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The determination of Fibonacci groups

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Fibonacci groups are the groups

$$F(2, r) = \text{gp}\left(a_1, a_2, \ldots, a_{r+2} : a_{i+2} = a_{i+1}a_i ; i = 1, \ldots, r ; a_{r+1} = a_1, a_{r+2} = a_2\right) ,$$

where r is a natural number. The groups $F(2, 8)$ and $F(2, 10)$ are shown to be infinite, thus leaving $F(2, 9)$ as the only Fibonacci group whose finiteness or infiniteness has not been determined.

1. Introduction

Fibonacci groups are the groups

$$F(2, r) = \text{gp}\left(a_1, a_2, \ldots, a_{r+2} : a_{i+2} = a_{i+1}a_i ; i = 1, \ldots, r ; a_{r+1} = a_1, a_{r+2} = a_2\right) , \tag{1}$$

where r is a natural number.

According to [1] the finite groups $F(2, r)$ are known, except when $r = 8, 9$ and 10. This note will establish that $F(2, 8)$ and $F(2, 10)$ are infinite.

The results now known are as follows: $F(2, 0)$ is the free group of rank 2; $F(2, 1)$ and $F(2, 2)$ are the trivial group; $F(2, 3)$ is the

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quaternion group of order 8 ; $F(2, 4)$ is cyclic of order 5 ; $F(2, 5)$ is cyclic of order 11 ; $F(2, 6)$ is the infinite metabelian group $\mathrm{gp}\left(a, b; b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2}\right)$; $F(2, 7)$ is cyclic of order 29 (established using an electronic computer by John Cannon (1971) and confirmed by Jane Watson); $F(2, 8)$ and $F(2, 10)$ are infinite; and, when $n \geq 11$ the groups $F(2, n)$ are infinite (Lyndon; see [1]).

In [1] Problem 4 asks whether the factor group of $F(2, 10)$ obtained by adding the relation $a_6 = a_1$ to the relations in (1) is finite. The group turns out to be $\mathrm{gp}\left(a, b; a^{11} = b^{23} = a^{-4}b^{-1}ab = 1\right)$ and has order $11 \cdot 23$. A proof is not given here as it is long, and an easy method for showing $F(2, 10)$ to be infinite is given below.

2. The group $F(2, 8)$

THEOREM 2.1. *The group* $F(2, 8)$ *is infinite: an infinite epimorph is the group* $\mathrm{gp}\left(c, d; c^2 = d^5 = (cd)^5 = \left(cd^2\right)^5 = 1\right)$.

The proof of the theorem follows from Lemmas 2.2 and 2.3 below.

LEMMA 2.2. *The group* $H = \mathrm{gp}\left(c, d; c^2 = d^5 = (cd)^5 = \left(cd^2\right)^5 = 1\right)$ *is an epimorph of* $F(2, 8)$.

Proof. Firstly the elements d and cd^3c generate H ; for $\left(cd^3c\right)^2 = cd^3c^2d^3c = cdc$ as $c^2 = d^5 = 1$, and since $(cd)^5 = 1$, the element c is obtained as $c^{-1} = dcdcdcdcd = d\left(cd^3c\right)^2d\left(cd^3c\right)^2d$.

Now set $x_1 = d$, $x_2 = cd^3c$ and $x_{n+2} = x_{n+1}x_n$ for $n = 1, 2, \ldots$. An easy calculation using the relations of H shows that $x_3 = cd^3cd$, $x_4 = cd^3cdcd^3c$, $x_5 = cd^2cd^{-1}c$, $x_6 = cd^2cd^2cdcd^3c$, $x_7 = cd^2cd^2$, $x_8 = d^{-1}cd^3c$, $x_9 = d$ and $x_{10} = cd^3c$.

The required epimorphism is the one mapping a_i to x_i for $i = 1, 2, \ldots, 10$.

LEMMA 2.3. *The group* $H = \mathrm{gp}\left(c, d; c^2 = d^5 = (cd)^5 = \left(cd^2\right)^5 = 1\right)$ *is*

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infinite.

Proof. The group

$$A = \mathrm{gp}\left(c_0, c_1, c_2, c_3; c_0^2 = c_1^2 = c_2^2 = c_3^2 = 1, \right.$$
$$\left. c_0c_1c_2c_3 = c_1c_3c_0c_2, (c_0c_1c_2c_3)^2 = 1\right)$$

is infinite; indeed, the free product of two cyclic groups of order 2 is an epimorph, as can easily be seen by placing $c_0 = 1$ and $c_2 = c_3$. Moreover A has an automorphism χ of order 5, with $c_i\chi = c_{i+1}$ $(i = 0, 1, 2)$ and $c_3\chi = c_0c_1c_2c_3$.

Since A is infinite, an extension

$$\mathrm{gp}\left(c_0, c_1, c_2, c_3, d : d^5 = c_0^2 = 1; c_i^d = c_{i+1}, i = 0, 1, 2; \right.$$
$$\left. c_3^d = c_0c_1c_2c_3, c_0c_1c_2c_3 = c_1c_3c_0c_2\right),$$

of A by means of χ is infinite also. Rewritten in terms of its generators $c = c_0$, d this becomes

$$\mathrm{gp}\left(c, d; c^2 = d^5 = 1, c^{d^4} = cc^dc^{d^2}c^{d^3} = c^dc^{d^3}cc^{d^2}\right),$$

or,

$$\mathrm{gp}\left(c, d; c^2 = d^5 = (cd^{-1})^5 = (cd^{-2})^5 = 1\right),$$

which is H.

3. The group $F(2, 10)$

THEOREM 3.1. *The group $F(2, 10)$ is infinite.*

Proof. Let

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

be matrices over the integers, and set $A_{n+2} = A_{n+1}A_n$ for $n = 1, 2, \ldots$. A short calculation shows that $A_{11} = A_1$ and $A_{12} = A_2$, so there is an

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epimorphism of $F(2, 10)$ onto the matrix group generated by A_1 and A_2 mapping a_i to A_i for $i = 1, 2, \ldots, 12$.

Now

$$A_7 = A_1^{-2}A_2A_1^{-1}A_2 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so that

$$A_7^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

an element which clearly has infinite order. It follows that $F(2, 10)$ is infinite.

Reference

[1] D.L. Johnson, J.W. Wamsley and D. Wright, "The Fibonacci groups", *J. London Math. Soc.* (to appear).

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