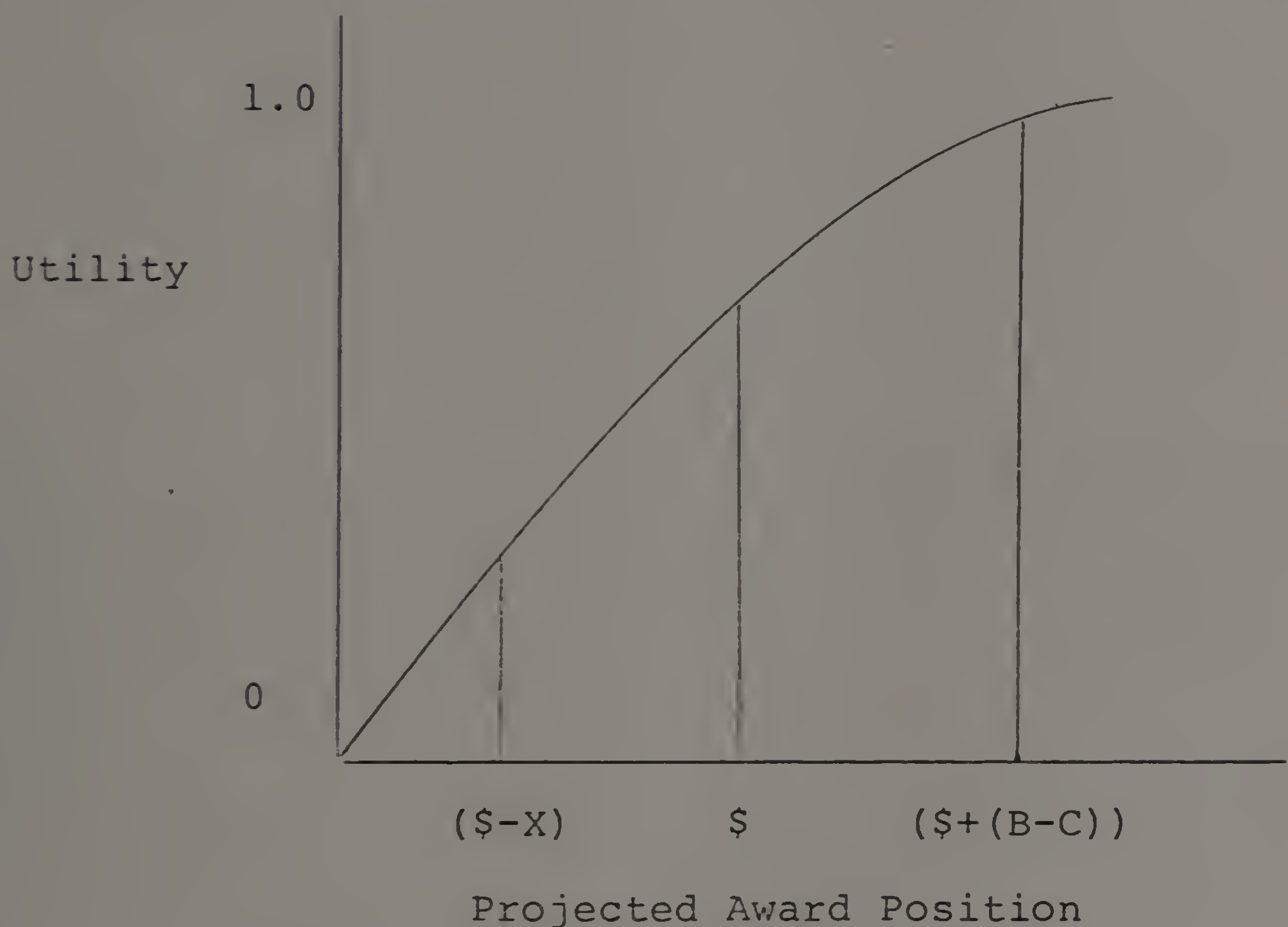
$$E(u(W)) = u((B_o - C) + \$) \prod_{i=1}^{n} G_{B_i/C}(B_o/C) - u(\$ - X)(1 - \prod_{i=1}^{n} G_{B_i/C}(B_o/C)$$

where u(W) is the firm's utility for contract W, \$ is the firm's current asset position, and X is the decrease in assets resulting from underutilizing the firm's resources. An example of a possible preference curve is shown in Figure 3 with values of the firm's asset position prior to the bid and after, if the contract is won and if the contract is lost. The utility values used in computing an optimal bid are therefore dependent upon the firm's current asset position and level of resource utilization.

Since the optimization models presented in this work are based upon the expected monetary value, introduction of a non-linear utility function into the model would require a variation of the optimization technique. If the utility function is determined empirically and described graphically, then utility values must be read directly from the curve.

FIGURE 3

Example utility function for a contractor's asset position



$ - Asset position before bid

($-X) - Asset position if the bid is lost

($+(B-C)) - Asset position if the bid is won

In order to determine the optimal bid with modeling approaches similar to the ones described in this work, it would be necessary to work with utility functions expressed in analytical form.

In each of the models discussed the bidder views each contract as if it were the only contract which the firm had to bid on, and bids to maximize the expected value of that contract. This treatment does not handle to problem of bidding on individual contracts whre the population of contracts available to the firm is much larger than the firm could execute, if won. This compound probability problem is a logical extension of the work here presented.

CHAPTER 7

SUMMARY AND CONCLUSIONS

The purpose of this dissertation was to develop quantitative models which could be applied to competitive bidding decisions in the construction industry. A central consideration was that the models developed would be applicable and suitable for implementation within the limitations of the actual business environment. Computer facilities are typically not available to construction company managers for data analysis related to bidding decisions. Required data manipulation therefore has to be relatively simple and necessary data must be available from existing sources.

Another important consideration was that the objective of the model had to be consistent with the objective of the firms involved in bidding. Maximization of expected profit was used as the objective in each of the three models developed. Although this objective may not be in precise agreement with the actual objective of the bidders, it provides the decision maker with an input to the bidding decision which can be acquired with minor computational effort.

With the maximization of expected value of the bid as the objective in choosing a bid for each contract, mathematical models of the bidding process were developed. Three separate approaches were taken in constructing probabilistic models. In each approach probability distributions of bid to cost ratios were assumed to be normal.

The first approach was to assess the probability distribution of the lowest competitor bid to cost ratios. This distribution was used in formulating an expected value expression. In order to determine the value of the bid which maximizes the expected value expression, the first derivative of the function was taken with respect to the bid and this function was set equal to zero to solve for an extreme value. If the second order conditions are satisfied, a root of this expression would yield a maximum expected value. In order to determine a root of this equation, a numerical approximation technique was employed. Applying the Newton-Raphson method to the function, an iterative expression was derived for approximating the root of the equation.

The second approach involved utilizing the probability distributions of individual competitors in deriving the joint probability distribution of competitor bid to cost ratios. This joint probability distribution of competitor bid ratios is utilized in formulating an expected value expression. In order to determine the bid which maximizes this expected value expression, the first derivative is set equal to zero and a numerical method was employed in approximating the root of the function.

The third approach involved utilizing the output of individual normal regression models for each competitor in estimating the mean and standard deviation of their respective bid to cost ratio distributions. An expected value expression was developed with these estimated parameters and the Newton-Raphson approximation method was employed in estimating the optimal bid as in model two.

For selected values of parameters of the assumed bid to cost ratio distributions, terms contained in the iterative expressions derived for approximating the optimal bid, were computed and displayed in a table. These tables enable the decision maker to compute an approximation of the optimal bid using model one or model two with a few simple calculations. It was shown that this approximation method yields an estiamte which is, on average, within four one hundredths of one percent of the bid which maximizes the expected value.

Actual bidding data was collected and used to test the assumptions under which the models were developed. The hypothesis of the normality of the distribution of lowest competitor bid to cost ratios was not rejected. Hypotheses of the normality of individual competitor bid to cost ratio distributions were tested for eleven competitors. Five of the eleven null hypotheses were rejected at a .05 level of significance. Similar hypotheses of the normality of bid to cost ratio distributions for union and non-union contractors were tested. The hypothesis of the normality of the of non-union bid to cost ratios was not rejected, while the same hypothesis for union contractors was rejected.

Sample regression equations were computed for elevan contractors using four independent variables. The sample regression coefficients were used to test the hypotheses that the population coefficients were equal to zero. Only four of the forty-four hypotheses were rejected.

The assumption of independence among the individual competitor bid to cost ratio distributions was examined by testing hypotheses about the population correlation coefficients. The sample correlation coefficients for six pairs of competitors were used to test the hypotheses that the individual population correlation coefficients equal zero. In the six cases the null hypothesis was not rejected, supporting the assumption of the independence of the individual distributions. When the bid ratios were standardized by dividing each ratio by the mean ratio for all competitors for the respective contract, two of the six hypotheses that the population correlation coefficient equalled zero were rejected.

Models one and two were applied to the sixty-eight sample bids to compute an approximation of the optimal bid using the estimated cost and the parameter estimates computed from the sample. The number of contracts won increased from the actual value of 17 to 27 for model one and 24 for model two. Total profits increased from \$220,807, to \$532,622, for model one and to \$398,346, for model two. The average profit per contract won was $15,772, under the existing system of bidding, $19,728, for model one, and $16,598 for model two.

The models are not restricted to the normal probability distribution, which is assumed in the analytical work. The approximation technique and use of computational tables could be adapted to any tractable probability distributions. A Bayesian approach to estimating the parameters of the probability distributions would add a dimension to the modeling approach in allowing the decision maker to utilize all available new information in pudating the parameters of the relevant probability distributions and in incorporating new competitors into the model. The models discussed are based upon the assumption of a linear utility function. Application of a non-linear preference for the potential payoffs may be more applicable in certain instances in the bidding decision when the potential payoff from the contract does not reflect the true value to the firm of winning the contract.

Both models one and two provide the decision maker with a means of computing an approximation of the bid which maximizes the expected value of the contract with little computational effort. The assumptions under which the models were developed appear to be valid and the accuracy of the approximation technique is significant, in the context of the bidding decision. The applicability of these models rests largely on the acceptance by the decision maker of the objective of maximization of expected value as a criterion in bid selection.

While models one and two provide a means for a contractor to utilize quantitative tools in the decision making process in a manner which is consistent with existing analytical resources, model three would require computer facilities for the required multiple regression problem. Although model three is intuitively appealing in that it incorporates factors into the decision model which do or should influence the bidding decision, its applicability is suspect within the framework of the decision process of most small contractors. When necessary parameters can be estimated by an outside agent with the required facilities, the contractor could proceed in applying model two in estimating optimal bids. Although the results of this research cast doubt on the appropriateness of the selected independent variables in predicting bid to cost ratios, selection of alternative independent variables could have contrasting results. The variables used were chosen from available data. Many factors with intuitively high potential correlation with the dependent variable were not recorded for historical bids.

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
-1.60	8.52299	-15.64140	-2.00	18.09628	-38.19257
-1.61	8.66576	-15.94298	-2.01	18.48393	-39.17747
-1.62	8.2123	-16.29138	-2.02	18.84970	-40.06259
-1.63	8.97257	-16.62605	-2.03	19.26772	-41.14226
-1.64	9.12981	-16.96762	-2.04	19.66466	-42.11905
-1.65	9.29130	-17.32201	-2.05	20.07787	-43.14316
-1.66	9.45826	-17.70108	-2.06	20.50836	-44.26093
-1.67	9.63094	-18.08728	-2.07	20.95726	-45.39227
-1.68	9.79959	-18.45686	-2.08	21.37691	-46.43040
-1.69	9.97388	-18.85242	-2.09	21.86414	-47.72476
-1.70	10.16383	-19.28932	-2.10	22.32045	-48.87292
-1.71	10.33946	-19.67209	-2.11	22.79814	-50.08237
-1.72	10.53136	-20.10837	-2.12	23.29384	-51.34761
-1.73	10.73013	-20.57646	-2.13	23.81113	-52.67790
-1.74	10.92369	-21.01971	-2.14	24.35149	-54.13869
-1.75	11.12283	-21.46173	-2.15	24.85353	-55.34721
-1.76	11.33020	-21.93474	-2.16	25.44185	-56.95966
-1.77	11.54382	-22.42686	-2.17	25.98944	-58.36758
-1.78	11.76651	-22.95818	-2.18	26.56064	-59.84636
-1.79	11.98135	-23.44421	-2.19	27.15427	-61.39529
-1.80	12.20380	-23.95137	-2.20	27.77747	-63.03214
-1.81	12.45033	-24.53911	-2.21	28.42651	-64.83321
-1.82	12.68857	-25.10953	-2.22	29.10915	-66.74419
-1.83	12.91979	-25.62871	-2.23	29.73192	-68.26988
-1.84	13.17576	-26.25128	-2.24	30.38461	-69.96799
-1.85	13.42303	-26.81677	-2.25	31.16087	-72.18568
-1.86	13.70014	-27.50125	-2.26	31.87419	-74.07672
-1.87	13.96687	-28.12247	-2.27	32.62045	-76.17656
-1.88	14.24230	-28.79057	-2.28	33.28955	-77.77013
-1.89	14.50823	-29.41170	-2.29	34.10344	-80.08507
-1.90	14.80641	-30.14572	-2.30	34.95760	-82.53830
-1.91	15.09163	-30.82407	-2.31	35.72563	-84.41396
-1.92	15.38924	-3.153664	-2.32	36.65926	-87.26674
-1.93	15.69677	-32.27957	-2.33	37.50378	-89.50883
-1.94	16.01643	-33.05818	-2.34	38.38759	-91.86865
-1.95	16.34898	-33.87503	-2.35	39.30952	-94.50220
-1.96	16.69521	-34.73288	-2.36	40.28049	-97.13400
-1.97	17.02617	-35.54718	-2.37	41.12448	-99.26540
-1.98	17.36832	-36.36578	-2.38	42.18297	-102.34166
-1.99	17.72595	-37.25885	-2.39	43.30132	-105.62062

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
-0.80	2.72040	-4.17670	-1.20	4.55664	-7.46703
-0.81	2.75226	-4.22939	-1.21	4.62168	-7.59225
-0.82	2.78561	-4.28420	-1.22	4.69024	-7.72234
-0.83	2.81818	-4.33868	-1.23	4.75801	-7.85347
-0.84	2.85230	-4.39642	-1.24	4.82693	-7.98602
-0.85	2.88597	-4.45308	-1.25	4.89814	-8.12401
-0.86	2.92126	-4.51212	-1.26	4.96785	-8.25938
-0.87	2.95717	-4.57292	-1.27	5.04211	-8.40385
-0.88	2.99225	-4.63327	-1.28	5.11775	-8.55292
-0.89	3.02905	-4.69512	-1.29	5.19297	-8.70061
-0.90	3.06614	-4.75964	-1.30	5.26955	-8.84980
-0.91	3.10429	-4.82529	-1.31	5.35127	-9.0.267
-0.92	3.14275	-4.89138	-1.32	5.43200	-9.17323
-0.93	3.81192	-4.95947	-1.33	5.51427	-9.33562
-0.94	3.22183	-5.02840	-1.34	5.59594	-9.49567
-0.95	3.26210	-5.09906	-1.35	5.68267	-9.67019
-0.96	3.30485	-5.17350	-1.36	5.77181	-9.85141
-0.97	3.34671	-5.24599	-1.37	5.85971	-10.02566
-0.98	3.38938	-5.32209	-1.38	5.95322	-10.22000
-0.99	3.43249	-5.39878	-1.39	6.04545	-10.40310
-1.00	3.47645	-5.47645	-1.40	6.14028	-10.59721
-1.01	3.52170	-5.55698	-1.41	6.23780	-10.79885
-1.02	3.56854	-5.64078	-1.42	6.33779	-10.99172
-1.03	3.61525	-5.72462	-1.43	6.43624	-11.20360
-1.04	3.66251	-5.80913	-1.44	6.53781	-11.41167
-1.05	3.71074	-5.89636	-1.45	6.64634	-11.64054
-1.06	3.76000	-5.98477	-1.46	6.75328	-11.85959
-1.07	3.81031	-6.07606	-1.47	6.86263	-12.09120
-1.08	3.86125	-6.16987	-1.48	6.97601	-12.32806
-1.09	3.91330	-6.26502	-1.49	7.08669	-12.55729
-1.10	3.96650	-6.36151	-1.50	7.20618	-12.81205
-1.11	4.02088	-6.46308	-1.51	7.32367	-13.06012
-1.12	4.07602	-6.56377	-1.52	7.44391	-13.31096
-1.13	4.13289	-6.67034	-1.53	7.56866	-13.57920
-1.14	4.19059	-6.77804	-1.54	7.69648	-13.85093
-1.15	4.24915	-6.88683	-1.55	7.82834	-14.13392
-1.16	4.30747	-6.99505	-1.56	7.95770	-14.40782
-1.17	4.36879	-7.11139	-1.57	8.09802	-14.71452
-1.18	4.42936	-7.22660	-1.58	8.23493	-15.01048
-1.19	4.49364	-7.34892	-1.59	8.37711	-15.32013

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
0.0	1.25345	-2.00000	-0.40	1.77953	-2.71171
-0.01	1.26347	-2.01267	-0.41	1.79689	-2.73678
-0.02	1.27350	-2.02554	-0.42	1.81440	-2.76192
-0.03	1.28385	-2.03863	-0.43	1.83228	-2.78792
-0.04	1.29453	-2.05164	-0.44	1.85032	-2.81402
-0.05	1.30497	-2.06518	-0.45	1.86852	-2.84070
-0.06	1.31567	-2.07897	-0.46	1.88688	-2.86799
-0.07	1.32638	-2.09298	-0.47	1.90593	-2.89587
-0.08	1.33744	-2.10694	-0.48	1.92518	-2.92441
-0.09	1.34886	-2.12154	-0.49	1.94432	-2.95292
-0.10	1.35970	-2.13597	-0.50	1.96303	-2.98169
-0.11	1.37150	-2.15081	-0.51	1.98401	-3.01211
-0.12	1.38298	-2.16585	-0.52	2.00430	-3.04212
-0.13	1.39459	-2.18120	-0.53	2.02452	-3.07270
-0.14	1.40648	-2.19686	-0.54	2.04582	-3.10479
-0.15	1.41850	-2.21286	-0.55	2.06707	-3.13692
-0.16	1.43082	-2.22884	-0.56	2.08886	-3.17613
-0.17	1.44329	-2.24529	-0.57	2.11059	-3.20312
-0.18	1.45580	-2.26223	-0.58	2.13227	-3.23687
-0.19	1.46835	-2.27883	-0.59	2.15513	-3.27173
-0.20	1.48159	-2.29632	-0.60	2.17797	-3.30665
-0.21	1.49462	-2.31409	-0.61	2.20139	-3.34263
-0.22	1.50770	-2.33182	-0.62	2.13366	-3.32284
-0.23	1.52124	-2.35006	-0.63	2.24916	-3.41716
-0.24	1.53457	-2.36820	-0.64	2.27284	-3.45417
-0.25	1.54823	-2.38716	-0.65	2.29783	-3.49324
-0.26	1.56235	-2.40628	-0.66	2.32284	-3.53312
-0.27	1.57629	-2.42573	-0.67	2.34892	-3.57430
-0.28	1.59098	-2.44544	-0.68	2.37429	-3.61461
-0.29	1.60549	-2.46549	-0.69	2.40108	-3.65723
-0.30	1.62008	-2.48594	-0.70	2.42715	-3.69893
-0.31	1.63519	-2.50707	-0.71	2.45437	-3.74204
-0.32	1.65040	-2.52821	-0.72	2.48197	-3.78712
-0.33	1.66570	-2.54979	-0.73	2.51080	-3.83298
-0.34	1.68154	-2.57168	-0.74	2.53922	-3.87889
-0.35	1.69723	-2.59394	-0.75	2.56858	-3.92798
-0.36	1.71329	-2.61677	-0.76	2.59752	-3.97356
-0.37	1.72929	-2.63952	-0.77	2.62778	-4.02355
-0.38	1.74569	-2.66310	-0.78	2.65817	-4.07379
-0.39	1.76278	-2.68756	-0.79	2.68904	-4.12453

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
0.0	1.25345	-2.00000	0.40	0.93565	-1.62579
0.01	1.24342	-1.98753	0.41	0.92939	-1.61892
0.02	1.23339	-1.97526	0.42	0.92308	-1.61237
0.03	1.22367	-1.96318	0.43	0.01724	-1.60556
0.04	1.21425	-1.95156	0.44	0.91135	-1.59907
0.05	1.20507	-1.93981	0.45	0.90541	-1.59263
0.06	1.19563	-1.92824	0.46	0.89941	-1.58625
0.07	1.18618	-1.91685	0.47	0.89362	-1.57996
0.08	1.17702	-1.90589	0.48	0.88776	-1.57372
0.09	1.16813	-1.89474	0.49	0.88214	-1.56766
0.10	1.15919	-1.88408	0.50	0.87617	-1.56204
0.11	1.15057	-1.87348	0.51	0.87068	-1.55583
0.12	1.14163	-1.86310	0.52	0.86514	-1.55018
0.13	1.13321	-1.85276	0.53	0.85982	-1.54442
0.14	1.12453	-1.84261	0.54	0.85441	-1.53860
0.15	1.11635	-1.83248	0.55	0.84923	-1.53291
0.16	1.10789	-1.82280	0.56	0.84370	-1.52496
0.17	1.09995	-1.81313	0.57	0.83840	-1.52208
0.18	1.09197	-1.80330	0.58	0.83333	-1.51661
0.19	1.08397	-1.79416	0.59	0.82816	-1.51130
0.20	1.07596	-1.78481	0.60	0.82323	-1.50611
0.21	1.06817	-1.77553	0.61	0.81793	-1.50114
0.22	1.06035	-1.76664	0.62	0.90401	-1.43952
0.23	1.05277	-1.75774	0.63	0.80801	-1.49089
0.24	1.04541	-1.74917	0.64	0.80314	-1.48615
0.25	1.03776	-1.74049	0.65	0.79814	-1.48133
0.26	1.03033	-1.73206	0.66	0.79339	-1.47635
0.27	1.02313	-1.72367	0.67	0.78883	-1.47131
0.28	1.01590	-1.71557	0.68	0.78427	-1.46667
0.29	1.00889	-1.70749	0.69	0.77958	-1.46193
0.30	1.00183	-1.69950	0.70	0.77490	-1.45760
0.31	0.99500	-1.69145	0.71	0.77040	-1.45319
0.32	0.98813	-1.68375	0.72	0.76583	-1.44857
0.33	0.98121	-1.67613	0.73	0.76145	-1.44411
0.34	0.97450	-1.66869	0.74	0.75676	-1.44004
0.35	0.96802	-1.66125	0.75	0.75257	-1.43538
0.36	0.96122	-1.65397	0.76	0.74808	-1.43162
0.37	0.95464	-1.64694	0.77	0.74376	-1.42726
0.38	0.94828	-1.63980	0.78	0.73972	-1.42290
0.39	0.94212	-1.63253	0.79	0.73562	-1.41881

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
0.80	0.73145	-1.41474	1.20	0.59269	-1.28890
0.81	0.72721	-1.41094	1.21	0.58936	-1.28686
0.82	0.72316	-1.40701	1.22	0.58681	-1.28406
0.83	0.71914	-1.40322	1.23	0.58387	-1.28171
0.84	0.71530	-1.39902	1.24	0.58140	-1.27899
0.85	0.71115	-1.39552	1.25	0.57831	-1.27695
0.86	0.70718	-1.39186	1.26	0.57539	-1.27502
0.87	0.70315	-1.38822	1.27	0.57271	-1.27261
0.88	0.69915	-1.38473	1.28	0.57053	-1.26947
0.89	0.69534	-1.38131	1.29	0.56740	-1.26788
0.90	0.69185	-1.37731	1.30	0.56476	-1.26588
0.91	0.68790	-1.37392	1.31	0.56239	-1.26301
0.92	0.68427	-1.37046	1.32	0.55962	-1.26100
0.93	0.68057	-1.36701	1.33	0.55738	-1.25852
0.94	0.67680	-1.36383	1.34	0.55412	-1.25776
0.95	0.67336	-1.36030	1.35	0.55175	-1.25528
0.96	0.66971	-1.35690	1.36	0.54930	-1.25278
0.97	0.66613	-1.35392	1.37	0.54644	-1.25157
0.98	0.66248	-1.35067	1.38	0.54451	-1.24816
0.99	0.65917	-1.34731	1.39	0.54216	-1.24640
1.00	0.65579	-1.34421	1.40	0.53975	-1.24428
1.01	0.65192	-1.34155	1.41	0.53726	-1.24215
1.02	0.64909	-1.33777	1.42	0.53434	-1.24143
1.03	0.64550	-1.33497	1.43	0.53240	-1.23868
1.04	0.64227	-1.33201	1.44	0.52933	-1.23799
1.05	0.63897	-1.32906	1.45	0.52726	-1.23521
1.06	0.63560	-1.32640	1.46	0.52475	-1.23389
1.07	0.63216	-1.32374	1.47	0.52289	-1.23110
1.08	0.62910	-1.32062	1.48	0.52024	-1.22978
1.09	0.62596	-1.31777	1.49	0.51787	-1.22851
1.10	0.62276	-1.31522	1.50	0.51583	-1.22606
1.11	0.61949	-1.31238	1.51	0.51332	-1.22479
1.12	0.61661	-1.30960	1.52	0.51153	-1.22273
1.13	0.61319	-1.30706	1.53	0.50889	-1.22146
1.14	0.61018	-1.30429	1.54	0.50697	-1.21937
1.15	0.60758	-1.30124	1.55	0.50500	-1.21725
1.16	0.60413	-1.29944	1.56	0.50254	-1.21643
1.17	0.60139	-1.29638	1.57	0.50043	-1.21429
1.18	0.59829	-1.29402	1.58	0.59869	-1.21211
1.19	0.59542	-1.29125	1.59	0.49601	-1.21132

APPENDIX I

q	u(q)	v(q)	q	u(q)	v(q)
1.60	0.41414	-1.20911	2.00	0.42222	-1.15556
1.61	0.49176	-1.20877	2.01	0.41966	-1.15592
1.62	0.48976	-1.20654	2.02	0.41811	-1.15572
1.63	0.48817	-1.20423	2.03	0.41732	-1.15221
1.64	0.48558	-1.20393	2.04	0.41566	-1.15198
1.65	0.48387	-1.20206	2.05	0.41393	-1.15177
1.66	0.48211	-1.19968	2.06	0.41213	-1.15073
1.67	0.48028	-1.19775	2.07	0.41026	-1.15056
1.68	0.47790	-1.19744	2.08	0.40959	-1.14870
1.69	0.47544	-1.19666	2.09	0.40757	-1.14764
1.70	0.47447	-1.19290	2.10	0.40682	-1.14568
1.71	0.47135	-1.19437	2.11	0.40371	-1.14855
1.72	0.46975	-1.19228	2.12	0.40284	-1.14658
1.73	0.46809	-1.18963	2.13	0.40194	-1.14455
1.74	0.46583	-1.18930	2.14	0.40099	-1.14144
1.75	0.46466	-1.18698	2.15	0.39899	-1.14358
1.76	0.46226	-1.18668	2.16	0.39793	-1.14038
1.77	0.46098	-1.18428	2.17	0.39578	-1.14161
1.78	0.45844	-1.18345	2.18	0.39353	-1.14293
1.79	0.45647	-1.18301	2.19	0.39394	-1.13833
1.80	0.45443	-1.18260	2.20	0.39155	-1.13969
1.81	0.45290	-1.18010	2.21	0.39193	-1.13369
1.82	0.45204	-1.17671	2.22	0.38938	-1.13394
1.83	0.44920	-1.17847	2.23	0.38855	-1.13395
1.84	0.44823	-1.17499	2.24	0.38462	-1.13964
1.85	0.44660	-1.17431	2.25	0.38486	-1.13316
1.86	0.44413	-1.17330	2.26	0.38387	-1.13196
1.87	0.44236	-1.17264	2.27	0.38284	-1.12945
1.88	0.44200	-1.16858	2.28	0.38407	-1.13401
1.89	0.43946	-1.16968	2.29	0.37931	-1.13151
1.90	0.43750	-1.16835	2.30	0.37809	-1.12892
1.91	0.43634	-1.16663	2.31	0.37545	-1.13389
1.92	0.43354	-1.16790	2.32	0.37778	-1.12132
1.93	0.43226	-1.16616	2.33	0.37500	-1.12500
1.94	0.43092	-1.16438	2.34	0.37209	-1.12890
1.95	0.42953	-1.16256	2.35	0.37302	-1.12223
1.96	0.42808	-1.16070	2.36	0.36992	-1.12633
1.97	0.42583	-1.16098	2.37	0.36929	-1.12656
1.98	0.42527	-1.15855	2.38	0.37021	-1.11937
1.99	0.42287	-1.15887	2.39	0.36681	-1.12221

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.00000	0.01000	1009.61743	9715.60156
1.00000	0.02000	259.27197	2404.33130
1.00000	0.03000	119.79405	1040.24365
1.00000	0.04000	70.55414	568.27954
1.00000	0.05000	47.45457	353.76440
1.00000	0.06000	34.68512	239.58818
1.00000	0.07000	26.82515	172.14040
1.00000	0.08000	21.60580	129.22214
1.00000	0.09000	17.93889	100.33736
1.00000	0.10000	15.24826	80.02769
1.00000	0.11000	13.20477	65.23619
1.00000	0.12000	11.60879	54.14795
1.00000	0.13000	10.33323	45.63263
1.00000	0.14000	9.29383	38.95799
1.00000	0.15000	8.84281	33.63333
1.00000	0.16000	7.70941	29.32025
1.00000	0.17000	7.09412	25.77957
1.00000	0.18000	6.56514	22.83844
1.00000	0.19000	6.10605	20.36966
1.00000	0.20000	5.70423	18.27774
1.00000	0.21000	5.34990	16.49022
1.00000	0.22000	5.03533	14.95103
1.00000	0.23000	4.75434	13.61640
1.00000	0.24000	4.50197	12.45188
1.00000	0.25000	4.27416	11.42982
1.00000	0.26000	4.06756	10.52801
1.00000	0.27000	3.87942	9.72836
1.00000	0.28000	3.70741	9.01606
1.00000	0.29000	3.54958	8.37891
1.00000	0.30000	3.40430	7.80671
1.00000	0.31000	3.27014	7.29098
1.00000	0.32000	3.14590	6,82451
1.00000	0.33000	3.03053	6.40127
1.00000	0.34000	2.92314	6.01610
1.00000	0.35000	2.82294	5.66455
1.00000	0.36000	2.72924	5,34286
1.00000	0.37000	2.64144	5.04773
1.00000	0.38000	2.55900	4.77633
1.00000	0.39000	2.48146	4.52619
1.00000	0.40000	2.40840	4.29515

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.01000	0.01000	910.68555	9729.80078
1.01000	0.02000	235.16898	2391.35742
1.01000	0.03000	109.41496	1030.12231
1.01000	0.04000	64.89713	561.16895
1.01000	0.05000	43.93744	348.74829
1.01000	0.06000	32.30481	235.98001
1.01000	0.07000	25.11562	169.48654
1.01000	0.08000	20.32265	127.22240
1.01000	0.09000	16.94257	98.79968
1.01000	0.10000	14.45355	78.82292
1.01000	0.11000	12.55687	64.27655
1.01000	0.12000	11.07094	53.37219
1.01000	0.13000	9.87988	44.99719
1.01000	0.14000	8.90672	38.43143
1.01000	0.15000	8.09855	33.19226
1.01000	0.16000	7.41796	28.94736
1.01000	0.17000	6.83782	25.46155
1.01000	0.18000	6.33805	22.56516
1.01000	0.19000	5.90347	20.13313
1.01000	0.20000	5.52243	18.07173
1.01000	0.21000	5.18565	16.30969
1.01000	0.22000	4.88657	14.79197
1.01000	0.23000	4.61884	13.47556
1.01000	0.24000	4.37805	12.32657
1.01000	0.25000	4.16039	11.31787
1.01000	0.26000	3.96276	10.42759
1.01000	0.27000	3.78256	9.63794
1.01000	0.28000	3.61763	8.93436
1.01000	0.29000	3.46615	8.30486
1.01000	0.30000	3.32655	7.73938
1.01000	0.31000	3.19752	7.22957
1.01000	0.32000	3.07792	6.76838
1.01000	0.33000	2.96676	6.34980
1.01000	0.34000	2.86320	5.96879
1.01000	0.35000	2.76650	5.62098
1.01000	0.36000	2.67600	5.30264
1.01000	0.37000	2.59114	5.01053
1.01000	0.38000	2.51140	4.74184
1.01000	0.39000	2.43635	4.49417
1.01000	0.40000	2.36559	4.26536

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.02000	0.01000	811.98657	9733.91797
1.02000	0.02000	211.23767	2373.89014
1.02000	0.03000	99.15199	1017.61963
1.02000	0.04000	59.31841	552.75781
1.02000	0.05000	40.47444	343.00024
1.02000	0.06000	29.96291	231.93456
1.02000	0.07000	23.43394	166.55537
1.02000	0.08000	19.06035	125.04312
1.02000	0.09000	15.96220	97.14063
1.02000	0.10000	13.67128	77.53366
1.02000	0.11000	11.91884	63.25653
1.02000	0.12000	10.54103	52.55215
1.02000	0.13000	9.43303	44.32870
1.02000	0.14000	8.52498	37.87953
1.02000	0.15000	7.76878	32.73166
1.02000	0.16000	7.13030	28.55905
1.02000	0.17000	6.58475	25.13132
1.02000	0.18000	6.11372	22.28203
1.02000	0.19000	5.70328	19.88857
1.02000	0.20000	5.34270	17.85909
1.02000	0.21000	5.02362	16.12366
1.02000	0.22000	4.73942	14.62833
1.02000	0.23000	4.48476	13.33086
1.02000	0.24000	4.25538	12.19802
1.02000	0.25000	4.04775	11.20315
1.02000	0.26000	3.85896	10.32477
1.02000	0.27000	3.68661	9.54547
1.02000	0.28000	3.52868	8.85088
1.02000	0.29000	3.38345	8.22923
1.02000	0.30000	3.24948	7.67067
1.02000	0.31000	3.12551	7.16695
1.02000	0.32000	3.01050	6.71115
1.02000	0.33000	2.90351	6.29738
1.02000	0.34000	2.80374	5.92064
1.02000	0.35000	2.71049	5.57665
1.02000	0.36000	2.62316	5.26174
1.02000	0.37000	2.54120	4.97272
1.02000	0.38000	2.46414	4.70681
1.02000	0.39000	2.39155	4.46165
1.02000	0.40000	2.32307	4.23514

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.03000	0.01000	713.60889	9713.74219
1.03000	0.02000	187.53127	2350.15259
1.03000	0.03000	89.03265	1002.08569
1.03000	0.04000	53.83282	542.79761
1.03000	0.05000	37.07362	336.39722
1.03000	0.06000	27.66411	227.39236
1.03000	0.07000	21.78323	163.32161
1.03000	0.08000	17.82091	122.66968
1.03000	0.09000	14.99912	95.35211
1.03000	0.10000	12.90239	76.15529
1.03000	0.11000	11.29135	62.17305
1.03000	0.12000	10.01958	5.168590
1.03000	0.13000	8.99304	43.62575
1.03000	0.14000	8.14891	37.30157
1.03000	0.15000	7.44372	32.25090
1.03000	0.16000	6.84661	28.15501
1.03000	0.17000	6.33506	24.78859
1.03000	0.18000	5.89229	21.98883
1.03000	0.19000	5.50558	19.63585
1.03000	0.20000	5.16514	17.63976
1.03000	0.21000	4.86329	15.93211
1.03000	0.22000	4.59392	14.46005
1.03000	0.23000	4.35215	13.18228
1.03000	0.24000	4.13402	12.06615
1.03000	0.25000	3.93626	11.08558
1.03000	0.26000	3.75621	10.21954
1.03000	0.27000	3.59160	9.45089
1.03000	0.28000	3.44057	8.76558
1.03000	0.29000	3.30152	8.15204
1.03000	0.30000	3.17310	7.60059
1.03000	0.31000	3.05414	7.10313
1.03000	0.32000	2.94366	6.65286
1.03000	0.33000	2.84078	6.24401
1.03000	0.34000	2.74476	5.87165
1.03000	0.35000	2.65494	5.53158
1.03000	0.36000	2.57074	5.22019
1.03000	0.37000	2.49165	4.93430
1.03000	0.38000	2.41724	4.67125
1.03000	0.39000	2.34709	4.42866
1.03000	0.40000	2.28086	4.20446

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.04000	0.01000	615.71997	9682.41016
1.04000	0.02000	164.12070	2317.73047
1.04000	0.03000	79.09081	982.68262
1.04000	0.04000	48.45695	530.95801
1.04000	0.05000	33.74411	328.81396
1.04000	0.06000	25.41370	222.29855
1.04000	0.07000	20.16663	159.75687
1.04000	0.08000	16.60632	120.08727
1.04000	0.09000	14.05467	93.42596
1.04000	0.10000	12.14779	74.68266
1.04000	0.11000	10.67506	61.02313
1.04000	0.12000	9.50705	50.77159
1.04000	0.13000	8.57028	42.88724
1.04000	0.14000	7.77876	36.69679
1.04000	0.15000	7.12359	31.74950
1.04000	0.16000	6.56705	27.73480
1.04000	0.17000	6.08885	24.43303
1.04000	0.18000	5.67383	21.68536
1.04000	0.19000	5.31046	19.37482
1.04000	0.20000	4.98982	17.41362
1.04000	0.21000	4.70490	15.73490
1.04000	0.22000	4.45014	14.28708
1.04000	0.23000	4.22105	13.02973
1.04000	0.24000	4.01399	11.93094
1.04000	0.25000	3.82598	10.96519
1.04000	0.26000	3.65452	10.11187
1.04000	0.27000	3.49755	9.35422
1.04000	0.28000	3.35332	8.67847
1.04000	0.29000	3.22037	8.07325
1.04000	0.30000	3.09743	7.52911
1.04000	0.31000	2.98341	7.03808
1.04000	0.32000	2.87741	6.59350
1.04000	0.33000	2.77859	6.18970
1.04000	0.34000	2.68627	5.82182
1.04000	0.35000	2.59983	5.48576
1.04000	0.36000	2.51873	5.17794
1.04000	0.37000	2.44249	4.89529
1.04000	0.38000	2.37069	4.63514
1.04000	0.39000	2.30296	4.39516
1.04000	0.40000	2.23896	4.17335

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.05000	0.01000	518.55054	9618.49609
1.05000	0.02000	141.10922	2273.09546
1.05000	0.03000	69.37074	958.35620
1.05000	0.04000	43.21187	516.89038
1.05000	0.05000	30.49664	320.10986
1.05000	0.06000	23.21768	216.59204
1.05000	0.07000	18.58772	155.83224
1.05000	0.08000	15.41887	117.28104
1.05000	0.09000	13.13034	91.35397
1.05000	0.10000	11.40051	73.11128
1.05000	0.11000	10.07069	59.80421
1.05000	0.12000	9.00397	49.80763
1.05000	0.13000	8.13515	42.11209
1.05000	0.14000	7.41484	36.06439
1.05000	0.15000	6.80862	31.22696
1.05000	0.16000	6.29181	27.29817
1.05000	0.17000	5.84630	24.06447
1.05000	0.18000	5.45849	21.37151
1.05000	0.19000	5.11801	19.10536
1.05000	0.20000	4.81680	17.18056
1.05000	0.21000	4.54853	15.53202
1.05000	0.22000	4.30812	14.10938
1.05000	0.23000	4.09151	12.87320
1.05000	0.24000	3.89535	11.79237
1.05000	0.25000	3.71692	10.84193
1.05000	0.26000	3.55393	10.00175
1.05000	0.27000	3.40448	9.25544
1.05000	0.28000	3.26696	8.58952
1.05000	0.29000	3.14002	7.99288
1.05000	0.30000	3.02248	7.45623
1.05000	0.31000	2.91335	6.97181
1.05000	0.32000	2.81176	6.53306
1.05000	0.33000	2.71696	6.13442
1.05000	0.34000	2.62830	5.77114
1.05000	0.35000	2.54520	5.43917
1.05000	0.36000	2.46716	5.13503
1.05000	0.37000	2.39373	4.85567
1.05000	0.38000	2.32452	4.59848
1.05000	0.39000	2.25917	4.36118
1.05000	0.40000	2.19738	4.14179

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.06000	0.01000	422.47583	9495.71875
1.06000	0.02000	118.63675	2211.01929
1.06000	0.03000	59.92586	927.73999
1.06000	0.04000	38.12073	500.17334
1.06000	0.05000	27.34254	310.13452
1.06000	0.06000	21.08206	210.20854
1.06000	0.07000	17.04997	151.51816
1.06000	0.08000	14.26061	114.23622
1.06000	0.09000	12.22745	89.12814
1.06000	0.10000	10.68542	71.43655
1.06000	0.11000	9.47883	58.51334
1.06000	0.12000	8.51077	48.79218
1.06000	0.13000	7.71792	41.29912
1.06000	0.14000	7.05736	35.40361
1.06000	0.15000	6.49896	30.68274
1.06000	0.16000	6.02100	26.84468
1.06000	0.17000	5.60748	23.68265
1.06000	0.18000	5.24633	21.04703
1.06000	0.19000	4.92829	18.82730
1.06000	0.20000	4.64614	16.94049
1.06000	0.21000	4.39420	15.32334
1.06000	0.22000	4.16790	13.92684
1.06000	0.23000	3.96354	12.71264
1.06000	0.24000	3.77810	11.65037
1.06000	0.25000	3.60910	10.71575
1.06000	0.26000	3.45444	9.88914
1.06000	0.27000	3.31240	9.15452
1.06000	0.28000	3.18150	8.49871
1.06000	0.29000	3.06048	7.91089
1.06000	0.30000	2.94827	7.38196
1.06000	0.31000	2.84395	6.90431
1.06000	0.32000	2.74672	6.47153
1.06000	0.33000	2.65588	6.07817
1.06000	0.34000	2.57083	5.71960
1.06000	0.35000	2.49103	5.39183
1.06000	0.36000	2.41601	5.09143
1.06000	0.37000	2.34536	4.81543
1.06000	0.38000	2.27870	4.56128
1.06000	0.39000	2.21572	4.32671
1.06000	0.40000	2.15611	4.10979

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.07000	0.01000	328.24170	9271.00000
1.07000	0.02000	96.91385	2123.82349
1.07000	0.03000	50.82744	889.19653
1.07000	0.04000	33.21266	480.34912
1.07000	0.05000	24.29556	298.73047
1.07000	0.06000	19.01408	203.08620
1.07000	0.07000	15.55752	146.78702
1.07000	0.08000	13.13411	110.93925
1.07000	0.09000	11.34767	86.74149
1.07000	0.10000	9.97964	69.65460
1.07000	0.11000	8.90028	57.14835
1.07000	0.12000	8.02800	47.72386
1.07000	0.13000	7.30904	40.44749
1.07000	0.14000	6.70665	34.71394
1.07000	0.15000	6.19485	30.11646
1.07000	0.16000	5.75482	26.37408
1.07000	0.17000	5.37255	23.28735
1.07000	0.18000	5.03747	20.71179
1.07000	0.19000	4.74139	18.54057
1.07000	0.20000	4.47792	16.69337
1.07000	0.21000	4.24200	15.10884
1.07000	0.22000	4.02953	13.73946
1.07000	0.23000	3.83720	12.54801
1.07000	0.24000	3.66229	11.50495
1.07000	0.25000	3.50255	10.58668
1.07000	0.26000	3.35610	9.77403
1.07000	0.27000	3.22135	9.05143
1.07000	0.28000	3.09695	8.40606
1.07000	0.29000	2.98176	7.82728
1.07000	0.30000	2.87481	7.30626
1.07000	0.31000	2.77523	6.83557
1.07000	0.32000	2.68230	6.40891
1.07000	0.33000	2.59537	6.02097
1.07000	0.34000	2.51388	5.66720
1.07000	0.35000	2.43734	5.34372
1.07000	0.36000	2.36531	5.04715
1.07000	0.37000	2.29740	4.77459
1.07000	0.38000	2.23327	4.52352
1.07000	0.39000	2.17262	4.29173
1.07000	0.40000	2.11516	4.07733

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.08000	1.01000	237.26978	8844.17969
1.08000	1.02000	76.24045	2000.68188
1.08000	1.03000	42.16373	840.83301
1.08000	1.04000	28.52100	456.94385
1.08000	1.05000	21.37068	285.74512
1.08000	1.06000	17.02142	195.16803
1.08000	1.07000	14.11466	141.61417
1.08000	1.08000	12.04196	107.37849
1.08000	1.09000	10.49262	84.18793
1.08000	1.10000	9.29226	67.76205
1.08000	1.11000	8.33577	55.70735
1.08000	1.12000	7.55619	46.60152
1.08000	1.13000	6.90887	39.55653
1.08000	1.14000	6.36298	33.99489
1.08000	1.15000	5.89653	29.52786
1.08000	1.16000	5.49343	25.88615
1.08000	1.17000	5.14165	22.87843
1.08000	1.18000	4.83202	20.36574
1.08000	1.19000	4.55740	18.24512
1.08000	1.20000	4.31221	16.43910
1.08000	1.21000	4.09196	14.88646
1.08000	1.22000	3.89305	13.54721
1.08000	1.23000	3.71253	12.37929
1.08000	1.24000	3.54795	11.35608
1.08000	1.25000	3.39732	10.45466
1.08000	1.26000	3.25892	9.65643
1.08000	1.27000	3.13134	8.94621
1.08000	1.28000	3.01334	8.31152
1.08000	1.29000	2.90390	7.74206
1.08000	1.30000	2.80211	7.22916
1.08000	1.31000	2.70721	6.76558
1.08000	1.32000	2.61851	6.34519
1.08000	1.33000	2.53544	5.96280
1.08000	1.34000	2.45746	5.61395
1.08000	1.35000	2.38413	5.29484
1.08000	1.36000	2.31505	5.00218
1.08000	1.37000	2.24985	4.73313
1.08000	1.38000	2.18821	4.48521
1.08000	1.39000	2.12986	4.25627
1.08000	1.40000	2.07455	4.04443

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
0.09000	0.01000	152.48480	8002.78906
0.09000	0.02000	57.04277	1827.77637
0.09000	0.03000	34,04326	780.67676
0.09000	0.04000	24.08411	429.51563
0.09000	0.05000	18.58464	271.04883
0.09000	0.06000	15.11246	186.40645
0.09000	0.07000	12.72604	135.97987
0.09000	0.08000	10.98692	103.54486
0.09000	0.09000	9.66408	81.46307
0.09000	0.10000	8.62446	65.75665
0.09000	0.11000	7.78614	54.18900
0.09000	0.12000	7.09594	45.42448
0.09000	0.13000	6.51786	38.62575
0.09000	0.14000	6.02670	33.24622
0.09000	0.15000	5.60424	28.91672
0.09000	0.16000	5.23705	23.38087
0.09000	0.17000	4.91493	22.45587
0.09000	0.18000	4.63010	20.00879
0.09000	0.19000	4.37644	17.94089
0.09000	0.20000	4.14910	16.17769
0.09000	0.21000	3.94419	14.66221
0.09000	0.22000	3.75854	13.35007
0.09000	0.23000	3.58958	12.20648
0.09000	0.24000	3.43514	11.20377
0.09000	0.25000	3.29343	10.31972
0.09000	0.26000	3.16294	9.53630
0.09000	0.27000	3.04239	8.83882
0.09000	0.28000	2.93069	8.21514
0.09000	0.29000	2.82690	7.65522
0.09000	0.30000	2.73020	7.15065
0.09000	0.31000	2.63990	6.69436
0.09000	0.32000	2.55538	6.28039
0.09000	0.33000	2.47609	5.90366
0.09000	0.34000	2.40158	5.55984
0.09000	0.35000	2.33142	5.24521
0.09000	0.36000	2.26524	4.95654
0.09000	0.37000	2.20272	4.69107
0.09000	0.38000	2.14355	4.44635
0.09000	0.39000	2.08748	4.22031
0.09000	0.40000	2.03426	4.01108

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.10000	0.01000	79.77246	6363.64453
1.10000	0.02000	39.88620	1590.90620
1.10000	0.03000	26.59082	707.07202
1.10000	0.04000	19.94312	397.72778
1.10000	0.05000	15.95449	254.54572
1.10000	0.06000	13.29541	176.76782
1.10000	0.07000	11.39607	129.87030
1.10000	0.08000	9.97156	99.43195
1.10000	0.09000	8.86361	78.56348
1.10000	0.10000	7.97725	63.63646
1.10000	0.11000	7.25204	52.59212
1.10000	0.12000	6.64771	44.19197
1.10000	0.13000	6.13634	37.65468
1.10000	0.14000	5.69803	32.46756
1.10000	0.15000	5.31816	28.28285
1.10000	0.16000	4.98578	24.85799
1.10000	0.17000	4.69250	22.01952
1.10000	0.18000	4.43180	19.64087
1.10000	0.19000	4.19855	17.62781
1.10000	0.20000	3.98862	15.90909
1.10000	0.21000	3.79869	14.43002
1.10000	0.22000	3.62602	13.14802
1.10000	0.23000	3.46837	12.02956
1.10000	0.24000	3,32385	11.04798
1.10000	0.25000	3.19090	10.18182
1.10000	0.26000	3.06817	9.41367
1.10000	0.27000	2.95453	8.72927
1.10000	0.28000	2.84901	8.11688
1.10000	0.29000	2.75077	7.56675
1.10000	0.30000	2.65908	7.07071
1.10000	0.31000	2.57330	6.62189
1.10000	0.32000	2.49289	6.21449
1.10000	0.33000	2.41735	5.84356
1.10000	0.34000	2.34625	5.50487
1.10000	0.35000	2.27921	5.19480
1.10000	0.36000	2.21590	4.91021
1.10000	0.37000	2.15601	4.64838
1.10000	0.38000	2.09927	4.40695
1.10000	0.39000	2.04545	4.18385
1.10000	0.40000	1.99431	3.97728

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.11000	0.01000	28.75339	3702.16187
1.11000	0.02000	25.45258	1284.16284
1.11000	0.03000	19.94554	619.44678
1.11000	0.04000	16.14267	361.47974
1.11000	0.05000	13.49870	236.21107
1.11000	0.06000	11.57927	166.24478
1.11000	0.07000	10.12965	123.28296
1.11000	0.08000	8.99877	95.03877
1.11000	0.09000	8.09304	75.48891
1.11000	0.10000	7.35184	61.40146
1.11000	0.11000	6.73433	50.91682
1.11000	0.12000	6.21209	42.90405
1.11000	0.13000	5.76474	36.64336
1.11000	0.14000	5.37731	31.65903
1.11000	0.15000	5.03854	27.62625
1.11000	0.16000	4.73983	24.31757
1.11000	0.17000	4.47449	21.56937
1.11000	0.18000	4.23724	19.26198
1.11000	0.19000	4.02384	17.30592
1.11000	0.20000	3.83087	15.63330
1.11000	0.21000	3.65554	14.19194
1.11000	0.22000	3.49554	12.94106
1.11000	0.23000	3.34894	11.84852
1.11000	0.24000	3.21414	10.88872
1.11000	0.25000	3.08976	10.04098
1.11000	0.26000	2.97464	9.28851
1.11000	0.27000	2.86776	8.61755
1.11000	0.28000	2.76833	8.01674
1.11000	0.29000	2.67554	7.47665
1.11000	0.30000	2.58876	6.98934
1.11000	0.31000	2.50744	6.54817
1.11000	0.32000	2.43106	6.14748
1.11000	0.33000	2.35920	5.78247
1.11000	0.34000	2.29146	5.44903
1.11000	0.35000	2.22751	5.14363
1.11000	0.36000	2.16702	4.86319
1.11000	0.37000	2.10973	4.60509
1.11000	0.38000	2.05539	4.36698
1.11000	0.39000	2.00378	4.14689
1.11000	0.40000	1.95470	3.94303

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.12000	0.01000	5.52316	1135.16235
1.12000	0.02000	14.37668	925.53906
1.12000	0.03000	14.24194	519.32739
1.12000	0.04000	12.72631	321.04150
1.12000	0.05000	11.23530	216.11641
1.12000	0.06000	9.97277	154.86159
1.12000	0.07000	8.93150	116.22762
1.12000	0.08000	8.07134	90.36998
1.12000	0.09000	7.35411	72.24165
1.12000	0.10000	6.74935	59.05278
1.12000	0.11000	6.23376	49.16374
1.12000	0.12000	5.78963	41.56122
1.12000	0.13000	5.40346	35.59210
1.12000	0.14000	5.06482	30.82072
1.12000	0.15000	4.76560	26.94710
1.12000	0.16000	4.49939	23.75969
1.12000	0.17000	4.26106	21.10556
1.12000	0.18000	4.04652	18.87221
1.12000	0.19000	3.85239	16.97520
1.12000	0.20000	3.67591	15.35035
1.12000	0.21000	3.51481	13.94793
1.12000	0.22000	3.36716	12.72920
1.12000	0.23000	3.23136	11.66340
1.12000	0.24000	3.10604	10.72600
1.12000	0.25000	2.99004	9.89720
1.12000	0.26000	2.88237	9.16083
1.12000	0.27000	2.78215	8.50367
1.12000	0.28000	2.68865	7.91475
1.12000	0.29000	2.60121	7.38492
1.12000	0.30000	2.51927	6.90656
1.12000	0.31000	2.44232	6.47321
1.12000	0.32000	2.36991	6.07938
1.12000	0.33000	2.30167	5.72042
1.12000	0.34000	2.23724	5.39233
1.12000	0.35000	2.17632	5.09168
1.12000	0.36000	2.11862	4.81549
1.12000	0.37000	2.06389	4.56117
1.12000	0.38000	2.01192	4.32647
1.12000	0.39000	1.96249	4.10943
1.12000	0.40000	1.91544	3.90833

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.13000	0.01000	0.44373	133.31487
1.13000	0.02000	6.93824	568.50244
1.13000	0.03000	9.58486	411.36157
1.13000	0.04000	9.73268	277.21094
1.13000	0.05000	9.18116	194.46654
1.13000	0.06000	8.48434	142.68629
1.13000	0.07000	7.80630	108.73146
1.13000	0.08000	7.19203	85.43761
1.13000	0.09000	6.64858	68.82767
1.13000	0.10000	6.17099	56.59387
1.13000	0.11000	5.75117	47.33484
1.13000	0.12000	5.38093	40.16455
1.13000	0.13000	5.05293	34.50168
1.13000	0.14000	4.76090	29.95320
1.13000	0.15000	4.49959	26.24571
1.13000	0.16000	4.26463	23.18463
1.13000	0.17000	4.05237	20.62823
1.13000	0.18000	3.85977	18.47166
1.13000	0.19000	3.68431	16.63585
1.13000	0.20000	3.52384	15.06029
1.13000	0.21000	3.37656	13.69811
1.13000	0.22000	3.24093	12.51248
1.13000	0.23000	3.11566	11.47422
1.13000	0.24000	2.99960	10.55986
1.13000	0.25000	2.69179	9.75052
1.13000	0.26000	2.79140	9.03068
1.13000	0.27000	2.69769	8.38766
1.13000	0.28000	2.61001	7.81089
1.13000	0.29000	2.52782	7.29158
1.13000	0.30000	2.45061	6.82237
1.13000	0.31000	2.37796	6.39701
1.13000	0.32000	2.30946	6.01020
1.13000	0.33000	2.24478	5.65741
1.13000	0.34000	2.18360	5.33479
1.13000	0.35000	2.12566	5.03899
1.13000	0.36000	2.07070	4.76711
1.13000	0.37000	2.01850	4.51665
1.13000	0.38000	1.96885	4.28541
1.13000	0.39000	1.92158	4.07148
1.13000	0.40000	1.87652	3.87319

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.14000	0.01000	0.01338	5.35256
1.14000	0.02000	2.76185	283.81201
1.14000	0.03000	6.01451	303.48560
1.14000	0.04000	7.18857	231.38948
1.14000	0.05000	7.34976	171.61507
1.14000	0.06000	7.12109	129.83310
1.14000	0.07000	6.75801	100.83803
1.14000	0.08000	6.36323	80.26088
1.14000	0.09000	5.97794	65.25636
1.14000	0.10000	5.61771	54.02940
1.14000	0.11000	5.28720	45.43277
1.14000	0.12000	4.98642	38.71555
1.14000	0.13000	4.71347	33.37286
1.14000	0.14000	4.46578	29.05699
1.14000	0.15000	4.24069	25.52243
1.14000	0.16000	4.03596	22.59254
1.14000	0.17000	3.84849	20.13750
1.14000	0.18000	3.67707	18.06044
1.14000	0.19000	3.51965	16.28784
1.14000	0.20000	3.37469	14.76320
1.14000	0.21000	3.24083	13.44248
1.14000	0.22000	3.11689	12.29095
1.14000	0.23000	3.00186	11.28098
1.14000	0.24000	2.89482	10.39030
1.14000	0.25000	2.79502	9.60092
1.14000	0.26000	2.70174	8.89804
1.14000	0.27000	2.61438	8.26950
1.14000	0.28000	2.53242	7.70518
1.14000	0.29000	2.45537	7.19665
1.14000	0.30000	2.38280	6.73678
1.14000	0.31000	2.31436	6.31958
1.14000	0.32000	2.24969	5.93991
1.14000	0.33000	2.18851	5.59343
1.14000	0.34000	2.13054	5.27639
1.14000	0.35000	2.07553	4.98553
1.14000	0.36000	2.02326	4.71805
1.14000	0.37000	1.97355	4.47152
1.14000	0.38000	1.92620	4.24380
1.14000	0.39000	1.88105	4.03303
1.14000	0.40000	1.83796	3.83759

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.15000	0.01000	0.00015	0.07431
1.15000	0.02000	0.88169	110.98940
1.15000	0.03000	3.48159	205.54393
1.15000	0.04000	5.10458	185.57529
1.15000	0.05000	5.75082	148.08870
1.15000	0.06000	5.88910	116.47496
1.15000	0.07000	5.79040	92.61467
1.15000	0.08000	5.58733	74.86937
1.15000	0.09000	5.34375	61.54192
1.15000	0.10000	5.09057	51.36690
1.15000	0.11000	4.84261	43.46170
1.15000	0.12000	4.60666	37.21677
1.15000	0.13000	4.38549	32.20734
1.15000	0.14000	4.17976	28.13309
1.15000	0.15000	3.98913	24.77792
1.15000	0.16000	3.81276	21.98392
1.15000	0.17000	3.64959	19.63368
1.15000	0.18000	3.49853	17.63873
1.15000	0.19000	3.35852	15.93139
1.15000	0.20000	3.22855	14.45921
1.15000	0.21000	3.10768	13.18113
1.15000	0.22000	3.99509	12.06466
1.15000	0.23000	3.89001	11.08372
1.15000	0.24000	2.79177	10.21737
1.15000	0.25000	2.79975	9.44845
1.15000	0.26000	2.61341	8.76293
1.15000	0.27000	2.53227	8.14922
1.15000	0.28000	2.45589	7.59764
1.15000	0.29000	2.38387	7.10010
1.15000	0.30000	2.31586	6.64978
1.15000	0.31000	2.25154	6.24091
1.15000	0.32000	2.19064	5.86857
1.15000	0.33000	2.13289	5.52851
1.15000	0.34000	2.07806	5.21713
1.15000	0.35000	2.02593	4.93131
1.15000	0.36000	1.97632	4.66831
1.15000	0.37000	1.92905	4.42578
1.15000	0.38000	1.88396	4.20164
1.15000	0.39000	1.84091	3.99410
1.15000	0.40000	1.79976	3.80155

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.16000	0.01000	0.00000	0.00036
1.16000	0.02000	0.22183	33.32442
1.16000	0.03000	1.84118	126.13582
1.16000	0.04000	3.46899	142.12195
1.16000	0.05000	4.38780	124.56065
1.16000	0.06000	4.79233	102.83899
1.16000	0.07000	4.90634	84.15027
1.16000	0.08000	4.86627	69.30217
1.16000	0.09000	4.74737	57.79338
1.16000	0.10000	4.59053	48.61636
1.16000	0.11000	4.41806	41.42715
1.16000	0.12000	4.24214	35.67142
1.16000	0.13000	4.06934	31.00700
1.16000	0.14000	3.90312	27.18274
1.16000	0.15000	3.74512	24.01300
1.16000	0.16000	3.59599	21.35933
1.16000	0.17000	3.45579	19.11722
1.16000	0.18000	3.32427	17.20686
1.16000	0.19000	3.20100	15.56666
1.16000	0.20000	3.08548	14.14844
1.16000	0.21000	2.97718	12.91420
1.16000	0.22000	2.87557	11.83370
1.16000	0.23000	2.78016	10.88258
1.16000	0.24000	2.69045	10.04111
1.16000	0.25000	2.60602	9.29315
1.16000	0.26000	2.52646	8.62541
1.16000	0.27000	2.45138	8.02685
1.16000	0.28000	2.38044	7.48828
1.16000	0.29000	2.31334	7.00198
1.16000	0.30000	2.24979	6.56142
1.16000	0.31000	2.18952	6.16105
1.16000	0.32000	2.13231	5.79613
1.16000	0.33000	2.07792	5.46263
1.16000	0.34000	2.02618	5.15704
1.16000	0.35000	1.97688	4.87634
1.16000	0.36000	1.92988	4.61791
1.16000	0.37000	1.88502	4.37944
1.16000	0.38000	1.84215	4.15896
1.16000	0.39000	1.80115	3.95467
1.16000	0.40000	1.76192	3.76507

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.17000	0.01000	0.00000	0.00000
1.17000	0.02000	0.04364	7.63849
1.17000	0.03000	0.88257	69.42319
1.17000	0.04000	2.24652	103.33188
1.17000	0.05000	3.25700	101.80367
1.17000	0.06000	3.83228	89.20244
1.17000	0.07000	4.10777	75.55595
1.17000	0.08000	4.20164	63.60898
1.17000	0.09000	4.18995	53.76501
1.17000	0.10000	4.11844	45.79041
1.17000	0.11000	4.01419	39.33627
1.17000	0.12000	3.89332	34.08369
1.17000	0.13000	3.76540	29.77451
1.17000	0.14000	3.63614	26.20764
1.17000	0.15000	3.50889	23.22879
1.17000	0.16000	3.38558	20.71954
1.17000	0.17000	3.26725	18.58861
1.17000	0.18000	3.15440	16.76523
1.17000	0.19000	3.04718	15.19394
1.17000	0.20000	2.94557	13.83108
1.17000	0.21000	2.84938	12.64180
1.17000	0.22000	2.75840	11.59818
1.17000	0.23000	2.67234	10.67759
1.17000	0.24000	2.59093	9.86160
1.17000	0.25000	2.51387	9.13507
1.17000	0.26000	2.44089	8.48549
1.17000	0.27000	2.37172	7.90243
1.17000	0.28000	2.30611	7.37714
1.17000	0.29000	2.24381	6.90231
1.17000	0.30000	2.18462	6.47170
1.17000	0.31000	2.12831	6.07998
1.17000	0.32000	2.07471	5.72266
1.17000	0.33000	2.02363	5.39583
1.17000	0.34000	1.97490	5.09611
1.17000	0.35000	1.92839	4.82063
1.17000	0.36000	1.88395	4.56684
1.17000	0.37000	1.84145	4.33251
1.17000	0.38000	1.80077	4.11572
1.17000	0.39000	1.76180	3.91477
1.17000	0.40000	1.72445	3.72816

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.18000	0.01000	0.00000	0.00000
1.18000	0.02000	0.00669	1.33814
1.18000	0.03000	0.38125	34.03445
1.18000	0.04000	1.38092	70.95309
1.18000	0.05000	2.34657	80.59651
1.18000	0.06000	3.00725	75.87134
1.18000	0.07000	3.39521	66.95929
1.18000	0.08000	3.59428	57.84735
1.18000	0.09000	3.67226	49.75473
1.18000	0.10000	3.67488	42.90375
1.18000	0.11000	3.63145	37.19701
1.18000	0.12000	3.56055	32.45827
1.18000	0.13000	3.47391	28.51256
1.18000	0.14000	3.37900	25.20950
1.18000	0.15000	3.28057	22.42638
1.18000	0.16000	3.18162	20.06522
1.18000	0.17000	3.08403	18.04832
1.18000	0.18000	2.98897	16.31409
1.18000	0.19000	2.89612	14.81348
1.18000	0.20000	2.80885	13.50734
1.18000	0.21000	2.72433	12.36408
1.18000	0.22000	2.64360	11.35820
1.18000	0.23000	2.56659	10.46881
1.18000	0.24000	2.49321	9.67887
1.18000	0.25000	2.42331	8.97426
1.18000	0.26000	2.35673	8.34322
1.18000	0.27000	2.29331	7.77596
1.18000	0.28000	2.23289	7.26424
1.18000	0.29000	2.17528	6.80109
1.18000	0.30000	2.12034	6.38060
1.18000	0.31000	2.06791	5.99772
1.18000	0.32000	2.01784	5.64813
1.18000	0.33000	1.97000	5.32808
1.18000	0.34000	1.92424	5.03436
1.18000	0.35000	1.88046	4.76418
1.18000	0.36000	1.83853	4.51510
1.18000	0.37000	1.79836	4.28499
1.18000	0.38000	1.75982	4.07195
1.18000	0.39000	1.72285	3.87438
1.18000	0.40000	1.68734	3.69080

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.19000	0.01000	0.00000	0.00000
1.19000	0.02000	0.00080	0.17978
1.19000	0.03000	0.14790	14.81155
1.19000	0.04000	0.80314	45.82193
1.19000	0.05000	1.63752	61.63208
1.19000	0.06000	2.31269	63.16586
1.19000	0.07000	2.76804	58.50365
1.19000	0.08000	3.04462	52.08467
1.19000	0.09000	3.19491	45.70642
1.19000	0.10000	3.26043	39.97429
1.19000	0.11000	3.27031	35.01958
1.19000	0.12000	3.24419	30.80104
1.19000	0.13000	3.19517	27.22487
1.19000	0.14000	3.13196	24.19063
1.19000	0.15000	3.06036	21.60728
1.19000	0.16000	2.98427	19.39745
1.19000	0.17000	2.90627	17.49709
1.19000	0.18000	2.82810	15.85399
1.19000	0.19000	2.75090	14.42563
1.19000	0.20000	2.67540	13.17744
1.19000	0.21000	2.60209	12.08123
1.19000	0.22000	2.53122	11.11389
1.19000	0.23000	2.46295	10.25638
1.19000	0.24000	2.39733	9.49304
1.19000	0.25000	2.33437	8.81077
1.19000	0.26000	2.27401	8.19865
1.19000	0.27000	2.21618	7.64751
1.19000	0.28000	2.16080	7.14959
1.19000	0.29000	2.10777	6.69836
1.19000	0.30000	2.05699	6.28819
1.19000	0.31000	2.00834	5.91429
1.19000	0.32000	1.96173	5.57256
1.19000	0.33000	1.91605	5.25941
1.19000	0.34000	1.87420	4.97179
1.19000	0.35000	1.83309	4.70700
1.19000	0.36000	1.79364	4.46272
1.19000	0.37000	1.75574	4.23687
1.19000	0.38000	1.71932	4.02766
1.19000	0.39000	1.68430	3.83351
1.19000	0.40000	1.65062	3.65302

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.20000	0.01000	0.00000	0.00000
1.20000	0.02000	0.00007	0.01858
1.20000	0.03000	0.05142	5.71589
1.20000	0.04000	0.44085	27.74733
1.20000	0.05000	1.10472	45.40926
1.20000	0.06000	1.74079	51.38591
1.20000	0.07000	2.22411	50.33687
1.20000	0.08000	2.55229	46.39381
1.20000	0.09000	2.75807	41.65741
1.20000	0.10000	2.87541	37.02217
1.20000	0.11000	2.93109	32.81526
1.20000	0.12000	2.94455	29.11874
1.20000	0.13000	2.92943	25.91553
1.20000	0.14000	2.89520	23.15367
1.20000	0.15000	2.84843	20.77325
1.20000	0.16000	2.79366	18.71735
1.20000	0.17000	2.73408	16.93570
1.20000	0.18000	2.67187	15.38547
1.20000	0.19000	2.60859	14.03078
1.20000	0.20000	2.54529	12.84171
1.20000	0.21000	2.48269	11.79347
1.20000	0.22000	2.42130	10.86541
1.20000	0.23000	2.36145	10.04043
1.20000	0.24000	2.30333	9.30419
1.20000	0.25000	2.24708	8.64468
1.20000	0.26000	2.19274	8.05183
1.20000	0.27000	2.14035	7.51711
1.20000	0.28000	2.08968	7.03326
1.20000	0.29000	2.04130	6.59414
1.20000	0.30000	1.99456	6.19448
1.20000	0.31000	1.94961	5.82972
1.20000	0.32000	1.90637	5.49597
1.20000	0.33000	1.86479	5.18985
1.20000	0.34000	1.82479	4.90841
1.20000	0.35000	1.78631	4.64911
1.20000	0.36000	1.74927	4.40968
1.20000	0.37000	1.71360	4.18817
1.20000	0.38000	1.67926	3.98284
1.20000	0.39000	1.64617	3.79217
1.20000	0.40000	1.61427	3.61481

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.21000	0.01000	0.00000	0.00000
1.21000	0.02000	0.00001	0.00148
1.21000	0.03000	0.01601	1.95684
1.21000	0.04000	0.22798	15.72546
1.21000	0.05000	0.71936	32.16931
1.21000	0.06000	1.28104	40.78387
1.21000	0.07000	1.75985	42.60379
1.20000	0.08000	2.11624	40.85118
1.21000	0.09000	2.36161	37.64839
1.21000	0.10000	2.51997	34.06979
1.21000	0.11000	2.61403	30.59700
1.21000	0.12000	2.66185	27.41902
1.21000	0.13000	2.67690	24.58929
1.21000	0.14000	2.66891	22.10164
1.21000	0.15000	2.64492	19.92625
1.21000	0.16000	2.60994	18.02629
1.21000	0.17000	2.56756	16.36508
1.21000	0.18000	2.52039	14.90925
1.21000	0.19000	2.47029	13.62943
1.21000	0.20000	2.41857	12.50050
1.21000	0.21000	2.36621	11.50105
1.21000	0.22000	2.31390	10.61301
1.21000	0.23000	2.26213	9.82110
1.21000	0.24000	2.21124	9.11243
1.21000	0.25000	2.16147	8.47610
1.21000	0.26000	2.11296	7.90285
1.21000	0.27000	2.06583	7.38483
1.21000	0.28000	2.02013	6.91529
1.21000	0.29000	1.97588	6.48848
1.21000	0.30000	1.93309	6.09949
1.21000	0.31000	1.89174	5.74402
1.21000	0.32000	1.85180	5.41839
1.21000	0.33000	1.81324	5.11940
1.21000	0.34000	1.77603	4.84425
1.21000	0.35000	1.74010	4.59050
1.21000	0.36000	1.70543	4.35601
1.21000	0.37000	1.67197	4.13891
1.21000	0.38000	1.63965	3.93751
1.21000	0.39000	1.60845	3.75037
1.21000	0.40000	1.57832	3.57616

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.22000	0.01000	0.00000	0.00000
1.22000	0.02000	0.00000	0.00009
1.22000	0.03000	0.00446	0.59473
1.22000	0.04000	0.11092	8.33166
1.22000	0.05000	0.45150	21.87602
1.22000	0.06000	0.92062	31.53468
1.22000	0.07000	1.37017	35.43257
1.22000	0.08000	1.73453	35.53094
1.22000	0.09000	2.00484	33.72060
1.22000	0.10000	2.19393	31.14044
1.22000	0.11000	2.31914	28.37814
1.22000	0.12000	2.39619	25.70990
1.22000	0.13000	2.43767	23.25108
1.22000	0.14000	2.45319	21.03766
1.22000	0.15000	2.44992	19.06833
1.22000	0.16000	2.433.5	17.32561
1.22000	0.17000	2.40679	15.78618
1.22000	0.18000	2.37370	14.42589
1.22000	0.19000	2.33601	13.22202
1.22000	0.20000	2.29528	12.15411
1.22000	0.21000	2.25267	11.20422
1.22000	0.22000	2.20904	10.35681
1.22000	0.23000	2.16502	9.59850
1.22000	0.24000	2.12108	8.91787
1.22000	0.25000	2.07755	8.30510
1.22000	0.26000	2.03468	7.75177
1.22000	0.27000	1.99264	7.25070
1.22000	0.28000	1.95156	6.79569
1.22000	0.29000	1.91152	6.38141
1.22000	0.30000	1.87257	6.00325
1.22000	0.31000	1.83472	5.65722
1.22000	0.32000	1.79800	5.33983
1.22000	0.33000	1.76240	5.04808
1.22000	0.34000	1.72790	4.77931
1.22000	0.35000	1.69449	4.53120
1.22000	0.36000	1.66214	4.30172
1.22000	0.37000	1.63082	4.08907
1.22000	0.38000	1.60050	3.89167
1.22000	0.39000	1.57115	3.70810
1.22000	0.40000	1.54274	3.53711

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.23000	0.01000	0.00000	0.00000
1.23000	0.02000	0.00000	0.00000
1.23000	0.03000	0.00111	0.16064
1.23000	0.04000	0.05074	4.12560
1.23000	0.05000	0.27288	14.26399
1.23000	0.06000	0.64552	23.72696
1.23000	0.07000	1.04895	28.92944
1.23000	0.08000	1.40461	30.50417
1.23000	0.09000	1.68676	29.91655
1.23000	0.10000	1.89697	28.25906
1.23000	0.11000	2.04639	26.17366
1.23000	0.12000	2.14762	24.00056
1.23000	0.13000	2.21186	21.90666
1.23000	0.14000	2.24815	19.96542
1.23000	0.15000	2.26355	18.20192
1.23000	0.16000	2.26342	16.61699
1.23000	0.17000	2.25183	15.20012
1.23000	0.18000	2.23187	13.93626
1.23000	0.19000	2.20583	12.80913
1.23000	0.20000	2.17547	11.80299
1.23000	0.21000	2.14211	10.90328
1.23000	0.22000	2.10676	10.09707
1.23000	0.23000	2.07015	9.37285
1.23000	0.24000	2.03287	8.72064
1.23000	0.25000	1.99535	8.13176
1.23000	0.26000	1.95791	7.59865
1.23000	0.27000	1.92081	7.11481
1.23000	0.28000	1.88420	6.67453
1.23000	0.29000	1.84824	6.27297
1.23000	0.30000	1.81301	5.90580
1.23000	0.31000	1.77858	5.56933
1.23000	0.32000	1.74499	5.26031
1.23000	0.33000	1.71227	4.97590
1.23000	0.34000	1.68043	4.71359
1.23000	0.35000	1.64947	4.47121
1.23000	0.36000	1.61939	4.24681
1.23000	0.37000	1.59017	4.03868
1.23000	0.38000	1.56181	3.84532
1.23000	0.39000	1.53428	3.66537
1.23000	0.40000	1.50756	3.49764

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.24000	0.01000	0.00000	0.00000
1.24000	0.02000	0.00000	0.00000
1.24000	0.03000	0.00025	0.03860
1.24000	0.04000	0.02182	1.90948
1.24000	0.05000	0.15865	8.91126
1.24000	0.06000	0.44127	17.35522
1.24000	0.07000	0.78909	23.16811
1.24000	0.08000	1.12324	25.83252
1.24000	0.09000	1.40593	26.27673
1.24000	0.10000	1.62848	25.45065
1.24000	0.11000	1.79554	23.99884
1.24000	0.12000	1.91612	22.30045
1.24000	0.13000	1.99951	20.56203
1.24000	0.14000	2.05387	18.88889
1.24000	0.15000	2.08587	17.32964
1.24000	0.16000	2.10080	15.90217
1.24000	0.17000	2.10277	14.60814
1.24000	0.18000	2.09496	13.44120
1.24000	0.19000	2.07981	12.39139
1.24000	0.20000	2.05921	11.44756
1.24000	0.21000	2.03459	10.59860
1.24000	0.22000	2.00709	9.83404
1.24000	0.23000	1.97755	9.14432
1.24000	0.24000	1.94665	8.52091
1.24000	0.25000	1.91490	7.95622
1.24000	0.26000	1.88269	7.44361
1.24000	0.27000	1.85034	6.97721
1.24000	0.28000	1.81806	6.55190
1.24000	0.29000	1.78605	6.16320
1.24000	0.30000	1.75444	5.80718
1.24000	0.31000	1.72332	5.48041
1.24000	0.32000	1.69278	5.17987
1.24000	0.33000	1.66287	4.90268
1.24000	0.34000	1.63362	4.64713
1.24000	0.35000	1.60505	4.41055
1.24000	0.36000	1.57719	4.19130
1.24000	0.37000	1.55004	3.98775
1.24000	0.38000	1.52359	3.79848
1.24000	0.39000	1.49784	3.62220
1.24000	0.40000	1.47278	3.45777

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.25000	0.01000	0.00000	0.00000
1.25000	0.02000	0.00000	0.00000
1.25000	0.03000	0.00005	0.00826
1.25000	0.04000	0.00881	0.82634
1.25000	0.05000	0.08874	5.33205
1.25000	0.06000	0.29390	12.33216
1.25000	0.07000	0.58297	18.18588
1.25000	0.08000	0.88659	21.56564
1.25000	0.09000	1.16053	22.83817
1.25000	0.10000	1.38761	22.73961
1.25000	0.11000	1.56623	21.36925
1.25000	0.12000	1.70152	20.61946
1.25000	0.13000	1.80057	19.22356
1.25000	0.14000	1.87035	17.81219
1.25000	0.15000	1.91694	16.45430
1.25000	0.16000	1.94536	15.18309
1.25000	0.17000	1.95966	14.01157
1.25000	0.18000	1.96303	12.94166
1.25000	0.19000	1.95799	11.96948
1.25000	0.20000	1.94651	11.08636
1.25000	0.21000	1.93013	10.29051
1.25000	0.22000	1.91006	9.56799
1.25000	0.23000	1.88725	8.91313
1.25000	0.24000	1.86244	8.31881
1.25000	0.25000	1.83621	7.77861
1.25000	0.26000	1.80903	7.28678
1.25000	0.27000	1.78125	6.83799
1.25000	0.28000	1.75315	6.42782
1.25000	0.29000	1.72497	6.05215
1.25000	0.30000	1.69686	5.70742
1.25000	0.31000	1.66896	5.39047
1.25000	0.32000	1.64138	5.09852
1.25000	0.33000	1.61420	4.82906
1.25000	0.34000	1.58747	4.57995
1.25000	0.35000	1.56125	4.34923
1.25000	0.36000	1.53555	4.13519
1.25000	0.37000	1.51041	3.93629
1.25000	0.38000	1.48583	3.75116
1.25000	0.39000	1.46183	3.57859
1.25000	0.40000	1.43840	3.41750

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.26000	0.01000	0.00000	0.00000
1.26000	0.02000	0.00000	0.00000
1.26000	0.03000	0.00001	0.00157
1.26000	0.04000	0.00335	0.33453
1.26000	0.05000	0.04771	3.05542
1.26000	0.06000	0.19063	8.50861
1.26000	0.07000	0.42277	13.98356
1.26000	0.08000	0.69046	17.73827
1.26000	0.09000	0.94839	19.63310
1.26000	0.10000	1.17328	20.14909
1.26000	0.11000	1.35795	19.80040
1.26000	0.12000	1.50363	18.96785
1.26000	0.13000	1.61499	17.89799
1.26000	0.14000	1.69760	16.73982
1.26000	0.15000	1.75678	15.57891
1.26000	0.16000	1.79714	14.46184
1.26000	0.17000	1.82254	13.41188
1.26000	0.18000	1.83613	12.43869
1.26000	0.19000	1.84042	11.54416
1.26000	0.20000	1.83744	10.72598
1.26000	0.21000	1.82878	9.97947
1.26000	0.22000	1.81573	9.29926
1.26000	0.23000	1.79929	8.67951
1.26000	0.24000	1.78027	8.11456
1.26000	0.25000	1.75932	7.59908
1.26000	0.26000	1.73695	7.12814
1.26000	0.27000	1.71357	6.69725
1.26000	0.28000	1.68950	6.30237
1.26000	0.29000	1.66500	5.93989
1.26000	0.30000	1.64028	5.60659
1.26000	0.31000	1.61551	5.29957
1.26000	0.32000	1.59081	5.01629
1.26000	0.33000	1.56628	4.75447
1.26000	0.34000	1.54201	4.51207
1.26000	0.35000	1.51806	4.28729
1.26000	0.36000	1.49448	4.07851
1.26000	0.37000	1.47131	3.88431
1.26000	0.38000	1.44856	3.70337
1.26000	0.39000	1.42626	3.53456
1.26000	0.40000	1.40443	3.37684

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.27000	0.01000	0.00000	0.00000
1.27000	0.02000	0.00000	0.00000
1.27000	0.03000	0.00000	0.00027
1.27000	0.04000	0.00119	0.12673
1.27000	0.05000	0.02465	1.67665
1.27000	0.06000	0.12035	5.69790
1.27000	0.07000	0.30082	10.52701
1.27000	0.08000	0.53031	14.36768
1.27000	0.09000	0.76701	16.68614
1.27000	0.10000	0.98416	17.69931
1.27000	0.11000	1.16998	17.80659
1.27000	0.12000	1.32204	17.35521
1.27000	0.13000	1.44255	16.59180
1.27000	0.14000	1.53552	15.67616
1.27000	0.15000	1.60535	14.70644
1.27000	0.16000	1.65612	13.74043
1.27000	0.17000	1.69142	12.81048
1.27000	0.18000	1.71426	11.93328
1.27000	0.19000	1.72711	11.11614
1.27000	0.20000	1.73199	10.36078
1.27000	0.21000	1.73054	9.66581
1.27000	0.22000	1.72408	9.02809
1.27000	0.23000	1.71366	8.44368
1.27000	0.24000	1.70015	7.90831
1.27000	0.25000	1.68423	7.41774
1.27000	0.26000	1.66646	6.96792
1.27000	0.27000	1.64730	6.55505
1.27000	0.28000	1.62710	6.17562
1.27000	0.29000	1.60616	5.82646
1.27000	0.30000	1.58472	5.50470
1.27000	0.31000	1.56296	5.20773
1.27000	0.32000	1.54105	4.93323
1.27000	0.33000	1.51911	4.67911
1.27000	0.34000	1.49723	4.44350
1.27000	0.35000	1.47550	4.22472
1.27000	0.36000	1.45398	4.02128
1.27000	0.37000	1.43272	3.83181
1.27000	0.38000	1.41176	3.65511
1.27000	0.39000	1.39113	3.49010
1.27000	0.40000	1.37086	3.33579

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.28000	0.01000	0.00000	0.00000
1.28000	0.02000	0.00000	0.00000
1.28000	0.03000	0.00000	0.00004
1.28000	0.04000	0.00040	0.04494
1.28000	0.05000	0.01224	0.88123
1.28000	0.06000	0.07395	3.70288
1.28000	0.07000	0.20994	7.75616
1.28000	0.08000	0.40157	11.45549
1.28000	0.09000	0.61374	14.01530
1.28000	0.10000	0.81876	15.40803
1.28000	0.11000	1.00151	15.90156
1.28000	0.12000	1.15634	15.79146
1.28000	0.13000	1.28305	15.31184
1.28000	0.14000	1.38402	14.62591
1.28000	0.15000	1.46261	13.84015
1.28000	0.16000	1.52231	13.02117
1.28000	0.17000	1.56632	12.20899
1.28000	0.18000	1.59745	11.42662
1.28000	0.19000	1.61809	10.68627
1.28000	0.20000	1.63021	9.99357
1.28000	0.21000	1.63545	9.35005
1.28000	0.22000	1.63515	8.75490
1.28000	0.23000	1.63040	8.20592
1.28000	0.24000	1.62210	7.70026
1.28000	0.25000	1.61096	7.23476
1.28000	0.26000	1.59759	6.80621
1.28000	0.27000	1.58246	6.41150
1.28000	0.28000	1.56598	6.04765
1.28000	0.29000	1.54846	5.71194
1.28000	0.30000	1.53018	5.40182
1.28000	0.31000	1.51134	5.11499
1.28000	0.32000	1.49213	4.84935
1.28000	0.33000	1.47269	4.60301
1.28000	0.34000	1.45313	4.37426
1.28000	0.35000	1.43556	4.16155
1.28000	0.36000	1.41405	3.96350
1.28000	0.37000	1.39466	3.77882
1.28000	0.38000	1.37545	3.60641
1.28000	0.39000	1.35645	3.44523
1.28000	0.40000	1.33770	3.29436

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.29000	0.01000	0.0	0.0
1.29000	0.02000	0.00000	0.00000
1.29000	0.03000	0.00000	0.00001
1.29000	0.04000	0.00013	0101493
1.29000	0.05000	0.00584	0.44372
1.29000	0.06000	0.04421	2.33510
1.29000	0.07000	0.14367	5.59148
1.29000	0.08000	0.29971	8.98763
1.29000	0.09000	0.48575	11.63006
1.29000	0.10000	0.67542	13.28911
1.29000	0.11000	0.85160	14.09749
1.29000	0.12000	1.00600	14.28577
1.29000	0.13000	1.13619	14.06469
1.29000	0.14000	1.24293	13.59373
1.29000	0.15000	1.32850	12.98336
1.29000	0.16000	1.39567	12.30639
1.29000	0.17000	1.44722	11.60910
1.29000	0.18000	1.48571	10.91989
1.29000	0.19000	1.51338	10.25546
1.29000	0.20000	1.53211	9.62493
1.29000	0.21000	1.54353	9.03265
1.29000	0.22000	1.54897	8.48000
1.29000	0.23000	1.54953	7.96650
1.29000	0.24000	1.54613	7.49064
1.29000	0.25000	1.53953	7.05031
1.29000	0.26000	1.53033	6.64316
1.29000	0.27000	1.51906	6.26671
1.29000	0.28000	1.50614	5.91856
1.29000	0.29000	1.49192	5.59638
1.29000	0.30000	1.47667	5.29800
1.29000	0.31000	1.46066	5.02140
1.29000	0.32000	1.44406	4.76471
1.29000	0.33000	1.42704	4.52622
1.29000	0.34000	1.40973	4.30439
1.29000	0.35000	1.39226	4.09781
1.29000	0.36000	1.37470	3.90519
1.29000	0.37000	1.35713	3.72535
1.29000	0.38000	1.33962	3.55726
1.29000	0.39000	1.32222	3.39996
1.29000	0.40000	1.30496	3.25257

APPENDIX II

Mean	Std Dev	H(y)	H'(y)
1.30000	0.01000	0.0	0.0
1.30000	0.02000	0.00000	0.00000
1.30000	0.03000	0.00000	0.00000
1.30000	0.04000	0.00004	0.00465
1.30000	0.05000	0.00268	0.21410
1.30000	0.06000	0.02571	1.42903
1.30000	0.07000	0.09639	3.94353
1.30000	0.08000	0.22043	6.93701
1.30000	0.09000	0.38018	9.53176
1.30000	0.10000	0.55237	11.35251
1.30000	0.11000	0.71919	12.40473
1.30000	0.12000	0.87041	12.84663
1.30000	0.13000	1.00163	12.85680
1.30000	0.14000	1.11207	12.58431
1.30000	0.15000	1.20290	12.13942
1.30000	0.16000	1.27615	11.59852
1.30000	0.17000	1.33412	11.01256
1.30000	0.18000	1.37905	10.41441
1.30000	0.19000	1.41298	9.82463
1.30000	0.20000	1.43771	9.25557
1.30000	0.21000	1.45480	8.71417
1.30000	0.22000	1.46555	8.20384
1.30000	0.23000	1.47107	7.72576
1.30000	0.24000	1.47228	7.27969
1.30000	0.25000	1.46995	6.86460
1.30000	0.26000	1.46472	6.47889
1.30000	0.27000	1.45712	6.12081
1.30000	0.28000	1.44760	5.78842
1.30000	0.29000	1.43653	5.47987
1.30000	0.30000	1.42422	5.19332
1.30000	0.31000	1.41091	4.92701
1.30000	0.32000	1.39683	4.67933
1.30000	0.33000	1.38216	4.44877
1.30000	0.34000	1.36704	4.23393
1.30000	0.35000	1.35160	4.03351
1.30000	0.36000	1.33594	3.84637
1.30000	0.37000	1.32015	3.67143
1.30000	0.38000	1.30430	3.50770
1.30000	0.39000	1.28845	3.35430
1.30000	0.40000	1.27265	3.21043

APPENDIX III

Competitor bids, bidder A's cost estimate and bid to cost ratios for sixty eight sample bids

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
1	$ 17,600	R	$ 18,461	1.049
		E	19,985	1.136
2	52,900	F	59,296	1.121
		I	60,120	1.136
		D	62,210	1.176
		E	67,326	1.273
		C	68,350	1.292
3	248,100	E	243,153	.980
		C	273,933	1.104
		F	284,224	1.146
		B	343,947	1.386
4	274,200	E	317,801	1.159
		F	337,324	1.230
		D	358,581	1.308
		B	393,333	1.434
		C	417,211	1.522
5	24,900	B	24,871	.999
		C	27,326	1.097
		D	27,843	1.118
		E	28,808	1.157
6	311,000	D	276,311	.888
		E	363,735	1.170
		B	393,973	1.267
		C	414,240	1.332
7	298,000	Q	327,009	1.097
		C	328,086	1.101
		E	343,762	1.154
		B	370,408	1.243
8	32,400	E	34,746	1.072
		F	43,850	1.353
		C	44,891	1.386
		J	48,210	1.488
		AF	51.602	1.593

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
9	99,000	K	108,390	1.095
		E	119,285	1.205
		C	134,768	1.361
		L	150,395	1.519
		D	153,328	1.549
		H	160,784	1.624
10	50,300	J	57,864	1.150
11	27,400	F	29,680	.962
12	723,000	F	772,622	1.069
		E	794,489	1.099
		C	905,506	1.252
		B	1,081,078	1.495
13	85,800	N	99,734	1.162
		F	104,350	1.216
		B	110,465	1.287
		M	126,111	1.470
		E	149,313	1.740
14	54,400	H	57,934	1.065
		K	60,819	1.118
		LL	60,971	1.121
15	48,500	H	49,090	1.012
		LL	54,174	1.117
		K	54,568	1.123
16	8,750	AA	9,587	1.096
		G	10,105	1.155
		BB	13,560	1.549
17	47,800	LL	52,281	1.094
		K	54,024	1.130
		D	77,134	1.614
		G	80,343	1.681
18	54,500	LL	56,547	1.038
		K	63,343	1.162
		D	85,719	1.573
		G	91,939	1.687

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
19	83,800	UU	68,259	.815
		D	80,543	.961
		N	93,598	1.117
20	1,388,200	D	1,436,260	1.035
		B	1.580,638	1.139
21	220,800	C	277,737	1.258
		D	284,500	1.288
		B	290,726	1.317
		E	296,369	1.342
22	230,200	TT	217,568	.945
		C	208,111	1.338
23	176,100	R	196,369	1.115
		J	205,769	1.168
		N	217,021	1.232
		U	247,681	1.406
		E	296,370	1.683
24	202,700	C	220,315	1.087
		N	244,717	1.207
		B	253,737	1.252
25	31,000	K	33,456	1.079
		LL	34,180	1.103
		N	39,768	1.283
26	28,300	J	31,567	1.115
		UU	32,140	1.136
		V	32,224	1.269
27	935,000	B	1,034,639	1.107
		E	1,225,302	1.310
		C	1,368,578	1.464
28	43,300	LL	43,072	.995
		K	49,995	1.155
		YY	68,430	1.580
		M	71,425	1.650
29	50,900	LL	50,190	.986
		K	57,388	1.127
		YY	68,430	1.344
		M	81,103	1.593

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
30	150,200	M	172,898	1.151
		TT	174,979	1.165
		I	204,816	1.364
		V	234,335	1.560
31	56,000	H	58,190	1.039
		D	59,955	1.071
		M	62.999	1.125
		B	69,442	1.240
32	650,600	GG	723,337	1.112
		M	747,133	1.148
		B	785,516	1.207
		C	867,521	1.333
33	19,200	M	18,670	.972
		J	21,141	1.101
		DD	21,697	1.130
		TT	21,932	1.142
		I	23,251	1.211
34	185,700	J	206,035	1.110
		R	206,820	1.114
		N	211,530	1.139
		TT	226,738	1.221
35	935,000	F	971,367	1.039
		E	1,039,607	1.112
		V	1,096,013	1.172
		I	1,142,449	1.222
		B	1,169,401	1.251
36	128,100	N	127,744	.997
		M	137,427	1.073
		J	140,572	1.097
		TT	147,671	1.153
		C	178,736	1.395
37	20,500	V	19,784	.965
		M	22,625	1.104
		TT	22,641	1.104
		I	22,852	1.115
		C	23,356	1.139
38	34,500	G	26,777	.776
		H	38,635	1.120

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
39	47,600	T	44,838	.942
		M	49,792	1.046
		GG	53,166	1.117
		J	55,627	1.169
		B	56,870	1.195
40	10,200	J	13,531	1.327
		I	13,601	1.333
		B	14,936	1.464
		AC	16,108	1.579
41	75,400	M	78,633	1.043
		TT	83,803	1.111
		J	91,333	1.211
		C	104,014	1.379
42	403,700	Q	393,700	.975
		M	470,000	1.164
		B	621,496	1.539
43	31,500	CC	29,213	.927
		AG	29,987	.952
		DD	31,054	.986
		EE	34,819	1.105
		FF	35,967	1.142
		GG	48,879	1.552
44	20,400	G	16,355	.802
		YY	22,071	1.082
		K	24,577	1.205
		AA	24,744	1.213
		HH	25,314	1.241
45	48,800	SS	43,500	.891
		RR	46,047	.944
		D	46,800	.959
		QQ	51,541	1.056
		KK	54,575	1.116
		PP	57,795	1.184
		J	58,500	1.199
		C	60,500	1.240
		M	68,500	1.404
		AE	74,200	1.520
		U	74,700	1.531
		CC	82,021	1.681

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
46	63,800	K	66,101	1.036
		D	68,121	1.068
		CC	69,459	1.089
		PP	74,272	1.164
		L	76,305	1.196
		G	77,552	1.216
		HH	79,455	1.245
		M	84,879	1.330
		H	92,343	1.447
		C	94,753	1.485
47	180,000	C	214,244	1.190
		J	216,672	1.204
		N	228,619	1.270
		GG	229,955	1.278
		AB	259,606	1.442
48	143,000	F	161,124	1.127
		I	161,540	1.130
		V	163,967	1.147
		M	168,705	1.180
		II	177,558	1.242
49	79,600	DD	82,820	1.040
		R	87,925	1.105
		J	91,104	1.145
50	111,500	CC	125,583	1.126
		M	126,802	1.137
		II	129,366	1.160
		I	131,841	1.182
		DD	132,173	1.185
51	310,100	M	304,845	.983
		I	328,168	1.058
		DD	334,236	1.078
		F	352,108	1.135
		GG	355,658	1.147

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
52	723,200	BB	852,265	1.178
		H	868,780	1.201
		D	896,667	1.240
		P	984,605	1.361
		L	1,028,255	1.422
		K	1,163,372	1.602
53	531,304	O	594,820	1.119
		L	630,325	1.186
		H	764,074	1.438
		G	775,475	1.460
54	46,000	XX	51,277	1.115
		DD	51.469	1.119
		AA	51,958	1.129
		LL	55,861	1.214
		WW	60,438	1.314
		K	64,733	1.407
55	108,500	II	99,518	.917
		JJ	103,261	.952
		I	119,482	1.101
		KK	123,559	1.139
56	38,600	D	38,550	.999
		I	41,365	1.072
		E	42,605	1.104
		GG	43,235	1.120
		LL	43,473	1.126
		J	45,904	1.189
		II	46,952	1.216
		F	47,940	1.242
		DD	48,373	1.253
		C	51,020	1.322
57	99,500	G	110,705	1.113
		K	138,488	1.392
		D	151,317	1.521
		GG	152,195	1.530
		LL	161,181	1.620
58	26,600	DD	28,296	1.064
		XX	29,407	1.106
		LL	29,799	1.120
		K	32,246	1.212

APPENDIX III

Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
59	44,500	PP	43,994	.989
		GG	44,306	.996
		II	49,672	1.116
		K	49,815	1.119
		J	51,172	1.149
60	339,900	F	378,424	1.113
		M	379,923	1.118
		C	380,641	1.120
		I	384,201	1.130
		DD	394,766	1.161
61	30,200	XX	28,382	.940
		LL	33,808	1.119
		DD	35,721	1.183
		K	39,864	1.320
62	110,300	J	113,925	1.033
		M	118,713	1.076
		I	123,280	1.118
		DD	123,790	1.122
		D	127,775	1.158
		KK	132,760	1.204
		ZZ	134,755	1.222
		CC	146,112	1.325
63	20,700	DD	21,515	1.039
		K	21,615	1.044
		E	24,698	1.193
		LL	26,307	1.271
		MM	28,300	1.367
		JJ	29,701	1.435
64	233,200	I	241,844	1.037
		E	251,000	1.076
		M	257,000	1.102
		B	261,195	1.120
		DD	262,501	1.126
65	250,100	C	277,219	1.108
		J	282,457	1.129
		R	289,155	1.156
		Q	309,475	1.237
		B	324,214	1.296

APPENDIX III

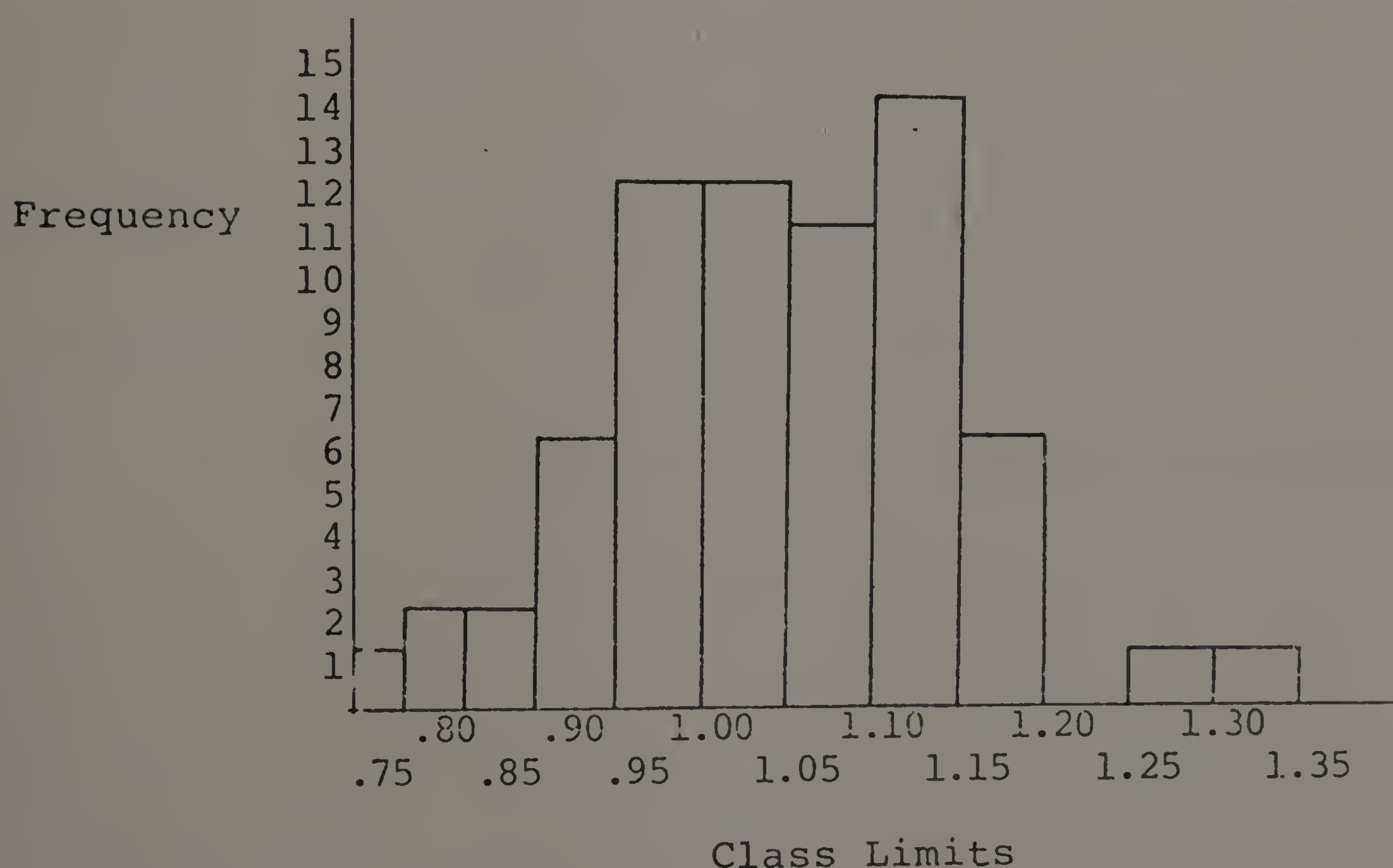
Contract	Contractor A's Cost Estimate	Bidder	Bid	Bid to Cost Ratio
66	137,500	DD	144,653	1.052
		JJ	155,494	1.131
		M	157,425	1.145
		KK	158,510	1.153
		NN	174,789	1.271
		OO	175,354	1.275
67	31,200	C	34,982	1.121
		Q	35,629	1.142
		B	36,394	1.166
		M	36,436	1.168
68	178,400	CC	165,572	.928
		QQ	183,345	1.028
		DD	192,983	1.082
		F	196,876	1.104
		VV	200,787	1.125
		V	201,342	1.129
		JJ	207,138	1.161
		C	220,023	1.233
		KK	233,268	1.308
		AD	246,133	1.380

APPENDIX IV

Frequency distribution and Histogram of lowest competitor bid to cost ratios for sixty-eight sample bids

Class	Class Frequency	Relative Frequency
.75- .80	1	.0147
.80- .85	2	.0294
.85- .90	2	.0294
.90- .95	6	.0882
.95-1.00	12	.1765
1.00-1.05	12	.1765
1.05-1.10	11	.1618
1.10-1.15	14	.2059
1.15-1.20	6	.0882
1.20-1.25	0	.0000
1.25-1.30	1	.0147
1.30-1.35	1	.0147

Mean	1.04529
Standard deviation	.09893

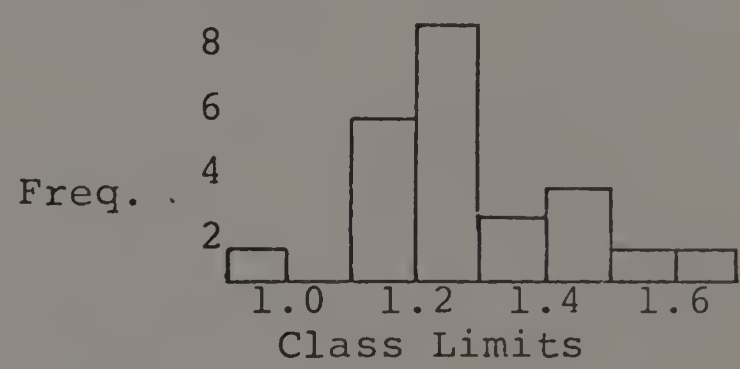


APPENDIX V

Frequency distribution and Histogram of bid to cost ratios for competitors with more than ten sample bid to cost ratios.

Competitor B

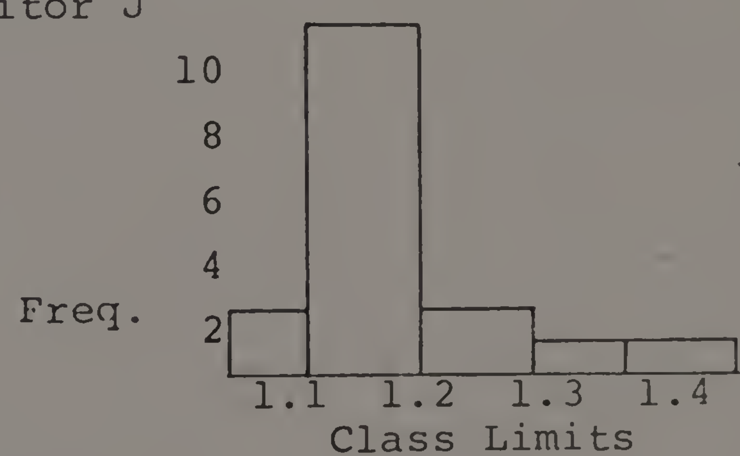
Interval	Frequency
0.9-1.0	1
1.0-1.1	0
1.1-1.2	5
1.2-1.3	8
1.3-1.4	2
1.4-1.5	3
1.5-1.6	1
1.6-1.7	1



Mean 1.2888
Standard deviation .1619

Competitor J

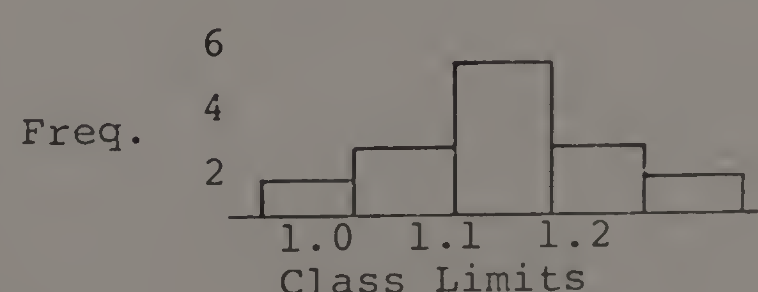
Interval	Frequency
1.0-1.1	2
1.1-1.2	11
1.2-1.3	2
1.3-1.4	1
1.4-1.5	1



Mean 1.1755
Standard deviation 1025

Competitor F

Interval	Frequency
0.9-1.0	1
1.0-1.1	2
1.1-1.2	5
1.2-1.3	2
1.3-1.4	1



Mean 1.1273
Standard deviation .0841

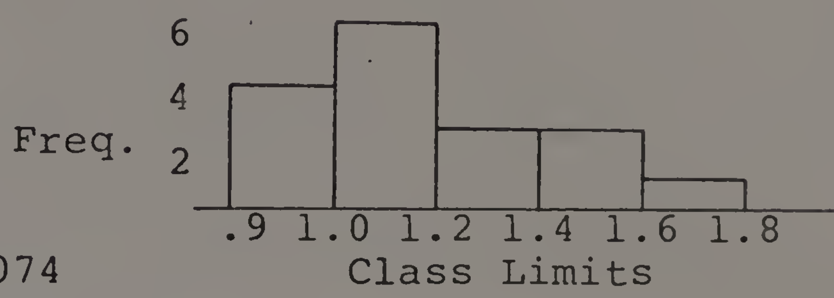
APPENDIX V

Frequency distribution and Histogram of bid to cost ratios for competitors with more than ten sample bid to cost ratios.

Competitor D

Interval	Frequency
0.9-1.0	4
1.0-1.2	6
1.2-1.4	3
1.4-1.6	3
1.6-1.8	1

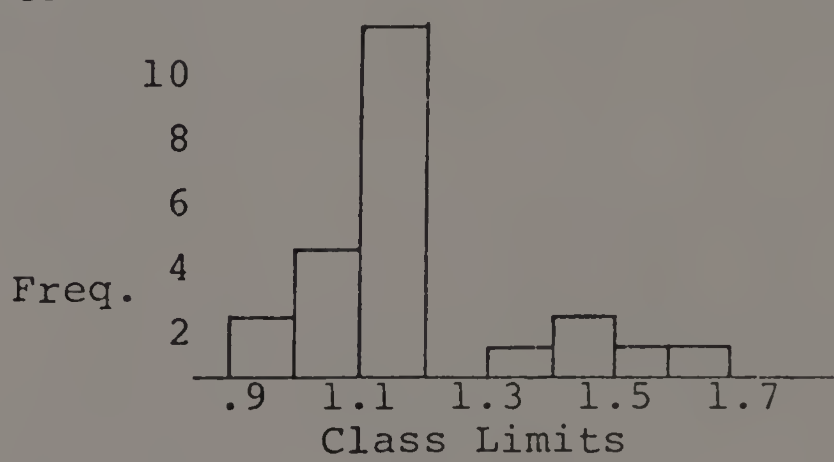
Mean 1.2074
Standard deviation .2344



Competitor M

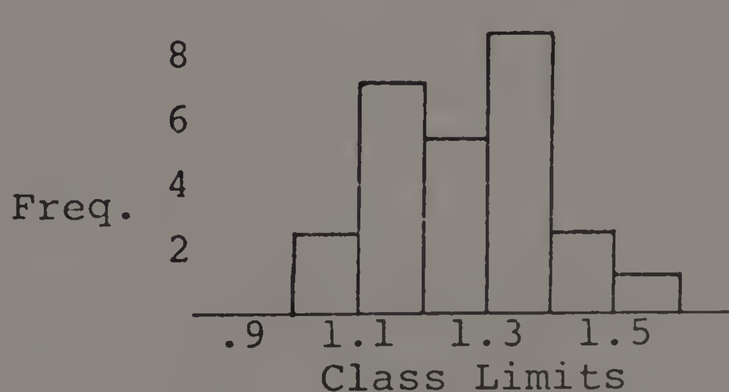
Interval	Frequency
0.9-1.0	2
1.0-1.1	4
1.1-1.2	11
1.2-1.3	0
1.3-1.4	1
1.4-1.5	2
1.5-1.6	1
1.6-1.7	1

Mean 1.1901
Standard deviation .1840



Competitor C

Interval	Frequency
0.9-1.0	0
1.0-1.1	2
1.1-1.2	7
1.2-1.3	5
1.3-1.4	8
1.4-1.5	2
1.5-1.6	1



Mean 1.2664
Standard deviation .1336

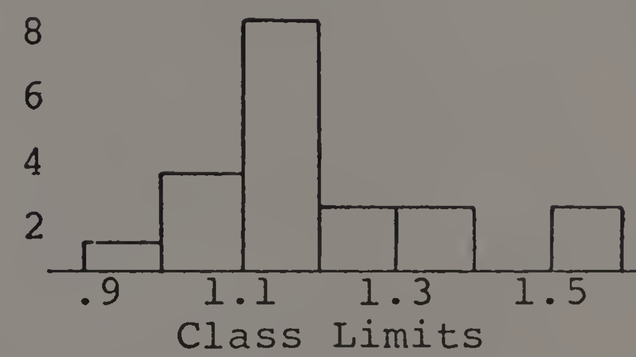
APPENDIX V

Frequency distribution and Histogram of bid to cost ratios for competitors with more than ten sample bid to cost ratios.

Competitor E

Interval	Frequency
0.9-1.0	1
1.0-1.1	3
1.1-1.2	8
1.2-1.3	2
1.3-1.4	2
1.4-1.5	0
1.5-1.6	2

Freq.

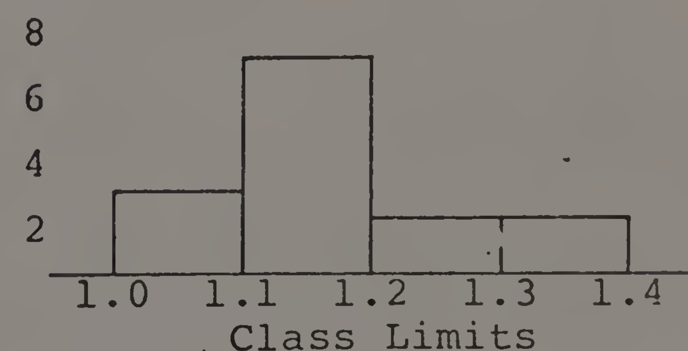


Mean	1.2203
Standard deviation	.1990

Competitor I

Interval	Frequency
1.0-1.1	3
1.1-1.2	7
1.2-1.3	2
1.3-1.4	2

Freq.

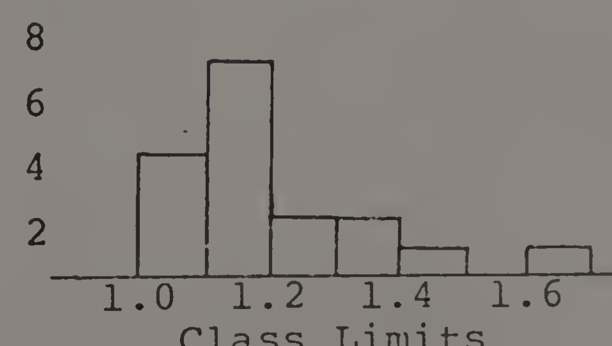


Mean	1.1578
Standard deviation	.0966

Competitor K

Interval	Frequency
1.0-1.1	4
1.1-1.2	7
1.2-1.3	2
1.3-1.4	2
1.4-1.5	1
1.5-1.6	0
1.6-1.7	1

Freq.



Mean	1.1961
Standard deviation	.1523

APPENDIX V

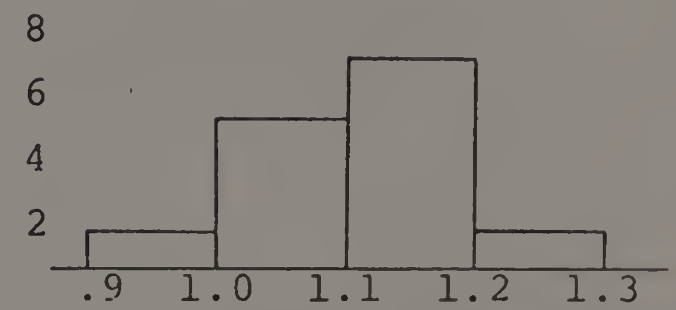
Frequency distribution and Histogram of bid to cost ratios for competitors with more than ten sample bid to cost ratios.

Competitor DD

Interval	Frequency
0.9-1.0	1
1.0-1.1	5
1.1-1.2	7
1.2-1.3	1

Mean 1.1129
Standard deviation .0692

Freq.



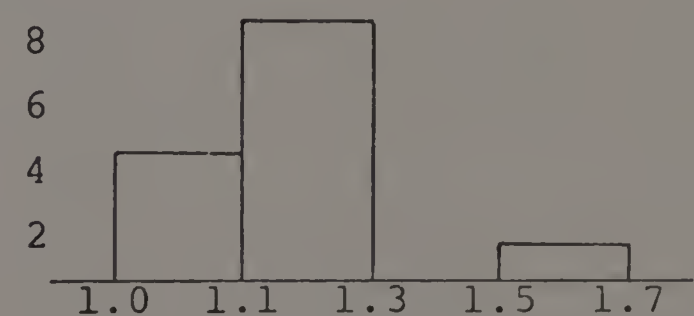
Class Limits

Competitor LL

Interval	Frequency
1.0-1.1	4
1.1-1.3	8
1.3-1.5	0
1.5-1.7	1

Mean 1.1583
Standard deviation .1642

Freq.



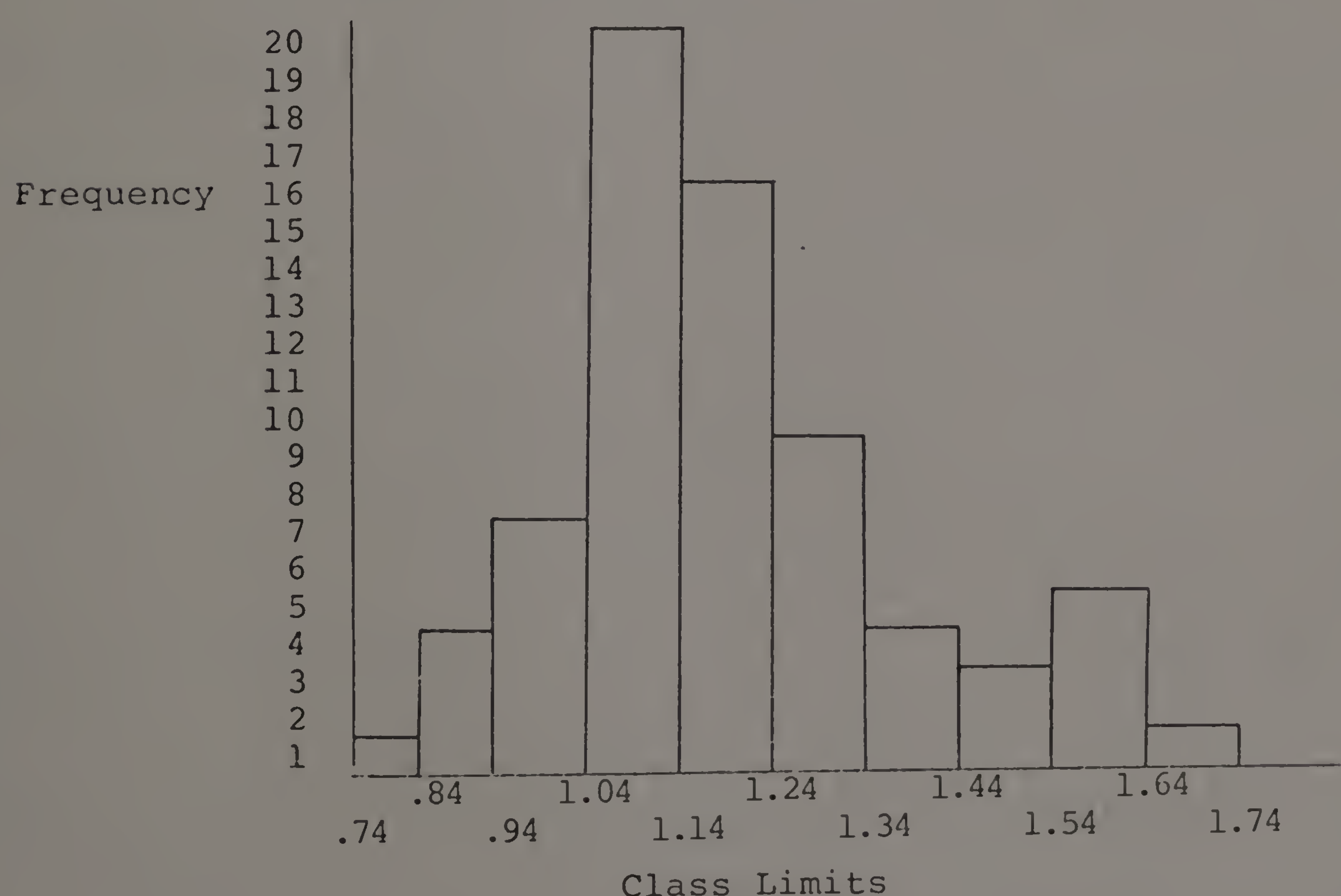
Class Limits

APPENDIX VI

Frequency distribution and histogram of bid to cost ratios of non-union contractors which competitor A bid against less than ten times in sixty-eight sample bids.

Class	Class Frequency	Relative Frequency
.74- .84	1	.014
.84- .94	4	.057
.94-1.04	7	.100
1.04-1.14	20	.286
1.14-1.24	16	.229
1.24-1.34	9	.129
1.34-1.44	4	.057
1.44-1.54	3	.043
1.54-1.64	5	.071
1.64-1.74	1	.014

Mean	1.1906
Standard deviation	.1877

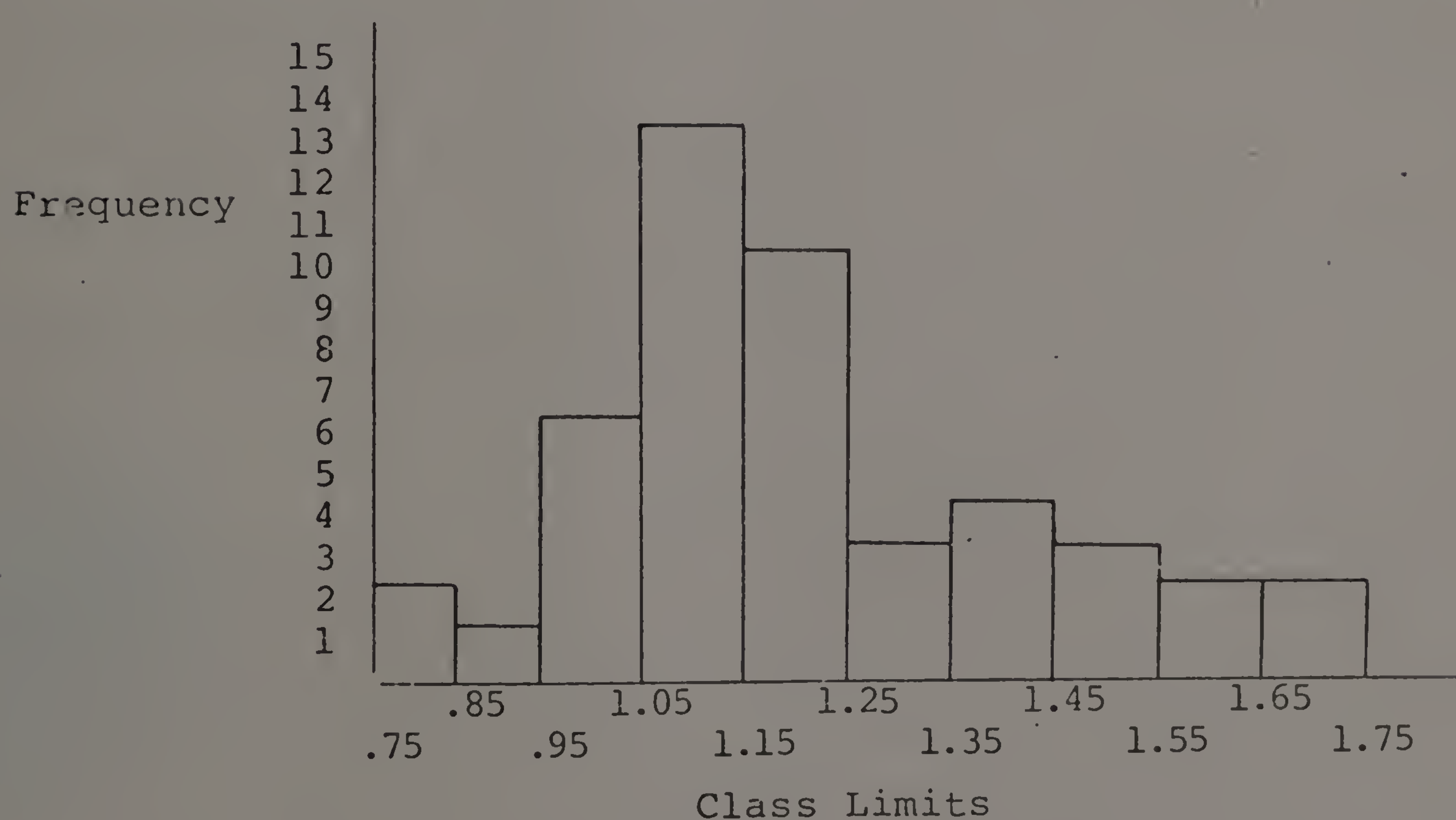


APPENDIX VII

Frequency distribution and histogram of bid to cost ratios of union contractors which competitor A bid against less than ten times in sixty-eight sample bids.

Class	Class Frequency	Relative Frequency
.75- .85	2	.043
.85- .95	1	.022
.95-1.05	6	.130
1.05-1.15	13	.283
1.15-1.25	10	.217
1.25-1.35	3	.065
1.35-1.45	4	.109
1.45-1.55	3	.065
1.55-1.65	2	.043
1.65-1.75	2	.043

Mean 1.2067
Standard deviation .2100



APPENDIX VIII

Computer programs used in generating the bidding table contained in Appendix II.

```
      DIMENSION X(550,10)
      BTC=1,1
      SQPI=SQRT(2.*(22./7.))
      SQ2=SQRT(2.)
      I-1
      NPRT=3
      DO 20 J1=1,11
      J=j1-1
      KL=100+J*10
      KU=110+J*10
      DO 10 M=KL,KU
      XM=M
      XMI=XM/100.
      DO 10 N=1,50
      ST=N
      STD=ST/100.
      X(I,1)=XM1
      X(I,2)=STD
      Y=(BTC-XM1)/STD
      X(I,3)=Y
      XP=-.5*Y*Y
      X(I,4)=(1./(SQPI*STD))*EXP(XP)
      X(I,5)=(-X(I,3)/STD)*X(I,4)
      X(I,6)=ERFC(Y/SQ2)/2.
      X(I,7)=X(I,4)/X(I,6)
      X(I,8)=(X(I,6)*X(I,5)+X(I,4)*X(I,4))/(X(I,6)*X(I,5))
   10 I=I=1
      WRITE (NPRT,3000)
      WRITE (NPRT,3010)((X(I,J),J=1,8),I=1,550)
      I=1
   20 CONTINUE
      STOP
 3000 FORMAT('1')
 3010 FORMAT(8(1X,F10.5,1X))
      END
```

APPENDIX IX

Computer program used in generating the bidding table contained in Appendix I.

```
      DIMENSION X(1000,10)
      NCRD=1
      NPRT=3
      READ (NCRD,1000)N
      READ (NCRD,1010) ((X(I,J),J=1,4),I=1,N)
      DO 10 K=1,N
     XK4=X(K,4)
     XK3=X(K,3)
     XK2=(1-X(K,2))
     XK1=-X(K,1)
     X(K,1)=XK1
     X(K,5)=XK2/XK3
  10 X(K,6)=(-2.-(XK2*XK4)/(XK3*XK3))
      WRITE (NPRT,3000)
      WRITE (NPRT,3010)((X(I,J),J=1,6),I=1,N)
      STOP
1000  FORMAT (I4)
1010  FORMAT (F4.2,1X,F5.4,1X,F5.4,1X,F5.4)
3000  FORMAT ('1')
3010  FORMAT (6(1X,F10.5,5X))
      END
```

APPENDIX X

Computer program used in the computation of the expected value of a contract, when bidding against three competitors, shown in Table 2.

```
      DIMENSION X(101,13)
      NPRT=3
      SQPI=SQRT(2.*(22./7.))
      SQ2=SQRT(2.)
      XM1=1.05
      XM2=1.10
      XM3=1.15
      XCST=80000
      SD1=.16
      SD2=.14
      SD3=.12
      I=1
      DO 10 M=1000,1100
      Y=M/1000.
      X(I,1)=XCST*Y
      X(I,2)=X(I,1)-XCST
      Z1=(Y-XM1)/SD1
      Z2=(Y-XM2)/SD2
      Z3=(Y-XM3)/SD3
      X(I,3)=ERFC(Z1/SQ2)/2.
      X(I,4)=ERFC(Z2/SQ2)/2.
      X(I,5)=ERFC(Z3/SQ2)/2.
      X(I,6)=X(1,3*X(I,4)*X(I,5)
      X(I,7)=X(8,6)*X(I,2)
      X(I,8)=(1./(SQPI*SD1))*EXP(Z1)
      X(I,9)=(1./(SQPI*SD2))*EXP(Z2)
      X(I,10)=(1./(SQPI*SD3))*EXP(Z3)
      X(I,11)=X(I,8)/X(I,3)
      X(I,12)=X(I,9)/X(I,4)
      X(I,13)=X(I,10)/X(I,5)
   10 I=1+1
      WRITE(NPRT,3000)
      WRITE(NPRT,3010)((X(I,J),J=8,13),I=1,101)
      STOP
 3000 FORMAT('1')
 3010 FORMAT(6(1X,F12.5,1X))
      END
```

APPENDIX XI

Paired sample bid to cost ratios for six pairs of competitors.

Competitors C and E

1.097	1.157
1.104	.980
1.522	1.159
1.252	1.099
1.258	1.342
1.464	1.310
1.361	1.205
1.292	1.273
1.332	1.170
1.332	1.104
1.386	1.072
1.101	1.154

Competitors K and LL

1.392	1.620
1.320	1.119
1.212	1.120
1.155	.995
1.127	.986
1.407	1.214
1.162	1.038
1.130	1.094
1.079	1.103
1.123	1.117
1.118	1.121
1.044	1.271

Competitors B and E

.999	1.57
1.386	.980
1.434	1.159
1.495	1.099
1.287	1.740
1.317	1.342
1.107	1.310
1.251	1.112
1.120	1.076
1.267	1.170
1.243	1.154

Competitors B and C

.999	1.097
1.386	1.104
1.434	1.522
1.495	1.252
1.317	1.258
1.252	1.087
1.107	1.464
1.207	1.333
1.296	1.108
1.166	1.121
1.267	1.332
1.243	1.101

Competitors C and M

1.333	1.148
1.379	1.043
1.120	1.118
1.121	1.168
1.139	1.104
1.240	1.404
1.485	1.330
1.395	1.073

Competitors B and M

1.287	1.470
1.670	1.151
1.240	1.125
1.207	1.148
1.195	1.046
1.120	1.102
1.166	1.168
1.539	1.164

APPENDIX XII

Paired sample standardized bid to cost ratios fox six pairs of competitors.

Competitors C and E

.997	1.052
.929	.825
1.177	.896
1.032	.906
1.003	1.070
1.165	1.043
1.008	.893
1.090	1.074
1.163	1.021
1.140	.952
1.032	.798
.962	1.008

Competitors K and LL

1.008	1.173
1.134	.962
1.083	1.001
.893	.769
.915	.801
1.158	.999
.878	.784
.849	.822
.930	.951
1.004	.999
.965	.968
.863	1.050

Competitors B and E

.908	1.052
1.386	.825
1.109	.896
1.232	.906
.967	1.307
1.050	1.070
.881	1.043
1.077	.957
1.021	.981
1.106	1.021
1.086	1.008

Competitors B and C

.908	.997
1.166	.929
1.109	1.177
1.232	1.032
1.050	1.003
1.063	.923
.881	1.165
1.017	1.123
1.103	.943
1.019	.979
1.106	1.163
1.086	.962

Competitors C and M

1.123	.967
1.160	.877
.990	.988
.979	1.020
1.036	1.005
1.017	1.152
1.215	1.038
1.204	.926

Competitors B and M

.967	1.104
1.256	.865
1.067	.968
1.017	.967
1.086	.951
1.021	1.004
1.019	1.020
1.256	.950

APPENDIX XIII

Computer program used in simulation of fifty bids against five competitors

```
      DIMENSION X(201,13)
      DIMENSION XMX(10),SDX(10),XM(5),SD(5),Z(5)
      NPRT=3
      NCRD=1
      WRITE(NPRT,3000)
      SQPI=SQRT(2.*(22./71))
      SQ2=SQRT(2.)
      XCST=800000
      READ(NCRD,1005)JSEED
      READ (NCRD,1010)(XMX(I),I=1,10)
      READ (NCRD,1010)(SDX(I),I=1,10)
      ISEED=JSEED
      DO 200 N=1,50
      XLAST=-1
      I=1
      DO 100 K=1,5
      CALL RANDX(ISEED,IRAND)
      XM(K)=XMX(IRAND)
      CALL RANDX(ISEED,IRAND)
      SD(K)=SDX(IRAND)
  100 CONTINUE
      DO 10 M=1000,1200
      XM1=M
      Y=XM1/1000.
      DO I1 M1=1.5
   11 Z(M1)=(Y-XM(MI))/SD(M1)
      X(I,1)=XCST*Y
      X(I,2)=X(I,1)-XCST
      DO 12 M2=3,7
      X(I,M2)=ERFC(Z(M2-2)/SQ2)/2.
   12 CONTINUE
      X(I,8)=X(I,3)*X(I,4)*X(I,5)*X(I,6)*X(I,7)
      X(I,9)=X(I,8)*X(I,2)
      XVALD=XLAST-X(I,9)
      IF(XVALD)4,4,20
    4 XLAST=X(I,9)
   10 I=1+1
   20 WRITE(NPRT,3010)(X(I-1,1),(XM(K),SD(K),K=1,5),XLAST)
  200 CONTINUE
      STOP
 1005 FORMAT(I4)
 1010 FORMAT(10(F4.2,1X))
 3000 FORMAT('1')
 3010 FORMAT(F7.0,5(3X,F4.2,1X,F4.2),F9.2)
      END
```

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