THE DIFFRACTION MATRIX FOR A DISCONTINUITY IN CURVATURE*

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Abstract— To enlarge the scope of the geometrical theory of diffraction, the diffraction matrix for a surface singularity where the curvature but not the slope is discontinuous is rigorously derived. The model that is employed consists of two parabolic cylinders of different latus recta joined together at the front, thereby creating a line discontinuity of the required form. For each of the two principal polarizations, asymptotic developments of the surface fields in the vicinity of the join are calculated, from which the diffraction coefficients are then obtained by integration. The results differ significantly from the physical optics estimates and are analogous to those for a wedge-like singularity. This analogy permits a trivial deduction of the complete diffraction matrix.

I INTRODUCTION

When a metallic object is illuminated by an electromagnetic wave, a powerful method for estimating its high frequency scattering behavior is the geometrical theory of diffraction, originated by Keller [1, 2]. The theory is basically an extension of ray techniques to include the concept of diffracted rays which arise from surface singularities of the body. The strength of each such ray contribution to the scattering is proportional to a diffraction coefficient which is determined, to the first order at least, by the local surface geometry

* This work was supported by the Northrop Corporate Laboratories, Hawthorne California under Purchase Order 12122.

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at the point of diffraction. In those cases where the diffraction coefficients are known, their expressions have been obtained from exact solutions of selected canonical problems displaying the geometry in question, and thus it is that the coefficients for an edge or wedge-like singularity (slope discontinuity) are deduced from the solution of the two-dimensional problem of plane wave scattering by a half-plane or wedge.

The diffraction coefficients are the key to the GTD method and one particular but important case where they are not yet known is when the surface slope (first derivative) is continuous, but the curvature (involving the second derivative) is discontinuous. Their derivation for this geometrical feature is vital for an adequate treatment of scattering by bodies such as a cone-sphere or hemispherically-capped cylinder, and in the absence of any exact canonical solution from which to deduce them, it has been necessary to rely (see, for example, [3]) on the crude estimates offered by physical optics. Automatically, therefore, the polarization dependence has been suppressed [4] .

Although an exact canonical solution would be desirable, it is not, in fact, essential to the determination of a diffraction coefficient, and an adequate description of the surface field in a vicinity of the geometric feature can suffice. For a discontinuity in curvature, we can obtain such a surface field description using the model that was employed by Weston [5, 6] in studying the creeping waves launched by the discontinuity. Weston considered only the case of a plane wave incident with its magnetic vector parallel to the line discontinuity (H polarization). This is treated in Section III and the initial part of the analysis follows closely that given in [5] . The analogous case of a plane wave incident with its electric vector parallel to the discontinuity (E polarization) is discussed in Section IV. The corresponding diffraction coefficients for H and E polarized waves are derived in Sections V and VI respectively, and the general diffraction matrix is constructed in Section VII. The results differ from the physical optics estimates for almost all angles of incidence and diffraction, and some of the consequences of these new and rigorous formulae are explored.

II PRELIMINARY CONSIDERATIONS

We consider a two-dimensional perfectly conducting surface consisting of two half parabolic cylinders of different latus recta joined at the front. In terms of the Cartesian coordinates (x, y, z) with the z axis coincident with the join, the surface is defined as:

$$x = -\frac{1}{2} a_2 y^2 , \quad y > 0 ; \qquad x = -\frac{1}{2} a_1 y^2 , \quad y < 0 , \tag{1}$$

so that the positive x axis is in the direction of the outward normal to the surface at the join. For convenience we shall henceforth write (1) as

$$x = -\frac{1}{2} a y^2 , \tag{2}$$

where $a = a_2$ ($y > 0$), $a = a_1$ ($y < 0$) . It is easily verified that the surface slope is continuous at the join (it is infinite there), and that the curvature is discontinuous at $y = 0$ unless $a_2 = a_1$.

A plane electromagnetic wave is incident with its propagation vector lying in the xy plane and making an angle α with the y axis, where* $0 < \alpha < \pi$ (see Fig. 1). If the wave has its magnetic vector in the z direction (H polarization), we can write

$$\begin{aligned} \underline{H}^i &= \hat{z}\, e^{-ik(x \sin\alpha + y \cos\alpha)} , \\ \underline{E}^i &= Z (\hat{x} \cos\alpha - \hat{y} \sin\alpha)\, e^{-ik(x \sin\alpha + y \cos\alpha)} , \end{aligned} \tag{3}$$

where $Z = 1/Y$ is the intrinsic impedance of free space and a time factor $e^{-i\omega t}$ has

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* In the derivation of the surface field about the join it is, in fact, necessary to assume that α is bounded away from zero and π to ensure that the shadow boundary is sufficiently far removed from the z axis; however, see Section VII.

been assumed and suppressed. Due to the presence of the perfectly conducting surface, a scattered field ($\underline{E}^s$, $\underline{H}^s$) will be generated satisfying the boundary condition

$$\hat{n} \wedge (\underline{E}^i + \underline{E}^s) = 0$$

at the surface, where $\hat{n}$ is a unit vector normal in the outwards direction. The initial task is to find the total (incident plus scattered) magnetic field at the surface, with particular reference to a region about the join.

Since the problem is two-dimensional (being independent of the coordinate z), it can be expressed as a scalar problem for the total magnetic field component, $H_z = u$, which is required to satisfy the Neumann boundary condition $(\partial u/\partial n) = 0$ at the surface, with $u-u_0$ obeying the radiation condition, where

$$u_0 = H_z^i = e^{-ik(x \sin \alpha + y \cos \alpha)} . \tag{4}$$

This is a hard-body problem and is treated in Section III.

If, on the other hand, the incident plane wave has its electric vector in the z direction, then

$$\begin{aligned} \underline{E}^i &= \hat{z}\, e^{-ik(x \sin \alpha + y \cos \alpha)} , \\ \underline{H}^i &= -Y (\hat{x} \cos \alpha - \hat{y} \sin \alpha) e^{-ik(x \sin \alpha + y \cos \alpha)} . \end{aligned} \tag{5}$$

The task is again to find the total magnetic field at the surface, and since the problem is two-dimensional, it can be expressed as a scalar one for the total electric field component $E_z = u$. This is required to satisfy the Dirichlet boundary condition $u=0$ at the surface, with $u-u_0$ obeying the radiation condition, where

$$u_0 = E_z^i = e^{-ik(x \sin \alpha + y \cos \alpha)} . \tag{6}$$

The resulting soft-body problem is treated in Section IV.

III SURFACE FIELD, H POLARIZATION

Maue's integral equation for the field on a two-dimensional acoustically hard surface at a point specified by the coordinate y is

$$u(y)=2u_o(y)+\frac{i}{2}\int u(y_1)\frac{\partial}{\partial n_1}H_o^{(1)}(kR)ds_1 \tag{7}$$

where

$$R=\left\{(x-x_1)^2+(y-y_1)^2\right\}^{1/2}$$

and $H_m^{(1)}(kR)$ is the Hankel function of the first kind of order m. For the particular surface defined by (2), it is convenient to take y_1 as the variable of integration. The line element ds_1 then becomes

$$ds_1=(1+\bar{a}^2y_1^2)^{1/2}dy_1$$

where $\bar{a}=a_2(y_1>0)$, $\bar{a}=a_1(y_1<0)$, and the integral equation takes the form

$$u(y)=2u_o(y)-\frac{ik}{2}\int_{-\infty}^{\infty}u(y_1)H_1^{(1)}(kR)\left\{\bar{a}y_1(y_1-y)-\frac{1}{2}(\bar{a}y_1^2-ay^2)\right\}\frac{dy_1}{R} \tag{8}$$

with

$$R=\left\{(y-y_1)^2+\frac{1}{4}(ay^2-\bar{a}y_1^2)\right\}^{1/2}. \tag{9}$$

Note that $\bar{a}$ has the value a_2 or a_1 depending on the sign of the variable of integration y_1, whereas a has the value a_2 or a_1 depending on the sign of the coordinate y of the field point.

Although an exact solution of eq. (8) is impossible, an asymptotic expansion of u(y) for large k/a can be found by making use of the particular character of the surface. To this end, we note that if the incident field (3) were to impinge on the complete and uniform parabolic surface $x=-\frac{1}{2}ay^2$, an asymptotic expansion of the surface field could be obtained by the Luneberg-Kline method, and is

$$u(y)=U(y,a)\,e^{ikf(y,a)} \tag{10}$$

with [7]

$$U(y, a) = 2- \frac{ia}{k}(\sin\alpha + ay\cos\alpha)^{-3} + O(k^{-2}) , \qquad (11)$$

$$f(y, a) = -y\cos\alpha + \frac{1}{2}ay^2 \sin\alpha . \qquad (12)$$

Equation (10) reduces to the result given in [8] for the special case of normal incidence, $\alpha = \pi/2$, and is valid only over the illuminated region of the surface. This includes the join by virtue of the restriction on α .

Following Weston [5] , the field on the conjoint surface of eq. (2) is now written as the sum of two parts: that which would exist on the section in question were the whole surface a continuation of it, plus a perturbation created by the join. Thus

$$u(y)=U(y, a)e^{ikf(y, a)} + \frac{1}{k} I(y, a)e^{iks(y, a)} \qquad (13)$$

where

$$s(y, a)=\int_0^{|y|} (1+a^2\tau^2)^{1/2} d\tau . \qquad (14)$$

The only unknown quantity in (13) is $I(y, a)$, and since the discontinuity in $U(y, a)$ is $O(k^{-1})$ at $y=0$, it is clear that $I(y, a)$ must be $O(k^0)$ for small y.

On substituting the expression for $u(y)$ into (8) and using the fact that the first part of (13) is, by definition, the field on a single parabolic cylinder formed by continuing that portion on which the field point lies, the integral equation reduces to

$$I(y, a)e^{iks(y, a)} = -\frac{ik}{2}\int_{-\infty}^{\infty} I(y_1, \bar{a})e^{iks(y_1, \bar{a})} K(y_1, y, a, \bar{a})dy_1 - \frac{ik^2}{2} Q \qquad (15)$$

where

$$K(y_1, y, a, \bar{a})=\left\{\bar{a}y_1(y_1-y)- \frac{1}{2}(\bar{a}y_1^2-ay^2)\right\} \frac{H_1^{(1)}(kR)}{R} , \qquad (16)$$

$$Q=\int_{-\infty}^{\infty}\left\{U(y_1, \bar{a})e^{ikf(y_1, \bar{a})} K(y_1, y, a, \bar{a})-U(y_1, a)e^{ikf(y_1, a)} K(y_1, y, a, a)\right\} dy_1. \qquad (17)$$

Although the integrand in (17) is a known function, so that in principle at least a precise evaluation of the integral is feasible, an asymptotic development suffices. Taking first the case $y<0$, we note that the integrand vanishes for $y_1<0$, and hence

$$Q=\int_0^\infty\left\{U(y_1,a_2)e^{ikf(y_1,a_2)}K(y_1,y,a_1,a_2)-U(y_1,a_1)e^{ikf(y_1,a_1)}K(y_1,y,a_1,a_1)\right\}dy_1 .$$

But

$$K(y_1,y,a_1,a_2)=\frac{1}{2k}H_1^{(1)}(\xi_1+\xi)\left\{a_2(\xi_1-\xi)+(a_1-a_2)\frac{\xi^2}{\xi_1-\xi}\right\}\left\{1+O(k^{-2})\right\}$$

where $\xi_1=ky_1$, $\xi=ky$, and since

$$U(y_1,a_2)=2+O(k^{-1})=U(y_1,a_1),$$

$$e^{ikf(y_1,a_2)}=e^{-i\xi_1\cos\alpha}\left\{1+O(k^{-1})\right\}=e^{ikf(y_1,a_1)},$$

we have

$$Q=\frac{a_2-a_1}{k^2}\int_{-\xi}^{\infty}e^{-i(\xi_1+\xi)\cos\alpha}H_1^{(1)}(\xi_1)\left(\xi_1-\frac{\xi^2}{\xi_1}\right)\left\{1+O(k^{-1})\right\}d\xi_1 . \tag{18}$$

A similar result holds for $y>0$ and this can be combined with (18) to give

$$Q=\mp\frac{a_2-a_1}{k^2}e^{-i\xi\cos\alpha}L(\xi)\left\{1+O(k^{-1})\right\} \tag{19}$$

where

$$L(\xi)=\int_{|\xi|}^{\infty}e^{\pm i\xi_1\cos\alpha}H_1^{(1)}(\xi_1)\left(\xi_1-\frac{\xi^2}{\xi_1}\right)d\xi_1 , \tag{20}$$

and the upper (lower) signs apply according as $y>0$ (<0) . In addition,

$$\int_{-\infty}^{\infty}I(y_1,\bar{a})e^{iks(y_1,\bar{a})}K(y_1,y,a,\bar{a})dy_1=O(k^{-2})$$

since $K=O(k^{-1})$ and $I=O(k^0)$, and the final expression for $I(y,a)$ is therefore

$$I(y,a) = \pm \frac{i}{2}(a_2 - a_1) e^{-ik\left\{s(y,a)+y\cos\alpha\right\}} L(\xi) + O(k^{-1}) . \tag{21}$$

The perturbation field is all that is needed to specify the total surface field u(y). From eq. (13),

$$u(y) = e^{-i\xi\cos\alpha} \left\{ U\left(\frac{\xi}{k}, a\right) e^{ia\frac{\xi^2}{2k}\sin\alpha} \pm \frac{i}{2k}(a_2 - a_1) L(\xi) + O(k^{-2}) \right\} \tag{22}$$

and this proves adequate for determining the diffraction coefficient to the leading order in k. We note in passing that expansion of the right hand side of (22) for $\xi \ll 1$ shows that for H polarization a discontinuity in curvature is characterized by a surface field 'singularity' of the form $y^2 \log |y|$.

IV SURFACE FIELD, E POLARIZATION

The problem of finding the surface field is now a soft body one but the basic character of the analysis is unchanged.

The appropriate form of Maue's integral equation is

$$\frac{\partial}{\partial n} u(y) = 2\frac{\partial}{\partial n} u_o(y) - \frac{i}{2}\int \frac{\partial}{\partial n_1} u(y_1) \frac{\partial}{\partial n} H_o^{(1)}(kR)\, ds_1 \tag{23}$$

where $u_o(y)$ and R are given in eqs. (6) and (9) respectively, and if we define

$$v(y) = (1+a^2y^2)^{1/2} \frac{\partial}{\partial n} u(y), \tag{24}$$

$$v_o(y) = (1+a^2y^2)^{1/2} \frac{\partial}{\partial n} u_o(y) , \tag{25}$$

the equation can be written as

$$v(y) = 2v_o(y) - \frac{ik}{2}\int_{-\infty}^{\infty} v(y_1) H_1^{(1)}(kR) \left\{ ay(y_1 - y) - \frac{1}{2}(\bar{a}\, y_1^2 - ay^2) \right\} \frac{dy_1}{R} . \tag{26}$$

We again postulate a representation of the surface field as the sum of that which would exist on a complete parabolic cylinder and a perturbation created by the join, viz.

$$v(y) = -ik\left\{V(y,a)e^{ikf(y,a)} + \frac{1}{k}\tilde{I}(y,a)e^{iks(y,a)}\right\} \tag{27}$$

where f(y, a) and s(y, a) are given in eqs. (12) and (14) respectively. The first term in (27) is the solution for a uniform parabolic cylinder of latus rectum $^1/a$, and by application of the Luneberg-Kline method we have [7]

$$V(y,a) = (\sin\alpha + ay\cos\alpha)\left\{2 + \frac{ia}{k}(\sin\alpha + ay\cos\alpha)^{-3} + O(k^{-2})\right\}, \tag{28}$$

which reduces to the result obtained by Keller et al [8] in the particular case of normal incidence, $\alpha = \pi/2$. The second term in (27) is due to the join, and it is evident that the unknown quantity $\tilde{I}(y,a)$ is $O(k^o)$ for small y.

On substituting (27) into (26) and invoking the fact that the first term in (27) is the solution for a uniform cylinder, the integral equation becomes

$$\tilde{I}(y,a)e^{iks(y,a)} = -\frac{ik}{2}\int_{-\infty}^{\infty}\tilde{I}(y_1,\bar{a})e^{iks(y_1,\bar{a})}\tilde{K}(y_1,y,a,\bar{a})dy_1 - \frac{ik^2}{2}\tilde{Q},$$

where

$$\tilde{K}(y_1,y,a,\bar{a}) = \left\{ay(y_1-y) - \frac{1}{2}(\bar{a}y_1^2 - ay^2)\right\}\frac{H_1^{(1)'}(kR)}{R}$$

and $\tilde{Q}$ is as shown in (17) but with U and K replaced by V and $\tilde{K}$. An asymptotic expansion of the expression for $\tilde{Q}$ is

$$\tilde{Q} = \pm\frac{a_2 - a_1}{k^2}\sin\alpha\, e^{-i\zeta\cos\alpha}M(\zeta)\left\{1 + O(k^{-1})\right\} \tag{29}$$

with

$$M(\zeta) = \int_{|\zeta|}^{\infty} e^{\pm i\zeta_1\cos\alpha}\,\overline{H_1^{(1)'}(\zeta_1)}\,\frac{(\zeta_1 - |\zeta|)^2}{\zeta_1}\,d\zeta_1, \tag{30}$$

where ζ_1 and ζ are as before, and the upper(lower) sign again holds according as $y > 0$ (< 0). In addition

$$\int_{-\infty}^{\infty}\tilde{I}(y_1,\bar{a})e^{iks(y_1,\bar{a})}\tilde{K}(y_1,y,a,\bar{a})dy_1 = O(k^{-2})$$

and hence

$$\tilde{I}(y,a) = \mp\frac{i}{2}(a_2-a_1)\sin\alpha\, e^{-ik\left\{s(y,a)+y\cos\alpha\right\}} M(\zeta)+O(k^{-1}) . \tag{31}$$

The total field v(y) on the surface is therefore

$$v(y) = -ike^{-i\zeta\cos\alpha}\left\{V\left(\frac{\zeta}{k}, a\right)e^{ia\frac{\zeta^2}{2k}\cos\alpha} \mp\frac{i}{2k}(a_2-a_1)\cos\alpha\, M(\zeta)+O(k^{-2})\right\} \tag{32}$$

and expansion of the right hand side for $|\zeta| \ll 1$ shows that for E polarization the join is characterized by a surface field 'singularity' of the form $y \log |y|$.

V DIFFRACTION COEFFICIENT, H POLARIZATION

For a two dimensional geometry such as that shown in Fig. 1, the scattered magnetic field at a point $\underline{r}'$ is

$$\underline{H}^s(\underline{r}') = \hat{z}\frac{i}{4}\int_S H_z(\underline{r})\frac{\partial}{\partial n}H_o^{(1)}(k|\underline{r}'-\underline{r}|)ds , \tag{33}$$

and in the far zone

$$H_z^s(\underline{r}') \sim \sqrt{\frac{2}{\pi k r'}}\; e^{ikr'-i\pi/4}\, P_H ,$$

where

$$P_H = \frac{k}{4}\int_S \hat{n}\cdot\hat{r}'\; H_z(\underline{r})e^{-ik\hat{r}'\cdot\underline{r}}\, ds$$

is the far field amplitude. For the specific geometry of Fig. 1,

$$P_H(\alpha,\theta) = \frac{k}{4}\int_{-\ell}^{\ell}(\sin\theta + ay\cos\theta)u(y)e^{-iky\cos\theta+ika\frac{y^2}{2}\sin\theta}\,dy$$

in which the integration has been limited to a region about the join since our concern is only with the scattering originating there. In terms of the variable ζ, however, the range of integration extends from $-k\ell$ to $k\ell$, and because k is large, it is sufficient to replace the limits by $\mp\infty$. Hence

$$P_H(\alpha,\theta)=\frac{1}{4}\int_{-\infty}^{\infty}(\sin\theta+a\frac{\zeta}{k}\cos\theta)u(\frac{\zeta}{k})e^{-i\zeta\cos\theta+ia\frac{\zeta^2}{2k}\sin\theta}\,d\zeta \tag{34}$$

and this is the integral to be evaluated.

An expansion for $u\,(\frac{\zeta}{k})$ accurate to two orders in k was given in eq. (22). When this is inserted into (34), expansion of $U\,(\frac{\zeta}{k}, a)$ as well as both quadratic exponentials in powers of k^{-1} yields

$$P_H(\alpha,\theta)=\frac{\sin\theta}{2}\int_{-\infty}^{\infty}e^{-ip\zeta}\left\{1-\frac{ia}{2k}\left[\operatorname{cosec}^3\alpha+2i\zeta\cot\theta-\zeta^2(\sin\alpha+\sin\theta)\right]\right.$$
$$\left.\pm\frac{i}{4k}(a_2-a_1)L(\zeta)+O(k^{-2})\right\}d\zeta \tag{35}$$

where

$p = \cos\alpha + \cos\theta$,

and since

$$\int_0^{\infty}\zeta^m e^{i\tau\zeta}\,d\zeta=(\frac{i}{\tau})^{m+1}\,m!\ ,$$

$$\int_{-\infty}^{0}\zeta^m e^{i\tau\zeta}\,d\zeta=-(\frac{i}{\tau})^{m+1}\,m!\ ,$$

it follows that

$$P_H(\alpha,\theta)=-\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha-\theta)}{p^3}+i\,\frac{a_2-a_1}{8k}\sin\theta\left\{\frac{2i}{p}\operatorname{cosec}^3\alpha+\left(\int_0^{\infty}-\int_{-\infty}^{0}\right)e^{-ip\zeta}\right.$$
$$\left.L(\zeta)d\zeta\right\}+O(k^{-2})\ . \tag{36}$$

The first term in (36) is contributed by the leading term in the expansion of $u(\frac{\zeta}{k})$ and is the physical optics result:

$$P_H^{p.o.}(\alpha,\theta)=-\frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha-\theta)}{p^3}\,. \tag{37}$$

To produce an explicit expression for $P_H(\alpha,\theta)$ accurate to the first order in k, it is necessary to evaluate precisely the integrals containing $L(\zeta)$. This is accomplished as follows. We first note that

$$\left(\int_0^\infty - \int_{-\infty}^0\right) e^{-ip\zeta} L(\zeta) d\zeta = \int_0^\infty \Bigg\{ e^{-ip\zeta} \int_\zeta^\infty e^{i\zeta_1 \cos\alpha} H_1^{(1)}(\zeta_1)(\zeta_1 - \frac{\zeta^2}{\zeta_1})\, d\zeta_1$$

$$- e^{ip\zeta} \int_\zeta^\infty e^{-i\zeta_1 \cos\alpha} H_1^{(1)}(\zeta_1)(\zeta_1 - \frac{\zeta^2}{\zeta_1}) d\zeta_1 \Bigg\} d\zeta \qquad (38)$$

and since

$$\int_0^\infty e^{\pm i\zeta_1 \cos\alpha} H_1^{(1)}(\zeta_1)\zeta_1 d\zeta_1 = \operatorname{cosec}^3\alpha \left\{ 1 \pm 1 \pm \frac{2}{\pi} (\sin\alpha\cos\alpha - \alpha) \right\} ,$$

integration by parts applied to the ζ integral in (38) changes the right hand side into

$$-\frac{2i}{p}\operatorname{cosec}^3\alpha + \frac{2i}{p} \int_0^\infty \Bigg\{ e^{-ip\zeta} \int_\zeta^\infty e^{i\zeta_1 \cos\alpha} H_1^{(1)}(\zeta_1) \frac{d\zeta_1}{\zeta_1} + e^{ip\zeta} \int_\zeta^\infty e^{-i\zeta_1 \cos\alpha}$$

$$H_1^{(1)}(\zeta_1) \frac{d\zeta_1}{\zeta_1} \Bigg\} \zeta\, d\zeta \ .$$

Each of the double integrals can now be reduced by a further integration by parts (see [7]). Thus

$$\int_0^\infty e^{-ip\zeta} \int_\zeta^\infty e^{i\zeta_1 \cos\alpha} H_1^{(1)}(\zeta_1) \frac{\zeta}{\zeta_1} d\zeta_1 d\zeta$$

$$= \frac{1}{p^2} \int_0^\infty \left\{ (1+ip\zeta) e^{-i\zeta\cos\theta} - e^{i\zeta\cos\alpha} \right\} H_1^{(1)}(\zeta) \frac{d\zeta}{\zeta}$$

with an analogous result for the other, and when these are inserted into (38) and thence (36), we obtain

$$P_H(\alpha, \theta) = P_H^{\text{p.o.}}(\alpha, \theta) - \frac{a_2 - a_1}{2k} \frac{1 + \cos(\alpha+\theta)}{p^3} + O(k^{-2}) , \qquad (39)$$

where we have used the formulae

$$\int_0^{\infty} \sin(\zeta \cos\gamma) H_1^{(1)}(\zeta) d\zeta = \cot\gamma$$

$$\int_0^{\infty} \left\{ \sin(\zeta\cos\gamma_1) - \sin(\zeta\cos\gamma_2) \right\} H_1^{(1)}(\zeta) \frac{d\zeta}{\zeta} = i(\sin\gamma_2 - \sin\gamma_1)$$

deduced from the discontinuous integrals of Weber and Schafheitlin [9]. The difference between P_H and its physical optics estimate is evident from eq. (39).

VI DIFFRACTION COEFFICIENT, E POLARIZATION

The derivation follows closely that in Section V, and only a brief description will suffice. In place of eq. (33) we now have

$$\underline{E}^s(\underline{r}') = \hat{z}\frac{i}{4} \int_S \frac{\partial}{\partial n} E_z(\underline{r}) H_o^{(1)}(k|\underline{r}'-\underline{r}|) ds \tag{40}$$

and in the far zone

$$E_z^s(\underline{r}') \sim \sqrt{\frac{2}{\pi k r'}}\; e^{ikr\tau - i\pi/4}\; P_E$$

where

$$P_E = -\frac{i}{4} \int_S \frac{\partial E_z}{\partial n} e^{-ik\hat{r}'\cdot\underline{r}} ds\ .$$

For the specific geometry of Fig. 1,

$$P_E(\alpha,\theta) = -\frac{i}{4k} \int_{-\infty}^{\infty} v\left(\frac{\zeta}{k}\right) e^{-i\zeta\cos\theta + ia\frac{\zeta^2}{2k}\sin\theta} d\zeta \tag{41}$$

where v(y) is defined in eq. (24), and when the expression (32) for v(y) is substituted into (41), expansion in powers of k^{-1} gives

$$P_E(\alpha,\theta) = \frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3} + i\,\frac{a_2-a_1}{8k}\,\sin\alpha\left\{\frac{2i}{p}\operatorname{cosec}^3\alpha + \left(\int_0^\infty - \int_{-\infty}^0\right) e^{-ip\zeta}\right.$$

$$\left. M(\zeta)d\zeta\right\} + O(k^{-2}) \ . \tag{42}$$

The first term is the physical optics approximation:

$$P_E^{p.o.}(\alpha,\theta) = \frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3} \tag{43}$$

and comparison with eq. (37) shows that

$$P_E^{p.o.}(\alpha,\theta) = -P_H^{p.o.}(\alpha,\theta) \ , \tag{44}$$

as expected [4] .

The integrals involving $M(\zeta)$ in (42) can be evaluated precisely using successive integrations by parts. The procedure is similar to that for H polarization, and when the results are inserted into eq. (42), we have

$$P_E(\alpha,\theta) = P_E^{p.o.}(\alpha,\theta) - \frac{a_2-a_1}{2k}\,\frac{1+\cos(\alpha+\theta)}{p^3} + O(k^{-2}) \ . \tag{45}$$

The shortcoming of the physical optics approximation is again evident.

VII THE DIFFRACTION MATRIX

A derivation of the diffraction matrix associated with a discontinuity in curvature is essential for the incorporation of our results within the general framework of the geometrical theory of diffraction. For this purpose, it is convenient to collect together the results obtained so far.

Using the model illustrated in Fig. 1 it has been shown that if

$$\underline{E}^i = \hat{z}\, e^{-ik(x\sin\alpha + y\cos\alpha)}$$

then

$$\underline{E}^s \sim \hat{z}\sqrt{\frac{2}{\pi k r'}}\; e^{i(kr' - \pi/4)}\; P_E$$

with

$$P_E = F - G \ ; \tag{46}$$

whereas if

$$\underline{H}^i = \hat{z}\, e^{-ik(x \sin\alpha + y \cos\alpha)},$$

implying

$$\underline{E}^i = Z\,(\hat{x}\cos\alpha - \hat{y}\sin\alpha)\, e^{-ik(x\sin\alpha + y\cos\alpha)},$$

then

$$\underline{H}^s \sim \hat{z}\sqrt{\frac{2}{\pi k r'}}\; e^{i(kr' - \pi/4)}\; P_H ,$$

implying

$$\underline{E}^s \sim -Z\,(\hat{x}\cos\theta - \hat{y}\sin\theta)\sqrt{\frac{2}{\pi k r'}}\; e^{i(kr' - \pi/4)}\; P_H$$

with

$$P_H = F + G . \tag{47}$$

In (46) and (47)

$$F = -\frac{a_2 - a_1}{2k}\,\frac{1 + \cos(\alpha+\theta)}{(\cos\alpha + \cos\theta)^3} + O(k^{-2}) , \tag{48}$$

$$G = -\frac{a_2 - a_1}{2k}\,\frac{1 + \cos(\alpha-\theta)}{(\cos\alpha + \cos\theta)^3} + O(k^{-2}) . \tag{49}$$

As demanded by the reciprocity condition concerning the interchange of receiver and transmitter, the expressions for F and G are unaffected if α and θ are interchanged.

The terms shown on the right hand sides of (48) and (49) are the leading terms in high frequency asymptotic developments of F and G, and are valid provided $\pi - (\theta + \alpha)$ is bounded away from zero (to separate the contribution of the specular point from that of the discontinuity). In the expansion of the surface field it was found necessary to assume that α is bounded away from 0 and π (to ensure that the discontinuity is fully illuminated), and on physical grounds it would also appear necessary to have θ similarly bounded so that the discontinuity will be directly visible to the field point. However, the expressions for F and G are finite and continuous in the limits $\alpha, \theta \to 0, \pi$; $\theta + \alpha \neq \pi$, allowing us to replace the conditions on α and θ individually by the less restrictive ones $|\alpha - \pi/2|, |\theta - \pi/2| \leq \pi/2$.

Equations (46) and (47) can be written more compactly as

$$P_{E,H} = F \mp G \, . \tag{50}$$

The physical optics approximation is

$$P^{p.o.}_{E,H} = \mp G$$

and under many circumstances F is small compared with G : for example, in the particular case of backscattering ($\theta = \alpha$),

$$F = -\frac{a_2 - a_1}{8k} \sec \alpha + O(k^{-2})$$

$$G = -\frac{a_2 - a_1}{8k} \sec^3 \alpha + O(k^{-2})$$

and F is less than G by a factor 10 or more for incidence within 18^o of normal. Nevertheless, F is the sole source of polarization dependence in the scattering. Its inclusion is therefore important in any analysis which seeks to reproduce the polarization characteristics, and is vital in any study aimed at the cross polarized components [4, 10] of the backscattered field.

It is instructive to compare the scattering from a discontinuity in curvature with that from a wedge-like singularity involving a discontinuity in slope. For the metallic wedge of half angle Ω shown in Fig. 2, the diffraction coefficients $P_{E,H}$ are [11]

$$P_{E,H} = \frac{i}{2} \, (X \mp Y), \tag{51}$$

where

$$X = \frac{1}{n} \sin \frac{\pi}{n} \left\{ \cos \frac{\pi}{n} - \cos \frac{1}{n} \, (\alpha - \theta) \right\}^{-1}$$

$$Y = \frac{1}{n} \sin \frac{\pi}{n} \left\{ \cos \frac{\pi}{n} + \cos \frac{1}{n} \, (\pi - \alpha - \theta) \right\}^{-1}$$

with $n = 2\,(1 - \Omega / \pi)$. On comparing (50) and (51), it is seen that the diffracted field produced by a discontinuity in second derivative is obtainable from that created by a first derivative discontinuity by replacing

$$X \quad \text{by} \quad -2iF = -\frac{a_2-a_1}{2ik} \frac{\sec^2 \frac{\alpha-\theta}{2}}{\cos\alpha+\cos\theta}$$

and (52)

$$Y \quad \text{by} \quad -2iG = -\frac{a_2-a_1}{2ik} \frac{\sec^2 \frac{\alpha+\theta}{2}}{\cos\alpha+\cos\theta} \quad .$$

This analogy between the two diffracted fields is rather interesting. In a physical sense a discontinuity in the second derivative is like a very 'subdued' version of a first derivative discontinuity, and the latter may occur in the form of a surface kink if insufficient care is taken in the fabrication of a model. From an examination of the surface in the immediate vicinity of the discontinuity, it could be hard to tell whether the discontinuity was in the second derivative (as, perhaps, intended) or was instead in the first derivative. Such a slight kink corresponds to a wedge of half angle close to $\pi/2$, and if the expressions for X and Y are particularized to this case, the resulting diffracted field is identical to that produced by a discontinuity in curvature if

$$\Omega = \frac{\pi}{2}\left\{1-\frac{a_2-a_1}{ik}(\cos\alpha+\cos\theta)^{-1}\right\} .$$

In addition to the expected dependence on ik, we observe that the effective wedge angle is also a function of α and θ.

A further consequence of the analogy between the two types of surface discontinuity is that we can write down immediately the complete diffraction matrix for a discontinuity in curvature by making the substitution (52) in the known matrix for a wedge-like discontinuity. For this purpose we introduce a set of base vectors $\hat{T}$, $\hat{N}$, $\hat{B}$, where these are parallel, normal and 'binormal' to the discontinuity respectively, with $\hat{B}$ pointed into the shadowed half space. The direction of $\hat{T}$ is chosen to make $\hat{T}$, $\hat{N}$, $\hat{B}$ a right-handed system, and in terms of the Cartesian coordinate system of Fig. 1,

$$\hat{T} = -\hat{z}, \qquad \hat{N} = \hat{x} , \qquad \hat{B} = -\hat{y} \quad .$$

Following Keller [1], we consider an incident plane wave having electric vector

$$\underline{E}^i = \hat{e}^i \, e^{ik\hat{i}\cdot\underline{r}}$$

where

$$\hat{i} = \hat{T} \cos\beta - \hat{N} \sin\beta \sin\alpha + \hat{B} \sin\beta \cos\alpha$$

with $0 < \beta < \pi$. At points far from the discontinuity in curvature and in a direction satisfying the previously-mentioned restrictions on α and θ, the diffracted electric field can be written as

$$\underline{E}^d = \hat{e}^d \left(- \frac{e^{i\pi/4}}{\sin\beta \sqrt{2\pi k r}} \right) e^{ik \hat{s} \cdot \underline{r}}$$

where

$$\hat{s} = \hat{T} \cos\beta + \hat{N} \sin\beta \sin\theta - \hat{B} \sin\beta \cos\theta,$$

and if $\hat{e}^i$ is treated as a column vector in the base $\hat{T}$, $\hat{N}$, $\hat{B}$, then

$$\hat{e}^d = \Delta \hat{e}^i$$

in which Δ is a 3 x 3 matrix. An adequate expression for Δ now follows by making the substitution (52) in Eq. (A. 13) of [12] and is

$$\Delta = \begin{pmatrix} 2i(F-G) & 0 & 0 \\ -2i(F-G)\cot\beta \sin\theta & -2i(F+G)\cos\theta\cos\alpha & -2i(F+G)\cos\theta\sin\alpha \\ 2i(F-G)\cot\beta \cos\theta & -2i(F+G)\sin\theta\cos\alpha & -2i(F+G)\sin\theta\sin\alpha \end{pmatrix} \quad (53)$$

where F and G are given in eqs. (48) and (49) respectively.

VIII CONCLUSION

The derivation of a valid diffraction matrix for a discontinuity in curvature enlarges the scope of geometrical diffraction theory and permits the application of GTD to surface singularities other than the wedge-type to which it was restricted heretofore. Two particular targets for which more complete treatments are now possible are the hemispherically-capped cylinder and the cone-sphere, and in the latter case some experimental confirmation of our analysis has already been obtained.

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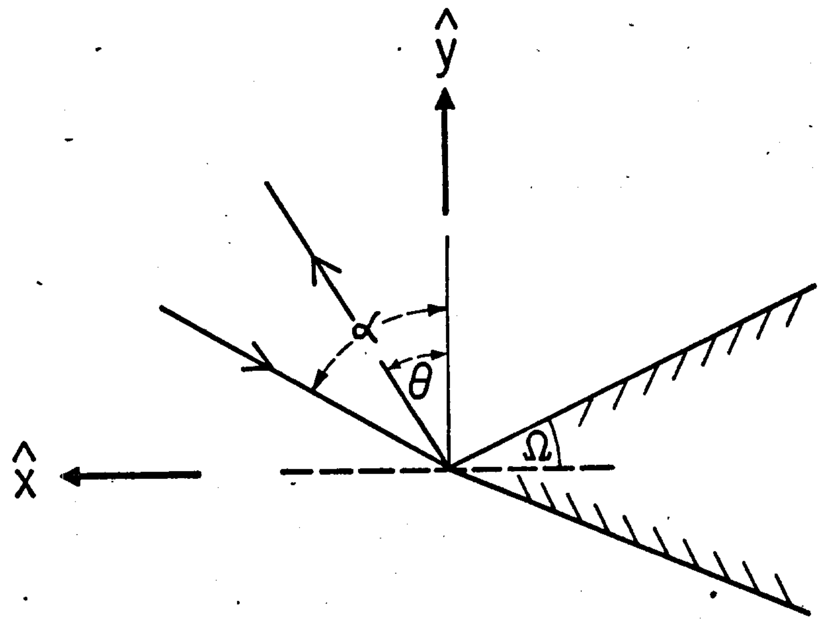

Fig. 2. The geometry for a first derivative (wedge-like) discontinuity

FIGURE CAPTIONS

1. The diffraction coefficient model for a discontinuity in curvature.

2. The geometry for a first derivative (wedge-like) discontinuity.