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ABSTRACT

This paper proposes a semantic analysis in which meanings of word problems are structures that include class and order relations, and suggests a hypothesis of developmental levels that can account for children's performance of these problems at various ages. The different kinds of problems vary in the complexity of semantic structures and the operations required to derive the meaning structures from the problem texts. A representational process in children's understanding of problems corresponding to the derivations in the semantic analysis is postulated which explains the relative difficulty of different kinds of word problems. The meaning structures can also be viewed as semantic interpretations of formal arithmetic sentences; this provides an analysis of children's achievements of more sophisticated understanding of arithmetic concepts and relationships. The first section reviews and presents empirical data for different categories of addition and subtraction word problems. The second section proposes developmental levels of word problem-solving ability that relate to growth in empirical, mathematical, and logical knowledge structures. The third section demonstrates how these developmental levels account for the accumulated data on children's performances of the arithmetic word problems presented earlier. (JN)

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THE DEVELOPMENT OF SEMANTIC CATEGORIES FOR ADDITION AND SUBTRACTION

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THE DEVELOPMENT OF SEMANTIC CATEGORIES FOR ADDITION AND SUBTRACTION

ABSTRACT. Research conducted in several countries has shown consistent patterns of performance on 'change', 'combine' and 'compare' word problems involving addition and subtraction. This paper interprets these findings within a theoretical framework that emphasizes the development of empirical, logical and mathematical knowledge.

Since Russell's (1971) famous axiomatic analysis, it has been accepted that the concept of numbers has two fundamental components: class and ordinal relations. Piaget (1965) analyzed the development of these concepts [illegible] children's performances of a variety of tasks requiring analysis of set relationships and ordered sequences.

In this paper, we propose a semantic analysis in which meanings of word problems are structures that include class and order relations, and we suggest a hypothesis of developmental levels that can account for children's performances of these problems at various ages. The different kinds of problems vary in the complexity of semantic structures and the operations required to derive the meaning structures from the problem texts. We postulate representational processes in children's understanding of problems corresponding to the derivations in our semantic analysis, and thereby explain the relative difficulty of different kinds of word problems. The meaning structures can also be viewed as semantic interpretations of formal arithmetic sentences. This provides an analysis of children's achievements of more sophisticated understanding of arithmetic concepts and relationships.

In the first section we review and present empirical data for different categories of addition and subtraction word problems. The second section proposes developmental levels of word problem-solving ability that relate to growth in empirical, mathematical and logical knowledge structures. The third section demonstrates how these developmental levels account for the accumulated data on children's performances of the arithmetic word problems presented earlier.

1. SEMANTIC CATEGORIES OF ADDITION AND SUBTRACTION WORD PROBLEMS

Previous analysis of addition and subtraction work problems have identified three main categroies of semantic structures: Change (join and separate),

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TABLE I

The three general semantic categories of addition and subtraction word problems

Current Name of the Category	Characteristics	Example	Previous Research and Titles for the Same Category
1. Combine	Involves Static relationship between sets. Asking about the Union set or about one of two disjoint sub-sets	There are 3 boys and 4 girls. How many children there are altogether?	COMBINE: Greeno (1980a, b), Heller & Greeno (1978); Riley (1979); Riley *et al.* (1981). PART-PART-W[illegible]rpenter and Moser (1981), Carpenter *et* [illegible] STATIC: Nesher [illegible] 1981) COMPOSITION OF TWO MEASURES: Vergnaud & Durand (1976), Vergnaud (1981).
2. Change	describes increase or decrease in some initial state to produce a final state.	John has 6 marbles. He lost 2 of them. How many marbles does John have now?	CHANGE: Greeno (1980a, b) JOINING AND SEPARATING: Carpenter & Moser (1981), Carpenter *et al.* (1981). DYNAMIC: Nesher & Katriel (1978), Nesher (1981). TRANSFORMATION LINKING TWO MEASURES: Vergnaud & Durand (1976), Vergnaud (1981).
3. Compare	Involves static comparison between two sets. Asking about the difference set or about one of the sets where the difference set is given	Tom has 6 marbles. Joe has 4 marbles. How many marbles does Tom have more than Joe?	COMPARE: Greeno (1980a, b), Carpenter & Moser (1981), Carpenter *et al.* (1981), Nesher & Katriel (1978), Nesher (1981). A STATIC RELATIONSHIP LINKING TWO MEASURE: Vergnaud & Durand (1976), Vergnaud (1981).

Combine, and Compare. A general description of each of these categories is given in Table I, along with examples and previous titles used in the past for the same categories.

When empirical data were collected according to the above three semantic categories (see Table II), it seemed that these categories had an explanatory and predictive power. For example, in addition word problems Change problems are the easiest, Combine problems are sometimes more difficult, and the Compare problems are the most difficult ones.

TABLE II

Performance on addition and subtraction word problems according to the general semantic categories (numbers are percentage of success)

	Riley *et al.* (1981)	Nesher & Katriel (1978)	
	+ and −	+	and −
'Change'	89	86	76
'Combine'	83	78	52
'Compare'	62	58	60

Further examination of the data, however, sheds some doubts on these general findings. Within each of the above categories, different problems can be formed by varying which item is the unknown. In Change problems, for example, the three items of information are the initial, change, and final sets. Any of these can be found if the other two are given, yielding three different cases. The unknown may be the initial, change or final set. Furthermore, the direction of change can either be a decrease or an increase, so there are a total of six kinds of Change problems. A similar set of variations exists for Compare problems. In Combine problems there are fewer variations; the unknown is either the union set or one of the subsets. Thus, if the position of the unknown is taken into account, we have 14 different categories of word problems, instead of three, as shown in Table III.

The main source of empirical evidence that the identity of the unknown set also influences children's performances comes from studies showing that problems within the same semantic category also vary in difficulty. Table IV shows empirical data from studies by several researchers. We do not attribute significance to the exact and absolute proportion of success on different tasks, since each one of the researchers used a different sample of word problems, and examined them in a different setting. We do, however, attribute to them a scale of relative difficulty.

Referring to Table IV, one can note that children had no difficulty solving Change 1 and Change 2 problems when the initial and change sets were given,

TABLE III

14 types of addition and subtraction word problems (semantic categories and position of the unknown)

Title	General Description
Combine 1	Question about the union set (whole).
Combine 2	Question about one subset (part).
Change 1	Increasing, question about the final set.
Change 2	Decreasing, question about the final set.
Change 3	Increasing, question about the change.
Change 4	Decreasing, question about the change.
Change 5	Increasing, question about the initial set.
Change 6	Decreasing, question about the initial set.
Compare 1	Mentioning 'more', question about the difference set.
Compare 2	Mentioning 'less', question about the difference set.
Compare 3	Mentioning 'more', question about the 'compared'.
Compare 4	Mentioning 'less', question about the 'compared'.
Compare 5	Mentioning 'more', question about the referent.
Compare 6	Mentioning 'less', question about the referent.

and they were asked to determine the final set. However, Change problems 5 and 6, in which the final and change sets were given and the initial set was unknown, were difficult at all grade levels.

As with Change problems, the difficulty of Combine and Compare problems also varied depending on which value in the problem was unknown. Combine 2 problems, for example, were significantly more difficult than Combine 1 problems. Compare problems in which the referent was unknown were more difficult than any of the other Compare problems.

These findings called for another hypothesis that would explain children's performances on a broader basis than semantic analysis of the verbal texts. In the following sections we will relate the differences in problem solving performance to a specific hypothesis concerning (1) differences in the schemes required to solve the various problems, and (2) differences in the availability of certain schemes to children of different ages. The hypotheses that we present here extend an analysis developed by Riley *et al.* ([illegible]) which postulates specific semantic processes corresponding to different levels of performance on each of the problem types. The schemes that we postulate here include operations for deriving class and order relations that cross the boundaries of the semantic categories as described in Table I.

TABLE IV

Percentage of success of 15 types of word problems in various empirical studies

	Carpenter *et al.* (1981) 1st	Fischer (1979) 2nd	Nesher & Teubel (1975) 5th	Nesher & Katriel (1978) 2nd–6th	Nesher (1981) 2nd–6th	Steffe & Johnson (1971) 1st	Riley *et al.* (1981) 1st	Riley *et al.* (1981) 2nd	Tamburino (1980) K	Vergnaud (1976) 2nd–6th
Combine 1	86	–	–	79	79	67	100	100	83	–
Combine 2	46	14	–	46	52	35	39	70	18	–
Change 1	79	–	–	87	82	67	100	100	89	–
Change 2	72	30	–	70	75	43	100	100	91	85
Change 3	51	–	–	62	72	41	56	100	8	–
Change 4	–	–	–	75	77	41	78	100	64	70
Change 5	–	5.5	–	–	48	67	28	80	32	–
Change 6	–	–	–	–	49	35	39	70	7	69
Compare 1	67	–	–	–	76	–	28	85	–	–
Compare 2	–	–	–		66	–	22	75	–	–
Compare 3	23	–	87	–	65	–	17	80	–	–
Compare 4	–	–	81	–	66	–	28	90	–	–
Compare 5	–	–	43	–	60	–	11	65	–	–
Compare 6	–	–	64	–	54	–	6	35	–	–

TABLE V

Aspects of development

The level	Empirical knowledge	Logical operations	Mathematical operations
1 Counting sets	Reference to sets, adding and removing members to sets. Understanding 'put' 'give', 'take', etc. as denoting change in location or possession	$x \in P$ $x \in R$ $n(P)$ $n(R)$ $H(a, b)$ ('Have')	Ability to count and find the cardinal number of a set. The order among numbers. $2 < 5 < 8$
2 Change	Ability to link events by cause and effect. Reference to the amount of change. Understand sequence of events ordered in time in a non-reversible manner	$G(J, o)$ becomes $C(H^-(J, o))$ $R(J, o)$ becomes $C(H^+(J, o))$ when: 'G': 'give' 'R': 'receive' 'C': 'cause'	Understanding addition and subtraction as procedures. '+' and '−' are distinct $a + b \to c$ $a - b \to c$

3 Part-Part-Whole	A reversible part-part-whole schema is available, and can be used to find unknown part in any slot in a sequence of events. Understanding class-inclusion.	Understanding the additive relation among three sets (P, Q, R). $P \cup R = Q$ $P \subset Q$ $R \subset Q$ $P \cap R = 0$ $x \in (P \cap Q)$ $n(P) + n(R) = n(P \cup R) = n(Q)$ $n(Q) - n(P) = n(R)$	Understanding the relation among three numbers in an equation (=). Connection between addition and subtraction: if $a + b = c$, then, $c - b = a$ and $c - a = b$
4 Directional relations	Reversibility of non-symmetrical relations. Ability to handle directional descriptions (more/less), and quantify a relational set (relative comparison).	$R(a, b) = R^{-1}(b, a)$ if $m < n$, then, $n > m$ Coordination of $n(a)$, $n(b)$, $R(a, b)$ and $n(a - b)$.	The ability to handle inequality and its relationship to equality, equalizing it by addition or substraction: if $a > b$, then $a - c = b$, and $b + c = a$

2. VARIOUS COMPONENTS (ASPECTS) IN GROWTH OF KNOWLEDGE

Our hypothesis suggests four developmental levels characterized by several components representing different aspects of knowledge. Each component has its own growth and is incorporated in the schemes underlying the child's strategies for solving simple addition and subtraction problems. The four levels will be first mentioned here, and later on, further articulated and connected to children's performance in word problems. The four levels are (see Table V):

Level 1: which includes the ability to identify sets by a variety of verbal descriptions (generic names, locations, points of time, possessions, etc.); perform simple operations such as adding or removing objects from sets; and understanding verbs denoting change in possession such as 'give' or 'take'. The arithmetic competence consists of the ability to count and find the cardinal number of a given set. The underlying schemes of this level are given schematically in Figure 1.

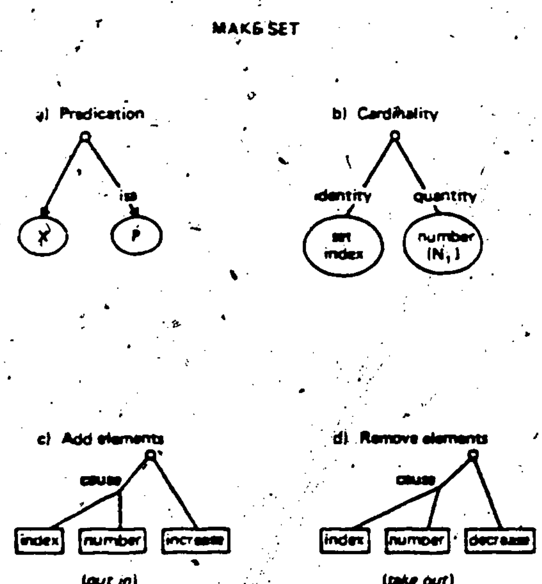

Fig. 1.

Level 2: includes the ability to link events by cause and effect and to anticipate results of actions described in ordinary language. It includes reference to the amount of change needed to transform a set into a larger or smaller set and the understanding of sequences of events ordered in time in a unidirectional and non-reversible manner. In arithmetic the + and − operations are distinct, not related, and the = sign (equality sign) is understood as a sign to perform a procedure. The Change scheme underlying this level is described in Figure 2. (Figure 2 described a 'change' which is 'decrease', but a corresponding structure exists for an 'increase' associated with addition.) Note that the separate schemes of level 1 are now incorporated into a temporarily-integrated scheme.

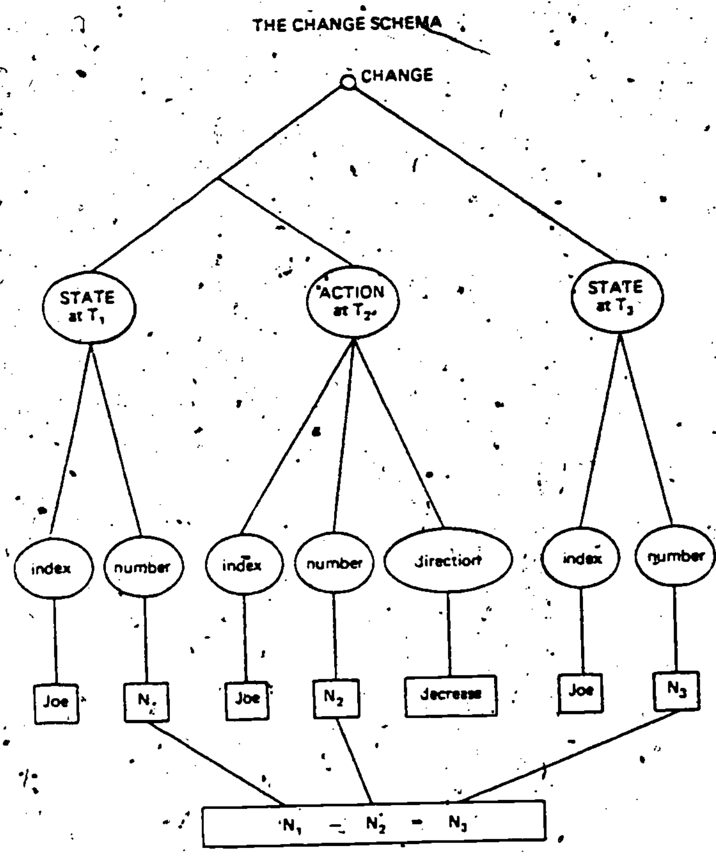

Fig. 2.

Level 3: includes an integrated Part-Part-Whole scheme that can be used to represent set relations with a slot for an unknown quantity if intentionally defined. A set can also be induced by means of relative comparison. The schemes at this level are related to the understanding of class inclusion and to the ability to quantify the same extension of objects even if there is a shift in the predication. In arithmetic at this level, the additive structure is reversible

and includes the = sign as denoting an equivalent relation. The underlying scheme for this level is schematically described in Figure 3. Note that the Part-Part-Whole scheme of level 3 is reversible (bi-directional arrows), and also incorporates the arithmetic additive relationship which now includes the operations + and − as related to inverse operations operating on the same structure.

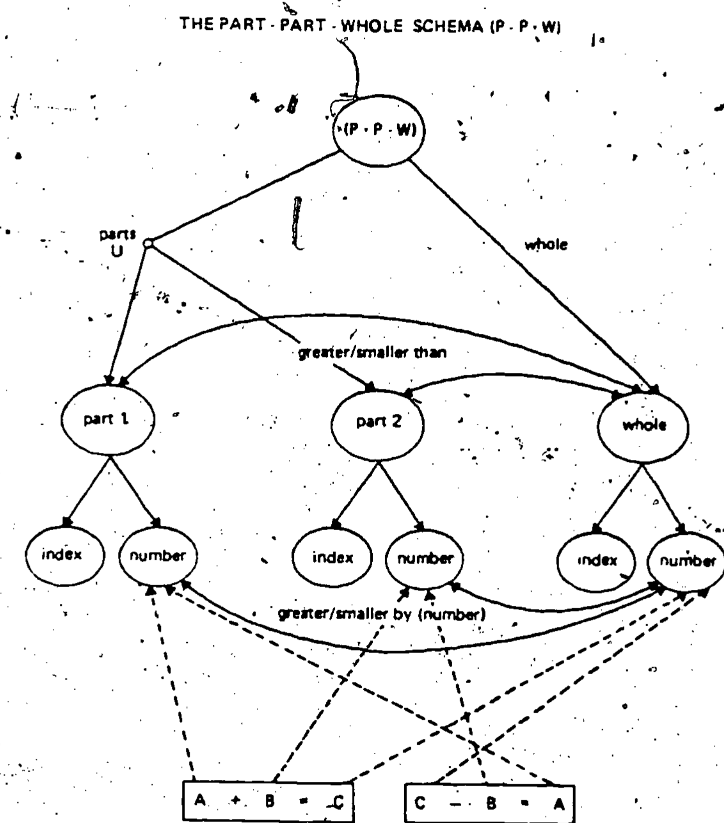

Fig. 3.

Level 4: includes the reversible scheme for non-symmetrical relations (that has already initiated at level 2). Directional (ordered) descriptions (i.e., 'more', 'less') can be handled in a flexible fashion. The arithmetic at this level includes the ability to handle inequality, and the ability to equalize inequality by addition or subtraction. See Figure 4 for the Directional scheme that can now handle ordered relations in a reversible manner and still maintain their directionality. We will refer to it in more detail in the next section.

COMPARISON SCHEMA

R(a, b) : a is more than b
R^{-1} (b, a) : b is less than a

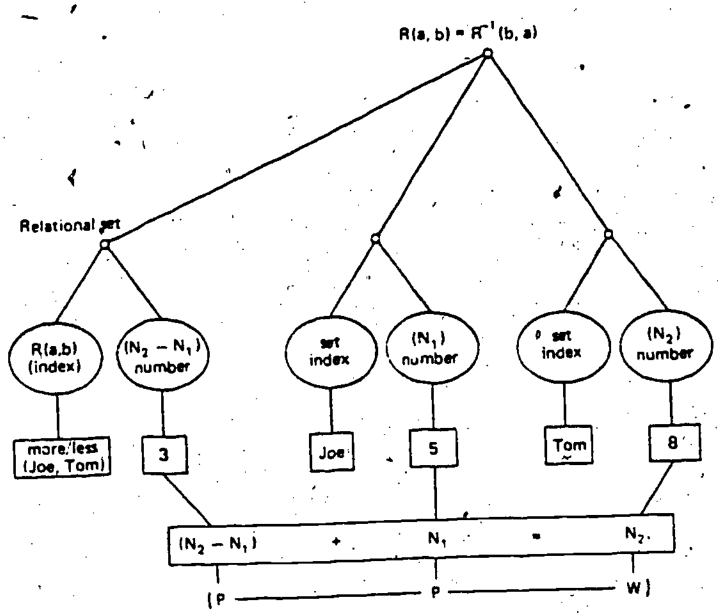

Fig. 4.

In the description of the above developmental levels we assume that there are at least two distinct structures of knowledge which are involved (Nesher, 1972)

(a) A child's knowledge of the world, and
(b) A child's knowledge of logico-mathmatical structures.

In both knowledge structures, a further distinction should be made between a child's knowledge of objects, events, relations etc., and his knowledge of how to express them in language. The objects of the first domain are real world objects and they are described by ordinary (natural) language (Lo); the objects in the second domain are numbers, operations on numbers, and relations among numbers and they are described by means of a special symbolic arithmetic language (La). If we regard *semantics* as the interpretation of an expression in the language that identifies the relevant reference, in the realm of objects and relations we are then faced with two different semantic structures.

The sources of these two knowledge structures, as was noted by Piaget, are not the same. The logico-mathematical growth of the child cannot, of course, be understood as divorced from his experience with physical objects. Yet the mechanism for that growth is different, as indicated by Piaget's reference to 'simple abstraction' and 'reflective abstraction' (Inhelder and Piaget, 1964; Piaget, 1967; 1970).

We go one step beyond Piaget in our attempt to be more specific about the growth in arithmetic knowledge. Arithmetic knowledge takes the form of learning operations and relations on numbers and, in turn, reflects on the comprehension of a physical situation as a closed system which, therefore, is susceptible to reversibility. We think that arithmetic knowledge is one of the mechanisms that facilitates the child's understanding of the necessary relationships which are involved in a given situation, and it plays a role in constructing class relations, including class-inclusion.

We shall use example of addition and subtraction to illustrate the distinction between the semantics of Lo and La. In La (arithmetic language) the semantics of the '+' and '−' signs as learned in the domain of cardinal numbers, is a simple one. If a sentence '$A + B = C$' is given, one knows many things such as: that C is a number greater than A and greater than B; that '$C - B = A$' describes essentially the same relationship among A, B and C as '$A + B = C$'; that addition increases the first number and subtraction decreases it; etc. This knowledge and similar kinds of knowledge are connected to the semantic knowledge of '+' and '−' which was attained by putting these operations in a broader mathematical structure within which the + and the − signs get their meanings in La.

In contrast, the semantic knowledge of Lo (ordinary language) is linked to the child's experience in his every day life. For example, when a child hears a sentence like the following: 'Dan had some marbles and he gave a few to Ruth . . .' he knows that Dan was left with fewer marbles. His comprehension of the situation is derived from his semantic understanding of the Lo expressions: 'had', 'gave to Ruth' and 'left with'. In fact, it is the understanding of Lo as formulated in a text of a word-problem, which enables the child to choose the correct La operation.

Therefore we, in our description, have differentiated between the logical and the arithmetic growth, whereas Piaget considered them to be one component (i.e., the logico-mathematical component). This distinction will also enable us to account more specifically for the levels of performance in arithmetic word problems.

Table V presents the four developmental levels in a more articulated manner in terms of Lo, La and the logical relations available at each level.

3. UNDERSTANDING LEVELS OF PERFORMANCE IN ARITHMETIC WORD PROBLEMS

In the last section we outlined the general kinds of knowledge that we assume underlie arithmetic problem-solving. We turn now to the empirical findings and show how they can be understood in the light of the above developmental levels.

A. Level 1

Referring to Table V and Figure 1, Level 1 is defined by the ability to represent and operate on single sets. The knowledge available to represent information about sets includes (1) Lo scheme for identifying sets and (2) the La schema for representing the cardinality of a set (see Riley *et al.* (1981) for more details).

These schemes are sufficient to solve Change problems 1 and 2 and Combine 1. These problems share two main characteristics: (1)The strategy required for solving the problem can be selected on the basis of partial and local information, and (2) the solution set is directly available for counting at the time the question is asked.

For example, consider how a level 1 child could solve a Combine 1 problem like

> Joe has 3 marbles.
> Tom has 5 marbles.
> How many marbles do Joe and Tom have altogether?

Understanding the first sentence requires that the child use his/her knowledge of the possession verb 'has' to represent a set of marbles belonging to Joe. On the basis of this representation, the child then selects an appropriate display and counts out a set of three objects. This procedure is repeated for the second sentence. To determine the answer, the child needs only to count the set. Thus solving Combine 1 involves three isolated actions of counting well-defined sets.

Change problems 1 and 2 can also be solved on the basis of local problem features that specify completely separate actions of counting. For example, consider the following Change 1 problem

Joe had 3 marbles.
Then Tom gave his 5 more marbles.
How many marbles does Joe have.

The first sentence of this problem is identical to the first sentence of Combine 1. The second sentence requires that the child first understand that 'gave' refers to an increase in this case, and then increase the initial set by the appropriate number of marbles. The answer again involves counting all members of the set described by a simple possessive phrase in the question, and this set is readily available to the child for counting it all, as a separate assignment.

In contrast, consider what happens when the solution set cannot be determined by reference to the final ownership alone, as in *Change 3*:

Joe had 3 marbles.
Then Tom gave him some more marbles.
Now Joe has 8 marbles.
How many marbles did Tom give Joe?

Solving this problem involves counting out an initial set of 3 blocks, then increasing that set by 5 blocks in response to 'Now Joe has 8 marbles'. At this point, the Level 1 child's representation of the problem is simply the final set of blocks belonging to Joe. Therefore when asked, 'How many marbles did Tom give to Joe?' the child answers, 'Eight'.

Thus, our analysis not only explains how Level 1 children solve certain problems successfully, but also why children at that level fail to solve other problems which require the ability to link events (as in the case of Change 3 problems). This is the knowledge that we attribute to children at Level 2.

Before discussing Level 2, we should mention that many Level 1 children solve Change 4 correctly, even though this problem also involves an unknown change set. This is because the effect of decreasing an initial set by some amount to get a specified final amount is that the *change* set and *final* set are now physically separate and both appear in the child's actual display.

B. Level 2

[illegible] Level 2 as presented in Table V and Figure 2, the child is able to relate the [illegible] that occurred in the initial set to an action in a causal chain. He is also [illegible] to estimate the direction of the change (increase or decrease). In *arithmetic* [illegible] this level there is an understanding of addition and subtraction operations as procedures to follow. Thus, for problems like Change 3, the Level 2 child knows that the change is the result of an action that he can evaluate qualitatively (the direction) and quantitatively (the amount).

In the same way, the child also solves problems of the type Compare 1 and Compare 2 (see Table III for a description of these problems). He regards them as '*make* this smaller one large' (for Compare 1) or '*make* the larger one smaller' (for Compare 2), which enables him to regard Compare 1 and 2 as Change 1 and 2 problems. (Interesting evidence for this hypothesis is provided by children's performances on 'equalizing' problems which appear in Carpenter and Moser (1981).) Note that, in solving Compare 1 and Compare 2 problems, at this level (Level 2) the child does not need to deal with an abstract set (the difference set) described by a relational term (i.e., 'x is *greater than y*'). He can ignore this expression and consider the two compared amounts as an initial state and final state, as was the case in the problems Change 3 and Change 4.

As to the mathematical action taken at this level, children who are merely able to count, can solve Change 3 and Change 4 problems. In particular, there will be no difficulty for those who have already experienced adding and subtracting of numbers (by counting on, or by memorizing facts). For children at this level the + sign means 'have more' and the − sign means 'take away'. This partial understanding of La, fully corresponds to their understanding of Lo expressions describing 'increasing', 'decreasing', 'more' or 'less'. The fact that children acquire limited semantic interpretation of La and Lo at this level cannot be differentiated by observing the child's performance on Change 3 and 4 problems. Therefore, problems in which such differentiation is crucial for the correct solution (i.e., Change 5, Change 6, Compare 5, Compare 6), cannot be solved at Level 2. The same observation was made by Kamii (1980) who found that children first employ the operations signs + and − in a very limited manner before employing the = sign.

C. Level 3

As seen in Table V and Figure 3. at level 3 the scheme of Part-Part-Whole is available to the child, as well as the additive structure among number-triples. This scheme enables reversible inferences about sets' relationships including the amount of a difference between two specified sets. Therefore, partial information can be represented with a slot for the unknown quantity. In arithmetic, at this level, the additive structure is reversible and includes the equality relation and understanding the necessary inference that *if* $a + b = c$, *then* $c - b = a$ or $c - a = b$.

This knowledge enables Level 3 children to solve problems like Combine 2 in which two quantified sets are given in the formulation of the problem, and the child must distinguish explicitly which are the subsets, and which is the union set in order to know what to do. It is only due to the inclusion relation which is expressed in Combine 2 problems that the child can understand that he should subtract. Otherwise, if only the previous mechanisms of levels 1 and 2 are available to him, then in the case of level 1 performance, he will add the two given sets, and in the case of level 2, he will not know what to do and will therefore fall back to a level 1 performance. (i.e., count any mentioned set).

The ability to solve problems such as Change 5 and Change 6 at level 3 brings in one of the most powerful predictions of our theoretical model. In these problems, the semantic schemes that originated in the child's experience with ordinary language *contradict* the newly learned semantics of addition and subtraction (+ and −). For example, let us consider a problem of the *Change 5* type:

> Dan had some marbles.
> He found 5 more marbles.
> Now he has 8 Marbles.
> How many marbles did he have to start with?

The child's experience with Lo language will direct him to add ('found' means 'adding'). Choosing to subtract (for the correct solution) can be achieved only if the semantics of La and Lo are differentiated as two autonomous systems, so that each one of them can be further elaborated to reach the necessary coordination between the two systems. For Lo this involves interpreting the 'initial state', the 'change' and the 'final state' of the above problem in a non-temporal manner as in a part-part-whole relationship. Since one part and the whole are given, finding the second part is achieved in La by subtraction, as one of the semantic interpretations of the '−' sign. Thus, at this level the child is able to make the mapping between Lo and La, *not* on the basis of isolated verbal cues, but rather on the basis of the understanding of the underlying semantics of Lo and La. Now he is able to impose the logical-mathematical structure, which is reversible and a-temporal on a sequential-temporal situation described in Lo (ordinary language).

Compare 3 and Compare 4 bring in another consideration. Compare problems include an inherent difficulty due to the fact that the task calls for comprehension of a *relation* between two quantities. In each Compare text, there is embedded an expression of the type 'A is n more than B' which conveys the following information:

(1) There is a quantity A: $n(A)$,
(2) There is a quantity B: $n(B)$,
(3) There is an order relationship between A and B: $R(A, B)$,
(4) There is a difference between A and B, that can be quantified: $n(A - B)$,

(5) The connection: If $n(A) > n(B)$ than: $n(B) + n(A - B) = n(A)$.

We have stressed the nature of an expression such as: 'Dan has 5 more marbles than Joe', since it is crucial for the solution of Compare problems, and it is a very condensed expression that conveys two kinds of information:

(a) that Dan had more than Joe, (a qualitative-order consideration), and
(b) that the difference set consists of 5 marbles (a quantitative consideration)

Usually it is the misunderstanding of (b) as a relative quantity and a difference set ($n(A - B)$) which causes the child most trouble. He wrongly interprets '5' as an absolute quantity which denotes what Joe has.

As we now examine the category of Compare problems, it becomes clear that Compare 1 and 2 problems, which do not contain such relative expressions in the given *numerical* information are easier and can be solved even at Level 2. Compare 3 and 4, however, can be solved at Level 3 and it is due to the fact that a child at that level is able to form any needed set (or subset) out of the given information, as to complete the Part-Part-Whole scheme. In particular, he is able to imagine and construct the difference set $A - B$, and therefore able to operate on three distinct sets: A, B, and $A - B$. Comprehending three distinct sets, in the case of Compare problems, is a necessary step for obtaining any missing numerical value. Let us examine the following Compare 3 example:

(1) Joe has 3 marbles.
(2) Tom has 5 more marbles than Joe.
(3) How many marbles does Tom have?

At *Level 2*, this problem will produce the answer '5'. (interpreting 5 as the absolute quantity that Tom has). At *Level 3*, however, the child is able to create a new set '$A - B$' and interpret the given information as:

I. $n(B) = 3$
II. $n(A - B) = 5$
III. $n(A) = ?\ (8)$

It is clear that this information by itself will not suffice to correctly solve the problem. The next step is to also consider the qualitative-order relationship between A and B, which is also conveyed in string II ('Tom has *more* than Joe'), and therefore the La apparatus for calculation is called upon to perform addition and not subtraction. The Lo understanding at Level 3 can be schematically described as shown in Figure 5. The reader should note that this structure of *problem 3* as expressed in Lo is a non-symmetrical structure and the order of the subject-predicate at the surface level is maintained. One should

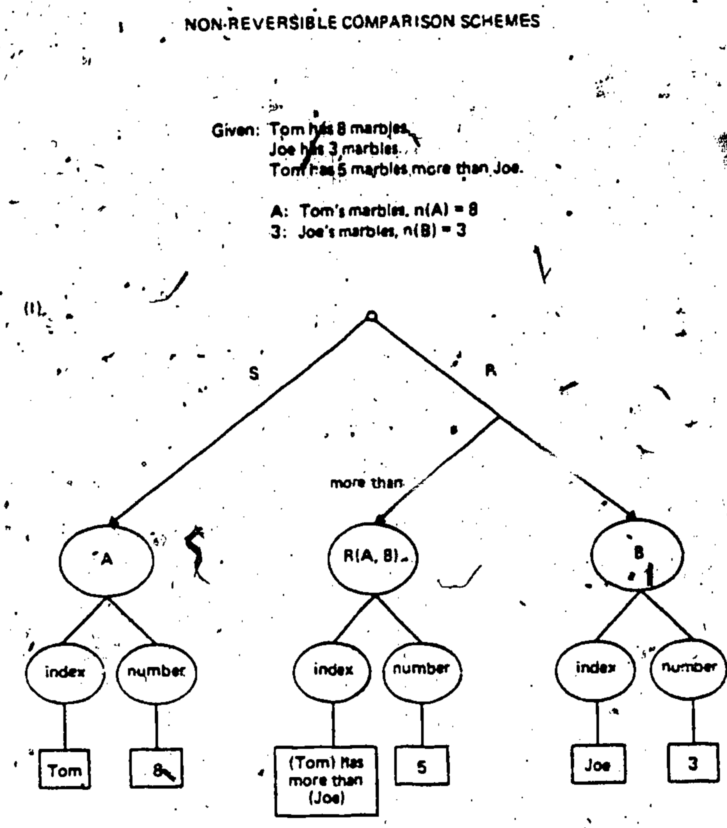

Fig. 5.

note, however, that the description of this relationship in Lo (ordinary language) is not unique. Given the above case that $n(A) > n(B)$, it is possible to express in Lo the same relationship in two ways: '*A* is *n* more than *B*', but also '*B* is *n* less than *A*'. Both describe the same underlying relationship between *A* and *B*. Thus, if *R* represents 'more', and R^{-1} represents 'less', it is always true that:

$$R(A, B) = R^{-1}(B, A)$$

Knowing the reversibility of the order relations is part of Lo knowledge, as well as part of La knowledge.

Thus, there are three more expressions 'similar' to '*A* is more than *B*' which are possible in Lo, two of which will describe a different underlying relationship between *A* and *B* (Figure 6).

We assume that at *Level 3* the child's representation of a relative description

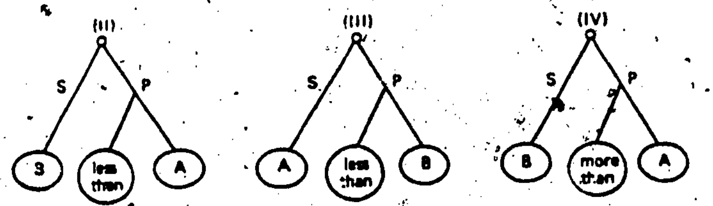

Fig. 6.

includes the ability to differentiate the relative quantity as a separate *set* (more/less than), but it is not flexible enough to invert the directional order relation which is necessary for solving Compare 5 and Compare 6 problems.

Therefore on the basis of the limited capacity as illustrated in Figures 5 and 6, the child is successful in solving Compare 3 and 4, but he fails on Compare 5 and 6. This is because the non-flexible representations include in their Lo description the terms 'more' and 'less' which are interpreted in Compare 3 and 4 problems, as + and − of La (Nesher and Teubal, 1975). No conflict can arise in Compare 3 and 4 between the semantic interpretations of Lo and La. The child adds for 'more' and subtracts for 'less'. This is not the case, however, with Compare 5 and 6 problems. Correct solution of these problems is delayed until Level 4.

D. Level 4

At Level 4 (see Table V and Figure 4) the scheme for non-symmetrical relations (which started at Level 2, in the description of a change, or comparison) is now available in a *reversible* manner. Directional and relative descriptions (i.e., 'more'/'less') can be handled in a flexible manner, and also a set can be induced by means of relative comparison (this already began at Level 3). In arithmetic this level will include the ability to handle inequality, and its relationship to equality: *If* $A > B$, *then* $A - C = B$ or $B + C = A$.

Unlike the schemes of Figures 5 and 6, the scheme described in Figure 4 is a more abstract representation, which incorporates all four special cases in one flexible structure. It does not ignore the fact that a Lo description is directional, ordered and non-symmetrical, but it is capable in that general form of making the necessary coordination between Lo and La even in cases in which Lo and La contradict on the surface level. i.e., the word 'more' is mentioned in Lo, and the child has to subtract (Compare 5) and vice versa (Compare 6), as in the following:

Compare 5 problem:

(1) Joe has 8 marbles.
(2) He has 5 more than Dan.

(3) How many marbles does Dan have?

At this level, Level 4, the child is therefore able to coordinate the directional-terms without ignoring them, by employing the reversed relation description which is needed for choosing the correct operation. Thus, he is able to read the word 'more' in the text, and yet perform a subtraction operation.

To sum up the detailed discussion in this section, we claim that our hypothesis concerning the developmental levels explains which kinds of problems can be solved by a child at a given level. This is summarized in Table VI.

TABLE VI

Type of Problem	Level 1	Level 2	Level 3	Level 4
Combine 1	x			
Combine 2			x	
Change 1	x			
Change 2	x			
Change 3		x		
Change 4		x*		
Change 5			x	
Change 6			x	
Compare 1			x*	
Compare 2			x*	
Compare 3			x	
Compare 4			x	
Compare 5				x*
Compare 6				x*

*In some empirical samples (Nesher and Katriel, 1978; Nesher, 1981) these problems fall in an earlier level, respectively.

4. FINAL COMMENTS

Research on early problem-solving repeatedly shows that kindergarten children and first graders, before learning arithmetic at school can solve simple addition and subtraction word problems but fail in some of them. (Carpenter, *et al.* 1981; Lindvall and Ibarra, 1979; Lindval, 1980; Riley *et al.*, 1981; Tamburino, 1980). We have tried to suggest a hypothesis that explains which kinds of problems can be solved without the aid of arithmetic, and for which ones the knowledge of arithmetic is crucial. For that purpose we needed to treat the growth of the child's knowledge-structure in a way that identified distinct main components (i.e., the empirical, the logical and the mathematical components).

The child's action schemes are, of course, integrated and growth in each component does not occur in isolation from the others. Yet, for understanding more sophisticated performance, it is important to analyze aspects of knowledge distinctively and to observe the contribution of each one of them in a given task.

We believe that our hypothesis, since it is detailed and narrow enough, has predictive power and can be examined empirically. At the moment it fits empirical data that has been found to be universal. The pedagogical implications of such an analysis are two fold. First, one can be more sensitive to the sequence of instruction when one understands the prerequisite knowledge structures for solving certain problems, and can adapt different strategies in teaching at different levels. Second, this analysis allows a better understanding of the difficulties that children encounter at different levels of performance. Similar analyses should, of course, be extended to other mathematical structures such as multiplication or place-value, in order to account for psychological factors affecting mathematical learning.

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