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THE DIFFERENTIAL OPERATOR RING OF AN AFFINE CURVE

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ABSTRACT. The purpose of this paper is to investigate the structure of the ring $D(R)$ of all linear differential operators on the coordinate ring of an affine algebraic variety X (possibly reducible) over a field k (not necessarily algebraically closed) of characteristic zero, concentrating on the case that $\dim X \leq 1$. In this case, it is proved that $D(R)$ is a (left and right) noetherian ring with (left and right) Krull dimension equal to $\dim X$, that the endomorphism ring of any simple (left or right) $D(R)$-module is finite dimensional over k, that $D(R)$ has a unique smallest ideal L essential as a left or right ideal, and that $D(R)/L$ is finite dimensional over k. The following ring-theoretic tool is developed for use in deriving the above results. Let D be a subalgebra of a left noetherian k-algebra E such that E is finitely generated as a left D-module and all simple left E-modules have finite dimensional endomorphism rings (over k), and assume that D contains a left ideal I of E such that E/I has finite length. Then it is proved that D is left noetherian and that the endomorphism ring of any simple left D-module is finite dimensional over k.

Introduction. In this paper, we will study the ring $D(R)$ of k-linear differential operators on a commutative k-algebra R, where k is a field of characteristic zero. Of special interest is the case where R is the coordinate ring of an affine algebraic variety X. When X is nonsingular, the ring $D(R)$ has been extensively studied and enjoys many nice properties; for example, $D(R)$ is noetherian. (We will use the term "noetherian" to indicate that a ring is both left and right noetherian.) When X is singular, $D(R)$ need not be noetherian, as shown by J. N. Bernstein, I. M. Gelfand and S. I. Gelfand [**3**]: if X is the normal cubic cone, i.e., the surface in complex 3-space given by $x^3 + y^3 + z^3 = 0$, then $D(R)$ is neither left nor right noetherian. Thus a major goal is to discover for which varieties X the ring $D(R)$ is noetherian. The main contribution of this paper is to prove that $D(R)$ is noetherian when $\dim X \leq 1$, and to develop some of the structure of $D(R)$ in this case.

The paper is organized as follows. §1 contains a number of basic results about the differential operators on commutative rings. In §2, the algebraic tool used in proving $D(R)$ is noetherian is developed. This result overlaps with the independent work of J. C. Robson and L. W. Small [**11**]. §§3 and 4 contain the main results on the structure of $D(R)$ when $\dim X \leq 1$. This work was motivated by the calculations of I. M. Musson [**10**]. These results were independently obtained by S. P. Smith and J. T. Stafford [**12**] in the case that X is an irreducible curve over an algebraically closed field of characteristic zero. §5 contains an example of a nonreduced k-algebra R with Krull dimension one, such that $D(R)$ is right but not left noetherian.

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1. Basic properties of differential operator rings. This section contains the basic properties of differential operator rings which we will need in §3. Most of the results are standard and stated here without proof. The reader is referred to [8]. Throughout this section k will be a field of characteristic zero and R will be a commutative k-algebra. Define $[\ ,\ ]$ on $\mathrm{End}_k(R)$ by $[f,g] = fg - gf$. It will be useful to identify $\mathrm{End}_R(R)$ with R. In order to avoid confusion, the evaluation of an element $f \in \mathrm{End}_k(R)$ at an element $r \in R$ will be denoted $f((r))$. This allows the composition of f with "scalar multiplication" by $r_1 + r_2 \in \mathrm{End}_R(R)$ to be denoted as $f(r_1 + r_2)$.

DEFINITION. Set $D_k^0(R) = R$, which we have identified with $\mathrm{End}_R(R) \subseteq \mathrm{End}_k(R)$. For $p > 0$, define

$$D_k^p(R) = \{f \in \mathrm{End}_k(R)\colon [f,r] \in D_k^{p-1}(R) \text{ for all } r \in R\}$$

and set

$$D_k(R) = \bigcup_{p=0}^{\infty} D_k^p.$$

Elements of $D_k(R)$ are called *k-linear differential operators on R*. When there is no confusion about the base field k, we will use the notations $D^p(R)$ and $D(R)$. The *order* of an operator $d \in D(R)$ is the least nonnegative integer m such that $d \in D^m(R)$, and we will write $\mathrm{ord}(d) = m$. An easy induction on order shows that $D^i(R)D^j(R) \subseteq D^{i+j}(R)$ for all i, j. Hence, $D(R)$ is a filtered k-subalgebra of $\mathrm{End}_k(R)$; it is called *the ring of k-linear differential operators on R*. Another induction on order will show that $[f,g] \in D^{i+j-1}(R)$ for all $f \in D^i(R)$ and $g \in D^j(R)$. Hence, the associated graded ring $\mathrm{gr}(D(R))$ of $D(R)$ is commutative.

An important property of differential operators is the following reduction formula, which may be proved by induction on m.

PROPOSITION (1.1). *Let $d \in D^m(R)$ and $n > m$. Then for all $r_1, \dots, r_n \in R$,*

$$dr_1 \cdots r_n = \sum_{s=0}^{m} (-1)^{m+s} \binom{n-1-s}{n-1-m} \times \sum_{i(1)<\cdots<i(s)} r_1 \cdots \hat{r}_{i(1)} \cdots \hat{r}_{i(s)} \cdots r_n d r_{i(1)} \cdots r_{i(s)}.$$

In particular, evaluating at $1 \in R$ yields

$$d((r_1 \cdots r_n)) = \sum_{s=0}^{m} (-1)^{m+s} \binom{n-1-s}{n-1-m} \times \sum_{i(1)<\cdots<i(s)} r_1 \cdots \hat{r}_{i(1)} \cdots \hat{r}_{i(s)} \cdots r_n d((r_{i(1)} \cdots r_{i(s)})). \qquad \square$$

REMARK. The case $n = m+1$ of the bottom formula

$$d((r_1 \cdots r_{m+1})) = \sum_{s=0}^{m} (-1)^{m+s} \sum_{i(1)<\cdots<i(s)} r_1 \cdots \hat{r}_{i(1)} \cdots \hat{r}_{i(s)} \cdots r_{m+1} d((r_{i(1)} \cdots r_{i(s)}))$$

is sometimes taken as the definition for a differential operator of order at most m.

COROLLARY (1.2). *If $\{r_\lambda\}_{\lambda\in\Lambda}$ generate R as a k-algebra, then any operator in $D^m(R)$ is determined by its values on* 1 *and the products $\{r_{\lambda(1)} \cdots r_{\lambda(s)} : \lambda(1), \dots, \lambda(s) \in \Lambda$ and $1 \le s \le m\}$.* □

EXAMPLE (1.3). Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a collection of independent indeterminates and $A = k[X_\Lambda]$ the polynomial ring over k in these indeterminates. A multi-index I is a function I from Λ to the nonnegative integers such that $I(\lambda) = 0$ except for a finite number of $\lambda \in \Lambda$. Define the degree of I by $\deg(I) = \sum I(\lambda)$ and set $x^I = \prod x_\lambda^{I(\lambda)}$. The derivations $\partial/\partial x_\lambda$ are easily seen to be commuting first order operators on A. Set $\partial^I = \prod(\partial/\partial x_\lambda)^{I(\lambda)} \in D(A)$. Fix a nonnegative integer m and consider the collection of formal sums

$$F_m = \left\{\sum f_I \partial^I : I \text{ is a multi-index with } \deg(I) \le m \text{ and each } f_I \in A\right\}.$$

Even though such a sum may have an infinite number of nonzero terms, it gives a well-defined differential operator of order at most m on A because all but finitely many terms vanish on any given element of A. By first evaluating at $1 \in A$, then at monomials of degree one, then degree two, etc., one can see that two sums $\sum f_I \partial^I$, $\sum g_I \partial^I \in F_m$ will induce the same operator on A if and only if $g_I = f_I$ for all I with $\deg(I) \le m$.

Now take any collection $\{h_I\}_{\deg(I)\le m}$ of elements of A. We claim that there is $\sum f_I \partial^I \in F_m$ such that $h_J = \sum f_I \partial^I((x^J))$ for all J with $\deg(J) \le m$. Solving for f_J results in the equation

$$f_J = \frac{1}{J!}\left(h_J - \sum_{\deg(I)<\deg(J)} f_I \partial^I((x^J))\right)$$

where $J! = \prod J(\lambda)!$. By inductively defining the f_I's in this way, we are able to construct a formal sum $\sum f_I \partial^I \in F_m$. It is easy to check that $h_J = \sum f_I \partial^I((x^J))$ for all J with $\deg(J) \le m$.

In particular, for each $d \in D^m(A)$, we can find $\sum f_I \partial^I \in F_m$ so that $d((x^J)) = \sum f_I \partial^I((x^J))$ for all J with $\deg(J) \le m$. From (1.2), it follows that $d = \sum f_I \partial^I$, and hence $D^m(A) = F_m$. Products of elements in $D(A)$ can be written in "standard form" (as in the description of F_m) by repeated use of the formulas

$$(\partial/\partial x_\lambda) f = f(\partial/\partial x_\lambda) + (\partial/\partial x_\lambda)((f)).$$

In the special case where $\Lambda = \{1, \dots, n\}$, we have that

$$D(k[x_1, \dots, x_n]) = A_n(k),$$

the nth Weyl algebra. □

LEMMA (1.4). *Let $A = k[X_\Lambda]$ be a polynomial ring in independent indeterminates $\{x_\lambda\}_{\lambda\in\Lambda}$, and let $R = A/B$ where B is an ideal of A. Then there is a k-algebra isomorphism*

$$\{d \in D(A)\colon d((B)) \subseteq B\}/\{d \in D(A)\colon d((A)) \subseteq B\} \cong D(R)$$

under which the coset of an operator $d \in D(A)$ satisfying $d((B)) \subseteq B$ corresponds to an operator in $D(R)$ mapping $r + B$ to $d((r)) + B$. In particular, any operator in $D^m(R)$ lifts to an operator $d \in D^m(A)$ such that $d((B)) \subseteq B$. $\square$

We will now consider the differential operators on a localization $S^{-1}R$ of R, following [**6**] where the case that S consists of non-zero-divisors is handled.

LEMMA (1.5). *Let S be a multiplicatively closed set in R. Let $d \in D^m(R)$ and suppose that $a, b \in R$ and $s, t \in S$ satisfy $at = bs$. Then in $S^{-1}R$, we have*

$$\sum_{p=0}^{\infty}(-1)^p[d,s]_p((a))/s^{p+1} = \sum_{p=0}^{\infty}(-1)^p[d,t]_p((b))/t^{p+1}$$

where $[d,s]_p$ denotes p successive brackets of d with s, and is defined inductively by $[d,s]_0 = d$ and $[d,s]_p = [[d,s]_{p-1}, s]$. (Notice that $[d,s]_p = 0$ when $p > m$.) $\square$

Lemma (1.5) allows us to "extend" operators from R to $S^{-1}R$. If $d \in D(R)$, define a function $\Phi(d)\colon S^{-1}R \to S^{-1}R$ by

$$\Phi(d)((a/s)) = \sum_{p=0}^{\infty}(-1)^p[d,s]_p((a))/s^{p+1}.$$

It is easy to see that $\Phi(d)$ is k-linear and to verify that

$$[\Phi(d), r/1] = \Phi([d,r]).$$

Using induction on the order of d, it follows that if $d \in D^m(R)$, then $\Phi(d) \in D^m(S^{-1}R)$. Notice that $\Phi(d)$ extends d in the sense that

$$\Phi(d)((r/1)) = d((r))/1$$

for all $r \in R$. Extending operators gives a map $\Phi\colon D(R) \to D(S^{-1}R)$, which is easily seen to be k-linear. In fact, viewing $D(R)$ and $D(S^{-1}R)$ as (R,R)-bimodules, we find that Φ is an (R,R)-bimodule homomorphism. Also note that the restriction of Φ to R gives the canonical map $R \to S^{-1}R$. That Φ is a ring homomorphism follows easily from the next lemma.

LEMMA (1.6). *Let $\delta \in D(S^{-1}R)$ and suppose $\delta((r/1)) = 0$ for all $r \in R$. Then $\delta = 0$.* $\square$

LEMMA (1.7). *Let R be a finitely generated k-algebra and S a multiplicatively closed subset of R.*

(a) *If $d \in D^m(R)$ and $\Phi(d)$ denotes the extension of d to $S^{-1}R$, then $\Phi(d) = 0$ if and only if $sd = 0$ for some $s \in S$.*

(b) *If $\delta \in D^m(S^{-1}R)$, then $\delta = \Phi(s)^{-1}\Phi(d)$ for some $s \in S$ and $d \in D^m(R)$.* $\square$

Recall the definition of localization for noncommutative rings. Let A be any ring with identity and S a multiplicatively closed subset of A. A *left ring of fractions* of A

with respect to S (if it exists) is a ring $[S^{-1}]A$ together with a ring homomorphism $\Phi\colon A \to [S^{-1}]A$ satisfying:

(1) $\Phi(s)$ is invertible for every $s \in S$;

(2) every element of $[S^{-1}]A$ is of the form $\Phi(s)^{-1}\Phi(a)$ with $s \in S$ and $a \in A$;

(3) $\Phi(a) = 0$ if and only if $sa = 0$ for some $s \in S$.

A right ring of fractions is defined similarly.

PROPOSITION (1.8). *Let R be a finitely generated k-algebra and S a multiplicatively closed subset of R. Then $D(S^{-1}R)$, via the map defined above, is both a left and right ring of fractions for $D(R)$ with respect to S.*

PROOF. Condition (1) is symmetric and immediate from $S^{-1}R \subseteq D(S^{-1}R)$. Conditions (2) and (3) for a left ring of fractions are in (1.7). For the right-handed version of (2), let $\delta \in D^m(S^{-1}R)$ and write $\delta = \Phi(s)^{-1}\Phi(d)$ for some $s \in S$ and $d \in D^m(R)$. Write

$$\Phi(s)^{-1}\Phi(d) = \Phi(d)\Phi(s)^{-1} - [\Phi(d), \Phi(s)^{-1}]$$

and use induction on order to find $t \in S$ and $d' \in D^{m-1}(R)$ with $[\Phi(d), \Phi(s)^{-1}] = \Phi(d')\Phi(t)^{-1}$. Thus

$$\delta = \Phi(d)\Phi(s)^{-1} - \Phi(d')\Phi(t)^{-1} = \Phi(dt - d's)\Phi(st)^{-1}$$

and the right-handed version of (2) holds. Finally, for the right-handed version of (3), let $d \in D(R)$ with $\Phi(d) = 0$. From (1.7), there is an $s \in S$ with $sd = 0$. Observe that

$$\Phi([d, s]) = [\Phi(d), \Phi(s)] = 0.$$

Using induction on order, there is a $t \in S$ with $[d, s]t = 0$. Thus

$$0 = sdt = dst - [d, s]t = dst$$

and the right-handed version of (3) holds. $\square$

The next proposition is simply a restatement of (1.8) in the version that most often appears in the literature.

PROPOSITION (1.9). *Let R be a finitely generated k-algebra and S a multiplicatively closed set in R. Extending operators gives an isomorphism $S^{-1}R \otimes_R D(R) \cong D(S^{-1}R)$ of left $S^{-1}R$-modules.* $\square$

In the special case that S consists of non-zero-divisors, we have the following proposition.

PROPOSITION (1.10). *Let S be a multiplicatively closed subset of non-zero-divisors of R. Identify R with its image under the embedding $R \hookrightarrow S^{-1}R$. Then extending operators gives an isomorphism*

$$D(R) \cong \{\delta \in D(S^{-1}R)\colon \delta((R)) \subseteq R\}. \quad \square$$

PROPOSITION (1.11). *Let R be a finitely generated k-algebra and L be an extension field of k. Then $L \otimes_k D_k(R) \cong D_L(L \otimes_k R)$ as L-algebras.*

PROOF. The isomorphism is simply the restriction to $L \otimes_k D_k(R)$ of the canonical L-algebra embedding $\psi\colon L \otimes_k \mathrm{End}_k(R) \hookrightarrow \mathrm{End}_L(L \otimes_k R)$. Identify R with

$1 \otimes R$ in $L \otimes_k R$; then $L \otimes_k R = LR$. Let $r \in R$, $\alpha \in L$, and $d \in D_k(R)$. As $\psi(d)$ is the L-linear extension of d, we have

$$[\psi(d), \alpha r] = \alpha[\psi(d), r] = \alpha\psi([d, r]).$$

Using additivity and induction on the order of d, it follows that $\psi(L \otimes_k D_k(R)) \subseteq D_L(LR)$.

Now let $\partial \in D_L^p(LR)$ and choose a basis $\{\varsigma_\lambda\}_{\lambda\in\Lambda}$ for L over k. For each $\lambda \in \Lambda$, let $\pi_\lambda \colon LR \to R$ denote the usual λth coordinate projection. Set $\partial_\lambda = \pi_\lambda \partial|_R$. Let $r \in R$, then

$$[\partial_\lambda, r] = [\pi_\lambda \partial|_R, r] = \pi_\lambda[\partial, r]|_R.$$

It follows from induction on p, that $\partial_\lambda \in D_k^p(R)$. Let $x_1, \ldots, x_n$ generate R as a k-algebra. The reduction formula (1.1) shows $\partial((R)) \subseteq \sum R\partial((x^I))$, where the sum is taken over all multi-indices I of deg $\leq p$. Thus all but a finite number of the ∂_λ are zero and $\sum \varsigma_\lambda \otimes \partial_\lambda \in L \otimes D_k(R)$. Using $\partial|_R = \sum \varsigma_\lambda \partial_\lambda$ and L-linearity, it follows that $\psi(\sum \varsigma_\lambda \otimes \partial_\lambda) = \partial$. Therefore $\psi(L \otimes_k D_k(R)) = D_L(L \otimes_k R)$. □

PROPOSITION (1.12). *Let R_1 and R_2 be commutative k-algebras, then there is a k-algebra isomorphism $D(R_1 \times R_2) \cong D(R_1) \times D(R_2)$.*

PROOF. Define $\psi \colon D(R_1) \times D(R_2) \to \operatorname{End}_k(R_1 \times R_2)$ by

$$\psi(d_1, d_2)((r_1, r_2)) = (d_1((r_1)), d_2((r_2))).$$

It is clear that ψ is a k-algebra embedding. From the formula

$$[\psi(d_1, d_2), (r_1, r_2)] = \psi([d_1, r_1], [d_2, r_2]),$$

it follows by induction on order that $\operatorname{Im}(\psi) \subseteq D(R_1 \times R_2)$.

Now let $\delta \in D(R_1) \times D(R_2)$ and consider $\pi_1\delta i_1 \colon R_1 \to R_1$, where i_1 and π_1 denote the standard injection and projection maps. Notice that

$$[\pi_1\delta i_1, r_1] = \pi_1\delta i_1 r_1 - r_1\pi_1\delta i_1 = \pi_1[\delta, (r_1, 0)]i_1$$

for all $r_1 \in R_1$. It follows by induction on order that $\pi_1\delta i_1 \in D(R_1)$, and similarly that $\pi_2\delta i_2 \in D(R_2)$. Using the formula

$$i_1\pi_1\delta i_2\pi_2 = i_1\pi_1\delta i_2\pi_2 i_2\pi_2 = i_1\pi_1(i_2\pi_2\delta + [\delta, i_2\pi_2])i_2\pi_2 = i_1\pi_1[\delta, i_2\pi_2]i_2\pi_2$$

and induction on order, it follows that $i_1\pi_1\delta i_2\pi_2 = 0$, and similarly $i_2\pi_2\delta i_1\pi_1 = 0$. From the equation

$$\delta = (i_1\pi_1 + i_2\pi_2)\delta(i_1\pi_1 + i_2\pi_2) = i_1\pi_1\delta i_1\pi_1 + i_2\pi_2\delta i_2\pi_2 = \psi(\pi_1\delta i_1, \pi_2\delta i_2),$$

we conclude that $\operatorname{Im}(\psi) = D(R_1 \times R_2)$. □

Let F be a field extension of k having finite transcendence degree over k. Let $\{x_1, x_2, \ldots, x_n\}$ be a transcendence basis for F over k and let ∂_i be the extension of $\partial/\partial x_i$ to F. It is well known (see [**8**]) that $D(F)$ is just the usual Ore extension of the field F by the n commuting derivations $\{\partial/\partial x_i\}$. Such Ore extensions have been extensively studied and are known to be noetherian and of Krull dimension n. (It is easy to see that the associated graded ring $\operatorname{gr}(D(F))$ is a polynomial ring over F in n indeterminates. In particular, $\operatorname{gr}(D(F))$ is noetherian of Krull dimension n, and it follows that $D(F)$ is noetherian with left and right Krull dimension at most n. The interested reader is referred to [**5**] for a result on the Krull dimension of Ore extensions.)

PROPOSITION (1.13). *Let F be a field extension of k having finite transcendence degree over k. Then the following hold.*

(a) *$D(F)$ is a simple domain.*

(b) *$D(F)$ is noetherian.*

(c) $\mathrm{l.K.dim}\, D(F) = \mathrm{r.K.dim}\, D(F) = \mathrm{tr.deg}(F/k)$. $\square$

Recall that a commutative ring is *reduced* if it does not contain any nonzero nilpotent elements.

PROPOSITION (1.14). *Let R be a finitely generated reduced k-algebra.*

(a) *$D(R)$ is a domain if and only if R is a domain.*

(b) *If* $\mathrm{r.K.dim}(D(R))$ *exists, then* $\mathrm{r.K.dim}\, D(R) \geq \mathrm{K.dim}\, R$. *If* $\mathrm{l.K.dim}(D(R))$ *exists, then* $\mathrm{l.K.dim}\, D(R) \geq \mathrm{K.dim}\, R$.

PROOF. (a) If $D(R)$ is a domain, then so is its subring R. Assume that R is a domain. Let F be the quotient field of R. From (1.10), $D(R)$ is isomorphic to a subring of the domain $D(F)$.

(b) Let K be the total quotient ring of R, i.e., the quotient ring with respect to the set S of regular elements. Then K is a product of fields $K_1 \times \cdots \times K_n$ and

$$\mathrm{K.dim}\, R = \max\{\mathrm{tr.deg}(K_i) \colon i = 1, \ldots, n\}.$$

From (1.8), $D(K)$ is a right ring of fractions for $D(R)$ with respect to S. Thus if $\mathrm{r.K.dim}\, D(R)$ exists, then $\mathrm{r.K.dim}\, D(K)$ exists and $\mathrm{r.K.dim}\, D(R) \geq \mathrm{r.K.dim}\, D(K)$. It follows from (1.12) and (1.13) that $\mathrm{r.K.dim}\, D(K) = \mathrm{K.dim}\, R$. Thus the inequality for right Krull dimension holds. As $D(K)$ is also a left ring of fractions for $D(R)$ with respect to S, the same argument works for left Krull dimension. $\square$

Finally, we consider the case of differential operators on a nonsingular variety. The differential operators on a regular finitely generated k-algebra R are well understood. Using (1.8), many questions can be reduced to the case of differential operators on the localization R_M of R at a maximal ideal M. It can be shown that the differential operators on R_M are an Ore extension of R_M by $n = \mathrm{K.dim}\, R_M$ commuting derivations.

THEOREM (1.15). *Let R be a regular finitely generated k-algebra. Then the following hold:*

(a) *$D^m(R)$ is equal to the left [right] R-submodule of $D(R)$ generated by all products of m or less k-derivations of R.*

(b) *$\mathrm{gr}(R)$ is a finitely generated commutative k-algebra.*

(c) *$D(R)$ is (left and right) noetherian.*

(d) $\mathrm{l.K.dim}\, D(R) = \mathrm{r.K.dim}\, D(R) = \mathrm{gl.dim}\, D(R) = \mathrm{K.dim}\, R$.

(e) *If R is a domain, then $D(R)$ is a simple domain.*

(f) *If M is any simple $D(R)$-module, then* $\dim_k \mathrm{Hom}_{D(R)}(M, M) < \infty$.

REMARK. If a ring D satisfies property (f), we will say that D has the finite dimensional property for simple modules.

PROOF OF (1.15). Statements (a)–(e) can be found in [8] in the case that R is a domain. The general statement follows from (1.12). Statement (f) follows from a theorem of L. W. Small (see [**11**]) that says:

THEOREM (L. W. SMALL). *If T is a k-algebra such that $L \otimes_k T$ is right noetherian for any extension field L of k and* $\mathrm{End}_T(N)$ *is algebraic over k for any right T-module N of finite length, then* $\mathrm{End}_T(M)$ *is finite dimensional over k for all simple right T-modules M.*

Any extension field L of k is separable over k and so $L \otimes_k R$ is a regular finitely generated commutative L-algebra (see [**9**, p. 208]). Using (3) and (1.11), we have that $L \otimes_k D_k(R) \cong D_L(L \otimes_k R)$ is a noetherian ring. That $D(R)$ has the finite dimensional property for simple modules is a result of

QUILLEN'S LEMMA [**7**]. *If N is a module of finite length over a nonnegatively filtered k-algebra T and the associated graded ring* $\mathrm{gr}(T)$ *is a commutative finitely generated k-algebra, then* $\mathrm{Hom}_T(N, N)$ *is algebraic over k.*

That $T = D(R)$ satisfies the conditions of Quillen's lemma is the result of (b). $\square$

2. A few algebraic preliminaries. The results of this section overlap with the independent work of J. C. Robson and L. W. Small. Their results, which include an improved version of Theorem (2.2), will appear in [**11**].

In this section the field k need not be of characteristic zero.

PROPOSITION (2.1). *Let D be a subalgebra of a left noetherian k-algebra E. Suppose that D contains a left ideal I of E and that E is finitely generated as a left D-module. If* $\mathrm{Hom}_E(E/I, E/J)$ *is finite dimensional over k for all left ideals J of E, then D is left noetherian.*

PROOF. Let B be a left ideal of D and consider the left D-module IB. Since IB is a left ideal of E, it is finitely generated over E and hence over D. Using the canonical inclusion $B/IB \hookrightarrow E/IB$, we see that as k-modules

$$B/IB \hookrightarrow \{x \in E/IB \mid Ix = 0\} \cong \mathrm{Hom}_E(E/I, E/IB).$$

Since $\dim_k \mathrm{Hom}_E(E/I, E/IB) < \infty$, it follows that B/IB is finitely generated as a k-vector space and hence also as a D-module. As both ends of the sequence $0 \to IB \to B \to B/IB \to 0$ are finitely generated D-modules, we conclude that B is a finitely generated left ideal of D. $\square$

THEOREM (2.2). *Let D be a subalgebra of a left noetherian k-algebra E such that D contains a left ideal I of E. Suppose the following hold.*

(1) *E/I is a left E-module of finite length.*

(2) $\dim_k \mathrm{Hom}_E(S, S) < \infty$ *for all simple left E-modules S.*

(3) *E is finitely generated as a left D-module.*

Then D is left noetherian and $\dim_k \mathrm{Hom}_D(M, M) < \infty$ *for all simple left D-modules M.*

REMARK. In [**11**], it is shown that hypotheses (1) and (2) are sufficient to show E/I is also of finite length as a D-module. From this it follows that E is finitely generated as a left D-module. Thus (3) is superfluous.

PROOF OF (2.2). It follows from (2) and induction on length that if M is a left E-module of finite length and N is a noetherian left E-module, then $\dim_k \mathrm{Hom}_E(M, N) < \infty$. It then follows from (2.1) that D is left noetherian.

As k-vector spaces

$$D/ID \hookrightarrow \{x \in E/ID | Ix = 0\} \cong \mathrm{Hom}_E(E/I, E/ID)$$

and so D/ID is finite dimensional. Let M be a simple left D-module. Then either $IM = 0$ or $IM = M$.

Case 1: $IM = 0$. There is a D-module isomorphism $M \cong D/J$ where J is a maximal left ideal of D. From $IM = 0$, it follows that $ID \subseteq J$ and so M is isomorphic to a factor of D/ID. Hence M is finite dimensional, and so $\mathrm{Hom}_D(M, M)$ is finite dimensional.

Case 2: $IM = M$. Consider the left E-module $ID \otimes_D M$. Let

$$\psi\colon ID \otimes_D M \to IDM = IM = M$$

denote the surjective left D-module homomorphism given by multiplication. Every $f \in \mathrm{Hom}_D(M, M)$ gives rise to the commutative diagram

$$\begin{array}{ccc} 1 \otimes f\colon ID \otimes_D D & \longrightarrow & ID \otimes_D M \\ \downarrow & & \downarrow \\ f\colon M & \longrightarrow & M \end{array}$$

It follows that $\mathrm{Hom}_D(M, M) \hookrightarrow \mathrm{Hom}_E(ID \otimes_D M, ID \otimes_D M)$ is an injective k-linear map.

To conclude that $\mathrm{Hom}_D(M, M)$ is finite dimensional, it will be sufficient to show that $ID \otimes_D M$ has finite length as a left E-module. Notice that $ID \otimes_D M$ is a noetherian left E-module and hence a noetherian left D-module. Consider the exact sequence of D-modules

$$0 \to \mathrm{Ker}\,\psi \to ID \otimes_D M \to M \to 0.$$

Let $\sum y_i \otimes m_i \in \mathrm{Ker}\,\psi$ where $y_i \in ID$ and $m_i \in M$. If $x \in ID$, then

$$x\left(\sum y_i \otimes m_i\right) = \sum xy_i \otimes m_i = x \otimes \left(\sum y_i m_i\right) = 0.$$

Thus $ID(\mathrm{Ker}\,\psi) = 0$ and so Ker ψ is a noetherian left D/ID-module. Since D/ID is finite dimensional over k, so is Ker ψ. Both ends of the exact sequence above are D-modules of finite length and so $ID \otimes_D M$ is of finite length as a D-module and hence necessarily as an E-module. $\square$

3. The differential operator rings of curves. The results of this and the following section were independently obtained by S. P. Smith and J. T. Stafford [**12**] under the additional hypotheses that R is a domain and that k is algebraically closed.

The goal of this section is to prove the following theorem.

THEOREM (3.1). *Let R be a reduced finitely generated commutative k-algebra of Krull dimension ≤ 1, where k is a field of characteristic zero. Then the following hold.*

(a) $D(R)$ *is a noetherian ring.*

(b) $\dim_k \mathrm{Hom}_{D(R)}(M, M) < \infty$ *for all simple $D(R)$-modules M.*

(c) $\mathrm{l.K.dim}\, D(R) = \mathrm{r.K.dim}\, D(R) = \mathrm{K.dim}\, R$.

Let R be as in (3.1) and write $R = A/B$ where A is a polynomial ring in finitely many independent indeterminates and B is a radical ideal of A. Letting

$Q_1, Q_2, \ldots, Q_n$ denote the primes minimal over B, we have $B = Q_1 \cap \cdots \cap Q_n$. Identify R with its image under the ring embedding $R \hookrightarrow R_1 \times \cdots \times R_n$ where $R_i = A/Q_i$. Let K_i denote the quotient field of R_i and $\overline{R}_i$ the integral closure of R_i in K_i. Since $\overline{R}_i$ is a finitely generated R_i-module, it is also a finitely generated k-algebra. We also have a chain of inclusions

$$R \subseteq R_1 \times \cdots \times R_n \subseteq \overline{R}_1 \times \cdots \times \overline{R}_n \subseteq K_1 \times \cdots \times K_n.$$

Set $\overline{R} = \overline{R}_1 \times \cdots \times \overline{R}_n$ and $K = K_1 \times \cdots \times K_n$. Consider the diagram of inclusions

$$\begin{tikzcd}
 & D(K) \arrow[dl, dash] \arrow[dd, dash] \\
E \arrow[d, dash] & \\
D \arrow[d, dash] & \overline{D} \arrow[dl, dash] \\
I &
\end{tikzcd}$$

where

$$D = \{\partial \in D(K) | \partial((R)) \subseteq R\}, \quad \overline{D} = \{\partial \in D(K) | \partial((\overline{R})) \subseteq \overline{R}\},$$
$$I = \{\partial \in D(K) | \partial((\overline{R})) \subseteq R\}, \quad E = \{\partial \in D(K) | \partial I \subseteq I\}.$$

Notice that I is a left ideal of D and E, and a right ideal of $\overline{D}$. The idea of using I in this context resulted from a study of the examples calculated by I. M. Musson [**10**].

We now present a series of lemmas showing that the hypotheses of (2.2) are satisfied.

LEMMA (3.2). *The following hold.*

(a) *K is the total quotient ring of R and $\overline{R}$, i.e., the quotient ring with respect to the set of regular elements.*

(b) *$D \cong D(R)$ and $\overline{D} \cong D(\overline{R})$.*

(c) *$D(K)$ is a finite product of simple noetherian domains, and as such has a (right $\cong$ left) ring of fractions with respect to the set of all of its regular elements. Denote this classical ring of quotients by Q. Then Q is also the classical quotient ring of $\overline{D}$, D, and E.*

PROOF. (a) An element of R is regular if and only if its coordinates are all nonzero if and only if it is invertible in K. We must show that every element of K is of the form $s^{-1}r$ where $s, r \in R$ and s is regular. Let $(\overline{r}_1/\overline{s}_1, \ldots, \overline{r}_n/\overline{s}_n) \in K = K_1 \times \cdots \times K_n$ where $r_i, s_i \in A$ with $s_i \notin Q_i$ and $\overline{r}_i, \overline{s}_i$ denote the images of r_i, s_i in R_i. For each i, choose $f_i \in A$ such that $f_i \notin Q_i$, but $f_i \in Q_j$ for all $j \neq i$. Denote by r and s the elements of R induced by the polynomials $r_1 f_1 + \cdots + r_n f_n$ and $s_1 f_1 + \cdots + s_n f_n$. As both s_i and f_i are not elements of Q_i, it follows that $s = (\overline{s}_1 \overline{f}_1, \ldots, \overline{s}_n \overline{f}_n)$ is a regular element of R. Finally,

$$s^{-1}r = (1/\overline{s}_1\overline{f}_1, \ldots, 1/\overline{s}_n\overline{f}_n)(\overline{r}_1\overline{f}_1, \ldots, \overline{r}_n\overline{f}_n) = (\overline{r}_1/\overline{s}_1, \ldots, \overline{r}_n/\overline{s}_n)$$

as required. Notice that the regular elements of $\overline{R}$ are invertible in K and contain the regular elements of R, so K is also the total quotient ring of $\overline{R}$.

(b) Follows from (1.10).

(c) From (1.12), it follows that $D(K) \cong D(K_1) \times \cdots \times D(K_n)$. Each $D(K_i)$ is a simple noetherian domain by (1.13). Since $D(K)$ is a finite product of simple noetherian domains, it has a classical (right $\cong$ left) ring of quotients. From (1.8), we know that $D(K)$ is the ring of fractions of $\overline{D}$ with respect to the set T of regular elements of $\overline{R}$. It is easy to check that the regular elements of $\overline{D}$ are also regular in $D(K)$, and so invertible in Q. Let $d \in Q$ and write $d = ab^{-1}$ where $a, b \in D(K)$ and b is regular. Because $D(K)$ is a right ring of fractions for $\overline{D}$ with respect to T, we may write $a = xs^{-1}$ where $x \in \overline{D}$ and $s \in T$. Then writing $bs = yt^{-1}$, where $y \in \overline{D}$ and $t \in T$, yields

$$d = xs^{-1}b^{-1} = x(bs)^{-1} = x(yt^{-1})^{-1} = (xt)y^{-1},$$

with $xt, y \in \overline{D}$ and y regular in $\overline{D}$. Thus Q is a classical right quotient ring of $\overline{D}$. Similarly, Q is also a classical left quotient ring for $\overline{D}$.

We know that $D(K)$ is the ring of fractions of D with respect to the set S of regular elements of R. It is easy to see that $D(K)$ is also the ring of fractions of E with respect to S. The same argument as above shows that Q is the classical quotient ring for both D and E. □

LEMMA (3.3). *The following hold.*

(a) *$\overline{D}$ is a finite product of simple noetherian domains and*

$$\text{l.K.dim}\,\overline{D} = \text{r.K.dim}\,\overline{D} = \text{K.dim}\,R.$$

(b) $\dim_k \operatorname{Hom}_{\overline{D}}(M, M) < \infty$ *for all simple $\overline{D}$-modules M.*

(c) *I contains an element $f \in R$ which is regular in R. (Notice that f is invertible in $D(K)$ and hence regular in $D, \overline{D}$, and E.)*

(d) *$\overline{D}I = \overline{D}$.*

(e) *The right $\overline{D}$-module I is a finitely generated projective generator.*

PROOF. We must make use of the fact that the normalization of an algebraic variety of dim ≤ 1 is smooth. The corresponding algebraic fact is that for a normal (integrally closed in its quotient field) commutative noetherian domain, the localizations at height one primes are regular local rings (see [**1**, Corollary (3.12), p. 135]). Since $\text{K.dim}\,\overline{R}_i = \text{K.dim}\,R_i \leq 1$, it follows that $\overline{R}_i$ is regular.

(a) From (1.12) and (3.2), we have that $\overline{D} \cong D(\overline{R}) = D(\overline{R}_1) \times \cdots \times D(\overline{R}_n)$. From (1.15), each $D(\overline{R}_i)$ is a simple noetherian domain having

$$\text{K.dim}\,D(\overline{R}_i) = \text{K.dim}\,\overline{R}_i = \text{K.dim}\,R_i.$$

It follows that

$$\text{l.K.dim}\,\overline{D} = \text{r.K.dim}\,\overline{D} = \text{K.dim}\,R.$$

(b) From (1.15), we see that each $D(\overline{R}_i)$ satisfies the finite dimensional property for simple modules. Hence $\overline{D}$ does also.

(c) Since $\overline{R}_i$ is a finitely generated R_i-module, there is a nonzero $\overline{s}_i \in R_i$ such that $\overline{s}_i\overline{R}_i \subseteq R_i$. As in the proof of (3.2), select $f_i \in A$ such that $f_i \notin Q_i$, but $f_i \in Q_j$ for all $j \neq i$. The polynomial $s_1 f_1 + \cdots + s_n f_n$ induces an element f of R. It is easy to see that $f = (\overline{s}_1\overline{f}_1, \ldots, \overline{s}_n\overline{f}_n)$ is a regular element of R and that $f\overline{R} \subseteq R$. Thus $f \in I$.

(d) As $\overline{D}$ is a product of simple rings, it follows that $\overline{D}I = \overline{D}$.

(e) From (1.15), we have that gl.dim $D(\overline{R}_i) = 1$. Thus $\overline{D}$ is a hereditary noetherian ring and hence I is a projective finitely generated right ideal of $\overline{D}$. It follows from the equation $\overline{D}I = \overline{D}$ that $I_{\overline{D}}$ is a generator. □

LEMMA (3.4). *The following hold.*

(a) $E \cong \mathrm{End}_{\overline{D}}(I)$.

(b) *E is a noetherian ring and (left = rt)* $\mathrm{K.dim}(E) = \mathrm{K.dim}(\overline{D})$.

(c) $\dim_k \mathrm{Hom}_E(M, M) < \infty$ *for all simple E-modules M.*

(d) *E is a finite product of simple rings. Furthermore $IE = E$ and E/I is a left E-module of finite length.*

(e) *E is a finitely generated left D-module.*

PROOF. (a) From (3.2), we know that Q is the classical ring of quotients for $\overline{D}$ and so Q is an injective right $\overline{D}$-module (see [**13**, p. 58]). It follows that if $\phi \in \mathrm{End}_{\overline{D}}(I)$, then there exists a $q \in Q$ with $\phi(x) = qx$ for every $x \in I$. The fact that $f \in I$ is invertible in Q allows us to conclude that $q = \phi(f)f^{-1}$ (so that q is uniquely determined by ϕ). It follows that $\mathrm{End}_{\overline{D}}(I) \cong \{q \in Q \colon qI \subseteq I\}$. However, if $q \in Q$ and $qI \subseteq I$, then $qf \in I$. Again, f is invertible in $D(K)$, and so $q \in If^{-1} \subseteq D(K)$.

(b) Because $I_{\overline{D}}$ is a finitely generated projective generator, we have that $\mathrm{End}_{\overline{D}}(I)$ is Morita equivalent to $\overline{D}$. It follows that $\mathrm{End}_{\overline{D}}(I)$, and hence E, is noetherian and of the same Krull dimension as $\overline{D}$.

(c) We need to explore the functors giving the equivalence between the module categories of $\overline{D}$ and E. Using an argument as above, we find that $\mathrm{End}_E(I) \cong \{d \in D(K) \colon Id \subseteq I\}$. The set on the right contains $\overline{D}$ as I is a right ideal of $\overline{D}$. Observe that if $Id \subseteq I$, then $\overline{D}Id \subseteq \overline{D}I$. Using the fact that $\overline{D}I = \overline{D}$ yields $d \in \overline{D}d = \overline{D}Id \subseteq \overline{D}I = \overline{D}$. Thus $\mathrm{End}_E(I) \cong \overline{D}$. This tells us that the bimodule ${}_EI_{\overline{D}}$ is balanced, and so $(-) \otimes_E I$ and $\mathrm{Hom}_E(I, E) \otimes_E (-)$ give equivalences between the categories of right and left E-modules with the right and left $\overline{D}$-modules respectively (see [**2**, p. 264]). Thus, if M is a simple right E-module, then $M \otimes_E I$ is a simple right $\overline{D}$-module, and the bijection $\mathrm{Hom}_E(M, M) \to \mathrm{Hom}_{\overline{D}}(M \otimes_E I, M \otimes_E I)$ is easily seen to be k-linear. As the vector space on the right is finite dimensional, so is the left. Thus E has the finite dimensional property for simple right modules and similarly also for simple left modules.

(d) From (3.3), $\overline{D}$ is a finite product of simple rings. This is a Morita invariant property, so E is also a finite product of simple rings. Since IE is an ideal of E containing a regular element, it must equal E. Because $\mathrm{K.dim}(E) \leq 1$, it follows that E/I is of finite length.

(e) Since $IE = E$, there is a finite sum $\sum x_i y_i = 1$ where $x_i \in I$ and $y_i \in E$. Observing that

$$E = E\left(\sum x_i y_i\right) = \sum E x_i y_i \subseteq \sum I y_i \subseteq \sum D y_i \subseteq E,$$

we conclude that E is finitely generated as a left D-module. □

PROOF OF THEOREM (3.1). Applying (2.2), we have that $D(R)$ is a left noetherian ring with the finite dimensional property for simple left modules. To obtain the right-handed version, consider $P = \{d \in D(K) \colon d((R)) \subseteq \overline{R}\}$. Using the regular element $f \in I$, we get a diagram of inclusions

$$\begin{tikzcd}[arrows=dash]
 & D(K) \arrow[dd] \\
E^* \arrow[ur] \arrow[d] & \\
D \arrow[d] & \overline{D}^* \\
I^* \arrow[ur] &
\end{tikzcd}$$

where $I^* = fP$, $\overline{D}^* = f\overline{D}f^{-1}$ and $E^* = \{\partial \in D(K)\colon I^*\partial \subseteq I^*\}$. Notice that I^* is a right ideal of both D and E^* and a left ideal of $\overline{D}^*$. As $1 \in P$, we have that $f \in fP = I^*$, and so I^* contains a regular element of R. The ring $\overline{D}^*$ is isomorphic as a k-algebra to $\overline{D}$, so it is a finite product of simple noetherian rings of Krull dimension ≤ 1 which satisfy the finite dimensional property for simple modules. As before, the left $\overline{D}^*$-module I^* is a finitely generated projective generator and $E^* \cong \mathrm{End}_{\overline{D}^*}(I^*)$. Thus E^* will be Morita equivalent to $\overline{D}^*$. Applying the symmetric version of (2.2) gives us that D is a right noetherian ring with the finite dimensional property for simple right modules. Thus parts (a) and (b) of (3.1) have been proved. For (c), observe that $D/ID \hookrightarrow \mathrm{Hom}_E(E/I, E/I)$ is an embedding of k-vector spaces, and so D/ID is finite dimensional. It follows from [**14**, Corollary (2.4)], that

$$\text{r.K.dim}(D(R)) = \text{r.K.dim}(E) = \text{r.K.dim}(\overline{D}) = \text{K.dim}\, R.$$

Similarly, D/DI^* is finite dimensional and

$$\text{l.K.dim}(D(R)) = \text{l.K.dim}(E^*) = \text{l.K.dim}(\overline{D}^*) = \text{l.K.dim}(\overline{D}) = \text{K.dim}\, R. \quad \square$$

4. The ideal structure of D of a curve. Throughout this section, R will be a reduced finitely generated commutative k-algebra of Krull dimension ≤ 1, where k is a field of characteristic zero. We will continue to use the notation K, D, E, I and Q from §3.

PROPOSITION (4.1). *Let $M = I((\overline{R}))$. Then the following hold.*

(a) *M is an ideal of R and contains a regular element of R.*

(b) *$E = \{\partial \in D(K)\colon \partial((M)) \subseteq M\}$.*

(c) *Set $L = \{\partial \in D(K)\colon \partial((R)) \subseteq M\}$. Then L is the smallest ideal of D which is essential as a left or right ideal of D.*

PROOF. (a) From the definition of I, we have that $M \subseteq R$. It follows from

$$RM = RI((\overline{R})) = I((\overline{R})) = M$$

that M is an ideal of R. From (3.3), I contains an element $f \in R$ which is regular in R. As $f = f((1))$, we see that $f \in M$.

(b) If $\partial \in E$, then $\partial((M)) = \partial I((\overline{R})) \subseteq I((\overline{R})) = M$. Conversely, if $\partial \in D(K)$ and $\partial((M)) \subseteq M$, then $\partial I((\overline{R})) = \partial((M)) \subseteq M$. Thus $\partial I \subseteq I$ by virtue of the definition of I and so $\partial \in E$.

(c) Using that $\partial((M)) \subseteq M$ for all $\partial \in D$, it is easy to see that L is an ideal of D and a left ideal of E. Notice that $I \subseteq L$. From (3.3), I, and hence L, contains an element $f \in R$ which is regular as an element of D. Thus L is essential as both a left and right ideal. From (3.2), we know that Q is the classical quotient ring of D

and so Goldie's theorem asserts that a left or right ideal is essential if and only if it contains a regular element. Thus an ideal J of D is essential as a left or right ideal if and only if it contains a regular element. Recall that an element of D is regular if and only if it is invertible in Q and hence also regular as an element of E. Let J be an ideal of D containing a regular element of D. Then LJ contains a regular element of D, and so LJE is nonzero ideal of E containing a regular element of E. From (3.4), E is a finite product of simple rings and so $LJE = E$. Thus

$$J \supseteq LJL = LJ(EL) = (LJE)L = EL = L. \quad \square$$

PROPOSITION (4.2). *The following are equivalent.*

(a) *$D(R)$ is a product of simple rings.*

(b) *$M = R$.*

(c) *$D(R)$ is Morita equivalent to $D(\overline{R})$.*

PROOF. From (3.2), $D(R) \cong D$ and $D(\overline{R}) \cong \overline{D}$, so we may substitute D and $\overline{D}$ into the statement of the proposition. If D is a product of simple rings, then $L = D$. In particular $1 \in L$, and so $1((1)) = 1 \in M$. Thus (a) implies (b). If $M = R$, it follows from (4.1) that $D = E$ and so (b) implies (c). Being a finite product of simple rings is a Morita invariant property, hence (c) implies (a), and the proof is complete. $\square$

The isomorphism $D(R) \cong D$ is given by extending operators as in (1.10), so we see that $d((M)) \subseteq M$ for all $d \in D(R)$. Thus there is a k-algebra homomorphism $\psi\colon D(R) \to \operatorname{Hom}_k(R/M, R/M)$ where $\psi(d)((r + M)) = d((r)) + M$. It is easy to check that $\operatorname{Im}(\psi) \subseteq D(R/M)$ and that $\operatorname{Ker}(\psi) = \{\partial \in D(R)\colon \partial((R)) \subseteq M\}$ corresponds to L under the isomorphism $D(R) \cong D$.

PROPOSITION (4.3). *The homomorphism ψ induces a k-algebra embedding $D(R)/\operatorname{Ker}(\psi) \hookrightarrow D(R/M)$. The kernel of ψ is the smallest ideal of $D(R)$ which is essential as a left or right ideal of $D(R)$. The k-algebra $D(R/M)$, and hence $D(R)/\operatorname{Ker}(\psi)$, is finite dimensional as a k-vector space.*

PROOF. In view of the remarks above, we need only verify that $D(R/M)$ is finite dimensional. The ring R has Krull dimension ≤ 1 and the ideal M contains a regular element, so R/M is artinian. As R/M is a finitely generated k-algebra, it follows that R/M is finite dimensional. Therefore $\operatorname{Hom}_k(R/M, R/M)$, and hence $D(R/M)$, is finite dimensional. $\square$

Although I, and hence M, is often difficult to determine, we always have that M contains the conductor $C = \{x \in \overline{R}\colon x\overline{R} \subseteq R\}$. In fact $C = I \cap D^0(K)$, and so $C((1)) = C \subseteq M$. It follows that

$$\dim_k D(R)/\operatorname{Ker}(\psi) \leq (\dim_k R/M)^2 \leq (\dim_k R/C)^2.$$

In the case that R is a domain, K is the quotient field of R and $D(K)$ is a domain. Thus all nonzero ideals contain a regular element and $D(R)$ has a unique minimal nonzero ideal.

The following example shows that the embedding of (4.3) need not be an isomorphism.

EXAMPLE (4.4). Let $k[x]$ be a polynomial ring in one indeterminate and let

$$R = k + k \cdot x(x-1)(x-2) + x^2(x-1)^2(x-2) \cdot k[x].$$

Then $D(R)/L$ is isomorphic to the ring of lower triangular matrices in $M_2(k)$.

PROOF. Using $k(x)$ as the quotient field of R, it is easy to check that $\overline{R} = k[x]$. The ring R is the coordinate ring of an affine curve which is unramified at its singular point. It is shown in [8] that $D(R) \subseteq D(\overline{R})$. Thus using (1.10), we know $D(R)$ is isomorphic to

$$D = \{\partial \in D(k[x])\colon \partial((R)) \subseteq R\}.$$

Let $d \in D$ and use (1.3) to write

$$d = a_n(\partial/\partial x)^n + a_{n-1}(\partial/\partial x)^{n-1} + \cdots + a_1(\partial/\partial x) + a_0$$

where $a_i \in k[x]$ for all i. Notice that $d((1)) = a_0 \in R$.

Now assume $n \geq 1$. Observe that $x^{n+2}(x-1)^{n+2}(x-2)^n \in R$ and so

$$d((x^{n+2}(x-1)^{n+2}(x-2)^n)) = a_n n! x^{n+2}(x-1)^{n+2} + (\text{something in } x^2(x-2)k[x])$$

is an element of $x^2k[x] \cap R = x^2(x-1)^2(x-2)k[x]$. It follows that $a_n \in (x-2)k[x]$. If $n > 1$, the same argument using $x^{n+2}(x-1)^{n+2}(x-2)^{n-1}$ shows that $a_{n-1} \in (x-2)k[x]$. Continuing in this manner, we find that $a_i \in (x-2)k[x]$ for $i = 1, \ldots, n$. For later use, note that

$$N = (x-2)k[x] \cap R = kx(x-1)(x-2) + x^2(x-1)^2(x-2)k[x]$$

satisfies $d((N)) \subseteq N$ for all $d \in D$.

Next we compute M. Let $d \in I = \{\partial \in D(k(x))\colon \partial((k[x])) \subseteq R\}$. In particular, $d((R)) \subseteq R$. It follows from the claim above that

$$d = a_n(\partial/\partial x)^n + a_{n-1}(\partial/\partial x)^{n-1} + \cdots + a_1(\partial/\partial x) + a_0$$

where $a_i \in k[x]$ for all i. Let C denote the conductor and observe that

$$C = x^2(x-1)^2(x-2)k[x].$$

From above, $a_i \in (x-2)k[x]$ when $i \geq 1$. As $d \in I$, we have that

$$d((x^2(x-1)^{n+1})) = a_0x^2(x-1)^{n+1} + (\text{something in } (x-1)(x-2)k[x])$$

is an element of $R \cap (x-1)k[x] = N$. Thus $a_0 \in R \cap (x-2)k[x] = N$. Evaluating d at x yields

$$d((x)) = a_1 + a_0x \in R \cap (x-2)k[x] = N \subseteq x(x-1)(x-2)k[x].$$

It follows that $a_1 \in x(x-1)(x-2)k[x]$. Evaluating d at x^2 now shows that $a_2 \in x(x-1)(x-2)k[x]$. Continuing in this manner, we find that $a_i \in x(x-1)(x-2)k[x]$ for $i = 0, 1, \ldots, n$.

If $n \geq 1$, then

$$d((x^n(x-1)^{n+1})) = a_n n!(x-1)^{n+1} + (\text{something in } x^2(x-1)^2k[x])$$

is an element of $(x-1)^2k[x] \cap R = C$. It follows that $a_n \in x^2k[x]$. Similarly, $a_n \in (x-1)^2k[x]$. Hence $a_n \in C$, and so $a_n(\partial/\partial x)^n \in I$. Thus

$$d - a_n(\partial/\partial x)^n = a_{n-1}(\partial/\partial x)^{n-1} + \cdots + a_1(\partial/\partial x) + a_0$$

is also an element of I. If $n-1 \geq 1$, we may use the same argument to show that $a_{n-1} \in C$. Continuing this process shows that $a_0 \in I$. From $a_0(x-1) \in (x-1)^2k[x] \cap R = C$, we see that $a_0 \in x^2k[x] \cap R = C$. Thus $a_i \in C$ for all i, and hence $M = C$.

Next we consider the derivation

$$\delta = (1-3x)x(x-1)(x-2)(\partial/\partial x).$$

It is easy to see that $\delta((M)) \subseteq M$. The equation

$$\delta((x(x-1)(x-2))) = 2x(x-1)(x-2) + (12-9x)x^2(x-1)^2(x-2)$$

shows that $\delta((x(x-1)(x-2))) \in R$. As $\delta((k)) = 0$, we conclude that $\delta \in D$.

Using $1+M$ and $x(x-1)(x-2)+M$ as a basis for the two dimensional vector space R/M, we may identify $\mathrm{Hom}_k(R/M, R/M)$ with $M_2(k)$ where

$$1+M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x(x-1)(x-2)+M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \delta + M = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

These three operators generate the lower triangular matrices, and so the image of D/L is either the lower triangular matrices or all of $M_2(k)$. The fact that $d((N)) \subseteq N$ for all $d \in D$ implies that $J = \{\partial \in D(k(x)) \colon \partial((R)) \subseteq N\}$ is an ideal of D. Notice that $N \subseteq J \neq D$ and $N \not\subseteq L$. Thus $L \subsetneq J$ and hence D/L is not a simple ring. Therefore the image of D/L cannot be $M_2(k)$. $\square$

5. Some counterexamples. We continue to assume that k is a field of characteristic zero.

The hypotheses that R be reduced and of Krull dimension ≤ 1 are both necessary in Theorem (3.1).

J. N. Bernstein, I. M. Gelfand and S. I. Gelfand have shown in [**3**] that the coordinate ring R of the normal cubic cone, i.e., the surface in complex 3-space given by $x^3 + y^3 + z^3 = 0$, has a differential operator ring which is neither left nor right noetherian. Here

$$R = \mathbf{C}[x, y, z]/(x^3 + y^3 + z^3)$$

and $R = \overline{R}$ is singular.

One might initially hope that the different operators on R would be noetherian when $\overline{R}$ was nonsingular, but S. P. Smith and J. T. Stafford have a nice counterexample in [**12**]. Namely, let R be the coordinate ring of a variety of dimension ≥ 2 which has only a finite number of singular points and whose normalization is nonsingular. Then $D(R)$ is right but not left noetherian.

Finally, we will compute an example which shows that $D(R)$ need not be noetherian when R is not reduced, but first we present some terminology and a lemma which is useful for computations.

Let $A = k[x_1, x_2, \ldots, x_t]$ be a polynomial ring in t indeterminates. We will say that the monomial $x_1^{m(1)} x_2^{m(2)} \cdots x_t^{m(t)}$ is of *degree* $m(1) + \cdots + m(t)$ and of *multidegree* $(m(1), m(2), \ldots, m(t))$. The ring A is graded by degree and multigraded by multidegree. An ideal B of A that is generated by homogeneous elements (with respect to degree) is called a *homogeneous* (*or graded*) *ideal* of A. It is well known that an element of A is in B if and only if its homogeneous pieces are in B. We will call the ideal C *multihomogeneous* if it is generated by multihomogeneous elements.

It is easy to show that an element of A is in C if and only if its multihomogeneous pieces are in C.

The ring $D(A)$ is also both graded and multigraded. Set $\partial_i = \partial/\partial x_i$. The differential operator

$$x_1^{m(1)} x_2^{m(2)} \cdots x_t^{m(t)} \partial_1^{n(1)} \partial_2^{n(2)} \cdots \partial_t^{n(t)}$$

has *degree* $m(1)+\cdots+m(t)-n(1)-\cdots-n(t)$ and *multidegree* $(m(1)-n(1),\ldots,m(t)-n(t))$. If $d \in D(A)$, then d has unique decompositions into finite sums $\sum d_i$ and $\sum d_{i(1),i(2),\ldots,i(t)}$ where d_i is a differential operator of degree i, and $d_{i(1),i(2),\ldots,i(t)}$ is a differential operator of multidegree $(i(1),i(2),\ldots,i(t))$.

LEMMA (5.1). *Let $A = k[x_1,x_2,\ldots,x_t]$ be a polynomial ring in t indeterminates and let $d \in D(A)$.*

(a) *If B is a homogeneous ideal of A, then $d((B)) \subseteq B$ if and only if $d_i((B)) \subseteq B$ for all i.*

(b) *If C is a multihomogeneous ideal of A, then $d((C)) \subseteq C$ if and only if $d_{i(1),i(2),\ldots,i(t)}((C)) \subseteq C$ for all $(i(1),i(2),\ldots,i(t))$.*

PROOF. (a) As B is a homogeneous ideal, we have that $d((B)) \subseteq B$ if and only if $d((f)) \in B$ for every homogeneous $f \in B$. Let n denote the degree of f. Then $d_i((f))$ is either zero or has degree $n+i$. It follows that $d((f)) = \sum d_i((f)) \in B$ if and only if $d_i((f)) \in B$ for all i.

(b) Similar. $\square$

EXAMPLE (5.2) Let $R = A/B$ where $A = k[x,y]$ is a polynomial ring in two indeterminates and $B = (x^2, xy)$ is the ideal of A generated by x^2 and xy. Then $D(R)$ is right but not left noetherian.

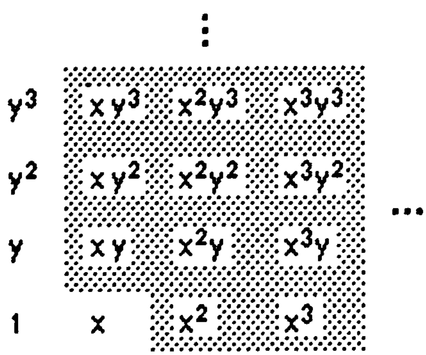
PROOF. From (1.4),

$$D(R) \cong \{d \in D(A)\colon d((B)) \subseteq B\}/\{d \in D(A)\colon d((A)) \subseteq B\}.$$

It is easy to see that

$$\{d \in D(A)\colon d((A)) \subseteq B\} = B \cdot D(A) = (x^2, xy)D(A).$$

We must find all $d \in D(A)$ with $d((B)) \subseteq B$. As B is bihomogeneous, we may use the lemma above to assume that d has bidegree (n,m). Below is a representation of the ideal B. The ideal B is given as the k-vector space spanned by the shaded monomials.



The effect of d is to map a given monomial to a scalar times the monomial which is n places to the right and m places up. If this location is off the diagram, i.e., below or to the left, then the scalar is automatically zero. The operator d will map B into B precisely when all of the monomials in the shaded area are sent either back into the shaded area or to zero. In other words, d must vanish on any monomials in the shaded region which get mapped into the nonshaded region.

We now consider all of the possible cases for (n, m).

Case 1 ($n \geq 1$). Monomials are mapped to the right $n \geq 1$ places and then either up or down according to m. Thus $d((B)) \subseteq B$ for any such d.

Case 2 ($n = 0,\ m \geq 0$). Monomials are mapped up $m \geq 0$ places. Thus $d((B)) \subseteq B$ for any such d.

Case 3 ($n = 0,\ m = -q < 0$). As the only monomial in B mapped to the nonshaded region is xy^q, we have that $d((B)) \subseteq B$ if and only if $d((xy^q)) = 0$. Subtracting off elements of $B \cdot D(A)$, we are left with a linear combination of $x(\partial/\partial x)(\partial/\partial y)^q$ and $\{y^i(\partial/\partial y)^{q+i} \colon i \geq 0\}$. Observing that $y^i(\partial/\partial y)^{q+i}$ vanishes on xy^q for all $i \geq 1$, it is left to determine if a linear combination of the form $\alpha x(\partial/\partial x)(\partial/\partial y)^q + \beta(\partial/\partial y)^q$ vanishes on xy^q, where $\alpha, \beta \in k$. Setting the value of this operator at xy^q equal to zero results in the equation $q!(\alpha + \beta)x = 0$, from which it follows that $\alpha = -\beta$. Hence d is in the span of $(x(\partial/\partial x) - 1)(\partial/\partial y)^q$ and $\{y^i(\partial/\partial y)^{q+i} \colon i \geq 1\}$.

Case 4 ($n = -p < 0,\ m \geq 1$). Modulo $B \cdot D(A)$, we have that d is in the span of $\{y^{m+i}(\partial/\partial x)^p(\partial/\partial y)^i \colon i \geq 0\}$ and must vanish on the monomials $x^p, x^p y, x^p y^2, \ldots$. The resulting equations show that any such d is zero.

Case 5 ($n = -p < 0,\ m = -q < 0$). Modulo $B \cdot D(A)$, we have that d is in the span of $x(\partial/\partial x)^{p+1}(\partial/\partial y)^q$ and $\{y^i(\partial/\partial x)^p(\partial/\partial y)^{q+i} \colon i \geq 0\}$, and must vanish on the monomials $x^{p+1}y^q$ and $x^p y^q, x^p y^{q+1}, x^p y^{q+2}, \ldots$. The resulting equations show that any such d is zero.

Combining the results from these five cases yields

$$\begin{aligned}\{d \in D(A) \colon d((B)) \subseteq B\} = {} & (x^2, xy)D(k[x,y]) + xk[\partial/\partial y] + yD(k[y]) \\ & + (x(\partial/\partial x) - 1)k[\partial/\partial y] + k.\end{aligned}$$

Using the following computations, it is easy to check that the vector spaces $k(x(\partial/\partial y)^p + B \cdot D(A))$, for $p \geq 0$, are actually left ideals of $D(R)$. It follows that $D(R)$ is not left noetherian.

$$xk[\partial/\partial y]x(\partial/\partial y)^p \subseteq x^2 k[\partial/\partial y](\partial/\partial y)^p \subseteq x^2 D(k[x,y]).$$
$$yD(k[y])x(\partial/\partial y)^p \subseteq xyD(k[y])(\partial/\partial y)^p \subseteq xyD(k[x,y]).$$
$$(x(\partial/\partial x) - 1)k[\partial/\partial y]x(\partial/\partial y)^p \subseteq (x(\partial/\partial x) - 1)xk[\partial/\partial y](\partial/\partial y)^p \subseteq x^2 D(k[x,y]).$$

To see that $D(R)$ is right noetherian, we will show that $D(R)$ is generated as a right module over the image of the subring $k + yD(k[y])$ by the cosets $1 + B \cdot D(A)$, $(x(\partial/\partial x) - 1)(\partial/\partial y) + B \cdot D(A)$, and $x(\partial/\partial y) + B \cdot D(A)$. That these three elements generate all of $D(R)$ can be seen from the equations

$$x(\partial/\partial y)yD(k[y]) = (xy(\partial/\partial y) + x)D(k[y])$$

and

$$\begin{aligned}&(x(\partial/\partial x) - 1)(\partial/\partial y)yD(k[y]) \\ &\quad = (x(\partial/\partial x) - 1)(y(\partial/\partial y) + 1)D(k[y]) \\ &\quad = (xy(\partial/\partial x)(\partial/\partial y) - y(\partial/\partial y) + (x(\partial/\partial x) - 1))D(k[y]).\end{aligned}$$

That the ring $k + yD(k[y])$ is right noetherian follows from the symmetric version of (2.2) with $E = D(k[y])$ and $D = k + yD(k[y])$. We conclude that $D(R)$ is noetherian as a right $k + yD(k[y])$-module and hence as a right $D(R)$-module. $\square$

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