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THE DIAMETER OF AN IMMERSED RIEMANNIAN MANIFOLD WITH BOUNDED MEAN CURVATURE

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Abstract

Let M be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below. Let $\bar{M}$ be an N-dimensional ($N > n$) complete, simply connected Riemannian manifold with nonpositive sectional curvature. We shall prove in this note that if there exists an isometric immersion φ of M into $\bar{M}$ with the property that the immersed manifold is contained in a ball of radius R and that the mean curvature vector H of the immersion has bounded norm $\|H\| \leqslant H_0$, ($H_0 > 0$) then $R \geqslant H_0^{-1}$.

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Generalizing a result by Aminov (1973), Hasanis and Koutroufiotis proved that if there exists an isometric immersion $\varphi: M \to E^N$ such that $\|H\| \leqslant H_0$ and $\varphi(M) \subset B_R$, then $R \geqslant H_0^{-1}$, where H is the mean curvature vector of the immersion and B_R a ball of radius R. In this paper we shall generalize the above result considering isometric immersions of M into $\bar{M}$. To prove the main theorem we need the following

LEMMA 1. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If a smooth function f on M has an upper bound, then for any $\varepsilon > 0$ there is a point where the Laplacian $\Delta f < \varepsilon$.*

This is obtained easily from Theorem A of Omori (1967). For a proof see Hasanis and Koutroufiotis (1979).

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LEMMA 2. *Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Let γ: $[0, 1] \to M$ be a geodesic parametrized proportional to arc length. If X is a Jacobi vector field along γ such that $X(0) = 0$, then*

$$\langle \nabla_T X, X \rangle \geqslant \langle X, X \rangle,$$

where ∇ is the Riemannian connection and T the tangent vector of γ.

PROOF. We consider the function

$$\psi = \langle t\nabla_T X - X, X \rangle. \tag{1}$$

We have

$$\begin{aligned} \frac{d\psi}{dt} &= t\langle \nabla_T \nabla_T X, X \rangle + \langle t\nabla_T X - X, \nabla_T X \rangle \\ &= t\langle -R(X, T)T, X \rangle + t\langle \nabla_T X, \nabla_T X \rangle - \langle X, \nabla_T X \rangle \\ &\geqslant t\langle \nabla_T X, \nabla_T X \rangle - \langle X, \nabla_T X \rangle. \end{aligned} \tag{2}$$

Since $\langle X, \nabla_T X \rangle^2 \leqslant \langle X, X \rangle \langle \nabla_T X, \nabla_T X \rangle$ we have

$$\langle \nabla_T X, \nabla_T X \rangle \geqslant \langle X, \nabla_T X \rangle^2 \langle X, X \rangle \quad \text{for } t > 0. \tag{3}$$

From (1), (2), (3) we get

$$\frac{d\psi}{dt} \geqslant \frac{\langle X, \nabla_T X \rangle}{\langle X, X \rangle} \psi. \tag{4}$$

We put $h = \langle X, X \rangle^{1/2}$. It follows that

$$\frac{1}{h}\frac{dh}{dt} = \frac{\langle X, \nabla_T X \rangle}{\langle X, X \rangle}.$$

Hence (4) gives

$$\frac{h\dfrac{d\psi}{dt} - \psi\dfrac{dh}{dt}}{h} \geqslant 0.$$

Thus

$$\frac{d}{dt}\left(\frac{\psi}{h}\right) \geqslant 0 \quad \text{for } t > 0. \tag{5}$$

But $\lim_{t\to 0} \psi/h = 0$ and so from (5) we have $\psi/h \geqslant 0$ and hence $\psi \geqslant 0$ which implies the desired inequality.

THEOREM. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let $\overline{M}$ be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. If there exist an isometric immersion φ: $M \to \overline{M}$ such that $\|H\| \leqslant H_0 = const > 0$ and $\varphi(M) \subset B_R$, then $R \geqslant H_0^{-1}$.*

PROOF. Let Q be the centre of the metric ball B_R in $\bar{M}$. Consider a vector X tangent to M at P and a geodesic $\beta(u)$ in M such that $\beta(0) = P$, $\beta'(0) = X$. Let $\sigma(t, u)$, $0 \leqslant t \leqslant 1$ be a parametrization of the unique geodesic in $\bar{M}$ from Q to $\beta(u)$ (proportional to arc length). Let T, X be the vector fields along σ defined by

$$T = \sigma_* \frac{\partial}{\partial t}, \qquad X = \sigma_* \frac{\partial}{\partial u}.$$

It follows easily from this construction that, for fixed u, X is a Jacobi field along the geodesic $t \to \sigma(t, u)$ with $X(0) = 0$.

Now, we consider a function F on M defined by

$$F(P) = \frac{1}{2}\{d(Q, P)\}^2,$$

where $d(Q, P)$ is the distance from Q to P in $\bar{M}$. We will estimate the Hessian form $\nabla^2 F$.

We put $f(u) = d(Q, \beta(u)) = \int_0^1 \|T\|\, dt$. We obtain easily that

$$f'(u) = \frac{1}{\|T\|}\langle T, X\rangle_{t=1}.$$

Since $\|T\| = f(u)$ we get $f(u) \cdot f'(u) = \langle T, X\rangle_{t=1}$, that is

$$F'(u) = \langle T, X\rangle_{t=1}.$$

If $\bar{\nabla}$ is the Riemannian connection of $\bar{M}$ then

$$F''(u) = \langle \bar{\nabla}_X T, X\rangle_{t=1} + \langle T, \bar{\nabla}_X X\rangle_{t=1} = \langle \bar{\nabla}_T X, X\rangle_{t=1} + \langle T, \bar{\nabla}_X X\rangle_{t=1}$$

since $[X, T] = 0$.

Using the Gauss equation, the fact $F''(u) = \nabla^2 F(X, X)$, and Lemma 2 we obtain

$$\nabla^2 F(X, X) \geqslant \langle X, X\rangle + \langle T, a(X, X)\rangle,$$

where a is the second fundamental form of immersion. Taking the trace of this, we have

$$\Delta F \geqslant n + n\langle T, H\rangle.$$

But by assumption $|\langle T, H\rangle| \leqslant RH_0$. Hence $\Delta F \geqslant n(1 - H_0 R)$. If we had $R < H_0^{-1}$ then $1 - H_0 R = c > 0$ and so $\Delta F \geqslant nc = \text{const} > 0$ but this contradicts Lemma 1, hence $R \geqslant H_0^{-1}$.

COROLLARY. *If M is a complete, simply connected Riemannian manifold with nonpositive sectional curvature, then a complete minimal submanifold of M, whose Ricci curvature is bound from below, is extrinsically unbounded.*

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