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THE DEVELOPMENT OF GENERAL DIFFERENTIAL AND GENERAL DIFFERENTIAL-BOUNDARY SYSTEMS

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I. ABSTRACT. The field of differential systems with general boundary conditions, differential-boundary systems is surveyed from 1900 to the present with special emphasis on the recent past. Results concerning dual systems, Green's matrices, eigenvalues and expansions, self-adjointness and applications (in particular to splines) are presented in such a way as to give a picture of the field as it has developed to the present. Finally several unsolved problems are listed.

II. **Introduction.** From various scattered places throughout the mathematical world certain problems, once thought of as sidelights to the field of ordinary boundary value problems, have recently coalesced into the small but vigorous new field of *general boundary value problems.* Consisting primarily of the study of ordinary differential systems under general boundary conditions and differential-boundary operators, the field has recently expanded to also include their applications to areas such as the calculus of variations, spline theory, hyperbolic dissipative systems and differential operators acting on subspaces of various L^2 spaces. While the field moved extremely slowly at the start (in fact it lay dormant for one 20 year period), progress has been quite rapid during the last decade. With new applications being rapidly discovered, the field promises to be even more impressive in the future.

The purpose of this article is to give a general picture of the field as it developed, especially to describe in detail the interesting results of the past few years. At the beginning results are somewhat disjointed, due in part to the complexities involved. However as time passes, the field settles down into a well defined related group of problems. Before describing its evolution, however, let us consider several instances in which differential boundary problems have arisen, and what it is about them which interests the mathematician.

The 1952 Feller [22], while examining diffusion processes, encountered an interesting generalization of the Fokker-Planck equation:

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On the interval (r_1, r_2) where r_1 is a natural boundary, and r_2 is an exit boundary (See Feller [22] for the appropriate definitions.), consider the equation

$$u_t = a(x)u_{xx} + b(x)u_x,$$

representing certain transition probabilities as functions of initial positions. Then the adjoint equation, the Fokker-Planck equation, representing probability densities, is given by

$$v_t = [(a(x)v)_x - b(x)v]_x - (\tau\tilde{p}(x)/p_2)\lim_{x\to r_2}[(a(x)v)_x - b(x)v].$$

The last term represents mass on the boundary. If $v = w(t)y(x)$, the equation in y resulting from separation of variables is

$$[(a(x)y)' - b(x)y]' - (\tau\tilde{p}(x)/p_2)[(a(r_2)y(r_2))' - b(r_2)y(r_2)] = \lambda y,$$

which is a differential-boundary equation.

In 1959 Phillips [52], while considering maximal dissipative operators, examined two methods of attack: Either begin with a minimal operator and examine its dissipative extensions, or begin with a maximal operator and examine its dissipative contractions. The first can lead to operators which are no longer merely differential operators (and frequently are differential-boundary operators), while the second can lead to differential operators with domains restricted by general boundary conditions.

Phillips gave an elegant example which illustrates both possibilities. In the space $\mathcal{L}^2(0,1)$ let L_0, L_1, L_α be defined by the expression $y' - y$ on the domains

$D(L_0) =$
$\{y : y \text{ is absolutely continuous}; y, y' \in \mathcal{L}^2(0,1); y(0) = 0, y(1) = 0\}$,

$D(L_\alpha) =$
$\{y : y \text{ is absolutely continuous}; y, y' \in \mathcal{L}^2(0,1); y(1) = \alpha y(0), |\alpha| \leqq 1\}$,

$D(L_1) = \{y : y \text{ is absolutely continuous}; y, y' \in \mathcal{L}^2(0,1)\}$.

Hence $L_0 \subset L_\alpha \subset L_1$. An easy calculation shows

$$(L_0 y, y) + (y, L_0 y) = -2(y,y) \leqq 0.$$

(The parentheses indicate the inner product in $\mathcal{L}^2(0,1)$.) Hence L_0 is dissipative. For the maximal operator L_1, the equation is

$$(L_1 y, y) + (y, L_1 y) = -2(y,y) + [|y(1)|^2 - |y(0)|^2].$$

From this it follows that maximal dissipative contractions of L_1 must have the form described by L_α. For such a contraction L_α,

$$(L_\alpha y, y) + (y, L_\alpha y) = -2(y, y) - |y(0)|^2[1 - |\alpha|^2] \leqq 0.$$

Let us now consider the operator L defined by $Ly = y' - y + y(0)h$, where $h \in \mathcal{L}^2(0, 1)$, $\|h\|^2 \leqq 2$, on the domain

$$D(L) = \{y : y \text{ is absolutely continuous; } y, y' \in \mathcal{L}^2(0, 1); y(1) = 0\}.$$

Then L is a maximal dissipative extension of L_0, but is not a contraction of L_1: The equation

$$(Ly, y) + (y, Ly) = -|y(0) - (y, h)|^2 - [2(y, y) - |(y, h)|^2] \leqq 0$$

shows L is dissipative. To show L is maximal is equivalent to showing $R(I - L) = \mathcal{L}^2(0, 1)$ (see Goldberg [24]), which is easily accomplished by using the Green's function.

Now since $L_0 \subset L$, we must have $L^* \subset L_0^*$, which is well known: $L_0^* z = -z' - z$ on

$$D(L_0^*) = \{z : z \text{ is absolutely continuous; } z, z' \in \mathcal{L}^2(0, 1)\}.$$

Since

$$\begin{aligned}(Ly, z) - (y, L_0^* z) &= -y(0)z(0)^* + y(0)\,(z, h)\\ &= 0\end{aligned}$$

for all y in $D(L)$, and since $y(0)$ is arbitrary, we must have $z(0) = (h, z)$. This completes the determination of $D(L^*)$: $L^* z = -z' - z$ on

$$(L^*) = \{z : z \text{ is absolutely continuous; } z, z' \in \mathcal{L}^2(0, 1); z(0) = (h, z)\}.$$

It is possible to show that L^* is a maximal dissipative contraction of L_0^*. It is an ordinary differential operator restricted by a *general boundary condition.*

The operator L represents a system in which energy is fed back with density $y(0)h$.

Another example of feedback-like phenomena occurs if a vibrating wire is affected by a magnetic field exerting a force per unit mass represented by $K(x)\,[cu(0, t) + du(1, t)]$, where c, d are constants and $u(x, t)$ represents the lateral displacement at x at time t. Under this circumstance the wave equation takes the form

$$u_{tt} = k^2 u_{xx} + K(x)\,[cu(0, t) + du(1, t)];$$

separation of variables then yields the equation

$$y'' + (K(x)/k^2)(cy(0) + dy(1)) = (-\lambda/k^2)y$$

in x, which is a differential-boundary equation.

In nuclear reactor construction the passage of atomic particles through laminated shields leads to ordinary differential operators with domains determined by boundary conditions of the form $y(a+) = y(a-)$ or $y'(a+) = y'(a-)$, etc., at the interfaces in the shielding. Generalization of such conditions lead to general boundary conditions

$$\sum A_i y(t_i) = 0,$$

$$\sum A_i y(t_i) + \int_a^b K(x)y(x)\,dx = 0,$$

or

$$\int_a^b d\nu(x)y(x) = 0$$

at various levels of abstraction.

Quite recently a number of problems in spline theory have been reformulated as differential systems with general boundary conditions. These are described in detail in section V.

In conclusion let us note that Coddington [17] has been studying self-adjoint differential operators on subspaces of various $\mathcal{L}^2$ spaces. Certain differential-boundary operators provide some of his examples. Still others [42] are more general than the operators considered by Coddington. The most reasonable setting for these operators seems to be as (multivalued) linear relations on $\mathcal{L}^2 \times \mathcal{L}^2$. Linear relations have their real beginning in a discussion by Arens [1] in 1961.

What is of interest concerning these problems? As indicated by Feller [22] and Phillips [52], the operator and its adjoint frequently have physical interpretations. In addition the adjoint is used to calculate the eigenvalues λ encountered in separation of variables. Further one cannot examine self-adjoint situations, such as is done by Coddington, unless the adjoint is known. Finally the solution to certain spline problems are best cast as the solutions of the adjoint problem. Hence the adjoint, the dual operator in the appropriate dual space, is of primary importance.

In addition, the solution of partial differential equations, such as encountered by Feller [22] and Phillips [52], require the use of eigenfunction expansions if separation of variables is contemplated. The parameters λ, introduced by separation of variables, constitute the spectrum of the operator involved. The corresponding solutions are the eigenfunctions associated with it. Hence a knowledge of the spectrum, eigenfunctions, and, in addition, the resolvent operator, Green's function — all those elements examined in spectral analysis — are also important.

The tools required in the field have changed with time. At the beginning the problems were classical in form, requiring the use of solutions to ordinary differential equations, Green's formula, the use of Green's functions and elementary spectral theory (primarily for discrete spectra). Recently, however, more and more functional analysis as well as more complex theories of integration have come into play. In particular, the use of more general spectral theory, the use of the resolvent operator, the density of domains satisfying certain general boundary conditions, the formulation of problems in abstract spaces with dual problems in the appropriate dual spaces, and in addition the use of Stieltjes integrals have become essential in recent years.

Let us now turn our attention to an examination of how the field developed.

III. **Early History, 1900–1962.** The history of general boundary value problems begins during the early part of this century with several exploratory articles, which attempted to properly define what reasonable problems were and to give some preliminary results. Because of their preliminary nature, the articles appear somewhat disjointed, especially at the beginning. We examine several briefly.

1. The first contribution to the field is Picone's consideration [53] of the problem

$$y^{(n)} = \sum_{i=1}^{n} p_i y^{(n-i)} + f, \sum_{k=1}^{n} \int_a^b a_{ik}(\tau) y^{(k-1)}(\tau)\, d\tau = 0,$$

$i = 1, \cdots, n$, notable primarily because of the integral boundary conditions. Picone first derived the Green's function for the case $p_i = 0, i = 1, \cdots, n$. Then by writing the differential equation as a Fredholm integral equation he discussed the general case.

2. A short time later Hilb [26] considered two systems which we have transcribed into notation a bit more modern:

$$Ly = Py' + Qy, \int_0^1 K(\xi)y(\xi)\,d\xi + \gamma y(0) - \Gamma y(1) = 0,$$

and

$$Ly = (p(x)y')' + qy,$$
$$k_1 y(1) - k_2 y'(1) = 0,$$

$$\int_0^1 K(\xi)y(\xi)\,d\xi + \beta y(0) - \alpha y'(0) = 0,$$

which are again of interest because of the integrals in the boundary conditions.

Hilb derived the Green's functions for these systems, showed that the spectrum is discrete (consists only of eigenvalues), and derived two nonself-adjoint eigenfunction expansions. The first takes the form

$$f(t) = \sum_1^\infty c_i y_i(t), c_i = \int_0^1 \bar{z}_i(t) f(t)\,dt,$$

where y_i satisfies $Ly_i = \lambda_i y_i$ and also the boundary conditions. The elements z_i were not clearly described, but he did show that they satisfy an adjoint type equation with the nonhomogeneous term K on the right side. We now know that the elements z_i satisfy an adjoint equation $L^* z_i = \bar{\lambda}_i z_i$ involving a differential-boundary operator as well as standard end point adjoint boundary conditions. The expansion is valid for all f in the domain of L.

The second expansion takes the form

$$f(t) = \sum_1^\infty d_i z_i(t), d_i = \int_0^1 f(t)\bar{y}_i(t)\,dt.$$

It is valid for all f in the domain of L^*.

3. Hilb's work was later extended to the interval $[0, \infty]$ by his student Betschler [2], and subsequently reworked by Krall [31] in his 1963 dissertation. The results in this instance involve not only a discrete sum, but also an integral due to a continuous spectrum on the positive real axis. Specifically the expansions look like

$$f(t) = \sum c_i y_i(t) + \int_0^\infty c(s) y(t, s)\,ds,$$

$$c_i = \int_0^\infty f(t)\,\overline{z_i(t)}\,dt, c(s) = \int_0^\infty f(t)\,\overline{z(t, s)}\,dt,$$

where $y_i(t)$ and $z_i(t)$, $y(t, s)$ and $z(t, s)$ are solutions to the differential equations generated by L and L^*. The expansion holds for all f in the intersection of the domain of L and $L^1[0, \infty)$. A similar adjoint expansion exists in the form

$$f(t) = \sum d_i z_i(t) + \int_0^\infty d(s)\, z(t, s)\, ds,$$

$$d_i = \int_0^\infty f(t)\overline{y_i(t)}\, dt, \quad d(s) = \int_0^\infty f(t)\overline{y(t, s)}\, dt.$$

This expansion holds for all f in the intersection of the domain of L^* and $L^1[0, \infty)$.

The primary unsolved problem left by Hilb and Betschler was the true nature of the adjoint system. (See Feller [22], page 470). This was found by Cole [20] in 1964 and Krall [32] in 1964–1965. It involves not only a differential operator, but a boundary value as well. It is, in fact, a differential boundary operator (see [39]).

4. In 1917 C. E. Wilder [69] discussed differential systems with multipoint boundary conditions:

$$u^{(n)} + \sum_{j=1}^{n} p_j u^{(n-j)} + \lambda u = 0,$$

$$W_i(u) = \sum_{j=1}^{k} W_{ji}(u) = 0, \quad i = 1, \cdots, n,$$

where

$$W_{ji}(u) = \sum_{\ell=0}^{n-1} \alpha_{ji}^{\ell} u^{(\ell)}(a_j),$$

and $a = a_1 < a_2 \cdots < a_k = b$.

Wilder derived the Green's function for the system, and achieved an eigenfunction expansion, which converges pointwise at places where the expanded functions are sufficiently smooth.

Although Wilder did not derive or define an adjoint system, he did comment that necessarily such a system would have to be discontinuous at the interior boundary points, a major departure from the smoothness requirements of all other earlier systems.

5. A major extension of the work of Hilb, Wilder, and also G. D. Birkhoff and R. E. Langer during the early 1920's, was made by J. D. Tamarkin [56], [57], [58]. In these papers covering a 13 year

span Tamarkin discussed the integro-differential equation

$$u^{(k)}(x) + \sum_{j=1}^{k} r_j(x)u^{(k-j)}(x) = r(x) + \sum_{\sigma=0}^{m} \int_a^b u^{(\sigma)}(\xi)R_\sigma(x, \xi)\, d\xi,$$

r_j continuous, $R_\sigma(x, \xi)$ continuous except along certain lines $\xi =$ constant and $x = \xi$, where discontinuities of the first kind were permitted. Tamarkin quickly reduced this equation to the form

$$\sum_{i=0}^{n} p_i(x)u^{(n-i)}(x) = f(x) + \sum_{j=1}^{\mu} \mathfrak{L}_j(u)\phi_j(x) + \int_a^b h(x, \xi)u(\xi)\, d\xi,$$

the terms $\mathfrak{L}_j(u)$ being linear in $u, \cdots, u^{(n-1)}$. In addition, boundary conditions

$$L_i(u) = A_i(u) + B_i(u) + \int_a^b \alpha_i(x)u(x)\, dx = 0, i = 1, \cdots, n,$$

where

$$A_i(u) = \sum_{k=1}^{n} \alpha_{ik}u^{(k-i)}(a), \quad B_i(u) = \sum_{k=1}^{n} \alpha_{ik}u^{(k-1)}(b)$$

were imposed. The solution to this system was exhibited in terms of a Green's function. Asymptotic results were derived in which the eigenvalues were shown to approach that of a similar problem with no integral part. An eigenfunction expansion was also derived. An adjoint system was defined, but only in the case of no integral part in the operator and end point boundary conditions (as did Birkhoff and later Langer).

6. A similar attempt was made in the early 1930's by K. Toyoda in two papers [59], [60]. The first considered the n-th order problem

$$Ly = y^{(n)} + \sum_{1}^{n} a_1 y^{(n-i)}, \quad U_i(y) = \sum_{k=1}^{m} U_{ik}(y(x_k)) = 0,$$

where

$$a \leqq x_1 < x_2 \cdots < x_m \leqq b, U_k(y(x_k)) = \sum_{j=0}^{n-1} \alpha_{ik}^{j} y^{(j)}(x_k).$$

The main result was the derivation of a Green's function.

The second considered n first order equations

$$y_i' + \sum_{j=1}^{n} a_{ij}y_j = 0, i = 1, \cdots, n$$

with boundary conditions

$$U_i(y) = \sum_{k=1}^{m} U_{ik}(y(x_k)) = 0, \quad \text{or} \quad U_i(y) = \int_a^b \phi_i(y(t))\, dt,$$

where

$$U_{ik}(y(x_k)) = \sum_{j=1}^{n} \alpha_{ik}{}^j y_j(x_k) \quad \text{and} \quad \phi_i(y(t)) = \sum_{j=1}^{n} \phi_{ij}(t)y_j(t).$$

Here the Green's function and adjoint Green's function were found, but the proper connection between the adjoint Green's function and the adjoint problem was not made.

7. From results arising in the calculus of variations, R. Mansfield [47] in the late 1930's recognized that if a problem was to be self-adjoint under interior point conditions, it must be discontinuous at those points. Consequently he considered the problem

$$y_i = \sum_{j=1}^{n} A_{ij}y_j, i = 1, \cdots, n.$$

under the boundary conditions

$$\alpha_\sigma(y) = \sum_{q=1}^{k-1} \sum_{j=1}^{n} [M_{\sigma jq}y_j(a_q{}^+) + N_{\sigma jq}y_j(a_{q+1}^-)] = 0,$$

$$\sigma = 1, \cdots, (k-1)n, \text{where } a = a_1 < a_2 \cdots < a_k = b.$$

This may be recast in terms of matrices as

$$y' = Ay, s(y) = \sum_{q=1}^{k=1} [M_q y(a_q{}^+) + N_q y(a_{q+1}^-)] = 0.$$

Mansfield reduced this problem to one in terms of a new independent variable in the form

$$u' = \mathcal{A}u, \mathcal{S}(u) = \mathcal{M}u(0) + \mathcal{N}u(1) = 0.$$

In this setting he then applied earlier results of Bliss [3] to discuss problems which were self-adjoint, self-adjoint under a transformation T, as well as their related eigenfunction expansions.

8. The major results of this period not yet mentioned are due to W. M. Whyburn [66], [67], [68]. His primary result was the reduction of the vector system

$$Ly = y' + Py = Q, \sum_{i=1}^{\infty} A_i y(s_i) + \int_a^b F(x)y(x)\,dx = C,$$

where the points $\{s_i\}_1^\infty$ are a set of the first species in $[a, b]$ (for some $m > 0$ the m-th derived set of $\{s_i\}_1^\infty$ is finite), to a system

$$Ly = y' + Py = Q, Ay(a) + By(b) + \int_a^b G(x)y(x)\,dx = D.$$

The function $G(x)$ depends upon the coefficients A_i and P, while the constant D depends upon C, the coefficients A_i, and the function Q. This means that if $Q = \lambda y$, then the parameter λ is introduced into the boundary condition, where it might not have been present earlier.

Whyburn also discussed extensively the Green's functions for the homogeneous system

$$Ly = y' + Py = Q,$$

$$Ay(a) + By(b) + \int_a^b G(x)y(x)\,dx = 0,$$

exhibiting its various properties, and then, he attempted to define an adjoint system in terms of the formal adjoint differential equation and some contrived boundary conditions. Specifically Whyburn's adjoint looks like

$$Mz = z' - zP = R,$$

$$-z(a) - z(b)B^{-1}A + \int_a^b z(x)L(W_1^{-1})W_1(a)\,dx = 0,$$

when B^{-1} exists, and when W_1 is a nonsingular solution of $MW = F$, $W(b) = B$. It looks like

$$Mz = z' - zP = R,$$

$$-z(a)A^{-1}B - z(b) + \int_a^b z(x)L(W_2^{-1})W_2(b)\,dx = 0,$$

when A^{-1} exists, and when W_2 is a nonsingular solution of $MW = F$, $W(a) = A$. Needless to say these conditions may not hold. Trouble arose immediately. The Green's function for these adjoints, when it exists, does not satisfy the usual symmetry formula $G(x, \xi) + G^*(\xi, x) = 0$, occurring in all other adjoint problems. It was, in fact, not the true adjoint (see Feller [22] page 470), which is a differential-boundary system, but a translate of it (see Jones [28]).

As one can see, these articles are not interconnected in any real sense. The results they contained are fragmentary and incomplete. They did, however, serve to exhibit a collection of interesting applicable problems, to provide the background for the more interesting, more complete work to follow, and to establish the field. The tools used in these articles were completely classical in nature. Systems, defined adjoints, Green's functions, eigenvalues, eigenfunction expansions were the rule. All this changed rapidly during the next eight year period.

IV. **Recent History, 1963–1971.** 1. After almost a twenty year gap, interest in multipoint boundary value problems was rekindled with the appearance of two papers by Cole [19], [20] in 1961 and 1964. The first discussed the multipoint system

$$Y' = \{\lambda R(x) + Q(x)\}Y, \sum_{u=1}^{m} W^{(u)}(\lambda)Y(a_u, \lambda) = 0,$$

where $a_1 < a_2 \cdots < a_m$, and where the components of $W^{(u)}$ are polynomials in λ. After deriving the Green's matrix and discussing its properties, an eigenfunction expansion was derived under certain regularity conditions.

The second considered the multipoint-integral system

$$Y' = A(x, \lambda)Y, \sum_{h=1}^{m} W^{(h)}(\lambda)Y(a_h, \lambda) + \int_a^b W(x, \lambda)Y(x, \lambda)\, dx = 0,$$

where $a = a_1 < a_2 \cdots < a_m = b$, $A(x, \lambda)$ is continuous in both x and λ, and $W^{(h)}(\lambda)$, $W(x, \lambda)$ are polynomials in λ; $W(x, \lambda)$ is infinitely differentiable in x. Again eigenvalues were found, the Green's matrix was discussed and an eigenfunction expansion was derived under certain regularity conditions. But, more important, *an adjoint system was defined which preserved those properties most closely associated with such systems*: Its existence does not depend upon the existence of an inverse matrix; Green's formula always holds; its Green's matrix G^* satisfies the formula

$$G(x, \xi) + G^*(\xi, x) = 0.$$

It was, however, no longer a purely differential system, but instead, took the form

$$Z' = -ZA(x, \lambda) + K(\lambda)W(x, \lambda), Z(a_h^+, \lambda) - Z(a_h^-, \lambda) = K(\lambda)W^{(h)}(\lambda),$$

$h = 1, \cdots, m$, where $Z(a_1^-, \lambda) = 0$, $Z(a_m^+, \lambda) = 0$, later called a differential-boundary system by Krall, since the parameter $K(\lambda)$ can be shown to depend upon the boundary terms $W^{(h)}(\lambda)$ and $Z(a_h^\pm, \lambda)$, $h = 1, \cdots, m$. This was the first time an adjoint system was not a purely differential system.

2. In a slightly different setting, Krall [31], [32], [33] derived similar results. Attempting to modernize the work of Hilb's student Betschler and at the same time extend the 1954 article of Naimark [49], Krall considered the following differential operator in $L^2[0, \infty)$: Let the differential expression ℓ be given by

$$\ell y = -y'' + q(x)y$$

on $[0, \infty)$, where q satisfies $\int_0^\infty |q(x)|\, dx < \infty$. Let D_0 denote those f defined on $[0, \infty)$ satisfying

1. f is in $L^2[0, \infty)$.
2. f' exists and is absolutely continuous on every finite subinterval of $[0, \infty)$.
3. ℓf is in $L^2[0, \infty)$.

Let $K(x)$ be in $L^2[0, \infty)$; let α and β be constants; let D denote those f in D_0 satisfying

$$\int_0^\infty K(x)f(x)\, dx - \beta f(0) + \alpha f'(0) = 0.$$

Define the operator L by setting

$$Lf = \ell f$$

for all f in D.

Krall showed that, when $|\alpha| + |\beta| \neq 0$, the domain D is dense in $L[0, \infty)$, and that the spectrum of L consists of a finite number of isolated eigenvalues and a continuous spectrum on the positive semiaxis $\lambda \geqq 0$.

As did Betschler, Krall derived, by expanding the Green's function, a spectral resolution involving both a discrete sum and integral, valid for a subet of D. In addition a second expansion was derived. At first the exact nature of this expansion was not recognized, largely because the adjoint operator L^* had not been successfully found. However in a second article [32], the adjoint operator was found (by using the Green's function). If we denote by E those g in D_0 satisfying

$$\bar{\beta}g(0) - \bar{\alpha}g'(0) = 0,$$

then $L^*g = -g'' + \bar{q}(x)g - \mu_g\bar{K}(x)$, where $\mu_g(0)/\bar{\alpha}$ or $\mu g = g'(0)/\bar{\beta}$ or both, depending upon whether or not α or β are zero or not. The second expansion was the spectral resolution of the adjoint operator.

This was the second appearance of a differential-boundary expression. In this instance, however, a new twist was introduced: *The adjoint was derived, not defined. It was determined by the setting.* The technique of deriving such objects has proven to be very useful.

3. During the next few years articles began to appear at an increasing rate. One natural question arising from the newly found adjoints was: What kind of generalization was necessary to permit self-adjointness? Krall [34] showed that if

$$LY = Y' + P(x)Y, L^+Z = -Z' + ZP(x)$$

$$U_1(Y) = AY(a) + BY(b),$$

$$U_2(Y) = CY(a) + DY(b),$$

satisfy Green's formula

$$\int_a^b [Z(LY) + (L^+Z)Y]\, dx = V_1(Z)U_1(Y) + V_2(Z)U_2(Y)$$

for some appropriate boundary functionals $V_1(Z)$, $V_2(Z)$, then the system

$$MY = LY + K_2(x)U_2(Y),$$

$$H(Y) = U_1(Y) + \int_a^b K_1(x)Y(x)\, dx = 0$$

has as its adjoint (either as an appropriate definition or in the proper setting by derivation) the system

$$M^+Z = L^+Z + V_1(Z)K_1(x),$$

$$J(Z) = V_2(Z) + \int_a^b Z(x)K_2(x)\, dx = 0.$$

Further an extended Green's formula holds:

$$\int_a^b [Z(MY) + (M^+Z)Y]\, dx = V_1(Z)H(Y) + J(Z)U_2(Y).$$

The Green's functions for these systems, when they exist, satisfy

$$G^+(t,x) + G(x,t) = 0.$$

Since the adjoint system has the same form as the original, trivial modifications easily yield self-adjoint systems (Krall [39]).

4. Another question concerned the possible self-adjointness of problems with interior boundary points. Neuberger [51] and Zettl [71] showed that the system

$$Y' = FY,\ A_1Y(a) + B_1Y(b) + C_1Y(c) = 0$$

could not ever be adjoint to a similar system.

5. Bryan [14] discussed the system

$$L = Y' - AY,\ \int_a^b dFY = 0,$$

where the auxilliary condition is generated by a function of bounded variation. Following Cole's lead, he discussed the Green's matrix, defined a (differential-boundary) adjoint system, and derived Green's formula.

6. Jones [28] showed that although the system

$$y' - A(x)y = \lambda B(x)y,\ My(a) + Ny(b) + \int_a^b F(x)y(x)\,dx = 0$$

cannot be classically self-adjoint, under certain conditions it will be self-adjoint under a transformation T, an idea which originated with Bliss [3] while studying the calculus of variations.

Jones' adjoint by definition is

$$-z - A^*(x)z = \lambda B^*z + F^*(x)V,$$
$$z(a) = -M^*V,\ z(b) = N^*V.$$

Here the boundary conditions are given in parametric form, and the operator is a differential-boundary operator. Jones transformed these systems into a new matrix form as follows: Let

$$u(x) = \int_a^x F(x)y(x)\,dx + My(a) \text{ and let } \phi = \begin{pmatrix} y \\ u \end{pmatrix}.$$

ϕ satisfies

$$\phi' - \mathcal{A}(x)\phi = \lambda\mathcal{B}(x)\phi,\ \mathcal{M}\phi(a) + \mathcal{N}\phi(b) = 0,$$

where

$$\mathcal{A}(x) = \begin{pmatrix} A(x) & 0 \\ F(x) & 0 \end{pmatrix}, \ \mathcal{B}(x) = \begin{pmatrix} B(x) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{M} = \begin{pmatrix} M & -I \\ 0 & 0 \end{pmatrix}, \ \mathcal{N} = \begin{pmatrix} 0 & 0 \\ N & I \end{pmatrix}.$$

This system is equivalent to the original. The adjoint of the new matrix system is

$$-\psi' - \mathcal{A}^*(x)\psi = \lambda\mathcal{B}^*(x)\psi, \ \mathcal{P}^*\psi(a) + \mathcal{Q}^*\psi(b) = 0,$$

where $\mathcal{P} = \begin{pmatrix} I & 0 \\ M & 0 \end{pmatrix}$, $\mathcal{Q} = \begin{pmatrix} 0 & I \\ 0 & -N \end{pmatrix}$, which is equivalent to the original adjoint. It is to this pair that the results of Bliss [3] were applied. Jones also showed that Whyburn's adjoint is a translate of his.

7. By using an extension of the matrix representation by Jones, Krall [35] showed that a differential-boundary system is also equivalent to an ordinary differential system with end point boundary conditions. Furthermore a system with a finite number of interior boundary points may also be reduced to such a system. The system under consideration has the form

$$L_bY = Y' + PY + \sum_{i+1}^{\ell} H_i \left[\sum_{j=0}^{m} C_{ij}Y(a_j+) + D_{ij}Y(a_j-) \right],$$

where $C_{im} = 0$, $D_{i0} = 0$,

$$M_iY = \int_a^b K_iY\,dx + \sum_{j=0}^{m} A_{ij}Y(a_j+) + B_{ij}Y(a_j-)],$$

$$\text{where } A_{im} = 0, \ B_{i0} = 0. \ i = 1, \cdots, k.$$

The adjoint to this system had already been found in $L_n^2[a, b]$ by Krall [36]. Here, however, the following change in notation was made: Let the interval $I = [a, b]$ be divided into subintervals by $I_j = [a_{j-1}, a_j]$, $j = 1, \cdots, m$, and let $\mathcal{Y}$ denote the $nm \times 1$ vector

$$\begin{pmatrix} Y(I_1) \\ \cdot \\ \cdot \\ \cdot \\ Y(I_m) \end{pmatrix},$$

where the first n components are evaluated in $I_1 = [a, a_1]$, etc. Then by similarly redefining the various coefficients, a new equivalent system

$$L_b \mathcal{Y} = \mathcal{Y}' + \mathcal{P}\mathcal{Y} + \mathcal{H}[\mathcal{C}\mathcal{Y}(A) + \mathcal{D}\mathcal{Y}(B)]$$

$$\mathcal{A}\mathcal{Y}(A) + \mathcal{B}\mathcal{Y}(B) + \int_A^B \mathcal{K}(\mathcal{X})\mathcal{Y}(\mathcal{X})\,d\mathcal{X} = 0$$

is found. The technique used is similar to and equivalent to the technique used by Mansfield [47]. The adjoint system has a similar equivalent form.

Now let

$$\mathcal{U}(\mathcal{X}) = \mathcal{A}\mathcal{Y}(A) + \int_A^{\mathcal{X}} \mathcal{K}\mathcal{Y}\,d\mathcal{X},$$

$$\mathcal{S} = \mathcal{C}\mathcal{Y}(A) + \mathcal{D}\mathcal{Y}(B),$$

then the transformed differential-boundary system is equivalent to

$$\begin{pmatrix} \mathcal{Y} \\ \mathcal{U} \\ \mathcal{S} \end{pmatrix}' = \begin{pmatrix} \mathcal{P} & 0 & -\mathcal{H} \\ \mathcal{K} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Y} \\ \mathcal{U} \\ \mathcal{S} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{A} & -I & 0 \\ 0 & 0 & 0 \\ \mathcal{C} & 0 & -\tfrac{1}{2}I \end{pmatrix} \begin{pmatrix} \mathcal{Y}\mathcal{A} \\ \mathcal{U}(A) \\ \mathcal{S}(A) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{B} & I & 0 \\ \mathcal{D} & 0 & -\tfrac{1}{2}I \end{pmatrix} \begin{pmatrix} \mathcal{Y}\mathcal{B} \\ \mathcal{U}(B) \\ \mathcal{S}(B) \end{pmatrix} = 0.$$

Again the adjoint is equivalent to the appropriate end point problem.

8. During the same period motivated by the work of Neuberger [51] and Zettl [71] on the three point boundary value problem, Loud [46] determined conditions under which a differential operator $ix' - A(t)x$, whose domain is smooth except at a finite number of points $a = c_0 < c_1 \cdots < c_m = b$, where jumps occur, can be self-adjoint.

9. Using the technique previously applied to differential-boundary operators under similar conditions, Krall [37] showed that in a Hilbert space the system

$$LY + A_1Y' + A_2Y, \quad M_iY = \sum_{i=0}^{m} [A_{ij}Y(a_j+) + B_{ij}Y(a_j-)] = 0,$$

$i = 1, \cdots, k$, has as its adjoint the system

$$L^*Z = -(A_1^*Z)' + A_0^*Z,$$

$$A_1^*(a_j-)Z(a_j-) = \sum_{i=1}^{k} B_{ij}^*\phi_i(Z),$$

$$-A_1^*(a_{j-1}+)Z(a_{j-1}+) = \sum_{i=1}^{k} A_{ij-1}^*\phi_i(Z),$$

$i = 1, \cdots, m.$

$$\text{If } \mathcal{A}_1 = \begin{pmatrix} A_1(x_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_1(x_m) \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} A_0(x_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0(x_m) \end{pmatrix},$$

$$\mathcal{A} = (A_{ij-1}),\ \mathcal{B} = (B_{ij}),$$
$$A = (a_0+, a_1+, \cdots a_{m-1}+),\ B = (a_1-, a_2-, \cdots a_m-),$$

then the system is self-adjoint if and only if

$$\mathcal{A}_i = -\mathcal{A}_1^*, \mathcal{A}_0 = \mathcal{A}_0^* - \mathcal{A}_1', \mathcal{A}\mathcal{A}_1(A)^{*-1}\mathcal{A}^* = \mathcal{B}\mathcal{A}_1(B)^{*-1}\mathcal{B}^*,$$

a result which has the same form as the classical result in simpler cases. (Locker [45] has recently extended these results to the n-th order operators.)

10. Zettl [72] followed by discussing the possibility of systems being adjoint under other than the usual weight functions. This was followed by a generalization due to Wong [70] and further discussion of Krall [38], who considered the possible adjoints for the operator $Ly = A_1y' + A_0y$ in the Hilbert space generated by the inner product

$$(y, z)_J = \int_a^b z^*Jy\, dt.$$

An adjoint exists for each nonsingular symmetric matrix J. Self-adjointness occurs if and only if

$$A_1^*J + JA_1 = 0, \; A_0^*J - JA_0 + (JA_1)' = 0.$$

11. Interest in self-adjointness was continued by Kim [30] who considered the system

$$Ly = y'' - h(x)(a_1 y(0) + a_2 y'(0)),$$

$$b_1 y(0) - b_2 y'(0) - \int_0^\infty K(x) y(x)\, dx = 0$$

in the setting $L^2(0, \infty)$, extending the results of Betschler [2] and Krall [31]. Kim showed that, except for a finite collection of isolated eigenvalues and a continuous spectrum on the positive real axis, a Green's function exists, generating solutions to $Ly - \lambda y = f$ which also satisfy the boundary condition. By using the Green's function he was able to show that the adjoint operator is given by

$$L^*z = z'' + \bar{K}(x) \left[\overline{\left(\frac{a_1}{a_1 b_2 + a_2 b_1} \right)} z(0) + \overline{\left(\frac{a_2}{a_1 b_2 + a_2 b_1} \right)} z'(0) \right],$$

with its domain restricted by the boundary condition

$$\overline{\left(\frac{b_1}{a_1 b_2 + a_2 b_1} \right)} z(0) - \overline{\left(\frac{b_2}{a_1 b_2 + a_2 b_1} \right)} z'(0) + \int_0^\infty \bar{h}(x) z(x)\, dx = 0.$$

Hence the operator L is self-adjoint if and only if $a_1 b_2 + a_2 b_1 = -1$; a_1, a_2, b_1, b_2 are all real; $h(x) = \bar{K}(x)$. The Green's function was then discussed in some detail, leading to a spectral resolution, similar to results previously stated.

The results during this period still appear to be somewhat disconnected. Indeed they were coming at a frightful rate from a great many mathematicians scattered throughout the world. Needless to say many of their ideas and results substantially overlapped.

12. The period came to a close with the appearance of the article by Krall [39] which gathered together and extended many of the results concerning regular differential and differential-boundary systems with interior and integral boundary conditions. The article begins by considering the system

$$LY = Y' + P(x)Y + \sum_{i=1}^{\ell} H_i(x) \sum_{j=1}^{m} [C_{ij} Y(a_{j-1}+) + D_{ij} Y(a_j-)] = 0,$$

$$\sum_{j=1}^{m} [A_{ij} Y(a_{j-1}+) + B_{ij} Y(a_j-)] + \int_a^b K_i(x) Y(x)\, dx = 0,$$

$i = 1, \cdots, k$, and defining its adjoint in parametric form by

$$L^+Z = -Z' + P^*(x)Z - \sum_{i=1}^{k} K_i^*(x)\phi_i = 0,$$

$$-Z(a_j-) + \sum_{i=1}^{k} B^*_{ij}\phi_i - \sum_{i=1}^{\ell} D^*_{ij} \int_a^b H_i^*(x)Z(x)\,dx = 0,$$

$$Z(a_{j-1}+) + \sum_{i=1}^{k} A^*_{ij}\phi_i - \sum_{i=1}^{\ell} C^*_{ij} \int_a^b H_i^*(x)Z(x)\,dx = 0,$$

$j = 1, \cdots, m$. These are immediately reduced to systems without interior boundary points, having the form

$$LY = Y' + PY + H[CY(a) + DY(b)] = 0,$$

$$AY(a) + BY(b) + \int_a^b K(x)Y(x)\,dx = 0,$$

and

$$L^+Z = -Z' + P^*Z - K^*\phi,$$

$$Z(a) + A^*\phi - C^*\int_a^b H^*(x)Z(x)\,dx = 0,$$

$$-Z(b) + B^*\phi - D^*\int_a^b H^*(x)Z(x)\,dx = 0.$$

By using various formulas involving A, B, C and D the parameter ϕ is eliminated, so the adjoint has the form

$$L^+Z = -Z' + P^*Z - K^*[\tilde{A}Z(a) + \tilde{B}Z(b)] = 0,$$

$$\tilde{C}Z(0) + \tilde{D}Z(b) + \int_a^b H^*(x)Z(x)\,dx = 0.$$

By using the matrix form of these systems, mentioned earlier, the equation $LY = 0$ is shown to have n linearly independent solutions (Y is an n-dimensional vector), and a fundamental matrix is produced for the matrix form. Then it is shown that the adjoint L^* (in $L_n^2[a, b]$) of the operator determined by L agrees with L^+, and that the boundary conditions are satisfied by elements in the domain of L^*.

A discussion of self-adjoint systems follows. The operator

$$TY = (1/i)Y' + P(x)Y + H(x)[CY(a) + DY(b)]$$

with domain restricted by the boundary condition

$$AY(a) + BY(b) + \int_a^b K(x)Y(x)\,dx = 0$$

is classically self-adjoint in $L_n^2[a, b]$ if and only if

1. $P = P^*$,
2. All matrices are $n \times n$ matrices,
3. $K(x) = -i[AC^* - BD^*]\,H^*(x)$,
4. $AA^* = BB^*$,
5. $H(x)[CC^* - DD^*] = 0$.

Since through the matrix form, the spectrum of T is seen to consist only of isolated eigenvalues, an eigenfunction expansion automatically follows by the technique exhibited in [**18**, chapter 7].

Systems self-adjoint under a transformation are also discussed.

Finally under the regularity assumptions that A and B are nonsingular an eigenfunction expansion is derived in the nonself-adjoint case by expanding the Green's matrix in a series of residues. The spectrum of L is shown to consist of only isolated eigenvalues; for each F in the domain of L,

$$F(x) = \sum_1^\infty Y_i(x) \int_a^b Z_i^*(\xi)F(\xi)\,d\xi,$$

where $\{Y_i\}$ and $\{Z_i\}$ are the corresponding eigenfunctions for L and L^* respectively.

This concluded the period in which a finite number of discontinuities of the first kind were permitted. As it drew to a close, mathematicians turned their attention to discontinuities of a more general nature: Those generated by functions of bounded variation seemed easiest to handle.

V. **Current Problems, 1970–1973.** 1. We mention first the article by Nersesian [**50**] which appeared in 1961, but since it was in Russian and was not translated, had little impact on the field. Nersesian studied the system

$$Ly = (y - a(x))' + \sum_{i=1}^m q_i(x)(x - \Delta_i(x)) + \int_0^x K(x, t)y(t)\,dt = \lambda y,$$

$$y(0) = \alpha, \beta y(\ell) + \int_0^\ell y(x)\,db(x) = A_0,$$

and its adjoint

$$L^+z = -(z + b(x))' + \sum_{i=1}^{m} q_i(x)z(x + \Delta_i^*(x)) + \int_x^{\ell} K(t, x)z(t)\, dt = \lambda z,$$

$$\alpha z(0) + \int_0^{\ell} z(x)\, da(x) = A_0, \quad z(\ell) = \beta,$$

where $\Delta_i(x) \geqq 0$, $0 \leqq x \leqq \ell$, and $\Delta_i(\ell) \leqq \ell$, $\Delta_i'(x) \leqq \theta < 1$, $0 \leqq x \leqq \ell$, $i = 1, \cdots, m$, where $q_i(x) = 0$ when $x < 0$, $\int_0^{\ell} |q_i(x)|\, dx < \infty$, $i = 1, \cdots, m$, where $\int_0^{\ell}\int_0^{\ell} |K(t, x)|\, dxdt < \infty$, and where $a(x)$ and $b(x)$ are continuous. He showed the existence of a countable collection of isolated eigenvalues and derived an eigenfunction expansion which converges in the classical sense:

$$\tfrac{1}{2}[f(x + 0) + f(x - 0)] = \sum_{-\infty}^{\infty} Y_k^0(x) \int_0^{\ell} Z_k^0(t)f(t)\, dt,$$

when f is of bounded variation on $[0, \ell]$.

2. The serious consideration of more general boundary value systems opened with the appearance of Bryan's article [**15**], which discussed differential-boundary systems suitably generalized to permit boundary operators and boundary conditions to be generated by functions of bounded variation. Bryan specifically considered the system

$$LY = (Y - H^*[CY(a) + DY(b)])' - PY,$$

$$AY(a) + BY(b) + \int_a^b dK(x)Y(x) = 0,$$

where H and K are of bounded variation, and an adjoint system defined by

$$L^+Z = -(Z - K^*[\tilde{C}Z(a) + \tilde{D}Z(b)])' - P^*Z,$$

$$\tilde{A}Z(a) + \tilde{B}Z(b) + \int_a^b dH(x)Z(x) = 0,$$

where

$$\begin{pmatrix} \tilde{C}^* & \tilde{A}^* \\ -\tilde{D}^* & -\tilde{B}^* \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}.$$

In this setting Green's formula takes the form

$$\int_a^b [Z^*(LY) - (L^*Z)^*Ydx =$$

$$-[\tilde{C}Z(a) + \tilde{D}Z(b)]^* \left[AY(a) + BY(b) + \int_a^b dKY \right]$$

$$- \left[\tilde{A}Z(a) + \tilde{B}Z(b) + \int_a^b dHZ \right]^* CY(a) + DY(b)].$$

The article concluded by deriving and discussing the Green's mattices for these systems, showing that

$$G(x, t) + G^*(t, x) = 0.$$

3. The results of Bryan were generalized considerably by Vejvoda and Tvrdý [63], [64]. In the first by calling upon techniques exhibited by Hildebrandt [27] and Krall [35], and appealing to Stieltjes-integral equations, they reduced the system

$$x' = A(t)x + K'(t)[\tilde{M}x(a) + \tilde{N}x(b)] + G'(t)\left[\int_a^b dH(\tau)x(\tau)\right] + f'(t),$$

$$Mx(a) + Nx(b) + \int_a^b dL_1(\tau)x(\tau) = 0, \int_a^b dL_2(\tau)x(\tau) = 0,$$

and its adjoint

$$y^{T\prime} = -y^T A(t) - (y^T(a)\tilde{P} + y^T(b)\tilde{Q})L'(t)$$

$$-\left[\int_a^b y^T(\tau)\, dG(\tau)\right] H'(t) - \rho^T L_2'(t),$$

$$y^T(a)P + y^T(b)Q - \int_a^b y^T(\tau)\, dK(\tau) = 0$$

(The coefficients satisfy

$$\begin{pmatrix} -\tilde{P} & -P \\ Q & Q \end{pmatrix} \begin{pmatrix} M & N \\ \tilde{M} & \text{N} \end{pmatrix} = I,$$

G, H, K, L_1, L_2 are of bounded variation, and hence differentiable a.e.) to

$$\xi(t) = \xi(a) + \int_a^t d\mathcal{A}(\tau)\xi(\tau) + \phi(t) - \phi(a), \mathcal{M}\xi(a) + \mathcal{N}\xi(b) = 0,$$

and

$$\eta^T(t) = \eta^T(a) - \int_a^t \eta(\tau)\, d\mathcal{A}(\tau), \eta^T(a)\, \mathcal{P} + \eta^T(b)\, \mathcal{Q} = 0.$$

If the operators L and L^+ are defined by

$$L\xi(t) = \xi(t) - \xi(a) - \int_a^t d\mathcal{A}(\tau)\xi(\tau),$$

$$L^+\eta^T(t) = -\eta^T(t) + \eta^T(a) - \int_a^t \eta^T(\tau)\, d\mathcal{A}(\tau),$$

then Green's formula is given by

$$\int_a^b [\eta^T(t)\, dL\xi(t) - dL^+\eta^T(t)\xi(t)] =$$

$$[\eta^T(a)\, \mathcal{P} + \eta^T(b)\mathcal{Q}]\, [\tilde{\mathcal{M}}\xi(a) + \tilde{\mathcal{N}}\xi(b)]$$
$$+ [\eta^T(a)\, \tilde{\mathcal{P}} + \eta^T(b)\mathcal{Q}]\, [\mathcal{M}\xi(a) + \mathcal{N}\xi(b)].$$

In the original setting, if

$$Lx = x' - Ax - K'(\tilde{M}x(a) + \tilde{N}x(b)),$$
$$L^+y^T = -y^{T'} - y^TA - (y^T(a)\tilde{P} + y^T(b)\tilde{Q})L_1{}' - \rho^TL_2{}',$$

it is given by

$$\int_a^b [y^TLx - L^+y^Tx]\, dt =$$

$$\left[y^T(a)P + y^T(b)Q - \int_a^b y^T\, dK\right]\left[\tilde{M}x(a) + \tilde{N}x(b)\right]$$

$$+\left[y^T(a)\tilde{P} + y^T(b)\tilde{Q}\right]\left[Mx(a) + Nx(b) + \int_a^b dL_1x\right] + \rho^T \int_a^b dL_2x.$$

The Green's matrix is derived and shown to have the usual properties.

The second [64] discusses in a similar manner the system

$$x' = Ax + C(t)x(a) + D(t)x(b) + \int_a^b d_sG(t, s)x(s) + f(t),$$

$$Mx(a) + Nx(b) + \int_a^b dL(s)x(s) = 0$$

(where C, D are in L^2 and G is in $L^2[BV]$), and its adjoint

$$y^T(t) = y^T(a) - \int_a^t y^T(s)A(s)\,ds - \gamma^T(L(t) - L(a))$$

$$- \int_a^b y^T(s)(G(s,t) - G(s,a))\,ds,$$

$$y^T(a) + \gamma^T M + \int_a^b y^T(s)C(s)\,ds = 0,$$

$$-y^T(b) + \gamma^T N + \int_a^b y^T(s)D(s)\,ds = 0.$$

Of particular interest, however, is section 5 in which the first system is set in $L^1(a, b)$ and the second is derived in $L^\infty(a, b)$ by a technique due to Wexler [**65**].

4. We are now at a turning point in the field. The tools of functional analysis, of course, were not used during the early development of the field. Furthermore, they were used at best sparingly during the period ending in 1971. The trend, however, is significant. They appear more and more often, and during the last few years have become to be used almost exclusively. Almost all of the most recent results have been found by considering an appropriate linear operator in a suitable space, or by considering an appropriate linear relation when the operator, which would normally be considered, does not possess a dense domain (see [1] and [**10**]).

The first consideration to operators generated by Stieltjes boundary conditions was given by Green and Krall [**23**], [**40**], [**41**], where the Stieltjes measure varied only by discrete jumps. Specifically, let $H = \mathcal{L}_n^2[0, 1]$, and denote by D_0 those elements y in H satisfying

1. y is absolutely continuous,
2. $\mathfrak{L}y = y' + Py$ is in H, where P is a continuous $n \times n$ matrix.

Denote by D the collection of all elements y in D_0 satisfying the condition

$$\sum_{i=0}^{\infty} A_i y(t_i) = 0,$$

where $\{t_i\}_0^\infty$ forms a (possibly somewhere) dense subset of $[0, 1]$, $t_0 = 0$, $t_1 = 1$, and where $\{A_i\}_0^\infty$ is a collection of $n \times n$ matrices satisfying

$$\sum_{i=0}^{\infty} \|A_i\| < \infty$$

for some convenient norm.

The operator L is then defined by setting $Ly = \mathcal{L}y$ for all y in D.

Assuming 0 is not an eigenvalue of L, Green and Krall [23] derived the Green's matrix, discussed its properties, and used it to show that D is dense in H. Another application of the Green's matrix yielded the adjoint operator: Let D^* denote those elements z in H satisfying

1. for some parametric vector ϕ,

$$z - \sum_{i=0}^{\infty} A_i^*\phi\, \chi(t_i, 1] \text{ is absolutely continuous } (\chi(t_i, 1] \textit{ is}$$

the characteristic function of the interval $(t_i, 1]$),

2. $\mathcal{L}^+z = -z' + P^*z$ exists a.e. in H,

3. $z(0-) = 0, z(1+) = 0$.

L^* is then defined by $L^*z = \mathcal{L}^+z$ for all z in D^*.

Finally the spectrum of L was shown to consist of isolated eigenvalues, which, when A_0 and A_1 are nonsingular, lie in a vertical strip in the complex plane.

Subsequent articles [40] and [41] derive both self-adjoint (under a transformation) and nonself-adjoint eigenfunction expansions, which are similar to those previously presented.

Of additional interest is the brief discussion of a one dimensional problem in which an infinite number of the coefficients A_i are bounded away from 0, and for which the corresponding t_i's are dense in a subset I of $[0, 1]$. In this situation the domain D is not dense in H and an adjoint *does not exist.*

5. Substantial improvements to the preceeding were made by Brown, Green and Krall [5], [12], [11]. First the setting was enlarged to be any of the Banach spaces $\mathcal{L}_n{}^p[0, 1], 1 < p < \infty$. Second the boundary condition was altered so the coefficients involved were $m \times n$ matrices satisfying

$$\sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 K(t)y(t)\, dt = 0.$$

By extensive use of various theorems from functional analysis (see Kelley and Namioka [29]), Brown was able to show that when

$$\bigcap_{i=0}^{\infty} \ker A_i^T \subset \ker K(t) \text{ a.e.}$$

the domain of L is dense in $\mathcal{L}_n^p[0, 1]$, $1 < p < \infty$. The dual operator in $\mathcal{L}_n^q[0, 1]$, $1 < q < \infty$, $1/p + 1/q = 1$, was shown in [12] to have the form

$$L^+z = -z' + P^*z + K^*\phi,$$

a differential boundary operator. Finally in [11] both self-adjoint (under a transformation) and nonself-adjoint eigenfunction expansions were derived, having the same form as previously derived expansions.

6. A second improvement was made by Brown and Krall in [13] and Brown in [6]. Since it forms a basis for several other articles and also is closely related to the earlier work of Bryan [14], Vejvoda and Tvrdý [63] and Tucker [61], we present it in some detail.

Let $X = \mathcal{L}_n^p[0, 1]$, $1 \leqq p < \infty$; let P be continuous $n \times n$ matrix; let ν be an $m \times n$ matrix valued measure of bounded variation. In X let D_p denote those elements satisfying

1. y is absolutely continuous,
2. $\ell y = y' + Py$ exists a.e. and is in X,
3. $U(y) = \int_0^1 d\nu(t)y(t) = 0.$

The operator L_p is then defined by setting $L_py = \ell y$ for all y in D_p.

The boundary condition $U(y) = 0$ is more general than the one previously mentioned, since if ν is decomposed, it will consist of a singular atomic part, generating a discrete sum, a singular continuous part, and an absolutely continuous part, generating an ordinary integral. Hence $U(y) = 0$ would look like

$$\int_0^1 s\nu_{sa}(t)y(t) + \int_0^1 d\nu_{sc}(t)y(t) + \int_0^1 \frac{d\nu_c}{dt}y(t)\,dt = 0,$$

where sa denotes singular-atomic, sc denotes singular-continuous, c denotes absolutely continuous. The first integral can be written as $\sum_{i=0}^{\infty} A_i y(t_i)$ if necessary.

Now let T be a dense subset of $[0, 1]$, and let K_ν^T denote $\bigcap_{t\in T} \ker \nu[0, t]$. The article shows that, when $1 \in T$, D_p is dense in X *if and only if*

$$K^T_{\nu_s^*} \subset K^T_{\nu_c^*},$$

where s denotes the singular portion of ν, and c denotes the portion of ν which is continuous with respect to Lebesgue measure.

Assuming that D_p is dense in X, the dual operator in $X^* = \mathcal{L}_n^q[0, 1]$, $1/p + 1/q = 1$, is then shown to have the following form: Let D_q^+ denote the collection of all elements z in X^* for which there is a parameter ϕ in $C^m/K^T_{\nu_s}$ such that

1. $\epsilon(t) = z(t) + \nu_s^*[0, t]\phi$ is absolutely continuous,

2. $\mathfrak{L}^+z = -z + P^*z + \dfrac{d\nu_c^*}{dt}\phi$ exists a.e. and is in X,

3. $z(0+) = -\nu_s^*[0]\phi$,

   $z(1-) = \nu_s^*[1]\phi$.

Then $L_q^*z = \mathfrak{L}^+z$ for all z in D_q^*, $1 \leqq p < \infty$. Brown has showed [**6**] that both L_p and L_q^* are normally solvable Fredholm operators which, when $n = m$, have a spectrum of discrete eigenvalues, accumulating only at ∞. Further ([**13**]) when $A_0 = \nu_s[0]$ and $A_1 = \nu_s[1]$ are nonsingular, spectral resolutions in the form of eigenfunction expansions exist.

An appeal to the theory of almost periodic functions [**44**] shows that the eigenvalues of L_p, given by

$$\det U(Y(t)e^{\lambda t}) = 0,$$

where $Y(t)$ is a fundamental matrix for $y' + Py = 0$, lie in a vertical strip $|\mathrm{Re}(\lambda)| < h$, and that the number of eigenvalues in a region bounded by $k < \mathrm{Im}(\lambda) < k + 1$ is bounded by a number M independent of k. Further for any $\delta > 0$, there is a number $m(\delta) > 0$ such that

$$|\det U(Y(t)e^{\lambda t})| > m(\delta)$$

for all λ outside circles of radius δ centered at the eigenvalues. This implies that outside these circles the Green's matrix for L_p exists and remains bounded. By using additional asymptotic estimates the Green's matrix is shown to be expandable in the form

$$G(\lambda_0; t, s) = \sum_{n=0}^{\infty} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} \sum_{\ell=0}^{m-1-j-k} u_j(t) U_\ell \nu_k(s)/(\lambda_0 - \lambda_n)^{m-j-k-\ell},$$

where each term is the residue of $G(\lambda, t, s)/(\lambda - \lambda_0)$ at an eigenvalue λ_n. The elements $u_j(t)$, $v_k(s)$ satisfy

$$(L_p - \lambda_n)u_0 = 0,$$

$$(L_p - \lambda_n)u_j = u_{j-1}, j = 1, \cdots, m - 1,$$

$$(L_q{}^+ - \bar{\lambda}_n)v_0 = \frac{dv_c{}^*}{dt},$$

$$(L_q{}^+ - \lambda_n)v_k = v_{k-1}, k = 1, \cdots, m-1.$$

Using this expansion it then follows that for each y in D_p,

$$y(t) = -\sum_{n=1}^{\infty} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} u_j(t)U_{m-1-j-k} \int_0^1 v_k{}^*(s)y(s)\,ds;$$

for each z in $D_q{}^+$,

$$z(s) = -\sum_{n=1}^{\infty} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} v_k(s)U^*_{m-1-j-k} \int_0^1 u_j{}^*(t)z(t)\,dt.$$

Expansions for operators which are self-adjoint (under a transformation) were also discussed. The hypotheses required to achieve such situations, however, are so strong that the existence of such operators is highly dubious.

7. The results of the preceding article have been extended in two ways. First Krall [42], [43] has considered Stieltjes differential-boundary operators with, surprisingly, a new kind of condition arising: an adjusting operator, inserted to permit differentiability, but free of boundary conditions, was found to be advantageous. In many respects it is similar to a condition introduced by Moyer [48] in a distributionally oriented study. The articles proceed as follows: Let A, B, C, D be $m \times n$, $m \times n$, $(2n - m) \times n$, $(2n - m) \times n$ matrices, $m \leqq 2n$, such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular, and let

$$\begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix}$$

be its inverse, where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are respectively $m \times n$, $m \times n$, $(2n - m) \times n$, $(2n - m) \times n$ matrices.

Let K be an $m \times n$ matrix valued measure of bounded variation, satisfying $dK(0) = A$, $dK(1) = B$. Let K_1 be an $r \times n$ matrix valued measure of bounded variation, satisfying $dK_1(0) = 0$, $dK_1(1) = 0$. Hence the boundary form $\int_0^1 \begin{pmatrix} dK \\ dK_1 \end{pmatrix} y$ represents the most general type of bounded variation. The "mass" at 0 and 1 is concentrated in the first m rows.

Finally let H and H_1 be $n \times (2n - m)$ and $n \times s$ matrices of bounded variation, and let P be a continuous $n \times n$ matrix.

Now in $\mathcal{L}_n{}^p[0, 1]$, $1 \leqq p < \infty$, denote by D those elements y satisfying

1. For each y there is an $s \times 1$ matrix valued constant ψ such that
$$y + H[Cy(0) + Dy(1)] + H_1\psi$$
is absolutely continuous,
2. $\mathfrak{L}y = (y + H[Cy(0) + Dy(1)] + H_1\psi)' + Py$ exists a.e. and is in $\mathcal{L}_n^p[0, 1]$,
3. $Ay(0) + By(1) + \int_{0+}^{1-} dK(t)y(t) = 0, \int_{0+}^{1-} dK_1(t)y(t) = 0.$

Define the operator L by setting $Ly = \mathfrak{L}y$ for all y in D.

The boundary conditions in 3 are assumed to satisfy the condition mentioned earlier guaranteeing the density of D in $\mathcal{L}_n^p[0, 1]$.

The dual operator L^+ in $\mathcal{L}_n^q[0, 1]$ was found to have the following description: Denote by D^+ those elements z satisfying

1. For each z there exists an $r \times 1$ matrix valued constant ϕ such that $z + K^*[\tilde{A}z(0) + \tilde{B}z(1) + K_1^*\phi$ is absolutely continuous,
2. $\mathfrak{L}^+z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1) + K_1^*\phi)' + P^*z$ exists a.e. and is in $\mathcal{L}_n^q[0, 1]$,
3. $\tilde{C}z(0) + \tilde{D}z(1) + \int_{0+}^{1-} dH^*(t)z(t) = 0, \int_{0+}^{1-} dH_1^*(t)z(t) = 0.$

The operator L^+ is given by $L^+z = \mathfrak{L}^+z$ for all z in D^+.

Note that while D is assumed to be dense in $\mathcal{L}_n^p[0, 1]$, D^+ may not be dense in $\mathcal{L}_n^q[0, 1]$ unless the measure generated by $\binom{H}{H_1}$ satisfies the same sort of acceptability condition as $\binom{K}{K_1}$. Further $\mathfrak{L}$ will not be uniquely defined, nor will D be dense, unless H_1 and K_1 possess a part which is singular with respect to Lebesgue measure. Such terms as $H_1\psi$ and $K_1\phi$ are called free, implying they are not dependent directly on a boundary form such as the other adjusting terms. These were alluded to by Moyer [48] and explicitly exhibited by Vejvoda and Tvrdý [63] in simpler situations.

Green's formula takes on the rather complicated form:
$$\int_0^1 [z^*(Ly) - (L^+z)^*y]\, dt =$$
$$\left[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z\right]^*[Cy(0) + Dy(1)]$$
$$+ \left[\tilde{A}z(0) + \tilde{B}z(1)\right]^* \left[Ay(0) + By(1) + \int_0^1 dKy\right]$$
$$+ \phi^* \left[\int_0^1 dK_1y\right] + \left[\int_0^1 dH_1^*z\right]^* \psi.$$

If, instead of considering L and L^+, the operators $T = (1/i)L$ and $T^* = -(1/i)L^+$ are examined in $\mathcal{L}_n^2[0, 1]$, it is found that $T = T^*$ if and only if

1. $P = -P^*$,
2. $m = n, r = s$,
3. $K = [BD^* - AC^*]H^*$ a.e.,
4. $AA^* = BB^*$,
5. $H[CC^* - DD^*] = 0$ a.e.,
6. $K_1 = MH_1^*$, where m is a nonsingular $r \times r$ matrix.

The second article [43] continues by deriving equivalent integral equation-boundary value systems, similar to those of [39] and [63], which are a bit too cumbersome to exhibit here. Fundamental matrices are exhibited, which, when inserted in the boundary conditions, show that the spectrum of L (and T) consists only of discrete eigenvalues, accumulating at ∞. This immediately implies that T has a standard eigenfunction expansion similar to that derived in [18] for ordinary differential operators.

A nonself-adjoint expansion was also derived while assuming that $H_1 = 0$, $K_1 = 0$, either H or K is continuous. In this case the results of [39] are easily adaptable. The general situation is still obscure.

8. The second extension, Brown [8], is to higher order ordinary operators under the assumption that the boundary measure has no part which is absolutely continuous with respect to Lebesgue measure. Specifically on $[a, b]$ let D_m^{pn} denote those elements y in $\mathcal{L}_m^p[a, b]$, $1 \leqq p < \infty$, satisfying

1. $y^{(n-1)}$ exists and is absolutely continuous,
2. $y^{(n)}$ is in $\mathcal{L}_m^p[a, b]$.

Hence the expression

$$\ell y = \sum_{i=0}^{n} A_i y^{(n-i)},$$

where the $m \times m$ matrices A_i are in $C^{(n-i)}$, and A_0 nonsingular on $[a, b]$, is also in $\mathcal{L}_m^p[a, b]$.

If ν_i are $\ell \times m$ matrix valued measures of bounded variation with no absolutely continuous part, let $D_m^{pn}[U]$ denote those elements y in $\mathcal{L}_n^p[a, b]$ satisfying

1. y is in D_m^{pn},

2. $U(y) = \sum_{i=1}^{n} d\nu_i(t) y^{(n-i)}(t) = 0.$

The operator L is then defined by setting $Ly = \ell y$ for all y in $D_m^{pn}[U]$. By using an equivalent first order vector form, Brown showed that the dual operator can be described in the following way: Define the expressions ℓ_j^+ by

$$\ell_0^+(z) = A_0^* z,$$

$$\vdots$$

$$\ell_j^+(z) = \sum_{i=0}^{j} (-1)^{j-i} (A_i^* z)^{(j-1)},$$

$$\vdots$$

$$\ell_n^+(z) = \sum_{i=0}^{n} (-1)^{n-i} (A_i^* z)^{(n-i)}.$$

Let $D_m^{qn+}[U]$ denote those elements z in $\mathcal{L}_m^q[a, b]$, $1/p + 1/q = 1$, satisfying

1. There exists a parameter ϕ such that $\ell_j^+(z) + \nu_{j+1}^*[0, t]\phi$ is absolutely continuous,

2. $\ell_j^+(z)[a] = -\nu_{j+1}[a]\phi, \ell_j^+(z)[b] = \nu_{j+1}[b]\phi.$

The dual operator is then given by $L^+z = \ell_n^+ z$ for all z in $D_m^{qn+}[U]$.

Green's formula has the form

$$\int_a^b [z^*(Ly) - (L^+z)^* y] \, ds = \phi^* U(y).$$

In conclusion Brown showed that both L and L^+ are normally solvable, and that the systems discussed by Wilder [69] and Loud [46] are special cases.

9. Almost immediately thereafter Brown [9] made a major advance by applying his results for the n-th order problem to the theory of Lg splines. The spline problem is to minimize the differential expression

$$\ell y = \sum_{i=0}^{n} A_i y^{(n-i)}$$

in $\mathcal{L}_m^2[a, b]$ subject to the constraints

$$\lambda(f) = \sum_{j=1}^{n} \int_a^b d\nu_j(t) y^{(n-j)}(t) = r,$$

where r is a $\ell \times 1$ vector. Such a minimizing function, if it exists, is an Lg-spline interpolating the data r with respect to the functionals λ.

The collection of elements y over which the minimizing takes place is the Sobolev class $W_m^{2,n}[r]$, consisting of elements y satisfying

1. y is in $\mathcal{L}_m^{2n}[a, b]$,
2. $y, y' \cdots y^{(n-1)}$ are absolutely continuous,
3. ℓy is in $\mathcal{L}_m^{2n}[a, b]$.

The nonlinear operator L_r, defined by $L_r y = \ell y$ for all y in $W_m^{2n}[r]$, is an "r translate" of the operator L discussed earlier.

Brown showed an Lg spline f exists for $\ell(y)$ in $W_m^{2n}[r]$. Furthermore f in W_m^{2n} is an Lg spline if and only if $\ell(f)$ is in the null space of L^+. This implies that an Lg spline f possesses the following degrees of smoothness:

1. f is in $C^{n-1}[a, b]$.
2. f is in $C^{n-1+j}[a, b]$ if and only if $\nu_1, \cdots, \nu_j$ are continuous.
3. f is in C^{2n} in every open set Q in the compliment of $\nu_1, \cdots, \nu_n$.

The proof of these results relies heavily on the theoretical studies concerning the operators generated by multipoint-integral boundary systems and certain facts from functional analysis. Not only are old results improved, but the hypotheses required are also substantially reduced.

10. Most recently Brown has written an article [**10**] which not only extends the results mentioned earlier concerning adjoints but also ties these together with the concepts of splines and generalized Green's matrices. Following [**1**] the article first considers abstract linear relations and adjoint relations. Next these concepts are applied in $\mathcal{L}_m^p[a, b]$ to the operator L given by the expressions $Ly = \ell y$, $\ell y = \sum_{i=0}^{m} A_i y^{(n-i)}$, where $\{A_i\}$ are measurable and in $\mathcal{L}_m[a, b]$, and $\det A_0 \neq 0$ on $[a, b]$, with the domain of L restricted by the boundary condition

$$U(y) = \sum_{i=0}^{n-1} \int_a^b d\nu_{n-i}(t) y^{(i)}(t) = 0.$$

When compared to [8], however, here the measures $\{\nu_i\}$ are permitted to have an absolutely continuous part. By using a generalized Green's matrix (see the next section) an adjoint relation L^* is found. As in [8], let

$$\ell_0^+ z = A_0^* z,$$
$$\vdots$$
$$\ell_j^+(z) = \sum_{i=0}^{j} (-1)^{j-i} (A_i^* z)^{(j-i)},$$
$$\vdots$$
$$\ell_n^+(z) = \sum_{i=0}^{n} (-1)^{n-i} (A_i^* z)^{(n-i)}.$$

Now rewrite the boundary condition $U(y)$ as follows: If $U(y)$ is decomposed into a singular part $U_s(y)$ and an absolutely continuous part $U_c(y)$, and the measures generating the absolutely continuous part are sufficiently differentiable, then

$$U_c(y) = \sum_{i=0}^{n-1} \int_a^b (-1)^i (\nu_{n-i}^c)^{(i+1)} y \, dt$$
$$+ \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} (-1)^{i-j} (\nu_{n-i}^c)^{(i-j)} y^{(j)} \Big|_b^a$$

upon integration by parts. Let

$$\xi_{n-j} = \sum_{i=j+1}^{n-1} (-1)^{i-j} (\nu_{n-i}^c)^{(i-j)} [\mu(a) - \mu(b)],$$

where $\mu(a) = 1$ when $t = a$, $\mu(a) = 0$ when $t \neq a$ and $\mu(b) = 1$ when $t = b$, $\mu(b) = 0$ when $t \neq b$, and let

$$\omega_{n-j} = \nu_{n-j} + \xi_{n-j}.$$

If

$$U_s'(y) = \sum_{j=0}^{n-1} \int_a^b d\omega_{n-j} y^{(j)},$$

and

$$U_c'(y) = \int_a^b \frac{d\omega^c}{dt} y\, dt,$$

where

$$\frac{d\omega^c}{dt} = \sum_{i=0}^{n-1} (-1)^i (\nu_{n-i}^c)^{(i+1)},$$

then

$$U(y) = U_s'(y) + U_c'(y).$$

The domain of L^* is then the set of all elements z in $\mathcal{L}_m^q[a, b]$, $1/p + 1/q = 1$, satisfying

1. There exists a parameter ϕ such that $\ell_j^+(z) + \nu_j^s[0, t]\phi$ is absolutely continuous,
2. $\ell_j^*(z)[a] = -\omega_j^*[a]\phi,$
   $\ell_j(z)[b] = \omega_j^*[b]\phi,$
3. $\ell^+(z) + \left(\frac{d\omega^c}{dt}\right)^* \phi$ is in $\mathcal{L}_m^q[a, b]$.

To actually describe L^*z is a bit more complicated than before: Let $y_1, \cdots, y_k$ be a fundamental set of solutions of $\ell y = 0$, and let

$$Y = \begin{pmatrix} y_1 & & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{pmatrix}.$$

If $\bar{\nu} = (\nu_n, \cdots, \nu_1)$, let

$$\dot{U}(Y) = \int_a^b d\bar{\nu} Y,$$

and let Δ be the first m components of

$$U(Y)^{*+} \int_a^b Y(s)^* \, ds,$$

where the symbol $+$ denotes the (Moore-Penrose) generalized inverse (see the next section). Finally let

$$\tilde{\omega} = \begin{pmatrix} \omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega_n \end{pmatrix},$$

and let ϕ' be in $K_{\tilde{\omega}^*}^{[a,b]}$ (see [13]). Then

$$L^*z = \ell_n^+(z) + \frac{d\omega^c}{dt}(\phi - \Delta + \phi').$$

The expression is unique (L^* is an operator) if and only if $K_{\tilde{\omega}^*}^{[a,b]} \subset N(d\omega^{c*}/dt)$ a.e., where N denotes null space. If L^*z is not unique, then it can be interpreted as a linear relation (Arens [1]).

The final part of the article ties these results to the theory of splines.

VI. **Generalized Inverses and Green's Matrices.** While it is beyond the scope of this article to broadly discuss the results concerning the application of generalized inverses to boundary value problems (see Reid [54] for such a survey), certain of the results do concern multipoint problems, and we would be negligent if we failed to mention them.

1. An excellent beginning for our purposes is the article by Bradley [4], which discussed the following:

Consider the formal differential expression $\ell y = A_1 y' + A_0 y_i$ and define the operator T by setting $Ty = \ell y$ for all absolutely continuous n-dimensional vectors y satisfying $My(a) + Ny(b) = 0$, where M and N are $k \times n$ matrices. The adjoint to T is determined by the formula $T^*z = -(A_1{}^*z)' + A_0{}^*z$, with the domain of T^* those absolutely continuous n-dimensional vectors satisfying the boundary condition $Rz(a) + Sz(b) = 0$, where R and S are $(2n - k) \times n$ matrices satisfying

$$MA_1^{*-1}(a)R^* = NA_1{}^{*-1}(b)S^*.$$

The nullity of T is the dimension of the null space of T.

A generalized Green's matrix for T is an essentially bounded, measurable $n \times n$ matrix G, defined on $\{(x, t) : a \leqq x \leqq b,\ a \leqq t \leqq b\}$ with the property that if f is in the range of T then

$$y = \int_a^b G(x, t) f(t)\, dt$$

is in the domain of T and $Ty = f$.

If $\mathcal{B}$ denotes the k-dimensional subspace of $2n$ dimensional complex space of end values $(y(a) : y(b))$ of elements in the domain of T; if

Y, Z are fundamental matrices for $A_1y' + A_0y = 0$, $-(A_1^*z)' + A_0^*z = 0$ satisfying $Z^*A_1Y = I_n$, the identity; if

$$s(Y) = MY(a) + NY(b),$$

$$s^-(Y) = MY(a) - NY(b);$$

and r is the nullity of T, then $S(Y)$ has rank $n - r$. If B is an $n \times r$ matrix satisfying $s(Y)B = 0$, $B^*B = I_r$, and C is a $k \times (k - n + r)$ matrix satisfying $C^*s(Y) = 0$ and $C^*C = I_{k-n+r}$, then the $(k + r) \times (k + r)$ matrix

$$\begin{pmatrix} s(Y) & C \\ B^* & 0 \end{pmatrix}$$

has an inverse of the form

$$\begin{pmatrix} R & B \\ C^* & 0 \end{pmatrix},$$

where R is the E. H. Moore generalized inverse of $s(Y)$. If $G_0(x, t)$ is defined by

$$G_0(x, t) = \tfrac{1}{2}\, Y(x) \left[\frac{|x - t|}{x - t} I_n + Rs^-(Y) \right] Z^*(t),$$

when $x \neq t$; $x, t \in [a, b]$,

$$G_0(x, x) = \tfrac{1}{2}\, Y(x)Rs^-(Y)Z^*,$$

when $x \in [a, b]$,

when $\dim \mathcal{B} < 2n$, or if

$$G_0(x, t) = \tfrac{1}{2} \frac{|x - t|}{x - t} Y^*(x)Z^*(t),$$

when $x \neq t$; $x, t \in [a, b]$,

$$G_0(x, x) = 0$$

when $x \in [a, b]$,

when $\dim \mathcal{B} = 2n$, then $G_0(x, t)$ is a generalized Green's matrix for T. Bradley showed further that a generalized Green's matrix is not unique. If the r columns of U form a basis for the null space of T, the p columns of the matrix V form a basis for the null space of T^*, and G_1 is a generalized Green's matrix, then the matrix G is also a generalized Green's matrix for T if and only if there exist $r \times n$ and $n \times p$ matrix functions Γ and Λ such that

$$G(x, t) = G_1(x, t) + U(x)\Gamma(t) + \Lambda(x)V^*(t).$$

Bradley also determined a unique principal generalized Green's matrix: If θ and Ω are integrable $n \times r$ and $n \times p$ matrices satisfying

$$\int_a^b \theta^* U dx \quad \text{and} \quad \int_a^b V^* \Omega dt$$

are nonsingular ($= I_r$ and $= I_p$), then

$$\Lambda(x) = -\int_a^b G_0(x, s)\Omega(s)\, ds,$$

$$\Gamma(x) = \int_a^b \int_a^b \theta(s)G_0(s, t)\Omega(t)V^*(x)\, dsdt - \int_a^b \theta^*(s)G_0(s, x)\, ds$$

determine a unique generalized Green's matrix $G_{\theta\Omega}$ satisfying

$$\int_a^b G_{\theta\Omega}(x, t)\Omega(t)\, dt = 0, \int_a^b \theta^*(x)G_{\theta\Omega}(x, t)\, dx = 0.$$

$G_{\theta\Omega}$ uniquely satisfies the following five conditions.

1. $G_{\theta\Omega}$ is continuous except on $x = t$. $G_{\theta\Omega}$ is differentiable in x on $[a, t) \cup (t, b]$.
2. If $t \in (a, b)$, then

$$\lim_{x \uparrow t} G_{\theta\Omega}(x, t), \lim_{x \downarrow t} G_{\theta\Omega}(x, t)$$

exist, and

$$G_{\theta\Omega}(t^+, t) - G_{\theta\Omega}(t-, t) = A_1^{-1}(t).$$

3. If $t \in [a, b]$, then

$$A_1 \frac{\partial}{Dx} G_{\theta\Omega}(x, t) + A_0 G_{\theta\Omega}(x, t) = -\Omega(x)V^*(t).$$

4. If $t \in (a, b)$, then

$M_{\theta\Omega}(a, t) + NG_{\theta\Omega}(b, t) = 0,$

and

$$\int_a^b \theta^*(x)G_{\theta\Omega}(x, t)\, dx = 0.$$

The article concludes by showing that the unique principal generalized Green's matrix for T^*, $H_{\Omega\theta}$, satisfies

$$H_{\Omega\theta}(x, t) = G^*_{\theta\Omega}(t, x).$$

2. Conti [21] considered the more abstract problem of solving

$$Dx = x' - A(t)x = y$$

subject to the constraint

$$Lx = \mathfrak{l},$$

where L is a linear operator mapping continuous n-dimensional vectors into C^n. If Y is a fundamental matrix of $D_x = 0$, $U(t, s)$ is the evolution operator $U(t, s) = Y(t)Y(s)^{-1}$, then every solution of $Dx = 0$ has the form $x = U(t, \tau)\xi$ for some $\xi \in C^n$, and every solution of $Dx = y$ is of the form

$$x = U(t, \tau)\xi + D_\tau{}^+y,$$

where

$$D_\tau{}^+y = \int_\tau^t U(t, s)y(s)\, ds.$$

If L_u denotes $L(U)$ with s fixed at τ, and $L_u{}^g$ is a generalized inverse of L_u, i.e., it satisfies

$$L_u L_u{}^g L_u = L_u,$$

then a necessary and sufficient condition that the problem $Dx = y$, $Lx = \mathfrak{l}$ have solutions is that

$$(I - L^u L_u{}^g]\,[\mathfrak{l} - LD_\tau{}^+y] = 0$$

for any generalized inverse $L_u{}^g$. Solutions are given by

$$UL_u{}^g\mathfrak{l} + U\xi_0 + [D_\tau{}^+ - UL_u{}^g LD_\tau{}^+]\,y,$$

ξ_0 in the null space of L_u.

If L_u has a right inverse $L_u{}^+$, then the expression above with $L_u{}^+$ replacing $L_u{}^g$ holds as a solution for all locally integrable y.

If L_u has an inverse $L_u{}^{-1}$ then the expression

$$UL_u{}^{-1}\mathfrak{l} + [D_\tau{}^+ - UL_u{}^{-1}D_\tau{}^+]\,y$$

is the unique solution to $Dx = y$, $Lx = \mathfrak{l}$.

These results were then applied to end point boundary conditions, duplicating in part the results of Bradley [4], to k-point boundary conditions, to conditions involving a countable set of points, and finally to conditions also involving integrals, either Stieltjes or Lebesgue.

3. Halanay and Moro [25] continued Conti's work in more depth, discussing the problem

$$Dx = x' - A(t)x = f$$

subject to the constraint

$$Lx = Mx(a) + Nx(b) + \int_a^b dF(t)x(t) = \ell,$$

where M, N are $m \times n$ matrix constants and F is an $m \times n$ matrix valued function of bounded variation continuous at a and b. Hence, letting $\tau = a$,

$$L_u = M + NU(b, a) + \int_a^b dF(t)U(t, a).$$

First defining an adjoint problem by

$$z' + zA(s) = -vF', z(a) = -vM, z(b) = vN,$$

v a parameter, and noting that solutions have the form

$$z(s) = -vMU(a, s) - v \int_a^s dF(t)U(t, s)$$

and satisfy $vL_u = 0$, they then showed that the problem $Dx = f$, $Lx = \ell$ has solutions if and only if

$$[I - L_u L_u^g]\ell = \int_a^b z(s)f(s)\, ds$$

where

$$z(s) = -[I - L_u L_u^g]\left[MU(a, s) + \int_a^b dF(t)U(t, s)\right]$$

is a solution matrix for the adjoint problem.

When solutions exist, Halanay and Moro also showed that they have the form

$$x(t) = U(t, a)(\xi_0 = L_u^g \ell) + \int_a^b G(t, s)f(s)\, ds,$$

where

$$G(t, s) = U(t, s) - U(t, a)L_u^g \left[(L_u - M)U(a, s) + \int_a^s dF(\sigma)U(\sigma, s)\right],$$

$$a \leqq s < t,$$

$$= -U(t,a)L_u^g \left[(L_u - M)U(a,s) + \int_a^s dF(\sigma)U(\sigma,s) \right],$$

$$t < s \leqq b,$$

and that $G(t, s)$ satisfies

1. $\dfrac{dG}{dt} - A(t)G = 0,$

2. $\dfrac{dG}{ds} + GA(s) = U(s,a)L_u^g F',$

3. $G(t,a) = U(t,a)[I - L_u^g(L_u - M)],\ G(t,b) = U(t,a)L_u^g N,$

4. $G(t,t+) - G(t,t-) = U(t,a)L_u^g F(t+) - F(t-) - I.$

4. Tucker [**61**] has substantially overlapped the results of Halanay and Moro. Due to notational difficulties, however, their articles are difficult to compare. At the risk of some duplication we present Tucker's main ideas also. Tucker considered the problem

$$Y' - AY = R,\ U(Y) = \int_a^b dF(x)Y(x) = K.$$

He defined the (Moore-Penrose) generalized inverse of a matrix X as a matrix X^+ satisfying

i. $XX^+X = X,$

ii. $X^+XX^+ = X^+,$

iii. $(XX^+)^* = XX^+,$

iv. $(X^+X)^* = X^+X,$

and adopted the following notation: If ϕ is a fundamental matrix for $Y' - AY = 0$, let

$$D = \int_a^b dF(x)\phi(x),$$

$$\gamma(x,t) = \tfrac{1}{2}\phi(x)\phi(t)^{-1},\ t < x,$$

$$= -\tfrac{1}{2}\phi(x)\phi(t)^{-1},\ x < t,$$

$$U_x[\gamma] = \int_a^b dF(x)\gamma(x,t).$$

Tucker then showed that the complete solution to $Y' - AY = R$, $U(Y) = K$ is given by

$$Y(x) = \int_a^b [\gamma(x, t) - \phi(x)D^+U_x[\gamma]]\, R(t)\, dt + \phi(x)[D^+K + N - D^+DN],$$

where N is an arbitrary matrix, provided $K - \int_a^b U_x[\gamma]\, R(t)\, dt$ is of the form DM for some matrix M.

He defined the same adjoint system as did Halanay and Moro, and also derived Green's formula

$$(LY, Z) - (Y, L^*Z) = -MU(Y),$$

which has already been discussed.

5. More recently the discussion of generalized Green's matrices has been considerably tightened by Brown [7], who not only put the results known previously in a more logical order, but also extended the older idea of a principal generalized Green's matrix to the more general boundary value problems. (See also Chitwood [16].)

The system studied is generated by

$$Ly = y' + Py,$$

with the constraint

$$U(y) = \int_0^1 d\nu(t) y(t) = 0,$$

where ν is an $m \times n$ matrix valued function of bounded variation; the setting is $\mathcal{L}_n^p[0, 1]$, $1 \leqq p < \infty$. The adjoint system in $\mathcal{L}_n^q[0, 1]$ is represented by the system

$$L^+Z = -z' + P^*z + \left(\frac{d\nu_c}{dt}\right)^* \phi$$

subject to the conditions

1. $z + \nu_s^*[0, t]\phi$ is absolutely continuous,
2. L^+z exists a.e. and is in $\mathcal{L}_n^q[0, 1]$,
3. $z(0+) = -\nu_s^*[0]\phi, z(1-) = \nu_s^*[1]\phi$.

A quasi-inverse of an operator $T : X \rightarrow Y$ is an operator $\tilde{T} : Y \rightarrow X$ such that for any y in $R(T)$, the range of T, $\tilde{T}y$ is defined and is in $D(T)$, the domain of T, and $T\tilde{T}y = y$.

It is easy to show that

1. $\tilde{T}T\tilde{T} = \tilde{T}$, $T\tilde{T}T = T$,
2. $R:\tilde{T}) \subset D(T)$ and $(I - \tilde{T}T)$ maps $D(T)$ into $N(T)$, the null space of T,
3. $\tilde{T}$ is $1 - 1$ on $R(T)$.

A standard generalized Green's matrix (for L) is an $n \times n$ matrix valued function $\tilde{G}(t, s)$, defined on $\{(t, s) : 0 \leqq t, s \leqq 1\}$ satisfying

1. $\tilde{G}(t, s) - Y(t)K(s)$ is continuous in the regions $t \geqq s$ and $s \geqq t$ except where $\nu[s] \neq 0$,
2. $\tilde{G}(t, s)$ is differentiable in t except when $t = s$, and is $\mathcal{L}^q$-summable in s,
3. $\frac{\partial}{\partial t}\tilde{G}(t, s) + P(t)\tilde{G}(t, s) = 0$ in $[0, s)$ and $(s, 1]$,
4. $\tilde{G}(t+, t) - \tilde{G}(t-, t) = I$ a.e.,
5. $\int_0^1 d\nu(t)\tilde{G}(t, s) = CN(s)Y(s)^{-1} + U(Y)K(s)$,

where Y is a fundamental matrix for $y' + Py = 0$, C is an $m \times m$ matrix with rows in $N(U(Y)^*)$,

$$N(s) = \int_s^1 d\nu(t)Y(t),$$

and $K(s)$ is an $n \times n$ matrix $\mathcal{L}^q$-summable kernel of a functional with range in $N(U(Y))$.

Brown showed that if $\tilde{G}(t, s)$ is a standard generalized Green's matrix, then

$$I(f) = \int_0^1 \tilde{G}(t, s)f(s)\, ds,$$

defined on $\mathcal{L}_n^p[0, 1]$, is a quasi-inverse for L. Further every bounded quasi-inverse of L has this representation.

In addition if $U(Y)^+$ denotes the Moore-Penrose generalized inverse for $U(Y)$, then

$$G_1(t, s) = Y(t)[\lambda_{[0,t]}(s)I - U(Y)^+N(s)]\,Y(s)^{-1},$$

($\lambda_{[0,t]}(s)$ is the characteristic function of $[0, t]$) is a standard Green's matrix for L. Further every standard generalized Green's matrix of L

has the representation

$$Y(t)\,[\lambda_{[0,t]}(s)I - U(Y)^{+}N(s) + K(s)Y(s)]\,Y(s)^{-1},$$

where $K(s)$ is an $n \times n$ matrix $\mathcal{L}^q$-summable kernel of a functional with range in $N(U(Y))$.

When $K(s)$ is of bounded variation, a standard generalized Green's matrix $G(t, s)$ satisfies

1. $\tilde{G}(s, s+) - \tilde{G}(s, s-) = -(I + Y(s)U(Y)^{+}\nu[s])$,

2. $$-\frac{\partial}{\partial s}G^*(T, s) + P^*(s)\tilde{G}^*(t, s) + \left[\frac{d\nu_c^*}{dt}U(Y)^{*+} + Y(s)^{*-1}Y^{*\prime}(s)K^*(s) + K^{*\prime}(s)\right]Y^*(t) = 0.$$

Now suppose that $\tilde{G}'(t, s)$ and $\tilde{G}(t, s)$ are equivalent in the sense that the operators they generate agree on $R(L)$, and let

$$\tilde{G}'(t, s) - \tilde{G}(t, s) = W(t, s),$$

the kernel of an operator annihilating $R(L)$. Then

$$\tilde{G}'(t, s) = G_1(t, s) + Y(t)K(s) + W(t, s).$$

In particular, if $W(t, s) = Q(t)V^*(s)$ and $K(s) = Y(t)^{-1}U(t)F(s)$, where $Q(t)$ and $F(s)$ are $n \times (m - r)$ and $(n - r) \times n$ essentially bounded matrices, $U(t)$ and $V(s)$ are $n \times (n - r)$ and $n \times (m - r)$ matrices whose columns span $N(L)$ and $N(L)^*$, then the resulting generalized Green's matrices include those studied by Bradley in the end point case.

Brown went on to show that if $\tilde{G}^*(t, s)$ is defined to be an adjoint generalized Green's matrix, then it induces a quasi-inverse for L^*.

Brown concluded by extending the notion of a principal generalized Green's matrix:

If $T : X \to Y$ is an operator with closed range and closed null space and P, Q are projection operators onto $R(T)$, $N(T)$, respectively, and $\tilde{T}$ is a bounded quasi-inverse for T, then by a generalized inverse $\hat{T}_{PQ}$ of T is meant the operator $(I - Q)\tilde{T}P$. If X and Y are Hilbert spaces and P, Q are orthogonal projections, $\hat{T}_{PQ}$ is denoted by $\hat{T}$ and satisfies

1. $T^* = (\hat{T}^*)$,
2. $(T\hat{T})^* = T\hat{T}$, the projection onto $R(T)$,
3. $(\hat{T}T)^* = \hat{T}T$, the projection onto $R(T^*)$,
4. $\hat{T} = (\hat{T}^*T)T^*$,
5. $\hat{T} = T^*(\hat{T}T^*)$,

6. If y is in $R(T)$, $\hat{T}y$ is the solution to $Tx = y$ with minimum norm,
7. $T\hat{T}y$ is the unique best approximation in $R(T)$ to y,
8. x in $D(T)$ minimizes $\|y - Tx\|$ if and only if $T^*Tx = T^*y$.

Now let U be an $n \times (n - r)$ matrix whose columns span $N(L)$, and let V be an $n \times (m - r)$ matrix whose columns span $N(L^*)$. Let W, Z be $n \times (n - r)$, $n \times (m - r)$ integrable matrices such that the integrals

$$\Delta = \int_0^1 W^*(t)U(t)\,dt \quad \text{and} \quad \nabla = \int_0^1 V^*(t)Z(t)\,dt$$

are nonsingular. By a principal generalized Green's matrix $G_{WZ}(t, s)$ with respect to W and Z is meant a generalized Green's matrix satisfying

$$\int_0^1 G_{WZ}(t, s)Z(s)\,ds = 0 \text{ a.e.},$$

$$\int_0^1 W^*(t)G_{WZ}(t, s)\,dt = 0 \text{ a.e.}$$

Then the generalized inverse $\hat{L}_{PQ}$ is an integral operator with a unique kernel

$$M(t, s) = H(t, s) - U(t)\Delta^{-1} \int_0^1 W^*(\tau)H(\tau, s)\,d\tau,$$

where

$$H(t, s) = \hat{G}(t, s) - \int_0^1 \hat{G}(t,\sigma)Z(\sigma)\,d\sigma\nabla^{-1}V^*(s),$$

and $\hat{G}(t, s)$ is an arbitrary standard generalized Green's matrix. Further, a generalized Green's matrix is a principal generalized Green's matrix with respect to W, Z if and only if it is the kernel of $\hat{L}_{PQ}$.

In addition to the properties listed previously, $G_{WZ}(t, s)$ also satisfies

1. $G_{WZ}(t, s)$ is continuous in the regions $t \geqq s$ and $s \geqq t$ except where $\nu[s] \neq 0$,
2. $G_{WZ}(t, s)$ is differentiable in t except when $t = s$,
3. $G_{WZ}(t+, t) - G_{WZ}(t-, t) = I$ a.e.,
4. $\dfrac{\partial}{\partial t} G_{WZ}(t, s) + P(t)G_{WZ}(t, s) = -Z(t)\nabla^{-1}V^*(s)$ in $[0, s)$ and $(s, 1]$,

5. $\int_0^1 d\nu(t)G_{WZ}(t, s) = 0$ a.e.

Finally when G_{WZ} exists, there exists a principal generalized Green's matrix for L^*, $H_{ZW}(t, s)$, given by

$$H_{ZW}(t, s) = G^*_{WZ}(s, t),$$

satisfying

$$\int_0^1 Z^* H_{ZW}(t, s)\, dt = 0 \text{ a.e.},$$

$$\int_0^1 H_{ZW}(t, s) W(s)\, ds = 0 \text{ a.e.}$$

6. Finally Brown [10] has extended some of these ideas to n-th order systems.

VII. **Problems Yet Unsolved.** Trying to predict the future is a dubious business at best. Who would have thought, prior to 1960, that the area would explode so? Nonetheless, certain problems do suggest themselves in a natural way. Let us list these quickly.

1. Generalized Green's matrices for differential-boundary operators have not been discussed in any of the situations they have been encountered. Surely this obvious omission can be quickly filled in.

2. Little work seems to have been done with Stieltjes integral operators and their associated boundary value problems. For example, the system

$$L\xi(t) = \xi(t) - \xi(a) - \int_a^t da(\tau)\xi(\tau),$$
$$\mathcal{M}\xi(a) + \mathcal{N}\xi(b) = 0,$$

discussed by Vejvoda and Tvrdý [63], has not been seriously examined. Problems concerning dual systems, Green's matrices, generalized Green's matrices, eigenfunction expansions, self-adjointness are all open.

3. Stieltjes differential-boundary operators which are self-adjoint under a transformation have not been examined. Although if the operator or its adjoint is purely differential, self-adjointness is unlikely; if both are differential-boundary, the probability of the existence of such transformations seems much greater.

4. The nature of general higher order differential-boundary operators has not been explored, except in one instance [30]. Here again questions concerning dual systems, Green's matrices, generalized Green's matrices, eigenfunction expansions, self-adjointness should be easily answered.

5. Additional application of the results of all these boundary value systems to the theory of splines should yield bountiful improvements and new results. Brown has indicated the way. Much more must surely follow.

6. The extension of differential-boundary operators in any form to systems involving more than one independent variable (partial differential-boundary operators) has been largely neglected. Only Feller [22] and Phillips [52] have considered them at all. Their true nature, dual systems, etc., all need to be worked out. The idea of generalized Green's matrices seems especially interesting. Furthermore, the application of such results to develop the theory of spline surfaces (see Varga [62]) seems to have tremendous possibilities. At this point in time, no one can even guess what will occur.

7. The application of Aren's work [1] on linear relations to systems with non-dense domains, similarly results concerning operators on quotient spaces, has only been touched by Brown [10]. Certainly, the whole theory of differential and differential-boundary systems when examined from this point of view will take on a radically different look. One can only surmise what that would be.

In conclusion let us apologize to those whose articles we have missed or misread. We are sure there are some. We hope that this brief survey proves useful. It is our firm conviction that the field, so long merely an off-shoot, is now coming into its own, and in fact, is really in its infancy with a very bright future before it.

VII. **Added in Proof.** Since this article was written several articles, which may be of interest to the reader, have either appeared or have been called to the author's attention.

First Brown [A], [C] has continued the work on the applications to splines. Both he and Krall [B], [D], [P] have further extended the results under Stieltjes boundary conditions. In addition Bruhns [E], [F], [G] has considered an n-th order problem and has derived an eigenfunction expansion for functions in the domain of the operator in question.

Coddington [H], [I], [J], [K], [L] with Dijksma [M] and de Snoo

[N], [O] have rapidly expanded their results on self-adjoint extensions of symmetric problems in a Hilbert space.

It is clear that the field is moving rapidly.

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