what vague and complex subject, and to place certain definite features in theoretical relation to one another, that the discussion here presented should be judged. Some of the views expressed have been stated before in other contexts,* and in a more detailed and complete memoir I hope to refer to these and other theories on the subject. In this account I have mentioned, very briefly, only those papers which I have had actual occasion to use in the present discussion; but it is hardly necessary to state how much such an investigation must owe to the labours of others who have previously studied the many-sided phenomena dealt with.

It is a pleasant duty to acknowledge the assistance which has been placed at my disposal, in the execution of the computations necessary for this paper, by the Government Grant Committee of the Royal Society and by the Astronomer Royal.

The Diffraction of Electric Waves by the Earth.

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During the last 15 years, the problem of determining the effect at a distant point of the earth's surface due to a Hertzian oscillator emitting waves of a definite frequency has been the subject of numerous theoretical investigations.

When certain assumptions of a physical character have been made, the problem is of a definitely mathematical type; it is in fact reduced to the problem of finding an approximate formula for the sum of a certain complicated series of an oscillatory nature; we shall summarise the principal methods which have been devised for dealing with this series.

The method of Poincaré† and Nicholson‡ is to replace the series by an integral and then to obtain an approximate value for the integral by means of the calculus of residues. The analysis employed by them, though

* Since writing this paper, for example, I have noticed that the symmetry of the disturbance variation about the solar meridian plane (§ 6) had been remarked by van Bemmelen in 1903 ('Terrestrial Magnetism,' vol. 8, p. 153), who also (*ibid.*, vol. 5, p. 123) refers to a theory of "current-vortices" (*cf.* § 11) by Schmidt ('Met. Zeitschrift,' 1889, p. 385).

† 'Palermo Rendiconti,' vol. 29, pp. 169–260 (1910).

‡ 'Phil. Mag.,' 1910, *passim*.

substantially sound, seems to be lacking in rigour in some points of detail; it is, moreover, exceedingly elaborate and their approximations become invalid in the neighbourhood of the antipodes of the transmitter.

The method employed by Macdonald* is to approximate to the terms of the oscillatory series, and then to replace the modified series by an integral. From the purely mathematical point of view this procedure seems open to question, as the sum of the series is considerably smaller than the differences between the larger terms in the original series and the corresponding terms of the modified series. It appears, however, that the sum of the differences is of the same order of smallness as the sum of the series, and consequently, as in many other physical problems, the end justifies the means. It should be pointed out that the reason for the chief discrepancy between the form of Macdonald's result and the results obtained by Poincaré and Nicholson is that Macdonald uses† the approximation

$$P_n(\cos\theta) \backsim J_0\{(2n+1)\sin\tfrac{1}{2}\theta\}$$

when n is large; but it is obvious from Laplace's approximation for $P_n(\cos\theta)$ that this formula ceases to be a valid approximation when θ is large enough for $n\theta^3$ to be appreciable.

The problem has been treated in a directly arithmetical manner by Love,‡ whose memoir contains a complete bibliography.

A completely different mode of procedure is adopted by March§ followed by Rybczynski ‖; these investigators work *ab initio* with a definite integral in place of a series; but it has been pointed out by Love that the whole of their analysis is fundamentally unsound, since the expression which they assume for the magnetic force has a line of singularities along the negative half of the axis of harmonics, and such an assumption is, of course, incorrect. The initial error seems to lie in an "inversion formula" which somewhat resembles Fourier's integral formula; the formula in question, according to March, is proved in his dissertation, but I have not succeeded in obtaining a copy.

Further criticisms could also be made as to the manner in which they approximate to the integral; for instance, while they profess to approximate to an integral involving a Legendre function of the first kind, it seems that they really obtain an approximation to an integral in which the function P_n is replaced by the function Q_n.

* 'Roy. Soc. Proc.,' A, vol. 90, pp. 50–61 (1914), and earlier papers.

† *Ibid.*, p. 55.

‡ 'Phil. Trans.,' A, vol. 215, pp. 105–131 (1915).

§ 'Ann. der Physik,' vol. 37, pp. 29–50 (1912).

‖ 'Ann. der Physik,' vol. 41, pp. 191–208 (1913).

In view of the discrepancies between the results obtained by different investigators, Dr. van der Pol has asked me to make a further examination of the problem, and to determine the magnetic force at the antipodes of the transmitter. This paper is the outcome of the analysis, and it appears to show that the physical assumptions do not account for the observed amount of diffraction. It therefore seems that further physical facts, such as the ionisation of the upper regions of the atmosphere, play a dominating part.

I must not omit to acknowledge gratefully the help which Dr. van der Pol has given me in connection with the physical aspects of the problem.

2. The essential advance in this paper is closely connected with the fundamental error of March and Rybczynski which was pointed out by Love.

In dealing with an oscillator on the positive half of the axis of harmonics, those writers express a Hertzian function by an integral of $P_s(\cos\theta)$, the integration being carried out with regard to the degree s of the Legendre function; such an integral has a line of singularities along the line $\theta = \pi$, and is regular along the line $\theta = 0$.

The fact is that, when harmonics of non-integral degree are introduced, the appropriate function to use is not $P_s(\cos\theta)$, but $P_s(-\cos\theta)$; this fundamental point is somewhat obscured by the equation

$$P_n(-\cos\theta) = (-)^n P_n(\cos\theta),$$

which holds between the functions whose degrees are integers. The failure of convergence of an integral involving $P_s(-\cos\theta)$ along the line $\theta = 0$ (when an oscillator is placed on the positive half of the axis of harmonics) is strictly analogous to the failure of convergence of the series $1+z+z^2+\ldots$, all round the circle $|z| = 1$ on account of the single singularity of the function $1/(1-z)$ at the point $z = 1$.

A simple electrostatic example is afforded by the potential of a unit charge at distance a from the origin. The potential near the origin is

$$\begin{aligned} V &= (1/a) \sum_{n=0}^{\infty} (r/a)^n P_n(\cos\theta) \\ &= (1/a) \sum_{n=0}^{\infty} (-)^n (r/a)^n P_n(-\cos\theta) \\ &= \frac{1}{2ia}\int (r/a)^s P_s(-\cos\theta)\frac{ds}{\sin s\pi}, \end{aligned}$$

where the contour starts from $+\infty$ and returns to $+\infty$ after encircling the points $s = 0, 1, 2, \ldots$, which are poles of the integrand.

On swinging round the contour* so as to surround the other poles of the

* *Cf.* Barnes, 'Lond. Math. Soc. Proc.' (2), vol. 6, pp. 141–177 (1908). Laplace's formula for P_s, valid when $|s|$ is large, has to be used to prove the convergence of the integral.

integrand, and evaluating the residues, we find the series for V in descending powers of r, valid when $r > a$.

3. We now give an outline of the analysis which leads to the series for the magnetic force, and with the aid of the calculus of residues we shall transform the series into a rapidly convergent series well adapted for numerical computation. The analysis of Poincaré leads to an approximation to the dominant term in this series, while the more powerful contour integrals of this paper give the complete series. The analysis is made more compact by using the Hertzian function and Bessel functions of the types employed by March rather than the functions previously used by English analysts.

It is supposed that the Earth is a homogeneous imperfectly conducting sphere of radius a surrounded by homogeneous dielectric. The sending apparatus is a Hertzian oscillator at distance b $(> a)$ from the centre of the sphere, its axis is along a radius of the sphere and is taken to be the axis of harmonics. The oscillator emits simple harmonic waves of period $2\pi/\omega$ and the electric and magnetic forces at any point are the real parts of the vectors $\mathbf{E}e^{i\omega t}$, $\mathbf{H}e^{i\omega t}$; $\mathbf{E}$ and $\mathbf{H}$ are connected by Maxwell's equations

$$\gamma \mathbf{H} = -rot\,\mathbf{E}, \qquad \beta \mathbf{E} = rot\,\mathbf{H},$$

where β, γ are constants of the medium transmitting the waves.

The spherical-polar co-ordinates are (r, θ, ϕ), the components of $\mathbf{E}$ are (E_r, E_θ, E_ϕ), and the components of $\mathbf{H}$ are (H_r, H_θ, H_ϕ); we also write $\cos\theta = \mu$.

Then the components of $\mathbf{E}$ and $\mathbf{H}$, in the case of symmetry about the axis of harmonics, are expressible in terms of the single Hertzian function Π by the equations*

$$E_r = -\frac{1}{rb}\left\{\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Pi}{\partial\mu}\right\}, \qquad E_\theta = \frac{1}{rb}\frac{\partial^2(r\Pi)}{\partial r\partial\theta}, \qquad E_\phi = 0;$$

$$H_r = 0, \qquad H_\theta = 0, \qquad H_\phi = -\frac{\beta}{b}\frac{\partial\Pi}{\partial\theta},$$

and Π satisfies the wave equation

$$(\nabla^2 + k^2)\Pi = 0,$$

where $k^2 = -\beta\gamma$.

Now the Hertzian function due to an oscillator in homogeneous infinite space is

$$\Pi_0 = e^{-ikR}/R,$$

* This function differs from March's function by the factor b; it is identical with Love's function in the case of an oscillator surrounded by infinite homogeneous dielectric; the presence of the factor b in the equations is explained by the equivalence of the operators $\partial/\partial\rho$, $\partial/b\partial\theta$ so far as functions of $r^2 - 2br\cos\theta + b^2$ are concerned if (ρ, ϕ, z) are cylindrical co-ordinates.

where R denotes the distance of (r, θ, ϕ) from $(b, 0, 0)$; and this function is expansible in the forms

$$\Pi_0 = -\frac{i}{krb} \sum_{n=0}^{\infty} (2n+1)\, \zeta_n(kb)\, \psi_n(kr)\, P_n(\mu),$$

$$\Pi_0 = -\frac{i}{krb} \sum_{n=0}^{\infty} (2n+1)\, \zeta_n(kr)\, \psi_n(kb)\, P_n(\mu),$$

according as $r < b$ or $r > b$, where*

$$\psi_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x), \qquad \zeta_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(x),$$

the symbol $H_n^{(2)}$ denoting the second of the two Hankel-Nielsen functions (which are sometimes called Bessel functions of the third kind).

We now have to take into account the fact that β, γ, k have not the same values when $r < a$ as when $r > a$; we accordingly denote the values of these constants for the interior of the Earth by the symbols β_i, γ_i, k_i.

We assume that the disturbance in the Hertzian function outside the sphere $r = a$ is Π_d, and that the disturbed function inside the sphere is Π_i. The appropriate series for Π_d and Π_i are

$$\Pi_d = -\frac{i}{krb} \sum_{n=0}^{\infty} (2n+1)\, a_n\, \zeta_n(kr)\, P_n(\mu),$$

$$\Pi_i = -\frac{i}{k_i rb} \sum_{n=0}^{\infty} (2n+1)\, b_n\, \psi_n(k_i r)\, P_n(\mu),$$

where the coefficients a_n, b_n are constants.

The boundary conditions are effectively

$$\beta(\Pi_0 + \Pi_d) = \beta_i \Pi_i, \qquad \frac{\partial}{\partial r}\{r\Pi_0 + r\Pi_d\} = \frac{\partial}{\partial r}\{r\Pi_i\},$$

when $r = a$. On substitution we find the value of a_n by solving two linear equations; the value is

$$a_n = -\frac{\zeta_n(kb)[\psi_n(k_i a)\, \psi_n'(ka) - \beta k_i \psi_n'(k_i a)\, \psi_n(ka)/(\beta_i k)]}{\psi_n(k_i a)\, \zeta_n'(ka) - \beta k_i \psi_n'(k_i a)\, \zeta_n(ka)/(\beta_i k)},$$

and we deduce that the value of Π just outside the surface of the earth is given by

$$\Pi(a, \theta) = -\frac{1}{krb} \sum_{n=0}^{\infty} \frac{(2n+1)\, P_n(\mu)\, \psi_n(k_i a)\, \zeta_n(kb)}{\psi_n(k_i a)\, \zeta_n'(ka) - \beta k_i \psi_n'(k_i a)\, \zeta_n(ka)/(\beta_i k)},$$

provided that we make use of the relation

$$\psi_n(ka)\, \zeta_n'(ka) - \zeta_n(ka)\, \psi_n'(ka) = -i.$$

We first discuss the limiting case in which k_i/β_i is negligible compared

* It should be noticed that this function ψ_n differs from the function which Love denotes by ψ_n.

with k/β; this assumption is equivalent to supposing that the earth is a perfect conductor.

The value of Π to be considered is now simply

$$\Pi_b(a, \theta) = -\frac{1}{kab} \sum_{n=0}^{\infty} (2n+1) P_n(\mu) \zeta_n(kb)/\zeta_n'(ka).$$

To examine the convergence of this series we observe that $|P_n(\mu)| \leq 1$ and, as $n \to \infty$ while k, a, b remain fixed,

$$\zeta_n(kb)/\zeta_n'(ka) \sim -ka(a/b)^n/n,$$

and so the series converges, since $b > a$.

Further, since* $|P_n'(\mu)| \leq n^2$, it is easy to see that the series obtained by differentiating term-by-term is uniformly convergent with regard to μ, and so no difficulties occur in obtaining H_ϕ and E_r from Π so long as $b > a$.

4. If we notice that $(2n+1) P_n(\mu) \zeta_n(kb)/\zeta_n'(ka)$ is the residue at $s = n+\frac{1}{2}$ of

$$2s\pi \frac{P_{s-\frac{1}{2}}(-\mu) \cdot \zeta_{s-\frac{1}{2}}(kb)}{\cos s\pi \cdot \zeta_{s-\frac{1}{2}}'(ka)},$$

we are led to the study of the integral

$$\int s \frac{P_{s-\frac{1}{2}}(-\mu) \cdot \zeta_{s-\frac{1}{2}}(kb)}{\cos s\pi \cdot \zeta_{s-\frac{1}{2}}'(ka)} ds$$

taken round a suitable contour.

Since $P_{s-\frac{1}{2}}(-\mu) = F(\frac{1}{2}-s, \frac{1}{2}+s; 1; \frac{1}{2}+\frac{1}{2}\mu)$ it is obvious that $P_{s-\frac{1}{2}}(-\mu)$ is an even integral function of s when μ has any assigned value such that $-1 \leq \mu < 1$. Also from the equation

$$H_s^{(2)}(x) = -\frac{1}{\pi i} \int_{-\infty}^{\infty - \pi i} e^{-s\theta + x \sinh\theta} d\theta$$

it is evident that $H_s^{(2)}(x)$ is an integral function of s when $R(x)$ is positive, and the relation

$$H_{-s}^{(2)}(x) = e^{s\pi i} H_s^{(2)}(x)$$

shows that $\zeta_{s-\frac{1}{2}}(kb)/\zeta_{s-\frac{1}{2}}'(ka)$ is an even function of s.

The integrand is therefore an odd function of s, whose only poles are at the zeros of the functions $\cos s\pi$, $\zeta_{s-\frac{1}{2}}'(ka)$.

We assume for the moment (see § 5) that $\zeta_{s-\frac{1}{2}}'(ka)$, *qua* function of s, has no zeros on the positive half of the real axis or on the imaginary axis, and then we take the contour of integration to be formed by a semicircle of large radius R, whose centre is at the origin which lies on the right of the imaginary axis, together with that part of the imaginary axis which joins the

* This follows without difficulty by induction from the equation

$$P_{n+1}'(\mu) - P_{n-1}'(\mu) = (2n+1) P_n(\mu).$$

points $\pm Ri$. It is going to be supposed that $R \to +\infty$ through such values that no poles of the integrand ever lie on the contour.

Since the integrand is an odd function of s, the integral along the part of the imaginary axis vanishes.

Next, consider the integral along the semicircle; so long as $\theta \neq \pi$, we may use Laplace's approximation

$$P_s(-\mu) \sim \frac{\cos\{s(\pi-\theta)+\frac{1}{4}\pi\}}{\sqrt{\{\frac{1}{2}s\pi\sin\theta\}}},$$

which is valid when $|s|$ is large and $R(s) \geq 0$; also the dominant terms of $H_s^{(2)}(x)$ are

$$\frac{i}{\sin s\pi}\left\{\frac{(\frac{1}{2}x)^{-s}}{\Gamma(1-s)} - \frac{e^{s\pi i}(\frac{1}{2}x)^s}{\Gamma(1+s)}\right\},$$

of which the latter part is negligible except near the ends of the semicircle; it follows that, since the path of integration avoids the infinities of $\zeta_{s-\frac{1}{2}}(kb)/\zeta_{s-\frac{1}{2}}'(ka)$, the value of this fraction on the contour has its modulus comparable with $(k/a)(b/a)^{\frac{1}{2}-s}/(\frac{1}{2}-s)$ and so, by an application of Jordan's Lemma, the integral round the semicircle tends to zero as the radius tends to infinity. Hence the sum of the residues of the integrand at its poles on the right of the imaginary axis is zero. If we assume (see § 6) that the zeros of $\zeta_{s-\frac{1}{2}}'(ka)$ are all simple, and if we call them $\nu_1, \nu_2, \ldots$, we get

$$\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\zeta_n(kb)/\zeta_n'(ka) + 2\pi\sum_{\nu}\frac{\nu P_{\nu-\frac{1}{2}}(-\mu)\zeta_{\nu-\frac{1}{2}}(kb)}{\cos\nu\pi\,[\partial\zeta_{s-\frac{1}{2}}'(ka)/\partial s]_{s=\nu}} = 0,$$

and so we find that when k_i/β_i is neglected

$$\Pi_b(a,\theta) = \frac{2\pi}{kab}\sum_{\nu}\frac{\nu P_{\nu-\frac{1}{2}}(-\mu)\zeta_{\nu-\frac{1}{2}}(kb)}{\cos\nu\pi\,[\partial\zeta_{s-\frac{1}{2}}'(ka)/\partial s]_{s=\nu}},$$

and this equation expresses the transformation of the series for the Hertzian function which the object of this paper is to obtain.

It will be found later (§ 7) that this series converges very rapidly except when θ is quite small, and that term-by-term differentiation with respect to θ is permissible, since the test for uniformity of convergence is satisfied.

Further, it will be shown in § 7 that convergence is uniform with respect to b when $b \geq a$; and so, by Abel's theorem, we may write a for b in the series when we wish to find the effect due to an oscillator placed on the Earth's surface.

The investigation which lies immediately before us consists of an intensive study of the zeros of $\zeta_{s-\frac{1}{2}}'(ka)$ *qua* function of s.

The investigation is simplified by the fact that k is real but is complicated by the fact that ka is large, so that asymptotic expansions for Bessel functions of large variables have to be employed. It may be stated here that for waves whose length is 5 kilom., ka is 8000; if the wave-length is 10 kilom. (about the greatest length which is practically realisable), ka is 4000.

5. We first prove two results stated in § 4, namely, that $\zeta_{s-\frac{1}{2}}'(ka)$ does not vanish when s is positive or when s is a pure imaginary. To prove them, we write

$$\eta_{s-\frac{1}{2}}(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} \mathrm{H}_s^{(1)}(x),$$

where x is written for brevity in place of ka, and use the relation

$$\eta_{s-\frac{1}{2}}'(x)\,\zeta_{s-\frac{1}{2}}(x) - \zeta_{s-\frac{1}{2}}'(x)\,\eta_{s-\frac{1}{2}}(x) = 2i.$$

When s and x are positive, $\eta_{s-\frac{1}{2}}'(x)$ and $\zeta_{s-\frac{1}{2}}'(x)$ are conjugate complex numbers, and so if one vanishes the other also vanishes; and this is impossible in view of the above relation.

If s is a pure imaginary, the complex conjugate to $\zeta_{s-\frac{1}{2}}'(x)$ is $\eta_{-s-\frac{1}{2}}'(x)$, and this is equal to $e^{s\pi i}\eta_{s-\frac{1}{2}}'(x)$; hence if $\zeta_{s-\frac{1}{2}}'(x)$ vanishes, so does $\eta_{s-\frac{1}{2}}'(x)$, and this is also impossible. Hence $\zeta_{s-\frac{1}{2}}'(x)$ never vanishes for positive values of s or for purely imaginary values.

6. We shall now examine the behaviour of the functions $\zeta_s(ka)$, $\zeta_s'(ka)$ by means of the method of steepest descents. On recalling Sommerfeld's integrals for Bessel functions of the third kind,

$$\mathrm{H}_n^{(1)}(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{x\sinh w - nw}\,dw, \qquad \mathrm{H}_n^{(2)}(x) = -\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i} e^{x\sinh w - nw}\,dw,$$

we see that we shall have to study contour integrals in which the integrand is $e^{x\sinh w - nw}$, the contours passing through stationary points of $x\sinh w - nw$.

In the electrical problem under consideration, x is large and positive, while n is an unrestricted complex variable. We write*

$$n = x\cosh(\alpha + i\beta) = x\cosh\gamma,$$

where α, β are real. In view of the fact that asymptotic expansions fail when $n > x$ or $n < -x$, we suppose that $0 < \beta < \pi$, while α assumes all real values; the requisite cuts in the n-plane are effected by this hypothesis.

The stationary points of $\sinh w - w\cosh\gamma$ are the points $\pm\gamma + 2m\pi i$, where m assumes all integral values; we therefore study the contours formed by parts of the curves

$$\mathrm{I}(\sinh w - w\cosh\gamma) = \pm\mathrm{I}(\sinh\gamma - \gamma\cosh\gamma).$$

The curve on which

$$\mathrm{I}(\sinh w - w\cosh\gamma) = +\mathrm{I}(\sinh\gamma - \gamma\cosh\gamma)$$

* There is no risk of confusing these variables β, γ with the physical constants β, γ introduced in § 3.

has a double point at $w = \gamma$ and the inclination of one of its branches there is $\frac{1}{4}\pi + \frac{1}{2}\tan^{-1}(\tanh\alpha\cot\beta)$.

This branch will be called C_1, and the other branch through the double point will be called C_1'.

We shall write

$$S_n^{(1)}(x) = \frac{1}{\pi i}\int_{C_1} e^{x\sinh w - nw}\,dw\,;$$

it is known* that, when $\alpha = 0$, the functions $S_n^{(1)}(x)$ and $H_n^{(1)}(x)$ are equal and that C_1 passes from $-\infty$ to $\infty + \pi i$.

If τ is defined by the equation

$$\sinh w - w\cosh\gamma = -\tau + \sinh\gamma - \gamma\cosh\gamma,$$

then τ increases from 0 to ∞ as w moves along C_1 in either direction from γ.

Similarly the curve

$$I(\sinh w - w\cosh\gamma) = -I(\sinh\gamma - \gamma\cosh\gamma)$$

has a double point at $w = -\gamma$; the branch of the curve which has inclination

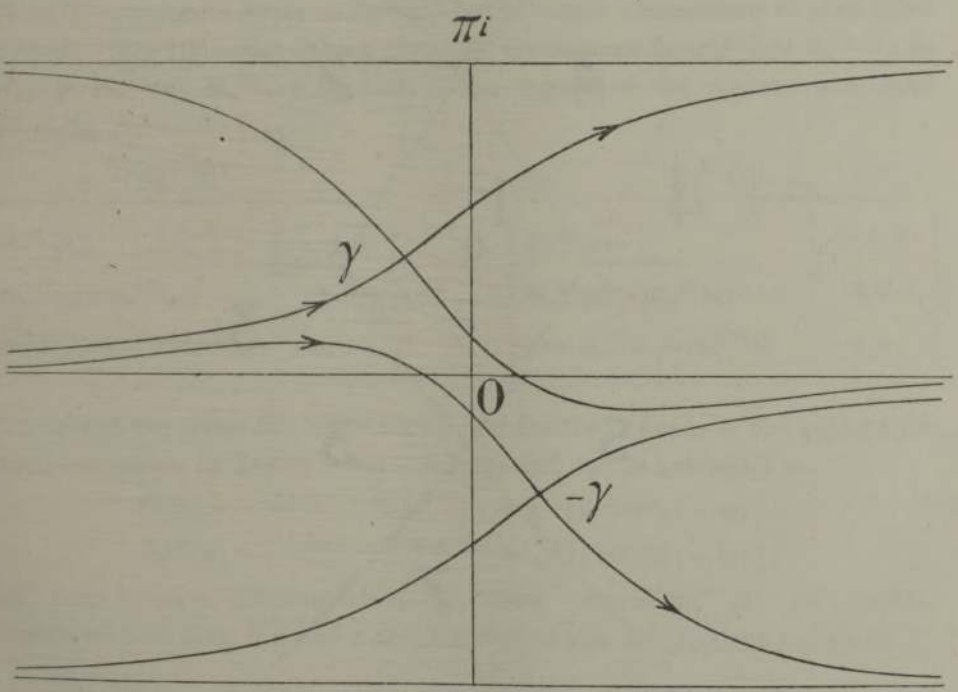

Fig. 1.

* *Cf.* 'Camb. Phil. Soc. Proc.,' vol. 19, p. 104 (1917).

$-\frac{1}{4}\pi+\frac{1}{2}\tan^{-1}(\tanh\alpha\cot\beta)$ at that point will be called C_2, and the other branch through this double point will be called C_2'.

We shall write

$$S_n^{(2)}(x) = -\frac{1}{\pi i}\int_{C_2} e^{x\sinh w - nw}\,dw$$

and, when $\alpha = 0$, $S_n^{(2)}(x) = H_n^{(2)}(x)$.

Now, if we vary γ continuously from a value for which $\alpha = 0$, the curve C_1 continues to pass from $-\infty$ to $\infty+\pi i$, until the curve changes its form by a process of bifurcation; such a change can only occur when the curve has a second double point; and so bifurcation can occur in two ways* according as the second double point on C_1 (or C_1') is the point $w=-\gamma$ or the point $w = 2\pi i-\gamma$. We consider the two types of bifurcation in turn.

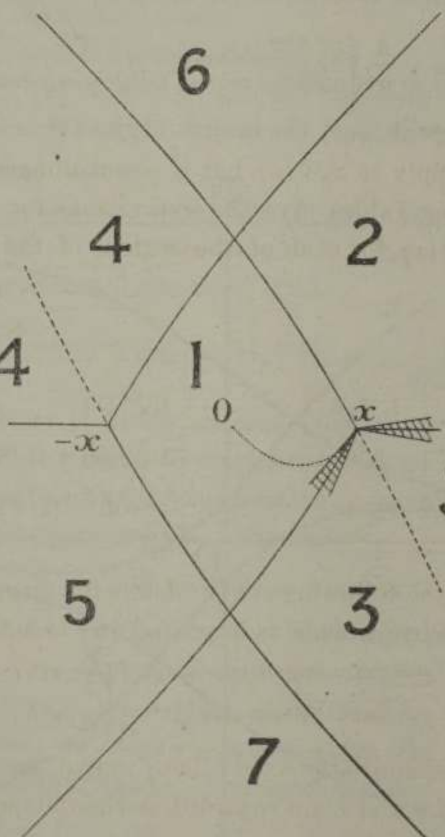

Fig. 2.

The first type occurs when $I(\sinh\gamma-\gamma\cosh\gamma) = 0$; and it can be shown that if $\alpha > 0$ the branch C_1 bifurcates, while if $\alpha < 0$ the branch C_1' bifur-

* In fig. 1 the contours are shown just before bifurcation takes place; the various modes of bifurcation are studied in detail by Debye, 'München Sitz.,' Abh. 5 (1910).

cates. In the latter case the bifurcation does not affect the form of C_1, but in the former case, after bifurcation, the curve C_1 starts at $\infty - \pi i$ and, after passing through γ, ends at $\infty + \pi i$, so that the value of $S_n^{(1)}(x)$ has changed abruptly to

$$\frac{1}{\pi i}\int_{\infty - \pi i}^{\infty + \pi i} e^{x \sinh w - nw}\, dw = 2J_n(x).$$

These bifurcations occur when n is on the curves shown in fig. 2 by continuous lines starting from the point $n = x$.

The second type of bifurcation occurs when

$$I\{\sinh\gamma - (\gamma - \pi i)\cosh\gamma\} = 0,$$

i.e. when n is on the curves shown in fig. 2 by continuous lines starting from $-x$. If $\alpha > 0$ the bifurcation does not affect C_1, but if $\alpha < 0$ the value of $S_n^{(1)}(x)$ becomes

$$\frac{1}{\pi i}\int_{-\infty}^{-\infty + 2\pi i} e^{x \sinh w - nw}\, dw,$$

and this is equal to $2e^{-n\pi i} J_{-n}(x)$.

Similar arguments apply to $S_n^{(2)}(x)$, but it seems unnecessary to give them in detail. The following Tables give the expressions for $S_n^{(1)}(x)$, $S_n^{(2)}(x)$, in terms of $H_n^{(1)}(x)$, $H_n^{(2)}(x)$, for each of the regions of the n-plane numbered 1–7 in fig. 2 :—

$S_n^{(1)}(x)$.

$H_n^{(1)}(x)$	1, 3, 4
$H_n^{(1)}(x) + H_n^{(2)}(x)$	2, 6
$H_n^{(1)}(x) + e^{-2n\pi i} H_n^{(2)}(x)$	5, 7

$S_n^{(2)}(x)$.

$H_n^{(2)}(x)$	1, 2, 5
$H_n^{(1)}(x) + H_n^{(2)}(x)$	3, 7
$e^{2n\pi i} H_n^{(1)}(x) + H_n^{(2)}(x)$	4, 6

Throughout the plane the following are the dominant terms of the asymptotic expansions given by Debye when x is large and $|\gamma|$ is not small :—

$$S_n^{(1)}(x) \sim e^{x(\sinh\gamma - \gamma\cosh\gamma) - \frac{1}{4}\pi i} \div \surd\{\tfrac{1}{2}\pi x \sin(-i\gamma)\},$$

$$S_n^{(2)}(x) \sim e^{-x(\sinh\gamma - \gamma\cosh\gamma) + \frac{1}{4}\pi i} \div \surd\{\tfrac{1}{2}\pi x \sin(-i\gamma)\};$$

and term-by-term differentiations of these expansions are permissible, whence we find that, if n and x are regarded as the independent variables,

$$\frac{\partial S_n^{(1)}(x)}{\partial n} \sim -\gamma S_n^{(1)}(x), \qquad \frac{\partial S_n^{(2)}(x)}{\partial n} \sim \gamma S_n^{(2)}(x),$$

$$\frac{\partial S_n^{(1)}(x)}{\partial x} \sim \sinh\gamma \,.\, S_n^{(1)}(x), \qquad \frac{\partial S_n^{(2)}(x)}{\partial x} \sim -\sinh\gamma \,.\, S_n^{(2)}(x),$$

$$\frac{\partial^2 S_n^{(1)}(x)}{\partial n \partial x} \sim -\gamma \sinh\gamma \,.\, S_n^{(1)}(x), \qquad \frac{\partial^2 S_n^{(2)}(x)}{\partial n \partial x} \sim -\gamma \sinh\gamma \,.\, S_n^{(2)}(x).$$

It follows that the large zeros of $H_n^{(1)}(x)$, $H_n^{(2)}(x)$ and their derivates are near the curve on which

$$R(\sinh\gamma-\gamma\cosh\gamma) = 0.$$

The portions of this curve near which zeros of $H_n^{(2)}(x)$ and its derivates lie are shown in fig. 2 by broken lines.

Since $\zeta_{n-\frac{1}{2}}'(x) \sim (\frac{1}{2}\pi x)^{\frac{1}{2}} \frac{d}{dx} H_n^{(2)}(x)$, we see that the large zeros of $\zeta_{n-\frac{1}{2}}'(x)$ are given approximately by the equation

$$e^{x(\sinh\gamma-\gamma\cosh\gamma)-\frac{1}{4}\pi i} = -e^{-x(\sinh\gamma-\gamma\cosh\gamma)+\frac{1}{4}\pi i},$$

i.e.
$$x(\sinh\gamma-\gamma\cosh\gamma)-\tfrac{1}{4}\pi i = -(m+\tfrac{1}{2})\pi i$$

where m is an integer large compared with x; and the corresponding value (ν_m) of ν which makes $\zeta_{\nu-\frac{1}{2}}'(x)$ have a simple zero is

$$\nu_m \sim -(m+\tfrac{1}{4})\pi i/\log m.$$

[A closer approximation shows that $R(\nu_m) > 0$].

7. We can now discuss the nature of the convergence of the series for $\Pi_b(a, \theta)$ obtained in § 4. It is obvious from the asymptotic expansion of $S_n^{(2)}(x)$ given in § 6 that $\zeta_{n-\frac{1}{2}}'(x)$ does not vanish when n lies in any of the regions of fig. 2 in which $H_n^{(2)}(x)$ is represented by a single series except when $|\gamma|$ is quite small; and so all the terms of high rank in the series for $\Pi_b(a, \theta)$ are comprised in the expression

$$\nu P_{\nu-\frac{1}{2}}(-\mu)\,\zeta_{\nu-\frac{1}{2}}(kb)\sec\nu\pi/[\partial\zeta_{s-\frac{1}{2}}'(ka)/\partial s]_{s=\nu}$$

where
$$\nu \sim -(m+\tfrac{1}{4})\pi i/\log m.$$

Also, at the large zeros of $\zeta_{n-\frac{1}{2}}'(x)$, it is found that the real parts of the exponents occurring in the asymptotic expansion of $\zeta_{n-\frac{1}{2}}(kb)$, $\zeta_{n-\frac{1}{2}}(ka)$ do not differ appreciably when b/a is nearly equal to unity; and so the general term in the series for $\Pi_b(a, \theta)$ is roughly equal to

$$\frac{\nu^2 P_{\nu-\frac{1}{2}}(-\mu)}{\sqrt{(\nu^2-x^2)}\,.\,\cos\nu\pi\,.\,\log\nu},$$

and this is comparable with $\nu^2 e^{-i\nu\theta}/[\sqrt{(\nu^2-x^2)}\,.\,\log\nu]$ except when θ is small. It is obvious from the last expression that the convergence properties stated at the end of § 4 are deducible from the test of Weierstrass for uniformity of convergence.

8. In order to obtain results of a numerical nature it is necessary to investigate those terms in the series for $\Pi_a(a, \theta)$ for which $I(\nu)$ is comparatively small; and to do this we shall require approximate formulæ for $\zeta_{n-\frac{1}{2}}'(x)$ when $|\gamma|$ is small.

It has been shown* that, when $\alpha = 0$, if $w = \pm\gamma + t$, and if $\pm\frac{1}{2}T^2\sinh\gamma + \frac{1}{6}T^3\cosh\gamma = -\tau$, then $|d(t-T)/d\tau|$ is a bounded function of τ and γ when $0 \leq \beta \leq \frac{1}{4}\pi$ and τ is positive. But the fact that this function is bounded is obviously not connected with the fact of $i\gamma$ being real, but must depend on the fact that, with the exception of $\tau = 0$, there is no stationary point of τ, *qua* function either of t or of T, which is in the neighbourhood of the curves in the t-plane, and in the T-plane on which τ is positive (≥ 0).

Suppose now that we confine our attention to those values of γ for which $|\gamma| \leq \frac{1}{2}$. The values of n for which C_2 has a second double point are represented by the curves which separate the region 3 from the regions 1 and 2 in fig. 2. The values of n for which the corresponding curve in the T-plane has a second double point are represented by the real axis on the right of x, and the curve passing from x to the origin (shown by the dotted line in fig. 2). The latter curve touches the boundary of the region 1 at x, and so, if we exclude from consideration the sectors which are shaded in fig. 2, then $|d(t-T)/d\tau|$ is a bounded function of x, γ and τ so long as $|\gamma| \leq \frac{1}{2}$.

Hence, when $|\gamma| \leq \frac{1}{2}$ and the shaded sectors are excluded from consideration, we find that

$$\left|\int e^{-x\tau}dt - \int e^{-x\tau}dT\right| < \pi A/x,$$

where A is a constant depending only on the angles of the sectors. Hence we have†

$$S_n^{(2)}(x) = e^{-x(\sinh\gamma-\gamma\cosh\gamma)}\{3^{-\frac{1}{2}}\tanh\gamma \,.\, e^{-\frac{1}{3}\pi i+i\xi}\,\Sigma_{\frac{1}{3}}^{(2)}(\xi) + A\theta_2 x^{-1}\},$$

where $\xi = \frac{1}{3}ix\sinh^3\gamma\,\mathrm{sech}^2\gamma$, the phase of ξ vanishing with α; the value of $\Sigma_{\frac{1}{3}}^{(2)}(\xi)$ is $H_{\frac{1}{3}}^{(2)}(\xi)$ when n lies in the regions 1 and 2, and it is $H_{\frac{1}{3}}^{(2)}(\xi) + e^{\frac{1}{3}\pi i}H_{\frac{1}{3}}^{(1)}(\xi)$ when n lies in the region 3, the transition taking place as n crosses the dotted curve (the equation of the dotted curve may be written $\arg\xi = \frac{1}{2}\pi$); and, finally, $|\theta_2| \leq 1$.

In a similar manner

$$S_n^{(1)}(x) = e^{x(\sinh\gamma-\gamma\cosh\gamma)}\{3^{-\frac{1}{2}}\tanh\gamma \,.\, e^{-\frac{1}{3}\pi i-i\xi}S_{\frac{1}{3}}^{(1)}(\xi) + A\theta_1 x^{-1}\}$$

so long as $|\gamma| \leq \frac{1}{2}$ and, in addition, n lies in the regions 1 and 3, and is not too close to the boundary which separates them from the region 2; and $|\theta_1| \leq 1$.

To discuss the approximate value of $H_n^{(2)}(x)$ we examine the function

$$\omega(n) \equiv e^{x(\sinh\gamma-\gamma\cosh\gamma)}H_n^{(2)}(x) - 3^{-\frac{1}{2}}\tanh\gamma \,.\, e^{-\frac{1}{3}\pi i+i\xi}H_{\frac{1}{3}}^{(2)}(\xi).$$

* 'Camb. Phil. Soc. Proc.,' vol. 19, pp. 103–110 (1917); in stating the following results, the notation has been modified by taking x and γ (instead of n and γ) as independent variables.

† *Cf.* 'Camb. Phil. Soc. Proc.,' vol. 19, p. 110 (1917).

Consider the value of $\omega(n)$ on the boundary of the shaded sector which separates the region 1 from the region 3.

On the arm of the sector which lies in the region 1, we have $\omega(n) = A\theta_2 x^{-1}$.

On the arm of the sector which lies in the region 3, we have

$$\omega(n) = Ax^{-1}(\theta_2 - \theta_1 e^{2x(\sinh\gamma - \gamma\cosh\gamma)}) + \tfrac{2}{3}\sqrt{3}\,.\,\tanh\gamma\,.\,e^{\frac{1}{6}\pi i + x(\sinh\gamma-\gamma\cosh\gamma)}$$
$$\times \sinh\{x(\sinh\gamma - \gamma\cosh\gamma + \tfrac{1}{3}\sinh^3\gamma\,\mathrm{sech}^2\gamma)\}\,H_{\frac{1}{3}}^{(1)}(\xi).$$

Now, on this arm of the sector, $R\{x(\sinh\gamma-\gamma\cosh\gamma)\} \leq 0$, and hence, using the appropriate approximation when $|\xi|$ is small or large (say less than $\frac{1}{2}$ or more than 8) for $H_{\frac{1}{3}}^{(1)}(\xi)$ and using the fact that it is bounded for intermediate values, we find that $|\omega(n)| < Bx^{-1}$, where B is an absolute constant.

On the part of the arc $|\gamma| = \frac{1}{2}$ which forms part of the boundary of the shaded region, by using the asymptotic expansions of $H_n^{(2)}(x)$, $H_{\frac{1}{3}}^{(2)}(\xi)$, we find the same inequality for $\omega(n)$.

Now $\omega(n)$ is analytic throughout the interior of the shaded region and so its real and imaginary parts have no true maxima or minima in the shaded region. Hence in the shaded region $|\omega(n)| < B\sqrt{2}\,.\,x^{-1}$.

Hence in the region 1 and in that part of the region 3 for which $R(\sinh\gamma - \gamma\cosh\gamma) \leq 0$, by using equations already given for $\omega(n)$, we find that

$$e^{x(\sinh\gamma-\gamma\cosh\gamma)}H_n^{(2)}(x) = 3^{-\frac{1}{2}}\tanh\gamma\,.\,e^{-\frac{1}{6}\pi i + i\xi}H_{\frac{1}{3}}^{(2)}(\xi) + O(1/x)$$

while, in the rest* of the region (3), the function $O(1/x)$ has to be replaced by $e^{2x(\sinh\gamma-\gamma\cosh\gamma)}O(1/x)$, the constant implied in the symbol O being independent of γ so long as $|\gamma| \leq \frac{1}{2}$.

It may be shown by purely formal analysis (involving a use of Cauchy's theorem) that it is permissible to differentiate this approximate formula with regard to n or x; and we find that

$$\frac{d}{dx}\{x^{\frac{1}{2}}H_n^{(2)}(x)\}$$
$$\sim -(3x)^{-\frac{1}{2}}\exp\{-\tfrac{2}{3}\pi i - x(\sinh\gamma-\gamma\cosh\gamma) + i\xi\}\,\mathrm{sech}\,\gamma\,\mathrm{cosech}\,\gamma$$
$$\times[H_{\frac{1}{3}}^{(2)}(\xi) + 3\xi\,dH_{\frac{1}{3}}^{(2)}(\xi)/d\xi].$$

The function on the right vanishes when $H_{-\frac{2}{3}}^{(2)}(\xi)$ vanishes. The three smallest zeros of this function have been calculated by Macdonald;† their values are

$$e^{\pi i}\times 0\cdot6854, \qquad e^{\pi i}\times 3\cdot90, \qquad e^{\pi i}\times 7\cdot05\,;$$

and from what is known concerning the zeros of Bessel functions of the first kind of high order, it is in the highest degree improbable that, when x has any

* The shaded area near the real axis has to be omitted, but it can easily be discussed in a similar manner.

† 'Roy. Soc. Proc.,' A, vol. 90, p. 54 (1914).

assigned value exceeding 4000 (this is the smallest value of x which is of practical importance), the zeros of $d\{x^{\frac{1}{2}}H_n^{(2)}(x)\}/dx$, *qua* function of ξ, differ from these values by more than about 1 per cent. at most.

The three most important values of n are therefore

$$x + \rho x^{\frac{1}{3}} e^{-\frac{1}{3}\pi i},$$

where $$\rho = 0\cdot8083, \quad 2\cdot577, \quad 3\cdot83.$$

Also, to a first approximation,

$$x^{\frac{1}{2}} H_n^{(2)}(x) \Big/ \frac{\partial^2}{\partial n \partial x} \{x^{\frac{1}{2}} H_n^{(2)}(x)\} \sim -\cosh^3\gamma \operatorname{cosech}^2\gamma$$
$$\sim -\tfrac{1}{2} x^{\frac{2}{3}} e^{\frac{1}{3}\pi i}/\rho.$$

Hence each of the dominant terms in the series for the Hertzian function is given by the formula

$$-\frac{\pi e^{\frac{1}{3}\pi i}}{a\rho\,(ka)^{\frac{1}{3}}} \frac{\nu P_{\nu-\frac{1}{2}}(-\mu)}{\cos\nu\pi}.$$

If λ be the wave-length (measured in kilometres) these terms may be written

$$3674\,a^{-1}\lambda^{-\frac{2}{3}} e^{\frac{1}{3}\pi i} \rho^{-1} P_{\nu-\frac{1}{2}}(-\mu) \sec\nu\pi,$$

approximately ; and ν takes the values

$$40000\lambda^{-1} + (13\cdot82 - 23\cdot94\,i)\,\lambda^{-\frac{1}{3}},$$
$$40000\lambda^{-1} + (43\cdot93 - 76\cdot32\,i)\,\lambda^{-\frac{1}{3}},$$
$$40000\lambda^{-1} + (65\cdot5 - 113\,i)\,\lambda^{-\frac{1}{3}}.$$

....

Hence by Laplace's approximation when θ is not nearly equal to 0 or π, and by Mehler's approximation* when θ is nearly equal to π, we find that the order of magnitude of the dominant term is exp $(-23\cdot94\,\lambda^{-\frac{1}{3}}\theta)$, and, compared with this, the other terms are negligible. This result is in substantial agreement with the approximations obtained by Nicholson and Macdonald. It would be easy to construct a table of values of the dominant term for various wave-lengths.

It is to be observed that in this theory there is not a "focus" at the antipodes of the oscillator; in fact the magnetic force vanishes at $\theta = \pi$.

For purposes of comparison it should be noticed that if the Earth were replaced by dielectric the Hertzian function at any point of the Earth's surface would be $\exp\{-2ika \sin\frac{1}{2}\theta\}/(2a\sin\frac{1}{2}\theta)$:

9. We shall now discuss the value of the Hertzian function on the hypothesis that the earth is not a perfect conductor.

* *Cf.* a forthcoming paper by the present writer in the 'Messenger of Mathematics.'

The contour integral to be discussed is

$$-\frac{2\pi}{kab}\int s\,\frac{P_{s-\frac{1}{2}}(-\mu)}{\cos s\pi}\cdot\frac{\zeta_{s-\frac{1}{2}}(kb)\,\psi_{s-\frac{1}{2}}(k_i a)\,ds}{\psi_{s-\frac{1}{2}}(k_i a)\,\zeta_{s-\frac{1}{2}}'(ka)-\beta k_i\psi_{s-\frac{1}{2}}'(k_i a)\,\zeta_{s-\frac{1}{2}}(ka)/(\beta_i k)}.$$

The sum of the residues at the poles $s=\frac{1}{2}$, $s=\frac{3}{2}$, ..., is $\Pi(a,\theta)$; the integral along the large semicircle vanishes as before; and so we find that

$$\Pi(a,\theta)=\frac{2\pi}{kab}\sum_{\nu}\frac{\nu P_{\nu-\frac{1}{2}}(-\mu)\,\zeta_{\nu-\frac{1}{2}}(kb)}{\cos\nu\pi\,[\partial\zeta_{s-\frac{1}{2}}'(ka)\,\phi(s)/\partial s]_{s=\nu}}+\frac{1}{kabi}\int_{-\infty i}^{\infty i}(\ldots)\,ds,$$

where $\phi(s)$ denotes

$$1-\beta k_i\{\psi_{s-\frac{1}{2}}'(k_i a)/\psi_{s-\frac{1}{2}}(k_i a)\}\{\zeta_{s-\frac{1}{2}}(ka)/\zeta_{s-\frac{1}{2}}'(ka)\}/(\beta_i k),$$

and the summation applies to the zeros of $\phi(s)$ for which $R(s)>0$.

Now the values of the constants β and γ are

$$\beta=(\sigma+i\epsilon\omega)/c,\qquad \gamma=i\mu\omega/c,\qquad k^2=-\beta\gamma,$$

where μ is the permeability, ϵ the dielectric constant, σ the conductivity, and c is the velocity of light, all measured in rational units.

The values of the constants are as follows:—

For air, $\epsilon=\mu=1$, $\sigma=0$.
For sea-water, $\epsilon=81$, $\mu=1$, $\sigma=4{\cdot}26\times10^{11}$.
For dry earth, $\epsilon=4$, $\mu=1$, $\sigma=10^7$.

It follows that for sea-water $k_i^2/k^2=-6000i+81$ at least, and that for dry earth k_i^2/k^2 varies between $-26i+4$ and $-52i+4$, while $\beta k_i/\beta_i k$ is k/k_i.

These values indicate that $\phi(s)$ differs little from unity in the case of sea-water when s is nearly equal to ka, while, even in the case of dry earth, the difference is not great. It appears that the series for $\Pi(a,\theta)$ is not much affected. We have, however, to consider the integral, which may be written in the form

$$\frac{1}{kabi}\int_0^{\infty i}\frac{sP_{s-\frac{1}{2}}(-\mu)\,\zeta_{s-\frac{1}{2}}(kb)}{\cos s\pi\,\phi(s)\phi(-s)\zeta_{s-\frac{1}{2}}'(ka)}\{\phi(-s)-\phi(s)\}\,ds.$$

Now

$$\phi(-s)-\phi(s)=\frac{k}{k_i}\frac{\zeta_{s-\frac{1}{2}}(ka)}{\zeta_{s-\frac{1}{2}}'(ka)}\left\{\frac{\psi_{-s-\frac{1}{2}}'(k_i a)}{\psi_{-s-\frac{1}{2}}(k_i a)}-\frac{\psi_{s-\frac{1}{2}}'(k_i a)}{\psi_{s-\frac{1}{2}}(k_i a)}\right\}$$

$$=-\frac{k}{k_i}\frac{\zeta_{s-\frac{1}{2}}(ka)}{\zeta_{s-\frac{1}{2}}'(ka)}\frac{2\sin s\pi}{J_s(k_i a)\,J_{-s}(k_i a)}.$$

Now when s is a pure imaginary and less than k, this is approximately

$$-\frac{k}{ik_i}\left(1-\frac{s^2}{x^2}\right)^{-\frac{1}{2}}\cdot\frac{2\sin s\pi}{J_s(k_i a)\,J_{-s}(k_i a)}.$$

The approximate value of $|J_s(k_i a) J_{-s}(k_i a)|$ when $s = 0$ is

$$\exp\{2R(k_i)a\} \div (2\pi |k_i a|),$$

and the exponent is $80000\lambda^{-1} R(k_i/k)$; by using the complete system of asymptotic expansions when $s \neq 0$, we find that the integral is convergent and is of order of magnitude $\exp\{-80000\lambda^{-1} R(k_i/k)\}$. It follows by easy arithmetic that the integral is negligible compared with the series. The effect of the waves which pass through the Earth is therefore trivial compared with that of the waves which travel round the Earth.

10. If we take the series for $\Pi_a(a, \theta)$ and expand $\sec \nu\pi$ into the series $2\{e^{-\nu\pi i} - e^{-3\nu\pi i} + e^{-5\nu\pi i} - \ldots\}$ and multiply the result by $e^{\nu(\pi-\theta)i}$, $e^{-\nu(\pi-\theta)i}$, which are the exponential factors in Laplace's approximation for $P_\nu(-\mu)$, the separate terms in the product denote the disturbances produced by waves travelling from the oscillator in either direction round the Earth after passing completely round 0, 1, 2, ..., times. There seems to be no similar physical significance for the separate terms corresponding to different values of ν.

Note on the Effect of Wind Pressure on the Pitch of Organ Pipes.

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In books on acoustics the pitch of organ pipes is treated as being dependent solely on the length of the pipe, while at the same time it is recognised that, as a consequence of the conditions existing at the open end or ends, the length in question is somewhat greater than that of the pipe itself. The "correction for open ends" has been calculated for particular cases, and for the open end of a pipe of circular section amounts to an addition to its length of about 0·82 times the radius. The conditions at the base, or speaking end, are more complex, and for this (as far as I am aware) no similar correction has been worked out. For these reasons the natural pitch of an organ pipe cannot be accurately determined from its material dimensions, but only by experiment; such, for instance, as noting the frequency of an exterior source of sound which produces the maximum resonance in the pipe.

When an organ pipe is made to "speak" in the ordinary way, it is a matter of common knowledge that the pitch of note produced is to some