

## THE DICHOTOMY OF HARMONIC MEASURES OF COMPACT HYPERBOLIC LAMINATIONS

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**Abstract.** Given a harmonic measure  $m$  of a hyperbolic lamination  $\mathcal{L}$  on a compact metric space  $M$ , a positive harmonic function  $h$  on the universal cover of a typical leaf is defined in such a way that the measure  $m$  is described in terms of these functions  $h$  on various leaves. We discuss some properties of the function  $h$ . We show that if  $m$  is ergodic and not completely invariant, then  $h$  is typically unbounded and is induced by a probability  $\mu$  of the sphere at infinity which is singular to the Lebesgue measure. A harmonic measure is called Type I (resp. Type II) if for any typical leaf, the measure  $\mu$  is a point mass (resp. of full support). We show that any ergodic harmonic measure is either of type I or type II.

**1. Introduction.** We call  $(M, \mathcal{L}, g)$  a compact  $C^2$  lamination if  $\mathcal{L}$  is an  $n$  dimensional lamination of class  $C^2$  on a compact metric space  $M$  and if  $g$  is a leafwise Riemannian metric of class  $C^2$ . (For the precise definition, see Section 2.) Then the leafwise Laplacian  $\Delta f$  is defined for any continuous leafwise  $C^2$  function  $f$ , on  $M$ . A probability measure  $m$  on  $M$  is called *harmonic* if for any such  $f$ , we have  $m(\Delta f) = 0$ . A harmonic measure always exists for any compact  $C^2$  lamination.

Given a harmonic measure  $m$ , there is a saturated conull set  $M^*$  such that a positive harmonic function  $h$ , called the characteristic harmonic function, is defined on the universal cover of each leaf in  $M^*$  up to a constant multiple. This function is obtained in the way of describing the measure  $m$  on each local chart. We first show (Theorem 3.13) that if  $m$  is ergodic and not completely invariant, then for any leaf in a saturated conull set, the characteristic harmonic function  $h$  is unbounded.

A compact  $C^2$  lamination  $(M, \mathcal{L}, g)$  of dimension  $d+1$  is called *hyperbolic* if the metric  $g$  has curvature  $-1$  on each leaf. The universal cover of each leaf is isometric to the hyperbolic space  $\mathbf{D}^{d+1}$ , and the characteristic harmonic function  $h$  corresponds to a probability measure  $\mu$  on the boundary at infinity  $\mathbf{S}_\infty^d$ . It depends upon the choice of a base point in  $\mathbf{D}^{d+1}$ , but its equivalence class is uniquely determined by the leaf. We show (Theorem 4.1) that if  $m$  is ergodic harmonic and not completely invariant, then for any leaf in a saturated conull set, the measure  $\mu$  is singular to the Lebesgue measure of  $\mathbf{S}_\infty^d$ .

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DEFINITION 1.1. A harmonic measure  $m$  is called of *Type I* if the measure  $\mu$  of  $S_\infty^d$  is a point mass for any leaf in a saturated conull set, and of *Type II* if the support of  $\mu$  is the whole  $S_\infty^d$ .

The main theorem of this paper is the following.

THEOREM 1.2. *An ergodic harmonic measure is either of Type I or of Type II.*

In Section 2, we prepare some prerequisites about harmonic measures. Especially the characteristic harmonic function  $h$  is defined. In Section 3, after a brief description of the leafwise Brownian motion, we study its reverse process. The reverse process plays a crucial role in the proof of the unboundedness of  $h$  (Section 3) and the singularity of  $\mu$  (Section 4). In Section 5, we study the leafwise unit tangent bundle  $N$  of a compact hyperbolic lamination  $\mathcal{L}$ . There is a naturally defined lamination  $\mathcal{H}$  on  $N$  of the same dimension as  $\mathcal{F}$ . Generalizing a result in [BM], we discuss one to one correspondance between harmonic measures on  $\mathcal{L}$  and pointed harmonic measures on  $\mathcal{H}$ , the latter being defined in Section 5. Finally the proof of Theorem 1.2, as well as some examples, is given in Section 6.

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**2. Harmonic measure.** Let  $M$  be a compact metric space, covered by a finite number of open sets  $E_i$ . Assume there is a homeomorphism  $\varphi_i : E_i \rightarrow U_i \times Z_i$ , where  $U_i$  is an open ball in  $\mathbf{R}^n$  and  $Z_i$  is a locally compact metric space. If  $E_i \cap E_j \neq \emptyset$ , then the transition function  $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$  is defined as a homeomorphism from  $\varphi_i(E_i \cap E_j)$  onto  $\varphi_j(E_i \cap E_j)$ . Assume that the transition function is of the form

$$\psi_{ji}(u, z) = (\alpha(u, z), \beta(z)),$$

where  $\alpha$  and  $\beta$  are continuous, and  $\alpha$  is of class  $C^2$  with respect to the first coordinate  $u$  and its first and second derivatives are continuous in  $z$ . A subset of  $M$  is called a *plaque* if it is of the form  $\varphi_i^{-1}(U_i \times z)$ , and a *transversal* if  $\varphi_i^{-1}(u \times Z_i)$ . A maximal pathwise connected countable union of plaques is called a *leaf*. This gives birth to a decomposition  $\mathcal{L}$  of  $M$  into leaves, which is called a lamination of class  $C^2$ .

A leaf naturally has a structure of  $n$ -dimensional  $C^2$  manifold. A field of leafwise metric tensors is called a *leafwise Riemannian metric of class  $C^2$*  if its leafwise derivatives up to order 2 (including order 0) are continuous on  $M$ . In this paper the triplet  $(M, \mathcal{L}, g)$  is simply referred to as a *compact  $C^2$  lamination*. By the compactness of  $M$ , each leaf of  $\mathcal{L}$  is complete and of bounded geometry. The leafwise volume defined by  $g$  is denoted by  $\text{vol}$ .

Henceforce we depress the homeomorphism  $\varphi_i$  and consider  $U_i \times Z_i$  as an open subset of  $M$ , which is called a *local chart*.

A function  $f : M \rightarrow \mathbf{R}$  is said to be *continuous leafwise  $C^2$*  if it is of class  $C^2$  in each leaf and its derivative up to order 2 is continuous on  $M$ . Then the leafwise Laplacian  $\Delta f$  with respect to  $g$  is defined, and is a continuous function on  $M$ .

DEFINITION 2.1. A probability measure  $m$  on  $M$  is called *harmonic* if  $m(\Delta f) = 0$  for any continuous leafwise  $C^2$  function.

REMARK 2.2. A harmonic measure always exists for any compact  $C^2$  lamination  $(M, \mathcal{L}, g)$ .

See [C, Theorem 3.5] for a simple proof using the Hahn-Banach theorem. Here is a structure theorem of a harmonic measure on a local chart.

THEOREM 2.3. Assume  $m$  is a harmonic measure on a compact  $C^2$  lamination. For any local chart  $U \times Z$ , there are a measure  $\nu$  on  $Z$  and a function  $h : U \times Z \rightarrow \mathbf{R}$  with the following properties.

- (1)  $h$  is positive and  $m$ -measurable.
- (2) For  $\nu$ -a.a.  $z$ , the restriction of  $h$  to the plaque  $U \times z$  is harmonic and  $h \text{vol}$  is a probability measure of the plaque.
- (3) For any continuous function with support in  $U \times Z$ , we have

$$m(f) = \int_Z \int_{U \times z} f(u, z) h(u, z) d\text{vol}(u) d\nu(z).$$

Furthermore, if a probability measure  $m$  on  $M$  is represented in this way in any local chart, then  $m$  is harmonic.

NOTATION 2.4. The theorem says that the measure  $m$  restricted to  $U \times Z$  is disintegrated in such a way that the conditional probability measure on each fiber  $U \times z$  is  $h(\cdot, z) \text{vol}$  and the push forward measure on the base  $Z$  is  $\nu$ . Henceforth this is denoted as

$$(2.1) \quad m|_{U \times Z} = \int_Z h \text{vol} d\nu.$$

PROOF. By the disintegration theorem, we have

$$m|_{U \times Z} = \int_Z m_z d\nu(z),$$

where  $m_z$  is a probability measure on  $U \times z$  and the assignment  $z \mapsto m_z$  is measurable. The measure  $\nu$  is the push forward of  $m$  by the projection  $p_2 : U \times Z \rightarrow Z$ , and is not necessarily a probability measure.

Denote the other projection by  $p_1 : U \times Z \rightarrow U$ . The leafwise Riemannian metric on each plaque  $U \times z$  is transferred to a Riemannian metric on  $U$ , and the corresponding Laplacian on  $U$  is denoted by  $\Delta_z$ . Consider any function  $f$  from the space  $C_c^2(U)$  of the  $C^2$  functions with compact support, and any continuous function  $g$  on  $Z$  with compact support. Then the product  $f \circ p_1 g \circ p_2$  is a continuous leafwise  $C^2$  function whose support is contained in  $U \times Z$ , and we have

$$\Delta(g \circ p_2 f \circ p_1) = g \circ p_2 \Delta(f \circ p_1) \text{ and } m_z(\Delta(f \circ p_1)) = m_z(\Delta_z f).$$

Since  $m(\Delta(g \circ p_2 f \circ p_1)) = 0$ , we have

$$\int_Z m_z(\Delta_z f) g(z) d\nu(z) = 0.$$

By the measurability of the assignment  $z \mapsto m_z$  and the boundedness of  $\Delta_z f$ ,  $m_z(\Delta_z f)$  is an integrable function on  $Z$  and thus  $m_z(\Delta_z f)v$  is a signed measure on  $Z$ , for which an arbitrary compactly supported continuous function  $g$  integrates to 0. This implies that for  $v$ -a.a.  $z$ ,  $m_z(\Delta_z f) = 0$ .

Since  $C_c^2(U)$  has a countable dense subset  $S$ , there is a  $v$ -conull set  $Z^*$  of  $Z$  such that  $m_z(\Delta_z f) = 0$  for any  $z \in Z^*$  and  $f \in S$ , and therefore for any  $f \in C_c^2(U)$ . But as is well known [N],  $m_z(\Delta_z f) = 0$  for any  $f \in C_c^2(U)$  if and only if  $m_z = h_z \text{vol}$  for a harmonic function  $h_z$  on  $U$  with respect to the Laplacian  $\Delta_z$ . Setting  $h(u, z) = h_z(u)$ , we obtain (2.1).

Next we are going to show that the function  $h$  is measurable. Consider another measure  $m_0$  on  $U \times Z$ , given by  $\int_Z \text{vol}/\text{vol}(U \times z)dv(z)$ . Clearly  $m$  and  $m_0$  are mutually equivalent and thus we have

$$m = km_0$$

for some  $m_0$ -measurable (equivalently  $m$ -measurable) function  $k$ . But the uniqueness of the disintegration implies that for  $v$ -a.a.  $z$ ,

$$h(u, z) = k(u, z)/\text{vol}(U \times z),$$

showing that  $h$  is measurable.

Finally the converse statement is easy to show. □

As an immediate consequence, we have the following corollary.

**COROLLARY 2.5.** *If a function  $f$  on  $M$  is  $C^2$  on each leaf and  $\Delta f$  is  $m$ -integrable, then  $m(\Delta f) = 0$ .*

**REMARK 2.6.** In [G], harmonic measures are defined by the property in Corollary 2.5, and the structure theorem is obtained. Our proof of Theorem 2.3 shows the equivalence of the two definitions.

Suppose two local charts  $U \times Z$  and  $U' \times Z'$  intersect and the harmonic measure  $m$  is decomposed on each local chart as

$$m|_{U \times Z} = \int_Z h \text{vol} dv \quad \text{and} \quad m|_{U' \times Z'} = \int_{Z'} h' \text{vol} dv',$$

then in the intersection of  $v$ -a.a. plaque  $U \times z$  and  $v'$ -a.a. plaque  $U' \times z'$ , we have

$$(2.2) \quad h'/h = dv/d\beta v',$$

where  $\beta$  is the holonomy map from (a part of)  $Z'$  to  $Z$ . On one hand this shows that  $v$  and  $v'$  are equivalent via the holonomy map. More generally we have the following proposition.

**PROPOSITION 2.7.** *A harmonic measure  $m$  is leafwise smooth, i.e.,*

- (1) *if a Borel set  $B \subset M$  satisfies  $\text{vol}(B \cap L) = 0$  for any leaf  $L$ , then  $m(B) = 0$ .*
- (2) *if a Borel set  $B$  is  $m$ -null, then the set  $\widehat{B}$  is also  $m$ -null, where  $\widehat{B}$  is the union of the leaves  $L$  such that  $\text{vol}(B \cap L) > 0$ .*

On the other hand, the equality (2.2) shows that on the intersection of two plaques, the function  $h'$  is a positive constant multiple of  $h$ . Dividing  $h'$  by that constant, one can continue  $h$  along a chain of plaques. Of course, this does not yield a function on a leaf since there will be a monodromy for  $h$ . However we will get a function on the holonomy cover of a leaf.

In what follows, when we say “an  $m$ -a.a. leaf  $L$ ”, this means “there exists a saturated conull set  $M^*$  and  $L$  is a leaf in  $M^*$ ”.

**PROPOSITION 2.8.** (1) *For an  $m$ -a.a. leaf  $L$ , the function  $h$  has a well-defined prolongation as a positive harmonic function on the holonomy cover  $\hat{L}$ . On  $\hat{L}$  two such functions which start from different plaques are unique up to a positive constant multiple.*

(2) *Given a path in  $L$ , the ratio of  $h$  at the initial point and the terminal point of any lift of the path to  $\hat{L}$  is constant.*

**PROOF.** To see (1), cover  $M$  with a finite number of local charts  $U_i \times Z_i$ . Then there is a  $\nu$ -conull set  $Z_i^*$  of each  $Z_i$  such that the harmonic function  $h$  is defined on  $U_i \times Z_i^*$ . The saturation of the union of  $Z_i \setminus Z_i^*$  is  $m$ -null by Proposition 2.7, and for any leaf  $L$  in the complement  $M^*$ , the function  $h$  has a prolongation on its holonomy cover  $\hat{L}$ .

The uniqueness part in (1), as well as (2), follows immediately from the construction.  $\square$

Of course, the harmonic function  $h$  has a lift to the universal covering space  $\tilde{L}$  of an  $m$ -a.a. leaf  $L$ , which will be denoted by the same letter  $h$ . The above statement (2) holds also for lifts of paths to the universal covering space. Let  $\Gamma$  be the deck transformation group of the covering map  $\tilde{L} \rightarrow L$ . Then we have the following corollary.

**COROLLARY 2.9.** *For any  $\gamma \in \Gamma$ ,  $h \circ \gamma$  is a constant multiple of  $h$ .*

**PROOF.** Join two points  $x \in \tilde{L}$  and  $y \in \tilde{L}$  by an arc  $c$ . Then  $\gamma x$  and  $\gamma y$  are joined by  $\gamma c$ . Two arcs  $c$  and  $\gamma c$  are lifts of the same arc in  $L$ . Therefore, we have

$$h(y)/h(x) = h(\gamma x)/h(\gamma y).$$

Since  $x$  and  $y$  are arbitrary, this shows that the function  $h(\gamma x)/h(x)$  is independent of  $x$ .  $\square$

**DEFINITION 2.10.** The function  $h$  in Proposition 2.8 is called the *characteristic harmonic function of  $m$* .

Notice that the characteristic harmonic function is defined only up to a positive constant multiple.

A harmonic measure  $m$  is called *completely invariant* if the characteristic harmonic functions are constant on (the holonomy covers of)  $m$ -a.a. leaves. In this case,  $m$  corresponds to a transverse invariant measure, i.e., an assignment of a finite measure to each transversal which is invariant by the holonomy maps. Conversely, a transverse invariant measure gives rise to a harmonic measure  $m$  whose characteristic harmonic function is constant on an  $m$ -a.a. leaf. Only a special class of laminations admit completely invariant measures.

**3. Brownian motion and its reverse process.** Let  $(M, \mathcal{L}, g)$  be a compact  $C^2$  lamination. Denote by  $\Omega$  the space of all the continuous leafwise paths  $\omega : [0, \infty) \rightarrow M$ , and

for  $t \geq 0$ , define a map  $X_t : \Omega \rightarrow M$  by  $X_t(\omega) = \omega(t)$ . Let  $\mathcal{B}$  be the  $\sigma$ -algebra of the Borel subsets of  $\Omega$  with respect to the compact open topology. It is well known, easy to show, that  $\mathcal{B}$  coincides with the minimal  $\sigma$ -algebra for which  $X_t$  ( $t \geq 0$ ) is Borel. In other words,  $\mathcal{B}$  is generated by the cylinder sets  $\{X_{t_1} \in B_1, \dots, X_{t_r} \in B_r\}$  ( $0 \leq t_1 < \dots < t_r, B_i$ ; a Borel subset of  $M$ ).

The leafwise Riemannian metric  $g$  gives the heat kernel  $p_t(x, y)$  ( $t > 0$ ) on each leaf. Define  $p_t(x, y)$  for any two points  $x, y \in M$  by setting  $p_t(x, y) = 0$  unless  $x$  and  $y$  lie on the same leaf. The heat kernel defines the Wiener probability measure  $W^x$  on  $\Omega$  ( $x \in M$ ). For a cylinder set  $\{X_{t_1} \in B_1, \dots, X_{t_r} \in B_r\}$  ( $t_1 > 0$ ), we define

$$(3.1) \quad \begin{aligned} &W^x\{X_{t_1} \in B_1, \dots, X_{t_r} \in B_r\} \\ &= \int_{B_1} \cdots \int_{B_r} p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_r-t_{r-1}}(y_{r-1}, y_r) d\text{vol}(y_r) \cdots d\text{vol}(y_1). \end{aligned}$$

Then  $W^x$  satisfies the dropping condition, and therefore it is defined not only for a cylinder set but also for any set in  $\mathcal{B}$ . That is,  $W^x$  is a probability measure on  $(\Omega, \mathcal{B})$ . It is concentrated on the subset  $\Omega_x = X_0^{-1}(x)$  since the probability measure  $p_t(x, \cdot)\text{vol}$  tends to the Dirac mass at  $x$  as  $t \rightarrow 0$ .

LEMMA 3.1. *The system of measures  $\{W^x\}_{x \in M}$  is Borel in the sense that for any  $S \in \mathcal{B}$ , the assignment  $x \mapsto W^x(S)$  is Borel.*

PROOF. Let  $\mathcal{C}$  be the family of the subsets  $S$  in  $\Omega$  for which  $M \ni x \mapsto W^x(S)$  is Borel, and let  $\mathcal{A}_0$  be the finite algebra formed by finite disjoint unions of cylinder sets. Then  $\mathcal{A}_0$  is contained in  $\mathcal{C}$ . For  $\{X_{t_1} \in B_{t_1}\} \in \mathcal{A}_0$ , see [CC, Lemma 2.3.1]. General case follows easily from this.

For an isolated ordinal  $\alpha > 0$ , define  $\mathcal{A}_\alpha$  to be the family of a subset which is obtained from subsets of  $\mathcal{A}_{\alpha-1}$ , by a finite sequence of two operations; one, taking a countable increasing union and the other, countable decreasing intersection. Then it is easy to show that  $\mathcal{A}_\alpha$  forms a finite algebra. Moreover  $\mathcal{A}_\alpha$  is contained in  $\mathcal{C}$  since a pointwise limit of Borel functions is Borel. For a limit ordinal  $\alpha$ , let  $\mathcal{A}_\alpha = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$ . Then again  $\mathcal{A}_\alpha$  is a finite algebra contained in  $\mathcal{C}$ .

The increasing sequence  $\{\mathcal{A}_\alpha\}$  stabilizes. Define  $\mathcal{A} = \mathcal{A}_{\alpha_0}$ , where  $A_\beta = A_{\alpha_0}$  for any ordinal  $\beta \geq \alpha_0$ . Then  $\mathcal{A}$  is contained in  $\mathcal{C}$ . On the other hand,  $\mathcal{A}$  is clearly a  $\sigma$ -algebra. Therefore, any Borel set, an element of the minimal  $\sigma$ -algebra which contains  $\mathcal{A}_0$ , belongs to  $\mathcal{A}$ , and hence to  $\mathcal{C}$ . □

The expectation of  $W^x$  is denoted by  $E^x$ . Applying Lemma 3.1, one can show that for any bounded Borel function  $f : M \rightarrow \mathbf{R}$ , its diffusion  $D_t f$  is bounded Borel, where

$$(D_t f)(x) = E^x[f(X_t)] = \int_M p_t(x, y) f(y) d\text{vol}(y).$$

More generally, the diffusion operator  $D_t$  defines a semigroup of contractions on the space  $L^p(M, m)$  ( $1 \leq p < \infty$ ) for a harmonic measure  $m$  and on  $C(M)$ , the space of continuous functions [C].

Since  $\{W^x\}$  is a Borel system of measures, by integrating  $W^x$  over any probability measure  $m$  on  $M$ , we get a probability measure  $\mathbf{P}_m$  on  $\Omega$ , i.e.,

$$\mathbf{P}_m = \int_M W^x dm(x).$$

Precisely, for any bounded Borel function  $F : \Omega \rightarrow \mathbf{R}$ ,

$$\mathbf{P}_m(F) = \int_M E^x[F] dm(x).$$

For  $t \geq 0$  let  $\theta_t : \Omega \rightarrow \Omega$  denote the shift map by  $t$ , i.e.,  $(\theta_t \omega)(t') = \omega(t + t')$ .

**THEOREM 3.2.** *The probability measure  $m$  is harmonic if and only if the probability measure  $\mathbf{P}_m$  is  $\theta_t$ -invariant for any  $t \geq 0$ .*

For the proof, see [CC, Theorem 2.3.7].

A harmonic measure  $m$  is called *ergodic* if whenever it is written as a nontrivial linear combination of two harmonic measures  $m_1$  and  $m_2$ , we have  $m = m_1 = m_2$ .

**THEOREM 3.3.** *Let  $m$  be a harmonic measure. Then the following conditions are equivalent.*

- (1)  $m$  is ergodic.
- (2) For any saturated Borel set  $M'$  in  $M$ , we have either  $m(M') = 0$  or  $m(M') = 1$ .
- (3) If  $f \in L^1(M, m)$  satisfies  $D_t f = f$  for any  $t \geq 0$ , then  $f$  is a constant.
- (4)  $\mathbf{P}_m$  is ergodic with respect to the semiflow defined by the shift map  $\theta_t$ , i.e., if a Borel subset  $S$  satisfies  $\theta_t^{-1}(S) = S$  for any  $t \geq 0$ , then either  $\mathbf{P}_m(S) = 0$  or  $\mathbf{P}_m(S) = 1$ .

**PROOF.** (1)  $\Rightarrow$  (2) follows from Corollary 2.5, and (4)  $\Rightarrow$  (1) is immediate. The other implications (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) can be shown in exactly the same way as the proof of [F, Theorem 3.1]. □

The diffusion operator  $D_t : L^2(M, m) \rightarrow L^2(M, m)$  ( $m$  a harmonic measure) is not self-adjoint unless  $m$  is completely invariant. Its adjoint  $D_t^*$  is first considered in [K]. Let  $h$  be the characteristic harmonic function defined on the holonomy cover  $\hat{L}$  of an  $m$ -a.a. leaf  $L$ . Denote by  $\hat{p}_t$  the heat kernel on  $\hat{L}$ . We have

$$p_t(x, y) = \sum_{\hat{y}} \hat{p}_t(\hat{x}, \hat{y}),$$

where the sum is taken for all the points  $\hat{y}$  over  $y$ , and is independent of the choice of  $\hat{x}$  over  $x$ .

We shall summarize well-known properties of the heat kernel  $\hat{p}_t$  on  $\hat{L}$  which follows from the bounded geometry of  $\hat{L}$ .

**LEMMA 3.4.** *For any harmonic function  $g : \hat{L} \rightarrow \mathbf{R}$ , we have*

$$g(\hat{x}) = \int_{\hat{L}} g(\hat{y}) \hat{p}_t(\hat{x}, \hat{y}) d\text{vol}(\hat{y}) \text{ and}$$

$$\hat{p}_{t+t'}(\hat{x}, \hat{z}) = \int_{\hat{L}} \hat{p}_t(\hat{x}, \hat{y}) \hat{p}_{t'}(\hat{y}, \hat{z}) d\text{vol}(\hat{y}).$$

Now define a new heat kernel on  $\hat{L}$  by

$$\hat{q}_t(\hat{x}, \hat{y}) = \frac{h(\hat{y})}{h(\hat{x})} \hat{p}_t(\hat{x}, \hat{y}).$$

The following lemma follows immediately from Lemma 3.4.

LEMMA 3.5. *We have*

$$(3.2) \quad \int_{\hat{L}} \hat{q}_t(\hat{x}, \hat{y}) \, d\text{vol}(\hat{y}) = 1 \text{ and}$$

$$(3.3) \quad \hat{q}_{t+t'}(\hat{x}, \hat{z}) = \int_{\hat{L}} \hat{q}_t(\hat{x}, \hat{y}) \hat{q}_{t'}(\hat{y}, \hat{z}) \, d\text{vol}(\hat{y}).$$

Define a heat kernel  $q_t$  on the leaf  $L$  by

$$q_t(x, y) = \sum_{\hat{y}} \hat{q}_t(\hat{x}, \hat{y}).$$

THEOREM 3.6. *The dual operator  $D_t^*$  is expressed for any  $f \in L^2(M, m)$  as*

$$(D_t^* f)(x) = \int_L q_t(x, y) f(y) \, d\text{vol}(y),$$

where  $L$  is the leaf through  $x$ .

Although this theorem is known to Vadim Kaimanovich, we shall include a proof, since there seems to be none in the literature.

Let  $\mathbf{G}$  denote the holonomy groupoid associated to the lamination  $\mathcal{L}$ , i.e.,  $\mathbf{G}$  is the space of leafwise paths modulo same end points and identical holonomy germs. Denote by  $r, s : \mathbf{G} \rightarrow M$  the range and the source maps. The fiber  $s^{-1}(x)$  is homeomorphic to the holonomy cover of the leaf through  $x$ , and the corresponding volume form of  $s^{-1}(x)$  is denoted by  $\text{vol}_x$ . Integrating these forms (seen as measures) over the harmonic measure  $m$  of  $M$ , we get a measure  $m_{\mathbf{G}}$  on  $\mathbf{G}$ . That is,

$$m_{\mathbf{G}} = \int_M \text{vol}_x \, dm(x).$$

Likewise we define a measure  $\text{vol}^y$  on  $r^{-1}(y)$ . Define a function  $\varphi : \mathbf{G} \rightarrow \mathbf{R}$  by  $\varphi([\gamma]) = h(\gamma(1))/h(\gamma(0))$ , where  $h$  is the characteristic harmonic function which is defined on the holonomy cover of an  $m$ -a.a. leaf. The function  $\varphi$  is well defined by Proposition 2.8 (2). Denote by  $J : \mathbf{G} \rightarrow \mathbf{G}$  the inverse map.

LEMMA 3.7. *We have  $Jm_{\mathbf{G}} = \varphi \cdot m_{\mathbf{G}}$ .*

PROOF. For an arbitrary  $[\gamma] \in \mathbf{G}$ , choose a neighbourhood  $U \times V \times Z$  of  $[\gamma]$  in  $\mathbf{G}$ , where  $U \times Z$  is a local chart containing  $\gamma(0)$  so that the holonomy along  $\gamma$  is defined on  $Z$  and  $V$  is a leafwise neighbourhood of  $\gamma(1)$ . Changing the notations slightly, we consider  $U$  (resp.  $V$ ) to be a neighbourhood of  $\tilde{\gamma}(0)$  (resp.  $\tilde{\gamma}(1)$ ) in the universal cover  $\tilde{L}$  of the leaf  $L$ , where  $\tilde{\gamma}$  is a lift of  $\gamma$  to  $\tilde{L}$ . Choosing  $Z$  smaller if necessary, we may assume that there is a precompact simply connected open set  $W$  of  $\tilde{L}$  such that  $U \cup V \cup \tilde{\gamma} \subset W$  and that there is a lamination preserving embedding of  $W \times Z$  into  $M$ .



Then by Theorem 2.3,

$$m|_{W \times Z} = \int_Z h \text{vol} \, d\nu$$

for a leafwise harmonic function  $h$  and a measure  $\nu$  on  $Z$ . For  $(u, v, z) \in U \times V \times Z \subset G$ , denote  $s(u, v, z) = (u, z) = x$  and  $r(u, v, z) = (v, z) = y$ . Restricted to  $U \times V \times Z$ ,  $\text{vol}_x$  is the volume form on  $u \times V \times z$  and  $\text{vol}^y$  on  $U \times v \times z$ .

On  $U \times V \times Z$  we have

$$m_G = \int_Z \text{vol}_x \cdot h(x) \text{vol}^y \, d\nu.$$

On the other hand on  $V \times U \times Z$ ,

$$Jm_G = \int_Z \text{vol}_y \cdot h(y) \text{vol}^x \, d\nu = \int_Z \varphi \cdot \text{vol}_y \cdot h(x) \text{vol}^x \, d\nu = \varphi \cdot m_G,$$

showing the lemma. □

REMARK 3.8. The measure  $m_G$  is defined not only for a harmonic measure, but also for any probability measure  $m$  on  $M$ . It is interesting to remark that the leafwise smoothness (Proposition 2.7) of  $m$  is equivalent to a basic notion in measured groupoids, the equivalence of  $Jm_G$  with  $m_G$  [AR].

PROOF OF THEOREM 3.6. The Riemannian heat kernel on the holonomy cover of the leaf yields a function  $\check{p}_t$  on  $G$  by

$$\check{p}_t([\gamma]) = \hat{p}_t(\gamma(0), \gamma(1)).$$

Notice that  $\check{p}_t \circ J = \check{p}_t$ . Likewise, a function  $\check{q}_t$  is defined from  $\hat{q}_t$ . They satisfy  $\check{q}_t = \varphi \check{p}_t$ . Clearly we have

$$(D_t f)(x) = \int_{s^{-1}(x)} \check{p}_t f \circ r \, d\text{vol}_x.$$

Thus

$$\begin{aligned} \langle D_t f, g \rangle &= \int_M \left( \int_{s^{-1}(x)} \check{p}_t f \circ r \, d\text{vol}_x \right) g(x) dm(x) \\ &= \int_G \check{p}_t f \circ r \circ s \, dm_G = \int_G \check{p}_t f \circ s \circ g \circ r \, \varphi \, dm_G \\ &= \int_G \check{q}_t g \circ r \circ f \circ s \, dm_G = \int_M \left( \int_{s^{-1}(x)} \check{q}_t g \circ r \, d\text{vol}_x \right) f(x) dm(x) = \langle f, D_t^* g \rangle. \end{aligned}$$

Therefore we have

$$(D_t^* g)(x) = \int_{s^{-1}(x)} \check{q}_t g \circ r \, d\text{vol}_x = \int_L q_t(x, y) g(y) \, d\text{vol}(y),$$

completing the proof. □

Now let us define the reverse process. First of all extend the new heat kernel  $q_t$  to  $M \times M$  by putting  $q_t(x, y) = 0$  unless  $x$  and  $y$  lie on the same leaf. Let  $\Omega_-$  be the space of continuous leafwise paths  $\omega$  from  $(-\infty, 0]$  to  $M$ , with the random variable  $X_{-t} : \Omega_- \rightarrow M$  defined by

$X_{-t}(\omega) = \omega(-t)$  ( $t \geq 0$ ). For  $x \in M$ , define the Wiener measure  $W_x^-$  on  $\Omega_-$  using the kernel  $q_t$ , that is, for example for  $0 < t_1 < t_2$  and for any Borel sets  $B_1$  and  $B_2$  of  $M$ ,

$$W_x^-\{X_{-t_2} \in B_2, X_{-t_1} \in B_1\} = \int_{B_2} \int_{B_1} q_{t_1}(x, y)q_{t_2-t_1}(y, z) d\text{vol}(y) d\text{vol}(z).$$

Lemma 3.5 implies that  $W_x^-$  is a probability measure, a probability because of (3.2), the dropping condition guaranteed by (3.3). The kernel  $q_t$  clearly satisfies the normal estimate of Cheng, Li and Yau [CLY] since the ratio to the Riemannian heat kernel is controlled by the Harnack inequality; the logarithm of any positive harmonic function defined on the holonomy cover of any leaf of  $\mathcal{L}$  is uniformly Lipschitz (due to the uniform boundedness of geometry of leaves). Therefore the reverse Wiener measure  $W_x^-$  is concentrated on the set of continuous paths. Moreover it is concentrated on the subspace  $\Omega_{-,x} = X_0^{-1}(x)$ .

Now let  $\bar{\Omega}$  be the space of biinfinite continuous leafwise paths  $\omega : \mathbf{R} \rightarrow M$ . Denote the like defined random variable by the same letter  $X_t : \bar{\Omega} \rightarrow M$  for  $t \in \mathbf{R}$ . Also denote  $\bar{\Omega}_x = X_0^{-1}(x)$ . Then by the natural identification of  $\Omega_{-,x} \times \Omega_x$  with  $\bar{\Omega}_x$ , the product measure  $W_x^- \times W_x^+$  is considered to be a measure on  $\bar{\Omega}_x$ , or on  $\bar{\Omega}$ .

Define a probability measure  $\bar{P}_m$  on  $\bar{\Omega}$  by

$$\bar{P}_m = \int_M W_x^- \times W_x^+ dm(x).$$

Denote its expectation by  $\bar{E}_m$ . Let  $\theta_t : \bar{\Omega} \rightarrow \bar{\Omega}$  be the shift map.

PROPOSITION 3.9. *The shift map  $\theta_t : \bar{\Omega} \rightarrow \bar{\Omega}$  preserves the measure  $\bar{P}_m$ .*

PROOF. We shall raise one example of computation.

$$\begin{aligned} \bar{P}_m\{X_{-t} \in B, X_{t'} \in B'\} &= \int_M dm(x) \int_B q_t(x, y)dy \int_{B'} p_{t'}(x, z)dz \\ &= \langle D_t^* \chi_B, D_{t'} \chi_{B'} \rangle_m = \langle \chi_B, D_t D_{t'} \chi_{B'} \rangle = \bar{P}_m\{X_0 \in B, X_{t+t'} \in B'\}. \quad \square \end{aligned}$$

THEOREM 3.10. *If  $m$  is an ergodic harmonic measure, then  $\bar{P}_m$  is ergodic with respect to the flow  $\{\theta_t\}$ .*

Before starting the proof, we recall the definition of conditional expectations. Denote by  $\bar{\mathcal{F}}$  the  $\sigma$ -algebra formed by the  $\bar{P}_m$ -measurable subsets. For  $t \in \mathbf{R}$ , let  $\bar{\mathcal{F}}_t$  be the minimal complete  $\sigma$ -algebra for which the map  $X_s$  is measurable for any  $s \geq t$ .

For example, in order to understand  $\bar{\mathcal{F}}_0$ , consider the measurable partition of  $\bar{\Omega}$  defined by the natural projection  $\pi : \bar{\Omega} \rightarrow \Omega$ . Then  $\bar{\mathcal{F}}_0$  consists of measurable subsets saturated by this partition. A function  $F$  is  $\bar{\mathcal{F}}_0$ -measurable if and only if there is a measurable function  $H$  on  $\Omega$  such that  $F = H \circ \pi$ .

For any integrable function  $F : \bar{\Omega} \rightarrow \mathbf{R}$ , denote by  $\bar{E}_m [F | \bar{\mathcal{F}}_t]$  the conditional expectation with respect to  $\bar{\mathcal{F}}_t$ . This is a unique  $\bar{\mathcal{F}}_t$ -measurable function on  $\bar{\Omega}$  such that for any bounded  $\bar{\mathcal{F}}_t$ -measurable function  $G$ ,

$$\bar{E}_m [G \bar{E}_m [F | \bar{\mathcal{F}}_t]] = \bar{E}_m [GF].$$

Let us explain it briefly for the convenience of the geometer readers.  $\bar{\mathcal{F}}_t$  defines a measurable partition of  $\bar{\Omega}$ : almost all classes of the partition admit the conditional probability measure. Integrating  $F$  by the conditional probability measure, we obtain a measurable function on the quotient space. But it is customary, more convenient, to consider it to be a  $\bar{\mathcal{F}}_t$ -measurable function  $\bar{E}_m [F | \bar{\mathcal{F}}_t]$  defined on the total space  $\bar{\Omega}$ .

PROOF OF THEOREM 3.10. For an integrable function  $F$  on  $\bar{\Omega}$ , define the Birkhoff average  $\mathbf{B}F$  by

$$\mathbf{B}F = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t F \circ \theta_s ds .$$

By the ergodic theorem, the operator  $\mathbf{B}$  is a well-defined contraction on  $L^1(\bar{\Omega}, \bar{\mathbf{P}}_m)$ , which is  $\theta_t$ -invariant.

Since by Theorem 3.3,  $\theta_t$  is ergodic in  $(\Omega, \mathbf{P}_m)$ , the Birkhoff average  $\mathbf{B}F$  is constant if  $F$  is  $\bar{\mathcal{F}}_0$ -measurable. Moreover, this holds for any  $\bar{\mathcal{F}}_{-t}$ -measurable function  $F$  for any  $t$ , since then  $F \circ \theta_t$  is  $\bar{\mathcal{F}}_0$ -measurable and  $\mathbf{B}F = \mathbf{B}(F \circ \theta_t)$ .

For any bounded  $\bar{\mathcal{F}}$ -measurable function  $F$ , the  $\bar{\mathcal{F}}_{-n}$ -measurable function  $F_{-n} = \bar{E}_m [F | \bar{\mathcal{F}}_{-n}]$  converges to  $F$  pointwise, by the martingale convergence theorem [O, Appendix C]. Thus we have  $\mathbf{B}F_{-n} \rightarrow \mathbf{B}F$ , and since  $\mathbf{B}F_{-n}$  is constant, the function  $\mathbf{B}F$  is also constant, showing the ergodicity. □

Applying the Birkhoff theorem to  $f \circ X_0 : \bar{\Omega} \rightarrow \mathbf{R}$  for a continuous function  $f : M \rightarrow \mathbf{R}$ , by virtue of Theorem 3.10, we have  $\bar{\mathbf{P}}_m$ -almost surely

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X_s) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{-t}^0 f(X_s) ds = m(f) .$$

Equivalently, denoting the Dirac mass by  $\delta_\cdot$ , we have  $\bar{\mathbf{P}}_m$ -almost surely

$$(3.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \delta_{X_s} ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{-t}^0 \delta_{X_s} ds = m ,$$

where the limit is taken in the space of the probability measures on  $M$  with the weak- $*$  topology.

Finally let us define an exponent for the biinfinite Brownian motion. Assume  $m$  is an ergodic harmonic measure of  $(M, \mathcal{L}, g)$  and  $h$  the characteristic harmonic function of  $m$ . Given  $\omega \in \bar{\Omega}$  and a positive number  $t$ , the ratio  $h(X_t(\omega))/h(X_0(\omega))$  is well defined by Proposition 2.8 since a path from  $X_0(\omega)$  to  $X_t(\omega)$  is specified by  $\omega$ . Define a random variable  $A_t : \bar{\Omega} \rightarrow \mathbf{R}$  by

$$A_t = \log h(X_t) - \log h(X_0) .$$

Let us show that  $A_t \in L^1(\bar{\Omega}, \bar{\mathbf{P}}_m)$ . Denote the expectation of  $W^x \times W_-^x$  by  $\bar{E}^x$ . Since  $A_t$  is  $\bar{\mathcal{F}}_0$ -measurable, we have  $\bar{E}^x[A_t] = E^x[A_t]$ , where  $E^x$  is the expectation of  $W^x$  defined before. By the Harnack inequality we have

$$E^x[|A_t|] \leq C_1 E^x[d(X_0, X_t)] \leq C_2 t ,$$

where  $d$  is the leafwise distance on the universal cover of the leaf induced from  $g$ . The last inequality follows from the bounded geometry of the leaf. Thus we have

$$\bar{E}_m [|A_t|] = \int_M E^x [|A_t|] dm(x) \leq C_2 t,$$

showing that  $A_t \in L^1(\bar{\Omega}, \bar{P}_m)$ .

Now  $A_t$  satisfies

$$(3.5) \quad A_{t+t'} = A_t + A_{t'} \circ \theta_t.$$

This shows that  $\bar{E}_m [A_t]$  is additive in  $t$ . Moreover, it is continuous in  $t$  at  $t = 0$  since  $E^x [d(X_0, X_t)] \rightarrow 0$  as  $t \rightarrow 0$ . That is,  $\bar{E}_m [A_t] = -\lambda t$  for some number  $\lambda$ .

PROPOSITION 3.11. *We have  $\lim_{t \rightarrow \infty} (1/t)A_t = -\lambda$  almost surely, and  $\lambda \geq 0$ . Furthermore,  $\lambda$  is positive unless  $m$  is completely invariant.*

PROOF. The first statement follows from (3.5) by the Birkhoff ergodic theorem.

To show  $\lambda \geq 0$  notice that

$$\int_M E^x [A_t] dm(x) = \bar{E}_m [A_t] = -\lambda t.$$

The expectation  $E^x [A_t]$  can be computed upstairs on the holonomy cover. Let  $\hat{x}$  be a lift of  $x$  and  $\hat{X}_t(\omega)$  the lift of  $X_t(\omega)$  starting at  $\hat{x}$  for  $\omega \in \Omega_x$ . Then

$$\begin{aligned} E^x [A_t] &= E^x [\log h(\hat{X}_t)] - \log h(\hat{x}) \\ &\leq \log E^x [h(\hat{X}_t)] - \log h(\hat{x}) = \log(\hat{D}_t h)(\hat{x}) - \log h(\hat{x}) = 0, \end{aligned}$$

where  $\hat{D}_t$  is the diffusion operator on the holonomy cover. The inequality follows from the concavity of  $\log$ , and the last equality from the harmonicity of  $h$ , showing  $\lambda \geq 0$ .

For the last statement, notice that  $\lambda = 0$  implies that for fixed  $t$ ,  $h(\hat{X}_t)$  is constant  $W^x$ -almost surely. This shows that  $h$  is constant for the holonomy cover of an  $m$ -a.a. leaf, completing the proof.  $\square$

For  $-t < 0$  define a random variable  $A_{-t} : \bar{\Omega} \rightarrow \mathbf{R}$  by

$$A_{-t} = \log h(X_{-t}) - \log h(X_0).$$

It satisfies

$$(3.6) \quad A_{-t-t'} = A_{-t} + A_{-t'} \circ \theta_{-t}.$$

Clearly  $\bar{E}_m [A_{-t}] = \lambda$ , and again by the Birkhoff ergodic theorem we have from (3.6) the following proposition.

PROPOSITION 3.12.  *$\bar{P}_m$ -almost surely,  $\lim_{t \rightarrow \infty} (1/t)A_{-t} = \lambda$ .*

Propositions 3.11 and 3.12 imply that for an  $m$ -a.a. point  $x$ , we have  $W^x \times W_-^x$ -almost surely

$$\lim_{t \rightarrow \infty} (1/t)A_t = -\lambda \quad \text{and} \quad \lim_{t \rightarrow \infty} (1/t)A_{-t} = \lambda,$$

showing the following theorem.

**THEOREM 3.13.** *For a non completely invariant ergodic harmonic measure, the characteristic harmonic function is unbounded on the holonomy cover of an  $m$ -a.a. leaf.*

**4. Hyperbolic laminations.** Henceforth in this paper, we only consider a compact hyperbolic  $C^2$  lamination  $(M, \mathcal{L}, g)$ , i.e., we assume throughout that the leafwise metric  $g$  has constant curvature  $-1$ , and denote the dimension of leaves by  $d+1$ . Let  $m$  be an ergodic harmonic measure for  $\mathcal{L}$ . The universal cover of an  $m$ -a.a. leaf  $L$  is identified with the simply connected complete hyperbolic space  $\mathbf{D}^{d+1}$ , and the characteristic harmonic function  $h$  of  $m$  is defined on  $\mathbf{D}^{d+1}$ . Choose a base point  $\tilde{x} \in \mathbf{D}^{d+1}$  and assume  $h(\tilde{x}) = 1$ . For any point  $\xi$  of the ideal boundary  $\mathbf{S}_\infty^d$ , let  $k_\xi$  denote the minimal positive harmonic function on  $\mathbf{D}^{d+1}$  corresponding to  $\xi$  normalized to take value 1 at  $\tilde{x}$ . In other words, we set  $k_\xi = \exp(-dB_\xi)$ , where  $B_\xi$  is the Buseman function corresponding to  $\xi$  such that  $B_\xi(\tilde{x}) = 0$ . Then there is a unique probability measure  $\mu_{\tilde{x}}$  on  $\mathbf{S}_\infty^d$  such that

$$(4.1) \quad h = \int_{\mathbf{S}_\infty^d} k_\xi d\mu_{\tilde{x}}(\xi).$$

See [AS] for details and related topics. Although the measure  $\mu_{\tilde{x}}$  depends on the choice of the point  $\tilde{x}$ , its equivalence class  $[\mu_L]$  is an invariant of the leaf  $L$ . Here two measures  $\mu_1$  and  $\mu_2$  on  $\mathbf{S}_\infty^d$  are said to be equivalent if for any Borel subset  $B$  of  $\mathbf{S}_\infty^d$ ,  $\mu_1(B) = 0$  if and only if  $\mu_2(B) = 0$ . In fact, for another point  $\tilde{y} \in \mathbf{D}^{d+1}$ , we have

$$(4.2) \quad h/h(\tilde{y}) = \int_{\mathbf{S}_\infty^d} k_\xi/k_\xi(\tilde{y})d\mu_{\tilde{y}}(\xi).$$

The uniqueness of the measure  $\mu_{\tilde{x}}$  implies by (4.1) and (4.2) that

$$\mu_{\tilde{x}} = (h(\tilde{y})/k_\xi(\tilde{y}))\mu_{\tilde{y}},$$

showing that  $\mu_{\tilde{x}}$  and  $\mu_{\tilde{y}}$  differ by a multiple of a bounded positive function, that is, they are equivalent.

**THEOREM 4.1.** *For a non completely invariant ergodic harmonic measure  $m$  on a compact hyperbolic lamination  $(M, \mathcal{L}, g)$  and for an  $m$ -a.a. leaf  $L$ , the measure class  $[\mu_L]$  is singular to the Lebesgue measure of  $\mathbf{S}_\infty^d$ .*

Before starting the proof, we need to study connections among the probability measures on  $\mathbf{S}_\infty^d$ , positive harmonic functions on  $\mathbf{D}^{d+1}$  and the Wiener measures.

Denote by  $\mathcal{P}(\mathbf{S}_\infty^d)$  the space of probability measures on  $\mathbf{S}_\infty^d$ , a compact metrizable convex space by the weak- $*$  topology. Denote by  $\mathcal{PH}$  the space of the positive harmonic function on  $\mathbf{D}^{d+1}$  taking value 1 at  $\tilde{x}$ , also a compact metrizable convex space by the compact open topology, (compact thanks to the Harnack inequality). The map  $\varphi_1 : \mathcal{P}(\mathbf{S}_\infty^d) \rightarrow \mathcal{PH}$  defined by

$$\varphi_1(\mu) = \int_{\mathbf{S}_\infty^d} k_\xi d\mu(\xi)$$

is an affine homeomorphism.

For any  $f \in \mathcal{PH}$ , define a heat kernel  $q_t$  on  $\mathbf{D}^{d+1}$  by

$$q_t(u, v) = \frac{f(v)}{f(u)} p_t(u, v),$$

where  $p_t$  is the Riemannian heat kernel and  $u$  and  $v$  are points of  $\mathbf{D}^{d+1}$ . The heat kernel defines a Wiener measure  $W_f^u$  for each point  $u \in \mathbf{D}^{d+1}$ . Denote by  $\Omega_{\tilde{x}}$  the space of continuous paths  $\omega : [0, \infty) \rightarrow \mathbf{D}^{d+1}$  such that  $\omega(0) = \tilde{x}$  and by  $\mathcal{P}(\Omega_{\tilde{x}})$  the space of probability measures on  $\Omega_{\tilde{x}}$ . Then easy calculation shows that the map  $\varphi_2 : \mathcal{PH} \rightarrow \mathcal{P}(\Omega_{\tilde{x}})$  defined by

$$\varphi_2(f) = W_f^{\tilde{x}}$$

is affine. (This is just for the base point  $\tilde{x}$  where  $\mathcal{PH}$  is normalized.)

Now let  $\Omega_{\tilde{x}}^\infty$  denote the subspace of  $\Omega_{\tilde{x}}$  consisting of those paths  $\omega$  in  $\Omega_{\tilde{x}}$  such that  $\lim_{t \rightarrow \infty} \omega(t)$  exists in  $\mathbf{S}^d_\infty$ . Let us show that for any  $f \in \mathcal{PH}$ , the set  $\Omega_{\tilde{x}}^\infty$  is  $W_f^{\tilde{x}}$ -conull. As is well known, this is true for  $f = k_\xi$  for any  $\xi \in \mathbf{S}^d_\infty$ , but any measure  $W_f^{\tilde{x}}$  is written as the convex integration

$$W_f^{\tilde{x}} = \int_{\mathbf{S}^d_\infty} W_{k_\xi}^{\tilde{x}} d\mu(\xi)$$

for some  $\mu \in \mathcal{P}(\mathbf{S}^d_\infty)$  since  $\varphi_1$  and  $\varphi_2$  are affine, showing the claim in the general case.

Denoting by  $X_\infty : \Omega_{\tilde{x}}^\infty \rightarrow \mathbf{S}^d_\infty$  the hitting map, we define an affine map  $\varphi_3 : \varphi_2(\mathcal{PH}) \rightarrow \mathcal{P}(\mathbf{S}^d_\infty)$  by  $\varphi_3(W_f^{\tilde{x}}) = X_\infty W_f^{\tilde{x}}$ .

Then the composite  $\varphi_3 \circ \varphi_2 \circ \varphi_1$  is the identity on  $\mathcal{P}(\mathbf{S}^d_\infty)$ , since this is true for the point masses, the map  $\varphi_3 \circ \varphi_2 \circ \varphi_1$  is affine, and any measure in  $\mathcal{P}(\mathbf{S}^d_\infty)$  is a convex integral of the point masses.

**PROOF OF THEOREM 4.1.** Let  $m$  be a non completely invariant ergodic harmonic measure of a compact hyperbolic lamination  $(M, \mathcal{L}, g)$ , and let  $\mathbf{D}^{d+1}$  be the universal cover of an  $m$ -a.a. leaf  $L$ . A base point  $\tilde{x} \in \mathbf{D}^{d+1}$  is chosen and the characteristic harmonic function  $h$  normalized at  $\tilde{x}$  is written as (4.1) using a probability measure  $\mu_{\tilde{x}}$ . The Wiener measure  $W_h^{\tilde{x}}$  defined by the characteristic harmonic function  $h$  corresponds to the measure  $W_{-}^{\tilde{x}}$  of the reverse process in Section 3. As before, denote by  $W^{\tilde{x}}$  the usual Riemannian Wiener measure. Then by Propositions 3.11 and 3.12, for an appropriate choice of  $\tilde{x}$  we have  $W^{\tilde{x}}$ -almost surely

$$(4.3) \quad \lim_{t \rightarrow \infty} (1/t) \log(h(X_t)) = -\lambda,$$

while  $W_h^{\tilde{x}}$ -almost surely

$$(4.4) \quad \lim_{t \rightarrow \infty} (1/t) \log(h(X_t)) = \lambda,$$

where  $\lambda$  is the characteristic exponent, positive in our case.

On one hand, the hitting measure  $X_\infty W^{\tilde{x}}$  of the Riemannian Wiener measure  $W^{\tilde{x}}$  coincides with the visible measure  $\mu_0$  at  $\tilde{x}$ , which is equivalent to the Lebesgue measure. On the

other hand, the other hitting measure  $X_\infty W_h^{\tilde{x}}$  is the measure  $\mu_{\tilde{x}}$ . Thus we have

$$W^{\tilde{x}} = \int_{S_\infty^d} W_{k_\xi}^{\tilde{x}} d\mu_0(\xi) \text{ and } W_h^{\tilde{x}} = \int_{S_\infty^d} W_{k_\xi}^{\tilde{x}} d\mu_{\tilde{x}}(\xi).$$

That is, for a  $\mu_0$ -a.a. point  $\xi$ , a  $W_{k_\xi}^{\tilde{x}}$ -a.a. path satisfies (4.3), while for a  $\mu_{\tilde{x}}$ -a.a. point  $\xi$ , a  $W_{k_\xi}^{\tilde{x}}$ -a.a. path satisfies (4.4), showing that the two measures  $\mu_0$  and  $\mu_{\tilde{x}}$  are mutually singular.  $\square$

**5. The leafwise unit tangent bundle of a hyperbolic lamination.** Associated with a compact hyperbolic lamination  $(M, \mathcal{L}, g)$ , there is defined the leafwise unit tangent bundle  $N$  of  $\mathcal{L}$  and the stable foliation  $\mathcal{H}$  on  $N$ . The space  $M$  is covered by open sets  $E_i$  on which the local charts  $\varphi_i : E_i \rightarrow U_i \times Z_i$  are defined. For a hyperbolic lamination, we can assume that each  $U_i$  is an open (precompact) ball in the hyperbolic space  $\mathbf{D}^{d+1}$  and the transition function  $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$  wherever defined is of the form

$$(5.1) \quad \psi_{ji}(u, z) = (g(z)u, \beta(z)),$$

where  $g(z)$  is an element of the Lie group  $G$  of the orientation preserving isometries of  $\mathbf{D}^{d+1}$ . The leafwise unit tangent bundle  $N$  of  $\mathcal{L}$  is defined from the collection of spaces  $T^1(U_i) \times Z_i$  by glueing them using the transition function  $\psi_{ji}$  defined by the same expression as (5.1), where  $T^1(U_i)$  is the unit tangent bundle of  $U_i$ .

Notice that the tangent bundle  $T^1(\mathbf{D}^{d+1})$  is  $G$ -equivariantly identified with  $\mathbf{D}^{d+1} \times S_\infty^d$  by assigning to a unit tangent vector the couple  $(\pi(v), v_\infty)$ , where  $\pi : T^1(\mathbf{D}^{d+1}) \rightarrow \mathbf{D}^{d+1}$  is the bundle projection and  $v_\infty \in S_\infty^d$  is the hitting point of the geodesic ray whose initial vector is  $v$ .

Thus a local chart  $T^1(U_i) \times Z_i$  is identified with  $U_i \times S_\infty^d \times Z_i$ . Then the transition function becomes

$$\psi_{ji}(u, \xi, z) = (g(z)u, g(z)\xi, \beta(z)).$$

The plaques of the form  $U_i \times \xi \times z$  are incorporated to a lamination  $\mathcal{H}$  of  $N$ , called the *stable foliation* of  $\mathcal{L}$ .

The canonical projection  $p : N \rightarrow M$  yields a submersion of a leaf of  $\mathcal{H}$  onto a leaf of  $\mathcal{L}$ , and thus the leafwise Riemannian metric  $g$  of  $\mathcal{L}$  can be pulled up to a leafwise Riemannian metric  $\check{g}$  of  $\mathcal{H}$ , the triplet  $(N, \mathcal{H}, \check{g})$  being a compact hyperbolic lamination. The leafwise volume form of  $\mathcal{H}$  is again denoted by  $\text{vol}$ .

As before,  $k_\xi$  denotes the minimal positive harmonic function associated to the point  $\xi \in S_\infty^d$  normalized at the point  $\tilde{x}$ .

**DEFINITION 5.1.** A harmonic measure  $\lambda$  on  $N$  is called *pointed harmonic* if for each local chart  $U \times S_\infty^d \times Z$ ,  $\lambda$  disintegrates on a plaque  $U \times \xi \times z$  to a probability measure which is a constant times  $k_\xi \text{vol}$ .

The purpose of this section is to establish a one to one correspondence between harmonic measures of  $\mathcal{L}$  and pointed harmonic measures of  $\mathcal{H}$ . We begin with a harmonic measure  $m$  of  $\mathcal{L}$ , and associate it to a pointed harmonic measure upstairs. Let  $x$  be a point on an  $m$ -a.a. leaf

$L$  of  $\mathcal{L}$ , and let  $\tilde{x}$  be a lift of  $x$  to the universal cover  $\mathbf{D}^{d+1}$  of  $L$ . Then a probability measure  $\mu_{\tilde{x}}$  on  $\mathbf{S}_{\infty}^d$  is defined using the characteristic harmonic function  $h$  normalized at  $\tilde{x}$  as in (4.1).

On the other hand, the unit tangent space  $T_x^1 L$  is identified with its lift  $T_{\tilde{x}}^1 \mathbf{D}^{d+1}$ , and the latter with  $\mathbf{S}_{\infty}^d$  by the visible map. By these identifications, the measure  $\mu_{\tilde{x}}$  on  $\mathbf{S}_{\infty}^d$  corresponds to a measure  $\mu_x$  on  $T_x^1 L$ , the notation being justified by the following lemma.

LEMMA 5.2. *The measure  $\mu_x$  is independent of the choice of a lift  $\tilde{x}$  of  $x$ .*

PROOF. We have only to prove that if  $\gamma$  is a deck transformation of the covering map  $\mathbf{D}^{d+1} \rightarrow L$ , then  $\mu_{\gamma\tilde{x}} = \gamma\mu_{\tilde{x}}$ . In this proof, we need a refined notation: the minimal positive harmonic function associated to  $\xi \in \mathbf{S}_{\infty}^d$  is denoted by  $k_{\xi, \tilde{x}}$  in order to keep in mind the point  $\tilde{x}$  where it is normalized. Clearly we have

$$k_{\gamma\xi, \gamma\tilde{x}} \circ \gamma = k_{\xi, \tilde{x}}.$$

On the other hand, by the definition of  $\mu_{\tilde{x}}$ , the characteristic harmonic function  $h$  normalized at  $\tilde{x}$  is given by

$$h = \int_{\mathbf{S}_{\infty}^d} k_{\xi, \tilde{x}} d\mu_{\tilde{x}}(\xi).$$

Therefore,

$$(5.2) \quad h \circ \gamma^{-1} = \int_{\mathbf{S}_{\infty}^d} k_{\gamma\xi, \gamma\tilde{x}} d\mu_{\tilde{x}}(\xi) = \int_{\mathbf{S}_{\infty}^d} k_{\xi, \gamma\tilde{x}} d(\gamma\mu_{\tilde{x}})(\xi).$$

Now by Corollary 2.9,  $h \circ \gamma^{-1}$  is a constant multiple of  $h$ , normalized at the point  $\gamma\tilde{x}$ . Therefore, we have

$$(5.3) \quad h \circ \gamma^{-1} = \int_{\mathbf{S}_{\infty}^d} k_{\xi, \gamma\tilde{x}} d\mu_{\gamma\tilde{x}}(\xi).$$

Comparing (5.2) with (5.3), the uniqueness of the probability measure shows that  $\mu_{\gamma\tilde{x}} = \gamma\mu_{\tilde{x}}$ . □

The inclusion  $T_x^1 \mathcal{L} \hookrightarrow N$  induces a map from  $\mathcal{P}(T_x^1 \mathcal{L})$  to  $\mathcal{P}(N)$  among the spaces of the probability measures. The image of  $\mu_x$  by this map is also denoted by the same letter, by abuse of notations.

Recall that if  $(X, \mu)$  is a measured space and  $(Z, \mathcal{B})$  is a Borel space, then a map  $\psi : X \rightarrow Z$  is called measurable if for any  $B \in \mathcal{B}$ ,  $\psi^{-1}(B)$  is a measurable set. Of course this depends only on the equivalence class of the measure  $\mu$ . If  $Z = \mathcal{P}(Y)$ , the space of the probability measures of a compact metric space  $Y$ , then  $\psi : X \rightarrow \mathcal{P}(Y)$  is said to be measurable if it is measurable with respect to the Borel structure of  $\mathcal{P}(Y)$  associated with the weak- $*$  topology. This is equivalent to saying that  $x \mapsto \psi(x)(f)$  is measurable for any continuous function  $f$  on  $Y$ .

LEMMA 5.3. *The assignment  $M \ni x \mapsto \mu_x \in \mathcal{P}(N)$  is measurable with respect to the harmonic measure  $m$ .*



PROOF. Since for any local chart  $U \times Z$  of  $\mathcal{L}$ ,  $U$  is assumed to be a domain in  $\mathbf{D}^{d+1}$ , the inclusion map of  $U \times Z$  into  $M$  can be extended using leafwise geodesics to a lamination preserving submersion  $\varphi : \mathbf{D}^{d+1} \times Z \rightarrow M$  in such a way that it is a local isometry on each leaf. The set  $\mathbf{D}^{d+1} \times Z$  is called a *prolonged local chart* of  $\mathcal{L}$ . Associated to it, we have a prolonged local chart  $\mathbf{D}^{d+1} \times \mathbf{S}_\infty^d \times Z$  for  $\mathcal{H}$ .

By Theorem 2.3, the harmonic measure  $m$  restricted to a local chart  $U \times Z$  is given by

$$m|_{U \times Z} = \int_Z h \text{ vol } dv,$$

where  $h$  is a measurable function defined on  $U \times Z$ , harmonic on a plaque  $U \times z$  for  $\nu$ -a.a.  $z$ . For the prolonged local chart  $\mathbf{D}^{d+1} \times Z$ , let  $m|_{\mathbf{D}^{d+1} \times Z}$  be the *lift* of  $m$  to  $\mathbf{D}^{d+1} \times Z$ , i.e., the integral of the counting measure on the fiber of the submersion  $\mathbf{D}^{d+1} \times Z \rightarrow M$  over  $m$ . Then we still have

$$(5.4) \quad m|_{\mathbf{D}^{d+1} \times Z} = \int_Z h \text{ vol } dv,$$

where  $h$  is an obvious extension. Notice that a slight generalization of Theorem 2.3 shows that  $h$  is measurable with respect to  $m|_{\mathbf{D}^{d+1} \times Z}$ .

Denote by  $\mathcal{PH}_u$  the space of positive harmonic functions taking value 1 at  $u \in \mathbf{D}^{d+1}$ . Then there is an affine homeomorphism of  $\mathcal{PH}_u$  with  $\mathcal{P}(\mathbf{S}_\infty^d)$ . Let  $(u, z) \in \mathbf{D}^{d+1} \times Z$  corresponds to  $x \in M$  by the submersion. The measure  $\mu_x = \mu_{(u,z)}$  of  $\mathbf{S}_\infty^d$  is associated to the function  $h(\cdot, z)/h(u, z) \in \mathcal{PH}_u$  by the above homeomorphism.

SUBLEMMA 5.4. The assignment  $\mathbf{D}^{d+1} \times Z \ni (u, z) \mapsto \mu_{(u,z)} \in \mathcal{P}(\mathbf{S}_\infty^d)$  is measurable with respect to  $m|_{\mathbf{D}^{d+1} \times Z}$ .

PROOF. The measure  $m|_{\mathbf{D}^{d+1} \times Z}$  is equivalent to  $\text{vol} \otimes \nu$ . Therefore by Fubini, there is a vol-conull subset  $\mathbf{D}_*^{d+1}$  such that for any poin  $u \in \mathbf{D}_*^{d+1}$ , the set  $\{z \in Z; h(u, z) < \alpha\}$  is  $\nu$ -measurable for any  $\alpha \in \mathcal{Q}$ . Then it is routine to show that for any  $u \in \mathbf{D}_*^{d+1}$  and  $\alpha \in \mathcal{R}$ , the set  $\{z \in Z; h(u, z) < \alpha\}$  is  $\nu$ -measurable.

For any  $u \in \mathbf{D}_*^{d+1}$ , the assignment to  $z \in Z$  of the harmonic function  $h(\cdot, z)/h(u, z)$  in  $\mathcal{PH}_u$  is  $\nu$ -measurable with respect to the  $\sigma$ -algebra  $\mathcal{B}(\mathcal{PH}_u)$  of the pointwise convergence topology on  $\mathbf{D}_*^{d+1}$ . In fact, for any  $v \in \mathbf{D}_*^{d+1}$  and  $a > 0$ , the set

$$\{z \in Z; h(v, z) > ah(u, z)\} = \bigcup_{\alpha \in \mathcal{Q}} (\{z \in Z; h(v, z) \geq \alpha\} \cap \{z \in Z; ah(u, z) < \alpha\})$$

is  $\nu$ -measurable.

The  $\sigma$ -algebra  $\mathcal{B}(\mathcal{PH}_u)$  coincides with the  $\sigma$ -algebra of the compact open topology. In fact, for  $(a, b) \in \mathcal{R}$  and a compact ball  $D$  of  $\mathbf{D}^{d+1}$ , the set

$$\mathcal{PH}_u(D, (a, b)) = \{f \in \mathcal{PH}_u; f(D) \subset (a, b)\}$$

belongs to  $\mathcal{B}(\mathcal{PH}_u)$  since, for a subset  $\{u_j\}_{j \in \mathbb{N}} \subset \mathbf{D}_*^{d+1} \cap D$  dense in  $D$ , we have

$$\begin{aligned} \mathcal{PH}_u(D, (a, b)) &= \bigcup_{n \in \mathbb{N}} \{f \in \mathcal{PH}_u; f(D) \subset [a + n^{-1}, b - n^{-1}]\} \\ &= \bigcup_{n \in \mathbb{N}} \bigcap_j \{f \in \mathcal{PH}_u; f(u_j) \in [a + n^{-1}, b - n^{-1}]\}, \end{aligned}$$

and this subset belongs to  $\mathcal{B}(\mathcal{PH}_u)$ . A general compact subset  $K \subset \mathbf{D}^{d+1}$  can be written as the decreasing intersection of finite unions of compact balls  $D_n$ , and the like defined set  $\mathcal{PH}_u(K, (a, b))$  also belongs to  $\mathcal{B}(\mathcal{PH}_u)$ , since

$$\mathcal{PH}_u(K, (a, b)) = \bigcup_n \mathcal{PH}_u(D_n, (a, b)).$$

The space  $\mathcal{PH}_u$  with the compact open topology is homeomorphic to the space  $\mathcal{P}(\mathbf{S}_\infty^d)$  with the weak- $*$  topology. This shows the  $\nu$ -measurability of  $\mu_{(u,z)}$  in the variable  $z$  for any  $u \in \mathbf{D}_*^{d+1}$ . On the other hand, the measure  $\mu_{(u,z)}$  is continuous in the variable  $u$  for any  $z \in Z$ .

Let  $f : \mathbf{S}_\infty^d \rightarrow \mathbf{R}$  be an arbitrary continuous function and fix it for a while. For any  $a \in \mathbf{R}$ , define

$$S(a) = \{(u, z) \in U \times Z; \mu_{(u,z)}(f) \geq a\}.$$

The proof of the sublemma is complete if we show that  $S(a)$  is a measurable set.

For any  $z \in Z$ , define the  $z$ -slice  $S(a)_z \subset U$  by

$$S(a) \cap (U \times z) = S(a)_z \times z.$$

Similarly, define the  $u$ -slice  $S(a)_u \subset Z$  for any  $u \in U$ . Then  $S(a)_u$  is  $\nu$ -measurable for any  $u \in \mathbf{D}_*^{d+1}$ , and  $S(a)_z$  is closed for any  $z \in Z$ . Moreover, since  $\mu_{(u,z)}(f)$  is a continuous function of  $u$ ,  $\{S(a_k)_z\}$  forms a (closed) neighbourhood system of  $\{S(a)_z\}$  for a sequence  $a_k \uparrow a$ . Choose a compact ball  $D \subset \mathbf{D}^{d+1}$  and define

$$S(a)_D = \{z \in Z; S(a)_z \cap D \neq \emptyset\}.$$

Then  $S(a)_D$  is  $\nu$ -measurable. In fact, for a dense subset  $\{u_j\}_{j \in \mathbf{N}}$  of  $D \cap \mathbf{D}_*^{d+1}$ , we have

$$S(a)_D = \{z \in Z; S(a_k)_z \cap \{u_j\} \neq \emptyset \text{ for all } k\} = \bigcap_k \bigcup_j S(a_k)_{u_j}.$$

Now let  $\{\mathcal{D}_i\}$  be a sequence of coverings of  $\mathbf{D}^{d+1}$  by countably many compact balls such that  $\text{mesh}(\mathcal{D}_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Define

$$S(a)^i = \bigcup_{D \in \mathcal{D}_i} D \times S(a)_D.$$

Then  $S(a)^i$  is measurable. On the other hand, since  $S(a)_z$  is closed, we have  $S(a)_z = \bigcap_i S(a)^i_z$ . That is,  $S(a) = \bigcap_i S(a)^i$  and  $S(a)$  is measurable, completing the proof.  $\square$

Sublemma 5.4 implies in particular for any local chart  $U \times Z$ , the assignment

$$U \times Z \ni (u, z) \mapsto \mu_{(u,z)} \in \mathcal{P}(\mathbf{S}_\infty^d)$$

is measurable. Define a map  $\iota_{(u,z)} : \mathbf{S}_\infty^d \rightarrow U \times \mathbf{S}_\infty^d \times Z$  by  $\iota_{(u,z)}(\xi) = (u, \xi, z)$ . Consider a map

$$\psi : U \times Z \times \mathcal{P}(\mathbf{S}_\infty^d) \rightarrow \mathcal{P}(U \times \mathbf{S}_\infty^d \times Z)$$

defined by  $\psi(u, z, \mu) = \iota_{(u,z)}\mu$ . Consider also a map  $\phi : \mathcal{P}(U \times \mathbf{S}_\infty^d \times Z) \rightarrow \mathcal{P}(N)$  induced by the inclusion. If  $(u, z) \in U \times Z$  corresponds to  $x \in M$ , then

$$(5.5) \quad \mu_x = (\phi \circ \psi)(u, z, \mu_{(u,z)}).$$

The proof of Lemma 5.3 is complete if we show that the RHS of (5.5) is a measurable function of  $(u, z)$ . Here we have the following sublemma.

**SUBLEMMA 5.5.** The map  $\psi : U \times Z \times \mathcal{P}(\mathbf{S}_\infty^d) \rightarrow \mathcal{P}(U \times \mathbf{S}_\infty^d \times Z)$  is continuous.

**PROOF.** Denote by  $C_c(U \times \mathbf{S}_\infty^d \times Z)$  the space of the continuous functions with compact supports. Consider a product  $f \circ p_1 g \circ p_2$  ( $f \in C_c(U \times Z)$ ,  $g \in C(\mathbf{S}_\infty^d)$ ,  $p_1 : U \times \mathbf{S}_\infty^d \times Z \rightarrow U \times Z$ ,  $p_2 : U \times \mathbf{S}_\infty^d \times Z \rightarrow \mathbf{S}_\infty^d$ , the canonical projections). Then clearly

$$\psi(u, v, \mu)(f \circ p_1 g \circ p_2) = f(u, v)\mu(g)$$

is a continuous function of  $(u, v, \mu)$ . On the other hand, finite sums of the products  $f \circ p_1 g \circ p_2$  form a dense subset of  $C_c(U \times \mathbf{S}_\infty^d \times Z)$  in the topology of the uniform convergence on compact sets. Standard argument shows that  $\psi(u, v, \mu)(F)$  is continuous for any  $F \in C_c(U \times \mathbf{S}_\infty^d \times Z)$ , finishing the proof.  $\square$

On the other hand,  $\phi$  is obviously continuous. The RHS of (5.5) is now shown to be a measurable function of  $(u, z)$ , completing the proof of Lemma 5.3.  $\square$

Integrating the measurable system of probability measures  $\{\mu_x\}_{x \in M}$  over  $m$ , we obtain a probability measure  $\lambda(m)$  of  $N$ , called the *canonical lift* of  $m$ .

**THEOREM 5.6.** For any harmonic measure  $m$  of  $M$ , the canonical lift  $\lambda(m)$  is pointed harmonic.

**PROOF.** Recall that on a prolonged local chart  $\mathbf{D}^{d+1} \times Z$ , the lift of the harmonic measure  $m$  is written as in (5.4), and the canonical lift  $\lambda(m)$  on the associated prolonged local chart  $\mathbf{D}^{d+1} \times \mathbf{S}_\infty^d \times Z$  disintegrates on  $\mathbf{D}^{d+1} \times \mathbf{S}_\infty^d \times z$  to (a constant multiple of)

$$(5.6) \quad \int_{\mathbf{D}^{d+1}} h(u)\mu_u \, d\text{vol}(u),$$

where  $\mu_u$  is the probability measure in  $\mathcal{P}(\mathbf{S}_\infty^d)$  determined by the equality

$$\frac{h(v)}{h(u)} = \int_{\mathbf{S}_\infty^d} \frac{k_\xi(v)}{k_\xi(u)} d\mu_u(\xi) \text{ for all } v \in \mathbf{D}^{d+1},$$

where  $k_\xi$  is the minimal harmonic function normalized at the base point  $\tilde{x}$ .

In order to disintegrate further the measure in (5.6) on  $\mathbf{D}^{d+1} \times \xi \times z$ , we have to transform the measure  $\mu_u$  which depends on  $u \in \mathbf{D}^{d+1}$  to a fixed measure  $\mu_{\tilde{x}}$ . First of all, we have

$$h(v) = \int_{\mathbf{S}_\infty^d} h(u) \frac{k_\xi(v)}{k_\xi(u)} d\mu_u(\xi) = \int_{\mathbf{S}_\infty^d} k_\xi(v) \frac{h(u)}{k_\xi(u)} \frac{d\mu_u}{d\mu_{\tilde{x}}}(\xi) d\mu_{\tilde{x}}(\xi).$$

Hence by the uniqueness of the probability measure, we have

$$\frac{h(u)}{k_\xi(u)} \frac{d\mu_u}{d\mu_{\tilde{x}}}(\xi) = 1,$$

showing that

$$\int_{\mathbf{D}^{d+1}} h(u)\mu_u \, d\text{vol}(u) = \int_{\mathbf{D}^{d+1}} k_\xi(u)\mu_{\tilde{x}} \, d\text{vol}(u).$$

This implies that the lift of the measure  $\lambda(m)$  disintegrates on  $\mathbf{D}^{d+1} \times \xi \times z$  to a constant multiple of  $k_\xi \text{vol}$ , completing the proof.  $\square$

Conversely, given any pointed harmonic measure on the leafwise unit tangent space  $N$ , its push down is a harmonic measure on  $M$  by Theorem 2.3. It is easy to show the following theorem by analogous computation.

**THEOREM 5.7.** *A harmonic measure on a compact hyperbolic lamination  $(M, \mathcal{L}, g)$  corresponds one to one to a pointed harmonic measure on its leafwise unit tangent bundle  $(N, \mathcal{H}, \check{g})$ , by the operations of taking the canonical lift and pushing down.*

**EXAMPLE 5.8.** If  $M$  is a closed oriented hyperbolic surface, considered as a single leaf lamination, then the unique harmonic measure  $m$  is the (normalized) area form. The canonical lift  $\lambda(m)$  on the unit tangent bundle  $T^1M$  is the (normalized) Haar measure.

**REMARK 5.9.** In case  $d = 1$ , the minimal parabolic subgroup  $P$  of  $G$  acts on the leafwise tangent bundle  $N$  of a compact 2-dimensional hyperbolic lamination from the right in such a way that the orbit lamination is the stable foliation  $\mathcal{H}$ , and a probability measure of  $N$  is pointed harmonic if and only if it is invariant by the action of  $P$ . Theorem 5.7 in this case is already obtained in [M] and [BM] by a somewhat different dynamical method. For higher dimension, we do not have such description of pointed harmonic measures.

**6. The dichotomy.** Let  $m$  be a harmonic measure on a compact hyperbolic lamination  $(M, \mathcal{L}, g)$ . For an  $m$ -a.a. leaf  $L$ , we have defined a measure class  $[\mu_L]$  on the boundary  $\mathbf{S}_\infty^d$  of the universal cover  $\mathbf{D}^{d+1}$  of  $L$ . In this section, we shall prove Theorem 1.2, i.e., for an ergodic harmonic measure  $m$ , either the support  $K_L = \text{Supp}([\mu_L])$ , called the *characteristic set* of  $L$ , is a singleton for any an  $m$ -a.a. leaf, or is the total space  $\mathbf{S}_\infty^d$ .

The argument closely follows the proof of [MV, Proposition 1], in which the authors attribute the original idea to Etienne Ghys.

To begin with, let us notice the following fact. Let  $\Gamma$  be the group of deck transformations of the covering map  $\mathbf{D}^{d+1} \rightarrow L$ . In the proof of Lemma 5.2, we have shown that  $\mu_{\gamma\tilde{x}} = \gamma\mu_{\tilde{x}}$  for any  $\gamma \in \Gamma$ . On the other hand, the equivalence class of the measure  $\mu_{\tilde{x}}$  does not depend on the choice of the particular point  $\tilde{x}$  from  $\mathbf{D}^{d+1}$ , as is explained in the beginning of Section 4. This shows that  $\gamma K_L = K_L$ .

Given a closed subset  $K$  of  $\mathbf{S}_\infty^d$  which is not a singleton, the *convex hull* of  $K$ , denoted by  $C(K)$ , is the convex hull in  $\mathbf{D}^{d+1}$  of the union of all the geodesics joining two points of  $K$ . It is a closed convex subset of  $\mathbf{D}^{d+1}$ , and the assignment  $K \mapsto C(K)$  is  $G$ -equivariant, where  $G$  is the group of all the orientation preserving isometries of  $\mathbf{D}^{d+1}$ . Therefore, we have the following lemma.

**LEMMA 6.1.** *Assume  $K_L$  is not a singleton. Then the convex hull  $C(K_L)$  of  $K_L$ , as well as its closed  $r$ -neighbourhood  $N_L(r)$  ( $r > 0$ ), is a  $\Gamma$ -invariant subset of  $\mathbf{D}^{d+1}$ .*

Choose a prolonged local chart  $\mathbf{D}^{d+1} \times Z$ , and denote the characteristic set of the leaf of  $\mathcal{L}$  corresponding to  $\mathbf{D}^{d+1} \times z$  by  $K_z$ . Denote by  $\mathcal{C}(\mathbf{S}_\infty^d)$  the space of nonempty closed subsets of  $\mathbf{S}_\infty^d$  equipped with the  $\sigma$ -algebra  $\mathcal{B}_{\mathcal{C}}$  of the Hausdorff topology.

LEMMA 6.2. *The assignment  $Z \ni z \mapsto K_z \in \mathcal{C}(\mathbf{S}_\infty^d)$  is  $\nu$ -measurable with respect to  $\mathcal{B}_{\mathcal{C}}$ .*

PROOF. For any open subset  $U$  of  $\mathbf{S}_\infty^d$ , define  $\mathcal{C}(\mathbf{S}_\infty^d)_U$  to be the open subset of  $\mathcal{C}(\mathbf{S}_\infty^d)$  consisting of those closed sets which intersects  $U$ . It is well known, easy to show, that the open sets  $\mathcal{C}(\mathbf{S}_\infty^d)_U$  generate the  $\sigma$ -algebra  $\mathcal{B}_{\mathcal{C}}$ . Therefore, it suffices to show that the set

$$Z_U = \{z \in Z; K_z \in \mathcal{C}(\mathbf{S}_\infty^d)_U\}$$

is  $\nu$ -measurable. Choose a countable family  $\{f_i\}$  of nonnegative continuous functions supported in  $U$  such that the union of their support is  $U$ , and take a base point  $\tilde{x} \in \mathbf{D}_*^{d+1}$ , where  $\mathbf{D}_*^{d+1}$  is the subset defined in the proof of Sublemma 5.4. Then the set  $Z_U$  consists of exactly those points  $z$  such that  $\mu_{(\tilde{x},z)}(f_i) > 0$  for some  $i$ . The  $\nu$ -measurable dependence of  $\mu_{(\tilde{x},z)}$  in the variable  $z$  established in the proof of Sublemma 5.4 completes the proof.  $\square$

DEFINITION 6.3. (1) Let  $M_I$  be the union of  $m$ -a.a. leaves  $L$  such that the characteristic set  $K_L$  is a singleton.

(2) Let  $M_{II}$  be the union of  $m$ -a.a. leaves  $L$  such that  $K_L = \mathbf{S}_\infty^d$ .

(3) Let  $M_{III} = M \setminus (M_I \cup M_{II})$ .

Lemma 6.2 implies that the three subsets are  $m$ -measurable. Since they are saturated and the harmonic measure  $m$  is ergodic, only one of them is conull. Henceforth we assume that  $M_{III}$  is conull and deduce a contradiction, which is sufficient for the proof of Theorem 1.2. For any  $m$ -a.a. leaf  $L$  and for  $r > 0$ , consider the image of  $N_L(r)$  by the covering map  $\mathbf{D}^{d+1} \rightarrow L$ . Taking their union for any  $m$ -a.a. leaf  $L$ , we get a subset of  $M$ , denoted by  $N(r)$ .

LEMMA 6.4. *The subset  $N(r)$  is measurable.*

PROOF. Denote by  $\mathcal{C}(\mathbf{D}^{d+1} \cup \mathbf{S}_\infty^d)$  the set of nonempty closed subsets of the compactification  $\mathbf{D}^{d+1} \cup \mathbf{S}_\infty^d$ , equipped with the Hausdorff topology. Then the map from  $\mathcal{C}(\mathbf{S}_\infty^d)$  to  $\mathcal{C}(\mathbf{D}^{d+1} \cup \mathbf{S}_\infty^d)$  which assigns to  $K$  the closure of the  $r$ -neighbourhood of the convex hull of  $K$  is clearly continuous.

Consider a prolonged local chart  $\mathbf{D}^{d+1} \times Z$  and again let  $K_z$  denote the characteristic set of the leaf in  $\mathcal{L}$  which corresponds to  $\mathbf{D}^{d+1} \times z$ . Also denote by  $N_z(r) \subset \mathbf{D}^{d+1}$  the closed  $r$ -neighbourhood of the convex hull of  $K_z$ . Then by the above observation and by Lemma 6.2, the map

$$Z \ni z \mapsto N_z(r) \cup K_z \in \mathcal{C}(\mathbf{D}^{d+1} \cup \mathbf{S}_\infty^d)$$

is measurable. In particular, for any open subset  $U$  of  $\mathbf{D}^{d+1}$ , the set

$$\{z \in Z; N_z(r) \cap U \neq \emptyset\}$$

is a measurable subset of  $Z$ .

Let us show that the union  $N_Z(r) = \bigcup_z N_z(r) \times z$  is a measurable subset of  $\mathbf{D}^{d+1} \times Z$ . Choose a sequence of open coverings of  $\mathbf{D}^{d+1}$ ,  $\mathcal{U}_1 \prec \mathcal{U}_2 \prec \dots$ , such that  $\text{mesh}(\mathcal{U}_i)$  tends to 0. Define

$$N_Z(r)^i = \bigcup_{U \in \mathcal{U}_i} U \times \{z; N_z(r) \cap U \neq \emptyset\}.$$

Then the set  $N_Z(r)^i$  is measurable, and hence  $N_Z(r) = \bigcap_i N_Z(r)^i$  is also measurable.

Now the image of  $N_Z(r)$  by the submersion of  $\mathbf{D}^{d+1} \times Z$  to  $M$  is measurable. In fact,  $N_Z(r)$  is a union of a null set and a Borel set. The image of a null set is null by the definition of the lift  $m|_{\mathbf{D}^{d+1} \times Z}$  of  $m$ . On the other hand, the image of a Borel set by a countable to one Borel map is Borel. This is a well-known fact about standard Borel spaces, and follows for example from [Ke, Corollary 15.2] and [S, Theorem 1.3]. Now the set  $N(r) \subset M$  is a finite union of measurable sets and is measurable.  $\square$

Let us finish the proof of Theorem 1.2. Recall we are assuming that  $M_{\text{III}}$  is conull in way of contradiction. Since  $M = \bigcup_r N(r) \bmod 0$ , we have  $m(N(r)) > 0$  for some  $r$ . By Theorem 3.3, the measure  $\mathbf{P}_m$  on the space  $\Omega$  of leafwise paths is ergodic with respect to the shift semiflow  $\theta_t$ . This means that, for  $\mathbf{P}_m$ -almost any path, the average time of stay in the set  $X_0^{-1}(N(r))$  is equal to  $\mathbf{P}_m(X_0^{-1}(N(r))) = m(N(r))$ . In other words, for an  $m$ -a.a.  $x$ ,  $W^x$ -almost surely we have

$$(6.1) \quad \lim_{t \rightarrow \infty} \frac{1}{t} dt \{s \in [0, t]; X_s \in N(r)\} = m(N(r)) > 0,$$

where  $dt$  denotes the Lebesgue measure on  $[0, \infty)$ .

But by Lemma 6.1, the inverse image  $p^{-1}(N(r))$  of the universal covering map  $p : \mathbf{D}^{d+1} \rightarrow L$  of an  $m$ -a.a. leaf  $L$  coincides with the set  $N_L(r)$ , the closed  $r$ -neighbourhood of the convex hull of the characteristic set  $K_L$ . Since  $K_L \neq S_\infty^d$ , there is a closed nondegenerate interval  $I$  contained in  $S_\infty^d \setminus K_L$ . For any point  $x$  on  $L$ , the set of paths whose lifts hit  $I$  has positive  $W^x$ -measure. On the other hand, for those paths the limit of (6.1) must be 0, since there is a neighbourhood of  $I$  in  $\mathbf{D}^{d+1} \cup S_\infty^d$  which does not intersect  $N(r)$ . A contradiction. Theorem 1.2 is now proved.

**EXAMPLE 6.5.** For any harmonic measure  $m$  of a compact hyperbolic lamination, the canonical lift  $\lambda(m)$  of  $m$ , a pointed harmonic measure of the leafwise unit tangent bundle, is of type I. Especially, the unique harmonic measure of the Anosov foliation on the unit tangent bundle of a closed oriented hyperbolic surface is of type I. See [G], [DK].

Ergodic completely invariant measures are typical examples of harmonic measures of Type II. But there are some more. An example is in order. Let  $\Sigma = \Gamma \backslash \mathbf{D}^2$ , where  $\Gamma < PSL(2, \mathbf{R})$  is a purely hyperbolic cocompact Fuchsian group.

Choose any homomorphism  $\rho : \Gamma \rightarrow \text{Homeo}(Z)$  to the group of the homeomorphisms of a compact metric space  $Z$  which satisfies the following conditions.

- (1) The homomorphism  $\rho$  is not injective.
- (2) There is no  $\rho(\Gamma)$ -invariant measure on  $Z$ .

Let  $M = \Gamma \backslash (\mathbf{D}^2 \times Z)$ , where the action of  $\Gamma$  is by deck transformation on the first factor and by  $\rho$  on the second. Then the horizontal lamination  $\{\mathbf{D}^2 \times z\}$  on  $\mathbf{D}^2 \times Z$  induces a lamination  $\mathcal{L}$  on  $M$ , called the *suspension* of  $\rho$ . Let  $m$  be any ergodic harmonic measure of  $\mathcal{L}$ , and notice that  $m$  is not completely invariant by (2).

PROPOSITION 6.6. *The above ergodic harmonic measure  $m$  is of type II.*

PROOF. By Theorem 2.3, the harmonic measure  $m$  determines the class of a probability measure  $\nu$  on  $Z$ . The measure  $\nu$  is quasi-invariant by the action of  $\rho(\Gamma)$ .

Assume for contradiction that  $m$  is of type I. Then for the prolonged local chart  $\mathbf{D}^2 \times Z$ , the characteristic set  $K_z$  ( $z \in Z$ ) is a singleton for  $\nu$ -a.a.  $z$ . This yields a measurable map  $k : Z \rightarrow \mathbf{S}_\infty^1$  by Lemma 6.2. The map  $k$  is  $\Gamma$ -equivariant with respect to  $\rho$  and the Fuchsian group action on  $\mathbf{S}_\infty^1$ , i.e., we have

$$k(\rho(\gamma)z) = \gamma k(z) \text{ for all } \gamma \in \Gamma, \nu\text{-a.a. } z \in Z.$$

The push forward measure  $k\nu$  is kept quasi-invariant by the Fuchsian group, and in particular its support is an infinite set. Choose a nontrivial  $\gamma \in \Gamma$  from the kernel of  $\rho$ , and let  $F$  be a Borel fundamental domain of  $\gamma$  for its action on  $\mathbf{S}_\infty^1 \setminus \text{Fix}(\gamma)$ . Then we have  $\nu(k^{-1}(F)) > 0$ . On the other hand, we have

$$k^{-1}\gamma F = \rho(\gamma^{-1})k^{-1}\gamma F = k^{-1}F \text{ mod } 0.$$

Thus we have  $\nu(\emptyset) = \nu(k^{-1}F \cap k^{-1}\gamma F) > 0$ . A contradiction. □

Finally let us pose some problems.

QUESTION 6.7. It is known [K2] that a compact hyperbolic lamination with a type I ergodic harmonic measure is an amenable measured foliation in the sense of [AR]. Is the converse true?

QUESTION 6.8. For an ergodic harmonic measure of type I of a compact hyperbolic lamination of dimension  $d+1$ , the characteristic exponent satisfies  $\lambda = d^2$ . For type II measure, is it true that  $\lambda$  is smaller than  $d^2$ ?

QUESTION 6.9. For an injective homomorphism from  $\Gamma$  (as above) to  $PSL(2, \mathbf{R})$  with dense image, is the harmonic measure of the suspension foliation type II?

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