BULL. AUSTRAL. MATH. SOC. 81A54, 81A78
VOL. 11 (1974), 311-313.

The development of the boson calculus for the orthogonal and symplectic groups

M.A. Lohe

The boson calculus has been used extensively in the study of the unitary groups $U(n)$ as a means of constructing explicitly all irreducible unitary representations. However, the boson calculus in this form cannot be applied directly to the subgroups $Sp(n)$ and $O(n)$; our aim is to develop the boson calculus in a form which is immediately applicable to all the classical groups.

The $U(n)$ representation space is constructed from tensors, which in the restriction to $Sp(n)$ and $O(n)$ must be traceless if the representations are to be irreducible. Instead of the usual boson operators we introduce "modified boson operators" a_i^α which satisfy in particular the traceless condition $\rho_{pq}a_p^\alpha a_q^\beta = 0$ (summation), where ρ is the metric for $Sp(n)$ or $O(n)$. With these operators, which behave as vectors under $Sp(n)$ or $O(n)$, we are able to construct manifestly traceless tensors (multivectors) of arbitrary symmetry. Furthermore, all objects to be studied may be defined in terms of these operators. In general, we find that the modified boson calculus which we develop has, for $Sp(n)$ and $O(n)$, a domain of application which not only includes that of ordinary boson operators, but is considerably larger.

We are now able to construct simply the irreducible spaces which carry all representations of $Sp(n)$ and $O(n)$. We calculate maximal and semi-maximal basis states, and all states in symmetric tensor representations of $O(n)$ and $Sp(n)$, and also general states for arbitrary tensor

Received 29 January 1974. Thesis submitted to the University of Adelaide, January 1974. Degree approved, July 1974. Supervisor: Professor C.A. Hurst.

representations of $O(3)$, $O(4)$. The $O(3)$ states appear as monomials and the $O(4)$ states as Jacobi polynomials in modified boson operators.

In the application to $Sp(n)$ we encounter the problem of state-labelling. We restrict our attention to $Sp(4)$, although keeping in mind the general problem, and we seek a solution using the parameters appearing in the branching theorem for $Sp(4)$ restricted to $Sp(2)$. We utilize modified boson operators in the construction of the non-orthogonal Weyl states, and carry out a suitable Gram-Schmidt orthogonalization to obtain explicitly the orthogonal basis states. Although in principle the extra labelling operator which is required may be found from these states, its form will necessarily be complicated. It is found that a satisfactory solution to the state-labelling problem, exhibiting the structure and simplicity which is apparent for $U(n)$ and $O(n)$, does not exist.

We carry out a further development of the boson calculus for $O(n)$ to enable the explicit construction of all spinor (double-valued) representations in spaces of traceless tensors. For the lower order groups this is done in such a way as to obtain the representation space of the covering group, by finding operators which satisfy the traceless condition, but are different from modified boson operators. Some of these operators satisfy simple triple commutation relations which are of interest for both group theory and field theory.

In order to enable the construction to be made of all spinor representations of $O(n)$ in general in a space of traceless tensors, or equivalently, harmonic homogeneous polynomials, we establish firstly the relation between the methods of the boson calculus and of Zhelobenko [1]. This latter method uses polynomials over a homogeneous space defined by a certain triangular subgroup, and we show the two methods can be directly related, so that one construction can be mapped into the other. Zhelobenko's formalism includes the spinor representations in a natural way, and we show how to transfer to the boson calculus so as to retain this construction; this is achieved ultimately by finding realizations of the Lie algebra of $O(n)$ which are new. Results are written out explicitly for $O(3)$.

Reference

[1] D.P. Zhelobenko, "The classical groups. spectral analysis of their finite-dimensional representations", *Russian Math. Surveys* **17** (1) (1962), 1-94.