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THE DIFFERENTIAL AND THE ROMAN DOMINATION NUMBER OF A GRAPH

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Let $G = (V, E)$ be a graph of order n and let $B(S)$ be the set of vertices in $V \setminus S$ that have a neighbor in the vertex set S. The differential of a vertex set S is defined as $\partial(S) = |B(S)| - |S|$ and the maximum value of $\partial(S)$ for any subset S of V is the differential of G. A Roman dominating function of G is a function $f : V \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u\in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is the Roman domination number of G, written $\gamma_R(G)$. We prove that $\gamma_R(G) = n - \partial(G)$ and present several combinatorial, algorithmic and complexity-theoretic consequences thereof.

1. INTRODUCTION

The Roman domination number is a variant of the domination number suggested by I. Stewart [**37**], motivated by a problem from military history. A *Roman dominating function* (RDF) of a graph $G = (V, E)$ is a (total) function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman dominating function is the value $f(V) = \sum_{u\in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. This parameter (as well as related ones) has been studied by many authors, both from the viewpoint of combinatorics and from the viewpoint of the algorithmic complexity. We only refer to the papers [**7, 10, 16, 23**] and the literature

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quoted therein. Alternatively, an RDF can be presented in terms of the ordered partition of V induced by f, i.e., $f = (V_0, V_1, V_2)$ with $V_i = \{v \in V : f(v) = i\}$. A partition (V_0, V_1, V_2) of V yields an RDF if V_2 dominates the set V_0, i.e., every vertex in V_0 has a neighbor in V_2. Then, $f(V) = \sum_{u \in V} f(u) = 2|V_2| + |V_1|$. An RDF $f = (V_0, V_1, V_2)$ is a γ_R-function if $f(V) = \gamma_R(G)$.

The differential of a graph was introduced in [**26**] in 2006 and studied by several authors [**2, 4, 3, 5**], motivated by its applications to information diffusion in social networks. More formally, the *differential of a vertex set* S is defined as $\partial(S) = |B(S)| - |S|$, where $B(S)$ is the set of vertices in $V \setminus S$ that have a neighbor in the vertex set S, and the *differential of a graph* is defined as $\partial(G) = \max\{\partial(S) : S \subseteq V\}$. We will say that $S \subseteq V$ is a *∂-set* or *differential set* if $\partial(S) = \partial(G)$.

These two parameters have been independently studied, as no relationship between them was known before. In this paper, we show that the sum $\gamma_R(G)+\partial(G)$ equals the order of G, and we present several consequences of this relationship. This type of Gallai identity result complements what was previously known for the domination number and the enclaveless number of a graph (defined below) and several other domination parameters [**1**].

We are going to present the consequences of this previously unknown relationship in two sections, one devoted to the more combinatorial results, and the other one to algorithms and complexity. In Section 5, we discuss the "edge versions" of our parameters. To a graph $G = (V, E)$, we can associate its *line graph* $\mathcal{L}(G) = (E, E')$, whose vertex set is the edge set of G, and ef is an edge in E' if $e, f \in E$ and e and f share exactly one end-vertex in G. The *Roman edge domination number* $\gamma_{Re}(G)$ of a graph G as introduced in [**13, 36**] equals the Roman domination number of its line graph $\mathcal{L}(G)$. Similarly, the *edge differential* of a graph G, written $\partial_e(G)$, equals the differential of its line graph $\mathcal{L}(G)$. This concept has not been investigated before, but our main result clearly gives relations between the two edge versions. As a main result of Section 5 we show that the question whether a graph admits an edge differential of size at least k is NP-complete, even if the graph is planar and if it has a maximum degree of four.

Finally, we fix some general notation and terminology. $G = (V, E)$ denotes a simple graph of *order* $n = |V|$ and *size* $m = |E|$. We denote two adjacent vertices u and v by $u \sim v$. For a vertex $v \in V$ we write $N(v) = \{u \in V : u \sim v\}$ for its open *neighborhood*. The *degree* of a vertex $v \in V$ will be denoted by $\delta(v) = |N(v)|$. For a non-empty subset $S \subseteq V$ we write $C(S) = V \setminus (S \cup B(S))$, and the subgraph in G induced by the vertex set S will be denoted by $G\langle S\rangle$. Given a graph H, the graph G is called *H-free* if G has no induced subgraph isomorphic to H.

We will discuss several basic graph parameters: the order n and size m of a graph were already mentioned; $\delta(G)$ denotes the *minimum degree* of any vertex of G, while $\Delta(G)$ refers to the *maximum degree*. The *distance* between two vertices x, y is the length of the shortest path between x and y, and the *diameter* $D(G)$ refers to the largest distance between any two vertices from G. There is a wealth of graph parameters related to domination and some of these will come into play in our exposition. Recall that a set D of vertices in $G = (V, E)$ is a *dominating set* if $B(D) = V \setminus D$. The size of the smallest dominating set is also known as the *domination number* of the graph G, written as $\gamma(G)$. A vertex set $D \subseteq V$ is a *k-dominating set* if every vertex not in D has at least k adjacent vertices in D. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set. Clearly, a 1-dominating set is just a dominating set. The *enclaveless number* of a graph $G = (V, E)$, denoted $\Psi(G)$, is defined as $\Psi(G) := \max\{|B(D)| : D \subseteq V\}$. As mentioned above, $\gamma(G) + \Psi(G) = n$ for any graph G of order n. The *edge domination number* of G, written $\gamma_e(G)$, equals $\gamma(\mathcal{L}(G))$.

2. RELATING THE ROMAN DOMINATION NUMBER AND THE DIFFERENTIAL OF A GRAPH

For every graph G with connected components $G_1, \ldots, G_k$, $\gamma_R(G) = \gamma_R(G_1) + \cdots + \gamma_R(G_k)$ and $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. Therefore, it is unnecessary to consider disconnected graphs. Most of the results in this paper will be obtained by using the following theorem.

Theorem 1. *If G is a graph of order n, then $\gamma_R(G) = n - \partial(G)$.*

Proof. For every RDF $f = (V_0, V_1, V_2)$ we can consider $f' = (V_0', V_1', V_2')$, where $V_2' = V_2$, $V_0' = B(V_2)$ and $V_1' = C(V_2)$, to obtain $f(V) \geq f'(V)$. Therefore,

$$\gamma_R(G) = \min\{f(V) : f = (V_0, V_1, V_2) \text{ is an RDF and } V_0 = B(V_2)\}.$$

Finally, using that $|V_2| = |B(V_2)| - \partial(V_2)$, we have

$$\begin{aligned}\gamma_R(G) &= \min_{V_2 \subseteq V}\{2|V_2| + |C(V_2)|\} = \min_{V_2 \subseteq V}\{|B(V_2)| - \partial(V_2) + |V_2| + |C(V_2)|\} \\ &= \min_{V_2 \subseteq V}\{n - \partial(V_2)\} = n - \max_{V_2 \subseteq V}\{\partial(V_2)\} = n - \partial(G).\end{aligned}$$

Corollary 1. *Given a graph G, $f = (V_0, V_1, V_2)$ is a γ_R-function of G if and only if V_2 is a ∂-set of G and $V_0 = B(V_2)$.*

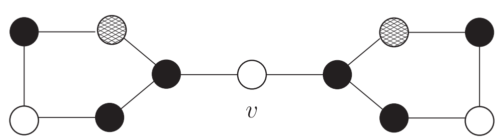

Figure 1. A subgraph also encountered in Theorem 6.

We are going to clarify this relation (and also the notions) with a small example. Consider the graph depicted in Fig. 1. The white vertices comprise a set that yields the differential of that graph. The white vertices are assigned two and the vertices with the diamond pattern are assined one by a γ_R-function.

3. NEW BOUNDS FOR THE ROMAN DOMINATION NUMBER OR FOR THE DIFFERENTIAL OF A GRAPH

In this section, we show new bounds for the Roman domination number of a graph which can be directly obtained from results about the differential of a graph. Occasionally, we also get new combinatorial bounds for the differential from published results on the Roman domination number.

3.1. Relations to basic graph parameters

Here, we focus on the order, size, minimum and maximum degree and the diameter of a graph. Sometimes, other parameters come into play, as well, refining previously made assertions.

A simple lower bound on the Roman domination number appeared in [**10**] : For any graph G of order n with $\Delta(G) \geq 1$, $\gamma_R(G) \geq \frac{2n}{\Delta(G)+1}$. Using Theorem 1 and Theorem 2.16 from [**6**], the same result follows. Moreover, an infinite family of graphs attaining this bound was also given in [**6**].

To get an upper bound for the Roman domination number in terms of the order and the minimum degree of a graph, we can consider the lower bound of the differential given in Theorem 2.15 in [**2**] together with Theorem 1 to conclude:

Theorem 2. *Let G be a graph of order n. Then* $\gamma_R(G) \leq \left\lfloor \frac{2n\delta(G)}{3\delta(G)-1} \right\rfloor$.

Lower bounds on the differential of a graph in terms of its order, size and maximum degree as proved in [**2**] translate as follows to so far unknown upper bounds on the Roman domination number:

Theorem 3. *If G is a graph of order n and size m, then*

$$\gamma_R(G) \leq \min\left\{ \left\lfloor \frac{3\Delta(G)n-2m}{3\Delta(G)-1} \right\rfloor, \left\lfloor \frac{(3\Delta(G)+4)n-2m}{3\Delta(G)+4} \right\rfloor \right\}.$$

Moreover, if G is a C_5-free graph, then $\gamma_R(G) \leq \left\lfloor \frac{(3\Delta(G)+2)n-2m}{3\Delta(G)+2} \right\rfloor$.

Conversely, we can obtain a new tight lower bound on the differential of a graph in terms of its minimum degree and its order, based on [**39**, Theorem 12].

Theorem 4. *For any graph G with $\delta(G) > 0$,* $\partial(G) \geq \left(\frac{2^{1+1/\delta(G)} \cdot \delta(G)}{(1+\delta(G))^{1+1/\delta(G)}} - 1 \right) n.$

In [**7**], it was proved that, for every sufficiently large connected graph G with $\delta(G) \geq 1$, it holds that $\gamma_R(G) \leq \frac{4n}{5}$ and, if $\delta(G) \geq 2$, then $\gamma_R(G) \leq \frac{8n}{11}$. This also follows from Theorems 3.1 and 3.4 in [**3**] by making use of Theorem 1. Observe that different techniques were used in the proofs of these theorems. Moreover, it was proved in [**24**] that $\gamma_R(G) \leq \frac{2n}{3}$ if $\delta(G) \geq 3$, which provides the unknown lower bound $n/3$ for the differential in such a case.

Moreover, from results in [**3**] and [**5**], we can improve these upper bounds for the Roman domination number in two cases: (a) if we consider the number of vertices having exactly two so-called hairs connected to it as an additional parameter and (b) if we consider special pendant subgraphs in a graph of maximum degree three. Let us turn to Case (a) first. A *hair* is a sequence of two vertices $w - v$, where w is a vertex of degree one and v has degree two. We denote by $p_5(G)$ the set of vertices in V which have exactly two hairs connected to them. Notice that these are centers of certain induced paths on five vertices. Some infinite families of graphs attaining the following bound were given in [**5**].

Theorem 5. *Any connected graph G of order $n \geq 6$ satisfies* $\gamma_R(G) \leq \dfrac{3n + |p_5(G)|}{4}$.

Now we come to Case (b). We say that a (connected) graph $G_1 = (V_1, E_1)$ is a *pendant subgraph* of $G = (V, E)$ from $v \in V_1$, if $V_1 \subseteq V$, $G_1 = G\langle V_1 \rangle$ and there exist $u \in V \setminus V_1$ and $e = uv \in E$ such that e is a bridge. For instance, a hair (described by $w - v$) can be viewed as a pendant subgraph of the form $(\{w, v\}, \{wv\})$. We are particularly interested in the graph G_1 depicted in Fig. 1 as being pendant from v.

Theorem 6. *Let G be a graph of order $n \geq 12$ with $\delta(G) \geq 2$ and $\Delta(G) \leq 3$. If G has t pendant subgraphs isomorphic to G_1, then* $\gamma_R(G) \leq \dfrac{5n + t}{7}$.

Some infinite families of graphs attaining this bound were given in [**3**].

In [**6**] and [**35**], basically the same characterizations of graphs with high maximum degree in terms of a low Roman domination number were given. For instance, for any graph G of order $n \geq 3$, $\Delta(G) = n - 1$ if and only if $\gamma_R(G) = 2$. Another consequence from [**6**] for Roman domination is the following one.

Proposition 1. *If G is a graph of order n and diameter $D(G)$, then*

$$\gamma_R(G) \leq n - \left(\left\lfloor \frac{D(G)}{3} \right\rfloor + 1 \right) (\delta(G) - 1).$$

The proposition above clearly improves Theorems 3 and 4 in [**27**] when $\delta(G) \geq 2$. Moreover, the authors in [**6**] gave some infinite families of graphs satisfying this upper bound. Notice that the simple bound $\gamma_R(G) \leq n - \Delta(G) + 1$ from [**7**] can be improved if the minimum degree or the diameter are not too small, based on Theorem 2.14 in [**2**] and Theorems 2.15 and 2.17 in [**6**], stated in terms of the differential of G.

3.2. Relations to other graph parameters

Here, we will consider several other domination parameters and their relation to the Roman domination number. The following result appears in [**10**] and it has been used by most of the authors who have worked on the Roman domination number.

Proposition 2. *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

In [**10**], the authors also characterize the graphs satisfying $\gamma_R(G) = \gamma(G)+1$ or $\gamma_R(G) = \gamma(G) + 2$, and they state, for connected graphs, that $\gamma_R(G) = \gamma(G)$ only when G is an isolated vertex. This readily transfers to a characterization of graphs whose enclaveless number is close to its differential. It was proved in [**14**] that, for any graph G of order $n \geq 3$, $\gamma_R(G) \leq n - \frac{\gamma(G)}{2}$, and the authors even characterized the graphs attaining this bound. The same upper bound and even a lower bound can be obtained from Theorem 1 and Theorem 2.4 in [**2**], where the authors proved that $\frac{\gamma(G)}{2} \leq \partial(G) \leq \gamma(G)(\Delta(G) - 1)$ using so-called big star packings. Namely, we derive:

Theorem 7. *If G is a graph of order $n \geq 3$, then*

$$n - \gamma(G)(\Delta(G) - 1) \leq \gamma_R(G) \leq n - \frac{\gamma(G)}{2}.$$

The lower bound above is sharp for the cycles $G = C_{3n}$ and for every graph G of order n with a vertex v satisfying $\delta(v) = n - 1$. The upper bound is better than $2\gamma(G)$ only when $\frac{2n}{5} \leq \gamma(G)$. Now, let us see that this upper bound can be improved in some classes of graphs, using some result proved in [**2**].

Theorem 8. *For any connected graph G of order n, the following is true.*

(a) *If $n \geq 3$ and $\delta(G) \geq 1$, then $\gamma_R(G) \leq \frac{2n}{5} + \gamma(G)$.*

(b) *If $n \geq 9$ and $\delta(G) \geq 2$, then $\gamma_R(G) \leq \frac{4n}{11} + \gamma(G)$.*

The upper bounds given in items (a) and (b) of this theorem are tight on the families of graphs given in Proposition 3.2 and Proposition 3.5 in [**3**], respectively.

It is known that $\gamma(G) \leq \frac{2n}{5}$ for every graph of order n with $\delta(G) \geq 2$ if $n \geq 8$. The shortest proof appeared in [**28**]. Hence, (b) is better than the bound given in [**14**] when $n \geq 9$ and $\delta(G) \geq 2$. Moreover, using the definition of 2-dominating set, Theorem 1 and a result from [**2**], we can get another upper bound that is better than the one in [**14**] when $\delta(G) \geq 3$.

Theorem 9. *If G is a graph of order n, with minimum degree $\delta(G) \geq 3$, then*

$$\gamma_R(G) \leq \min\left\{n + \gamma(G) - \gamma_2(G), n - \frac{\gamma_2(G)}{2}\right\}.$$

A set $S \subseteq V$ is a *2-packing* if for all $u, v \in S$, $d(u, v) > 2$. The *2-packing number* of G, denoted by $\alpha_2(G)$, is the maximum cardinality of a 2-packing of G. Corollary 1 in [**10**] relates the Roman domination number and the 2-packing number of the graph. Using that $\partial(G) \geq \alpha_2(G)(\delta(G)-1)$ – proved in [**2**] – we give a new result relating the Roman domination number and the 2-packing number.

Proposition 3. *If G is a graph of order n, then $\gamma_R(G) \leq n - \alpha_2(G)(\delta(G) - 1)$.*

3.3. Results for special graph classes

Other known results could be used to get a lower bound for the differential and, in consequence, for the Roman domination number of certain classes of graphs. A graph is said to be *k-connected* if there does not exist a set of $k-1$ vertices whose removal disconnects the graph. A *cut-vertex* is any vertex whose removal increases the number of connected components. A *block* in G is a maximal connected subgraph without a cut-vertex. An *end-block* in G is a block with exactly one cut-vertex of G. We denote by $eb(G)$ the number of end-blocks in G. The graph $K_{1,3}$ (the complete bipartite graph with one vertex in one partition and three vertices in the other) is called a *claw* and we denote by Δ^* the graph obtained from a triangle, say $v_1v_2v_3$, by adding three new vertices u_1, u_2, u_3 and three new edges v_1u_1, v_2u_2, v_3u_3. By Theorem 2.19 in [**2**] and Theorem 1, we can directly obtain the following result.

Theorem 10. *Let G be a claw-free graph of order n.*

(a) *If G is a 2-connected graph, or it is a Δ^*-free graph or it has at most two end-blocks, then $\gamma_R(G) \leq \left\lceil \frac{2n}{3} \right\rceil$. (As $\gamma_R(P_n) = \lceil 2n/3 \rceil$, the bound is sharp.)*

(b) *If $eb(G) \geq 2$, then $\gamma_R(G) \leq \left\lceil \frac{2n + eb(G) - 2}{3} \right\rceil$.*

4. CONSEQUENCES FOR COMPLEXITY AND ALGORITHMS

The two graph parameters $\partial(G)$ and $\gamma_R(G)$ lead to two natural algorithmic decision problems:

- DIF: Given a graph G and an integer k, decide if $\partial(G) \geq k$;
- ROM: Given a graph G and an integer ℓ, decide if $\gamma_R(G) \leq \ell$.

Due to the relation proved in Theorem 1, a decision algorithm for DIF can be easily used for deciding ROM and vice versa. In this section, we assume basic knowledge from complexity theory and algorithms on side of the reader.

4.1. Classical complexity considerations

Here, we will review the question of NP-hardness (or polynomial-time solvability) of ROM and DIF, as undertaken so far in the literature. P. A. Dreyer showed that ROM is NP-hard by a reduction from 3-SAT [**12**, Theorem 2.42], while S. Bermudo and H. Fernau proved in [**4**] by a reduction from 3-*Dimensional Matching* that DIF is NP-hard. Due to Theorem 1, both results are equivalent. It is said in [**12**] that A. McRae showed that ROM is NP-complete even when restricted to chordal, bipartite, split, or planar graphs. However, no proofs of these results got published according to our knowledge. Conversely, it is shown in [**4**] that DIF remains NP-complete when restricted to split graphs or to cubic graphs. NP-hardness of ROM for planar graphs, split graphs, and bipartite graphs is shown in [**33**], and for unit disk graphs in [**34**]. Again, Theorem 1 can be used to deduce hardness results for all these graph classes both for ROM and for DIF. We will complement the known results by proving NP-hardness for ROM / DIF on line graphs below (in the formalization of Roman edge domination) in Section 5.

M. Liedloff et al. showed [**21**] that ROM can be solved in linear time in interval graphs, another simple model of geometric intersection graphs, cographs and distance-hereditary graphs; hence, DIF can be efficiently solved in these graph classes, as well. In [**25**], it is proved that ROM can be solved in polynomial time on strongly chordal graphs. In [**16, 21**], it is moreover shown that ROM (and hence DIF) can be solved in polynomial time provided that a tree decomposition or a k-expression is also a part of the input in order to testify bounded treewidth or bounded cliquewidth. The status on chordal graphs seems to be unknown so far. Similarly, polynomial-time solvability of minimum domination is known for cocomparability graphs and even for asteroidal-triple free graphs [**20**], which is open for DIF / ROM.

Having established NP-hardness of DIF and ROM in several graph classes, it becomes interesting to consider fixed-parameter tractability and approximability.

4.2. Fixed-parameter tractability

Let us first turn towards fixed-parameter tractability, now considering DIF and ROM together with their standard parameters, which are k and ℓ in our definitions above. In a nutshell, a *parameterized problem* is a decision problem that comes with an explicitly defined parameter function; it is *fixed-parameter tractable*, or in FPT, if each instance I can be solved in time $O(f(\kappa(I)) \cdot p(|I|))$, where f is some arbitrary function, p is some polynomial, and κ is the function that yields the parameter value for instance I. As polynomial factors do not play a vital role in this context, the O^*-notation was invented to suppress such factors; so we can write $O^*(f(\kappa(I)))$ to spell out fixed-parameter tractability. In our case, if (G, ℓ) is an instance of ROM, then $\kappa((G, \ell)) = \ell$ would yield the standard parameterization. FPT is the realm of tractability in parameterized complexity, while hardness is established by placing a parameterized problem higher in the so-called W-hierarchy. The question if FPT equals W[1], the second-lowest level of this hierarchy, is similar to the famous P versus NP question in classical complexity. Equivalent to being in FPT is the existence of a so-called (problem) kernel, which refers to a problem instance whose size is exclusively bounded by a function in the parameter, and such a problem instance can be obtained in polynomial-time from any problem instance. Traditionally, the size of instances comprising of graphs are measured in terms of their order.

The parameters k and ℓ are so-called *dual parameters*, as a vertex set testifying a differential of k also shows that the considered graph or order n also admits a Roman domination function of weight $\ell = n - k$, and vice versa. It has been observed many times that often, dual parameters behave contrastingly with respect to parameterized tractability. It has been shown in [**16**] that ROM is W[2]-complete on general graphs, meaning that no fixed-parameter algorithms are to be expected in this situation. Conversely, in [**5**] it has been shown that DIF admits a problem kernel of at most $4k$ vertices on general graphs, which immediately implies fixed-parameter tractability for DIF. Notice that this relation is very similar to what is known for domination (or the enclaveless number), see [**11**].

As it has been started with the paper by Ł. Kowalik for the enclaveless number [**19**], it is interesting to see if smaller kernels can be obtained for special graph classes where DIF / ROM is still NP-hard. For instance, it follows from the NP-hardness result for DIF on cubic graphs shown in [**4**] that DIF is NP-hard for graphs of minimum degree at least two or three. The combinatorial bounds obtained in [**3, 7, 24**] imply: (a) DIF, restricted to graphs of minimum degree two, admits a kernel with at most $\frac{11k}{3}$ many vertices. (b) DIF, restricted to graphs of minimum degree at least three, admits a kernel with at most $3k$ many vertices.

We furthermore know from [**24**], Corollary 11: For any connected cocomparability graph G on n vertices, $\gamma_R(G) \leq \lceil 2n/3 \rceil$. This implies a kernel of order $3k$ for DIF on co-comparability graphs; however, it would not appear to be too surprising if ROM would be solvable in polynomial time for these graphs, as this is known for the classical domination problem [**20**], even on asteroidal-triple-free graphs. From a construction in [**23**], we can conclude a kernel with at most $\frac{34}{11}k$ many vertices for DIF, restricted to 2-connected graphs. In this context, it should be mentioned that no fixed-parameter algorithms are to be expected for ROM on chordal graphs (with standard parameterization), where W[1]-hardness was shown in [**22**].

Also, subexponential algorithms have been obtained for ROM when restricted to planar graphs. More precisely, an $O^*(c_{ROM}^{\sqrt{\ell}})$ algorithm for ROM was presented in [**16**]. As [**18**, Theorem 5] shows, branchwidth and treewidth of a planar graph of order n are of size $O(\sqrt{n})$, and since the kernelization rules given in [**5**] do not violate planarity, we can conclude that there also exists some $O^*(c_{DIF}^{\sqrt{k}})$ algorithm for DIF on planar graphs. Currently, using a tree decomposition approach without further improvements, $c_{ROM} \approx 5^{9.55}$ and $c_{DIF} \approx 5^{12.73}$.

From the mentioned kernel of order $4k$ for DIF whose proof construction preserves planarity, we can also deduce that there is no kernel of order $(4/3-\epsilon)\ell$ for ROM, restricted to planar instance, unless P equals NP according to [**8**]. However, no explicit linear-order kernel has been shown for ROM so far to complement this lower-bound result. This can be a non-trivial task, as exemplified by the domination problem on planar graphs in [**8**]. Such a result would allow to establish lower-bounds on kernel sizes for DIF, as well.

An issue related to parameterized algorithms is that of exact exponential-time algorithms. More precisely, one could consider the order of a graph as the parameter of the problem. Of course, this interpretation would wash away any difference between DIF and ROM. For DIF on general graphs, S. Bermudo and H. Fernau developed in [**4**] an algorithm running in time $O^*(1.755^n)$ for graphs of order n. As the treewidth of planar graphs of order n is bounded by $3.182 \cdot \sqrt{n}$, according to Theorems 1 and 5 of [**18**] together, we can conclude $O^*(5^{3.182 \cdot \sqrt{n}})$ algorithms for solving DIF / ROM on planar graphs of order n.

4.3. Approximability

Let us now turn to the question of approximability. This means that we formally ask the question for a given graph to find a Roman domination function of smallest weight or a set yielding the largest differential. We start with a simple corollary from Proposition 2:

Corollary 2. *If MINIMUM DOMINATION can be approximated up to a factor of α in some graph class, then MINIMUM ROMAN DOMINATION can be approximated up to a factor of 2α in the same graph class.*
Similarly, if MINIMUM ROMAN DOMINATION can be approximated up to a factor of α in some graph class, then MINIMUM DOMINATION can be approximated up to a factor of 2α in the same graph class.

Proof. Let S be the solution returned by some factor-α approximation algorithm for MINIMUM DOMINATION on input $G = (V, E)$. Clearly, $(\overline{S}, \emptyset, S)$ is a valid RDF for G of weight $2|S|$. Furthermore, $2|S| \leq 2\alpha\gamma(G) \leq 2\alpha\gamma_R(G)$. Conversely, let (V_0, V_1, V_2) represent an RDF guaranteeing some factor-α approximation for MINIMUM ROMAN DOMINATION on input $G = (V, E)$. Then, $D = V_1 \cup V_2$ is a valid dominating set of G, with $|D| \leq |V_1| + 2|V_2| \leq \alpha\gamma_R(G) \leq 2\alpha\gamma(G)$. □

Presumably, ROM is MAX SNP complete on degree-bounded graphs, but this has not been shown so far. However, as MINIMUM DOMINATION is MAX SNP complete according to [**31**], we can conclude with Corollary 2:

Corollary 3. *Let $d > 2$ be fixed. ROM is constant-factor approximable on graph with maximum degree at most d. Moreover, there is no PTAS for ROM on the class of graphs with maximum degree at most d unless P equals NP.*

On general graphs, ROM on graphs of order n is approximable within $2 + 2\log(n)$, but not approximable within $c\log(n)$ for any $c > 0$ according to [**30**, Theorem 3.3] unless P equals NP. In the same paper, a PTAS for that problem on planar graphs is shown, and it is noticed that there is no FPTAS for ROM on planar graphs unless P equals NP. The positive algorithmic result can be also deduced by more general results from [**17**], from which also PTAS results for DIF on planar graphs would follow. Alternatively, we can use Corollary 2 together with [**32**] to obtain the hardness results. Moreover, we can deduce, based on [**9**, Theorem 1]:

Theorem 11. *ROM cannot be approximated to within a factor of $0.5 \cdot (1-\varepsilon)\ln(n)$ in polynomial time for any constant $\varepsilon > 0$ unless NP $\subseteq$ DTIME$(n^{O(\log(\log(n)))})$. The same results hold also in bipartite and split graphs.*

PTASs for ROM on geometric intersection graphs have been studied in [**29, 34**]. By providing MAX SNP hardness results, it was shown in [**4**] that DIF does not have a PTAS on split graphs or on degree-bounded graphs, although for the latter class of graphs, a constant-factor approximation is available. Many questions are still open, for example: Does DIF allow for a constant-factor approximation algorithm on general graphs? Or at least on split graphs?

5. EDGE VERSIONS OF ROM AND DIF

We now consider the Roman edge domination number of G, as introduced in [**13, 36**] and defined in Section 1, considering Roman domination on line graphs. This also means that many known bounds on Roman domination carry over immediately to Roman edge domination, adapting the parameters accordingly. For instance, Theorem 1 translates as follows: If G is a graph of size m, then $\gamma_{Re}(G) = m - \partial_e(G)$. Such translations would comprise (even strengthened versions of) the main results in [**13**], which were proved there without reference to the known results on Roman (vertex) domination. We can get more results in the same way, for instance, from Theorem 8, we deduce the following hitherto unknown relations:

Corollary 4. *For any connected graph G of size m, the following is true.*

(a) *If $m \geq 3$ and $\delta(G) \geq 1$, then $\gamma_{Re}(G) \leq \frac{2m}{5} + \gamma_e(G)$.*

(b) *If $m \geq 9$ and $\delta(G) \geq 2$, then $\gamma_{Re}(G) \leq \frac{4m}{11} + \gamma_e(G)$.*

We refrain from stating the corresponding results for the edge differential for all these corollaries. We would like to mention that several interesting combinatorial properties that cannot be obtained by a translation of other previously known result through the line graph interpretation can be found in [**36**]. This combinatorial parameter is interesting because of the following hardness result:

Theorem 12. *It is NP-complete to decide, given a graph $G = (V, E)$ and an integer ℓ, if $\gamma_{Re}(G) \leq \ell$. This is even true for planar graphs of maximum degree four.*

Proof. Membership in NP can be easily established by "guess-and-check". We prove hardness by providing a reduction from *Vertex Cover*, restricted to planar cubic graphs. Let $G = (V, E)$ be a planar cubic graph with $V = \{v_1, \ldots, v_n\}$. We fix an embedding of G in the plane. As in the proof of [**38**, Theorem 1], we replace each vertex v_i of G by a part F_i shown in Fig. 2, yielding the graph G'. As in the construction in [**38**], whenever there is an edge between v_i and v_j in G, then there is exactly one edge from $\{q_i m_j, q_i p_j, m_i q_j, m_i p_j, p_i q_j, p_i m_j\}$, and the degrees of m_i and p_i (within G') are equal to three, while the degree of q_i is equal to two. Hence, there is a bijection between the edges in E and the edges E'' that do not occur in any of the gadgets F_i, in the sense that $v_i v_j$ is mapped onto the unique edge between F_i and F_j. This yields the graph $G' = (V', E')$. G' is planar and has maximum degree 4. We claim: G has a vertex cover of size $\leq k$ if and only if G' admits a Roman edge domination function (REDF) of weight at most $5n + k$.

If C is a vertex cover of G, then we consider the REDF f that maps every edge from $E_2 = \{q_i w_i^1, m_i w_i^2, p_i w_i^3 : v_i \in C\} \cup \{w_i^0 w_i^1, w_i^2 w_i^3 : v_i \notin C\}$ to two, from $E_1 = \{p_i r_i : v_i \notin C\}$ to one and all other edges to zero. As the weight of f equals $2|E_2| + |E_1| = 6|C| + 5(n - |C|) = 5n + |C| \leq 5n + k$, f maintains the bound.

To see that a Roman edge domination function (REDF) of weight $5n + k$ in G' determines a vertex cover in G of size k, we first analyze the gadget F_i.

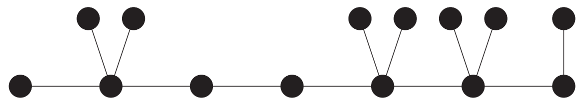

Figure 2. Replacements F_i for single vertices v_i in the construction of Theorem 12.

Claim 1. Let us show that we can assume, w.l.o.g., that every REDF of minimum weight satisfies the following three conditions:

1. At least one edge from $\{q_i w_i^1, w_i^1 w_i^0\}$ gets weight two.
2. At least one edge from $\{m_i w_i^2, w_i^2 w_i^3\}$ gets weight two.
3. At least one edge from $\{w_i^2 w_i^3, w_i^3 p_i\}$ gets weight two.

	$q_iw_i^1$	$w_i^1w_i^0$	$w_i^0m_i$	$m_iw_i^2$	$w_i^2w_i^3$	$w_i^3p_i$	p_ir_i	weight sum
Case 1	2	0	0	2	2	0	0/1	6/7
Case 2	2	0	0	2	0	2	0	6
Case 3	2	0	0/1	0	2	2	0	6/7
Case 4	2	0	0/1	0	2	0	0/1	4-6
Case 5	0	2	0	2	2	0	0/1	6/7
Case 6	0	2	0	2	0	2	0	6
Case 7	0	2	0	0	2	2	0	6
Case 8	0	2	0	0	2	0	0/1	4/5

Table 1. Possible situations for some minimal REDF, restricted to F_i.

Proof of Claim 1. As the arguments are similar, we only consider the first case. Assume that neither $q_iw_i^1$ nor $w_i^1w_i^0$ gets weight two. Then, in order to produce a valid REDF, either both of $w_i^1s_i^1$ and $w_i^1t_i^1$ get weight one, or one of them gets weight zero and the other one weight two. In either case, we can modify this REDF by letting both $w_i^1s_i^1$ and $w_i^1t_i^1$ get weight zero and either q_iwi^1 or $w_i^1w_i^0$ gets weight two. ◇

Claim 2. The previous item leaves as possible situations for any REDF with respect to F_i the ones listed in Table 1. Notice that Case 2 is the strongest solution in the sense that in this case, F_i-edges also dominate all edges from E'' that have F_i-vertices as endpoints. Therefore, and as we are looking for a solution with minimum weight, we can ignore all other cases that lead to a weight sum of six or bigger in F_i, replacing them by Case 2. Thus, more in detail, the remaining possible situations are the ones collected in Table 2. Here 0^* indicates that the corresponding edge has to be dominated by an edge in E'' with weight 2.

	$q_iw_i^1$	$w_i^1w_i^0$	$w_i^0m_i$	$m_iw_i^2$	$w_i^2w_i^3$	$w_i^3p_i$	p_ir_i	weight sum
Case 2	2	0	0	2	0	2	0	6
Case 4.1	2	0	0^*	0	2	0	0^*	4
Case 4.2	2	0	0^*	0	2	0	1	5
Case 4.3	2	0	1	0	2	0	0^*	5
Case 8.1	0	2	0	0	2	0	0^*	5
Case 8.2	0	2	0	0	2	0	1	5

Table 2. A reduced number of cases for some minimal REDF, restricted to F_i.

Claim 3. We can assume that, w.l.o.g., no edge in E'' carries weight two.

Proof of Claim 3. Assume that $e \in E''$ interconnects F_i and F_j and that it has weight two. At least one endpoint of e is adjacent to an edge that is labeled 0^* in Table 2. Otherwise, it serves no domination purpose with respect to neither F_i nor F_j and its weight can be readily reduced to one.

Assume first that only F_i (w.l.o.g.) has an endpoint of e that is also endpoint of an edge f labeled 0^*. Hence, there is only one reason for e carrying weight two (and not one), which is f. We can therefore transform this REDF into another one with the same weight, by labeling both e and f with one. We can further modify the REDF on F_i to be either in Case 2 (if the weight of F_i was five before the modifications) or in Case 4.2 or 4.3 (if the weight of F_i was four before).

Secondly, it might be that both endpoints are adjacent to an edge that is labeled 0^* in Table 2. If then the REDFs for F_i and for F_j are in one of the situations described by the Cases 4.2, 4.3 or 8.1, the weight assigned to F_i, F_j and e totals to 12, and this weight can be also obtained by replacing the settings of the REDF by reducing the weight of e to zero and putting F_i and F_j into Case 2. If, say, F_i is in Case 4.1, but F_j is not, then the overall weight of the REDF on F_i, F_j and e is 11, and this weight can be also obtained in the following way: we can modify F_i towards Case 4.2 or 4.3 (depending on which edge of F_i labeled 0^* is dominated by e) and F_j towards Case 2, finally setting the weight of e to zero. If both F_i and F_j are in Case 4.1, there exists another edge e', with weight 2, connecting F_i and F_ℓ which is adjacent to the other edge in F_i labeled 0^*. By the previous cases we can suppose that F_ℓ is also in Case 4.1 and e' is also adjacent to an edge in F_ℓ labeled 0^*. In such a case, the weight assigned to F_i, F_j, F_ℓ, e and e' totals to 16, and this weight can be also obtained by replacing the settings of the REDF by reducing the weights of e and e' to zero, putting F_i into Case 2, and F_j and F_ℓ into Cases 4.2 or 4.3. Notice that the modifications that we described might trigger further similar ones, as long as edges labeled 0^* prevail. $\diamond$

Claim 4. We can assume, w.l.o.g., that gadgets in an optimum REDF satisfy Case 2 or Case 8.2 in Table 2.

Proof of Claim 4. By item 3, the only possible "danger" are remaining Cases 4.2 or 4.3 that might be even introduced by the described modifications. But then, some edge labeled 0^* would exist, which necessitates the existence of some edge in E'' that is labeled two, contradicting item 3. $\diamond$

Claim 5. We can assume that, w.l.o.g., all edges in E'' carry weight zero.

Proof of Claim 5. By Claim 3, all these edges have weight at most one. Consider an edge $e \in E''$ of weight one that interconnects the gadgets F_i and F_j. If one of the gadgets F_i or F_j, say, F_i, contributes six to the weight, then e is dominated by an edge from F_i. Hence, we can assume that both F_i and F_j contribute five each to the overall weight, i.e., both are in situation of Case 8.2 according to item 4. We can change this REDF by having a weight of zero on e and putting F_i into the situation of Case 2. $\diamond$

The last two claims show that an optimum REDF corresponds to some optimum vertex cover in the original graph as required, since all edges from E'' have to be dominated by some edges from the gadgets. More precisely, given an optimum solution REDF satisfying the last two claims with weight $5n + k$ corresponds to a vertex cover C with $|C| = k$ of the original instance as follows: (a) There are exactly k gadgets F_i that contribute 6 to the overall weight, treated as in Case 2 of Table 2. (b) We collect all vertices v_i of the original graph whose gadgets F_i contribute 6 to the overall weight into the set C. (c) As each edge in E'' is dominated by some edge from a gadget in Case 2 of Table 2 and is not dominated by any gadget in Case 8.2 of Table 2, C is a vertex cover in the original graph.

Corollary 5. *It is NP-complete to decide, given a graph $G = (V, E)$ and an integer k, if $\partial_e(G) \geq k$. This is even true for planar graphs of maximum degree four.*

Theorem 13. *ROMAN EDGE DOMINATION is fixed-parameter tractable.*

Proof. The following procedure (similar to [**15**]) shows the claim. Let (G, ℓ) be the given ROMAN EDGE DOMINATION instance. 1. Enumerate all minimal vertex covers of size at most 2ℓ.2. At the leaves of the search tree, in the positive case, we have collected at most 2ℓ vertices in a set C that forms a minimal vertex cover of G; in the negative case, we have a proof that on this branch of the search tree, no sufficiently small vertex cover exists.3. In the positive case, do the following. (a) Go through all decompositions $\ell = \ell_1 + 2\ell_2$ for integers $\ell_1, \ell_2 \geq 0$. (b) Go through all possibilities to delete ℓ_1 edges from $G\langle C\rangle$, yielding the graph G'. In other words, we have chosen an edge set E'' of cardinality ℓ_1, and G' is obtained from $G\langle C\rangle$ by deleting the edges in E''. Let C' be obtained from C by deleting all endpoints of the edges from E''. (c) Check in polynomial-time if G' contains a set of at most ℓ_2 edges that covers all vertices in C'.

The correctness of this algorithm is based on the following observation: $G = (V, E)$ admits an REDS $f : E \to \{0, 1, 2\}$ of weight at most ℓ if and only if $\ell = \ell_1 + 2\ell_2$ for some non-negative integers ℓ_1, ℓ_2 and we can delete ℓ_1 edges from G such that the remaining graph admits an edge dominating set of size at most ℓ_2. □

The edge version of Corollary 2 shows that any constant-factor approximation algorithm for MINIMUM EDGE DOMINATING SET can be also seen as a constant-factor approximation algorithm for MINIMUM ROMAN EDGE DOMINATION. Hence, MINIMUM ROMAN EDGE DOMINATION can be approximated to a factor of four.

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