The Dilogarithm Function

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The dilogarithm function, defined in the first sentence of Chapter I, is a function which has been known for more than 250 years, but which for a long time was familiar only to a few enthusiasts. In recent years it has become much better known, due to its appearance in hyperbolic geometry and in algebraic K-theory on the one hand and in mathematical physics (in particular, in conformal field theory) on the other. I was therefore asked to give two lectures at the Les Houches meeting introducing this function and explaining some of its most important properties and applications, and to write up these lectures for the Proceedings.

The first task was relatively straightforward, but the second posed a problem since I had already written and published an expository article on the dilogarithm some 15 years earlier. (In fact, that paper, originally written as a lecture in honor of Friedrich Hirzebruch's 60th birthday, had appeared in two different Indian publications during the Ramanujan centennial year—see footnote to Chapter I). It seemed to make little sense to try to repeat in different words the contents of that earlier article. On the other hand, just reprinting the original article would mean omitting several topics which were either developed since it was written or which were omitted then but are of more interest now in the context of the appearances of the dilogarithm in mathematical physics.

The solution I finally decided on was to write a text consisting of two chapters of different natures. The first is simply an unchanged copy of the 1988 article, with its original title, footnotes, and bibliography, reprinted by permission from the book *Number Theory and Related Topics* (Tata Institute of Fundamental Research, Bombay, January 1988). In this chapter we define the dilogarithm function and describe some of its more striking properties: its known special values which can be expressed in terms of ordinary logarithms, its many functional equations, its connection with the volumes of ideal tetrahedra in hyperbolic 3-space and with the special values at $s = 2$ of the Dedekind zeta functions of algebraic number fields, and its appearance in algebraic K-theory; the higher polylogarithms are also treated briefly. The second, new, chapter gives further information as well as some more recent developments of the theory. Four main topics are discussed here. Three of them—functional equations, modifications of the dilogarithm function, and higher polylogarithms—are continuations of themes which were already begun in Chapter I. The fourth topic, Nahm's conjectural connection between (torsion in) the Bloch group and modular functions, is new and especially fascinating. We discuss only some elementary aspects concerning the asymptotic properties of Nahm's q-expansions, referring the reader for the deeper parts of the theory, concerning the (in general conjectural) interpretation of these q-series as characters of rational conformal field theories, to the beautiful article by Nahm in this volume.

As well as the two original footnotes to Chapter I, which are indicated by numbers in the text and placed at the bottom of the page in the traditional manner, there are also some further footnotes, indicated by boxed capital letters in the margin and placed at the end of the chapter, which give updates or comments on the text of the older article or else refer the reader to the sections of Chapter II where the topic in question is developed further. Each of the two chapters has its own bibliography, that of Chapter I being a reprint of the original one and that of Chapter II giving some references to more recent literature. I apologize to the reader for this somewhat artificial construction, but hope that the two parts of the paper can still be read without too much confusion and perhaps even with some enjoyment. My own enthusiasm for this marvelous function as expressed in the 1988 paper has certainly not lessened in the intervening years, and I hope that the reader will be able to share at least some of it.

The reader interested in knowing more about dilogarithms should also consult the long article [22] of A.N. Kirillov, which is both a survey paper treating most or all of the topics discussed here and also contains many new results of interest from the point of view of both mathematics and physics.

Chapter I. The dilogarithm function in geometry and number theory[1]

The dilogarithm function is the function defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for } |z| < 1 .$$

The definition and the name, of course, come from the analogy with the Taylor series of the ordinary logarithm around 1,

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \text{for } |z| < 1 ,$$

which leads similarly to the definition of the *polylogarithm*

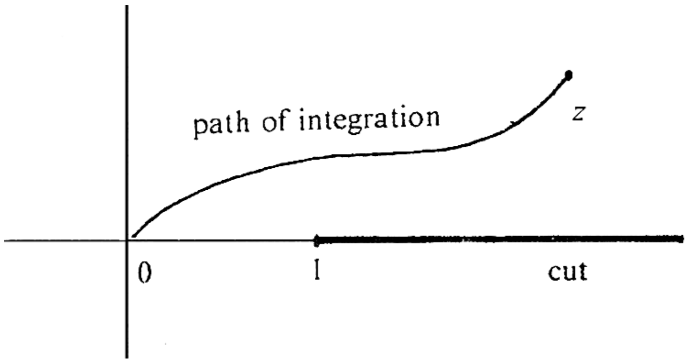
$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \text{for} \quad |z| < 1, \quad m = 1, 2, \dots .$$

The relation

$$\frac{d}{dz}\mathrm{Li}_m(z) = \frac{1}{z}\,\mathrm{Li}_{m-1}(z) \qquad (m \geq 2)$$

is obvious and leads by induction to the extension of the domain of definition of Li_m to the cut plane $\mathbb{C} \smallsetminus [1, \infty)$; in particular, the analytic continuation of the dilogarithm is given by

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\,\frac{du}{u} \qquad \text{for} \quad z \in \mathbb{C} \smallsetminus [1, \infty) .$$



[1] This paper is a revised version of a lecture which was given in Bonn on the occasion of F. Hirzebruch's 60th birthday (October 1987) and has also appeared under the title "The remarkable dilogarithm" in the Journal of Mathematical and Physical Sciences, 22 (1988).

Thus the dilogarithm is one of the simplest non-elementary functions one can imagine. It is also one of the strangest. It occurs not quite often enough, and in not quite an important enough way, to be included in the Valhalla of the great transcendental functions—the gamma function, Bessel and Legendre- functions, hypergeometric series, or Riemann's zeta function. And yet it occurs too often, and in far too varied contexts, to be dismissed as a mere curiosity. First defined by Euler, it has been studied by some of the great mathematicians of the past—Abel, Lobachevsky, Kummer, and Ramanujan, to name just a few—and there is a whole book devoted to it [4]. Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor. In this paper we wish to discuss some of these appearances and some of these formulas, to give at least an idea of this
A remarkable and too little-known function.

1 Special values

Let us start with the question of special values. Most functions have either no exactly computable special values (Bessel functions, for instance) or else a countable, easily describable set of them; thus, for the gamma function

$$\Gamma(n) = (n-1)!\,, \qquad \Gamma\bigl(n+\tfrac{1}{2}\bigr) = \frac{(2n)!}{4^n n!}\sqrt{\pi}\,,$$

and for the Riemann zeta function

$$\begin{array}{llll}
\zeta(2) = \dfrac{\pi^2}{6}, & \zeta(4) = \dfrac{\pi^4}{90}, & \zeta(6) = \dfrac{\pi^6}{945}, & \ldots\,, \\
\zeta(0) = -\dfrac{1}{2}, & \zeta(-2) = 0, & \zeta(-4) = 0, & \ldots\,, \\
\zeta(-1) = -\dfrac{1}{12}, & \zeta(-3) = \dfrac{1}{120}, & \zeta(-5) = -\dfrac{1}{252}, & \ldots\,.
\end{array}$$

Not so the dilogarithm. As far as anyone knows, there are exactly eight
B values of z for which z and $\mathrm{Li}_2(z)$ can both be given in closed form:

$$\begin{aligned}
\mathrm{Li}_2(0) &= 0, \\
\mathrm{Li}_2(1) &= \frac{\pi^2}{6}, \\
\mathrm{Li}_2(-1) &= -\frac{\pi^2}{12}, \\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),
\end{aligned}$$

$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{-1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{10} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right).$$

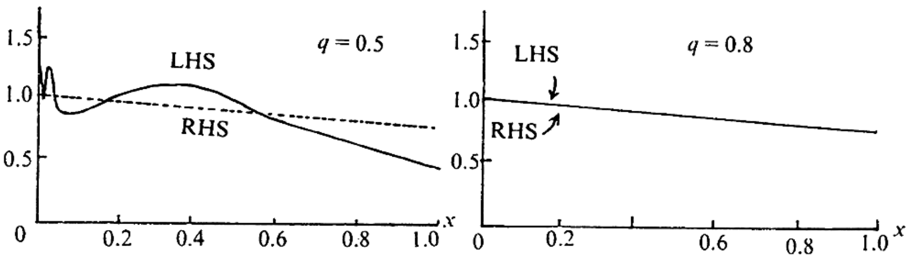
Let me describe a recent experience where these special values figured, and which admirably illustrates what I said about the bizarreness of the occurrences of the dilogarithm in mathematics. From Bruce Berndt via Henri Cohen I learned of a still unproved assertion in the Notebooks of Srinivasa Ramanujan (Vol. 2, p. 289, formula (3.3)): Ramanujan says that, for q and x between 0 and 1,

$$\cfrac{q}{x+\cfrac{q^4}{x+\cfrac{q^8}{x+\cfrac{q^{12}}{x+\cdots}}}} = 1 - \cfrac{qx}{1+\cfrac{q^2}{1-\cfrac{q^3x}{1+\cfrac{q^4}{1-\cfrac{q^5x}{1+\cdots}}}}}$$

"very nearly." He does not explain what this means, but a little experimentation shows that what is meant is that the two expressions are numerically very close when q is near 1; thus for $q = 0.9$ and $x = 0.5$ one has

$$\mathrm{LHS} = 0.7767340194\cdots, \qquad \mathrm{RHS} = 0.7767340180\cdots,$$

A graphical illustration of this is also shown.



The quantitative interpretation turned out as follows [9]: The difference between the left and right sides of Ramanujan's equation is $O\bigl(\exp(\frac{\pi^2/5}{\log q})\bigr)$ for $x=1$, $q\to 1$. (The proof of this used the identities

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} \;=\; \prod_{n=1}^{\infty}\bigl(1-q^n\bigr)^{\left(\frac{n}{5}\right)} \;=\; \frac{\sum(-1)^r q^{\frac{5r^2+3r}{2}}}{\sum(-1)^r q^{\frac{5r^2+r}{2}}}\,,$$

which are consequences of the Rogers-Ramanujan identities and are surely among the most beautiful formulas in mathematics.) For $x\to 0$ and $q\to 1$ the difference in question is $O\bigl(\exp(\frac{\pi^2/4}{\log q})\bigr)$, and for $0<x<1$ and $q\to 1$ it is $O\bigl(\exp(\dfrac{c(x)}{\log q})\bigr)$ where $c'(x)=-\dfrac{1}{x}\,\operatorname{arg\,sinh}\dfrac{x}{2}=-\dfrac{1}{x}\,\log\bigl(\sqrt{1+x^2/4}+x/2\bigr)$. For these three formulas to be compatible, one needs

$$\int_0^1 \frac{1}{x}\log(\sqrt{1+x^2/4}+x/2)\,dx = c(0)-c(1) = \frac{\pi^2}{4}-\frac{\pi^2}{5} = \frac{\pi^2}{20}\,.$$

Using integration by parts and formula A.3.1 (6) of [4] one finds

$$\begin{aligned}\int \frac{1}{x}\,\log\bigl(\sqrt{1+x^2/4}+x/2\bigr)\,dx &= -\frac{1}{2}\,\mathrm{Li}_2\Bigl(\bigl(\sqrt{1+x^2/4}\,-\,x/2\bigr)^2\Bigr)\\ &\quad -\frac{1}{2}\,\log^2(\sqrt{1+x^2/4}+x/2)\,+\,(\log x)\log(\sqrt{1+x^2/4}+x/2)+C\,,\end{aligned}$$

so

$$\begin{aligned}\int_0^1 \frac{1}{x}\log(\sqrt{1+x^2/4}+x/2)\,dx &= \frac{1}{2}\,\mathrm{Li}_2(1)-\frac{1}{2}\Bigl(\mathrm{Li}_2\bigl(\frac{3-\sqrt5}{2}\bigr)+\log^2\bigl(\frac{1+\sqrt5}{2}\bigr)\Bigr)\\ &= \frac{\pi^2}{12}-\frac{\pi^2}{30}=\frac{\pi^2}{20}\ \ !\end{aligned}$$

2 Functional equations

In contrast to the paucity of special values, the dilogarithm function satisfies a plethora of functional equations. To begin with, there are the two reflection properties

$$\begin{aligned}\mathrm{Li}_2\bigl(\frac{1}{z}\bigr) \;&=\; -\,\mathrm{Li}_2(z)\;-\;\frac{\pi^2}{6}\;-\;\frac{1}{2}\log^2(-z)\,,\\ \mathrm{Li}_2(1-z)\;&=\;-\,\mathrm{Li}_2(z)\;+\;\frac{\pi^2}{6}\;-\;\log(z)\,\log(1-z)\,.\end{aligned}$$

Together they say that the six functions

$$\mathrm{Li}_2(z),\ \ \mathrm{Li}_2\big(\frac{1}{1-z}\big),\ \ \mathrm{Li}_2\big(\frac{z-1}{z}\big),\ \ -\mathrm{Li}_2\big(\frac{1}{z}\big),\ \ -\mathrm{Li}_2(1-z),\ \ -\mathrm{Li}_2\big(\frac{z}{z-1}\big)$$

are equal modulo elementary functions, Then there is the duplication formula

$$\mathrm{Li}_2(z^2) = 2\left(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)\right)$$

and more generally the "distribution property"

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z) \qquad (n = 1,\, 2,\, 3, \dots).$$

Next, there is the two-variable, five-term relation

$$\begin{aligned} &\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2\left(1-xy\right) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right) \\ &= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right) \end{aligned}$$

which (in this or one of the many equivalent forms obtained by applying the symmetry properties given above) was discovered and rediscovered by Spence (1809), Abel (1827), Hill (1828), Kummer (1840), Schaeffer (1846), and doubtless others. (Despite appearances, this relation is symmetric in the five arguments: if these are numbered cyclically as z_n with $n \in \mathbb{Z}/5\mathbb{Z}$, then $1 - z_n = \left(z_{n-1}^{-1} - 1\right)\left(z_{n+1}^{-1} - 1\right) = z_{n-2}z_{n+2}$.) There is also the six-term relation

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \quad \Rightarrow \quad &\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2(z) \\ &= \frac{1}{2}\big[\mathrm{Li}_2\big(-\frac{xy}{z}\big) + \mathrm{Li}_2\big(-\frac{yz}{x}\big) + \mathrm{Li}_2\big(-\frac{zx}{y}\big)\big] \end{aligned}$$

discovered by Kummer (1840) and Newman (1892). Finally, there is the strange many-variable equation

$$\mathrm{Li}_2(z) = \sum_{\substack{f(x)=z \\ f(a)=1}} \mathrm{Li}_2\left(\frac{x}{a}\right) + C(f)\,, \tag{1}$$

where $f(x)$ is any polynomial without constant term and $C(f)$ a (complicated) constant depending on f. For f quadratic, this reduces to the five-term relation, while for f of degree n it involves $n^2 + 1$ values of the dilogarithm.

All of the functional equations of Li_2 are easily proved by differentiation, **C**
while the special values given in the previous section are obtained by combining suitable functional equations. See [4].

3 The Bloch-Wigner function $D(z)$ and its generalizations

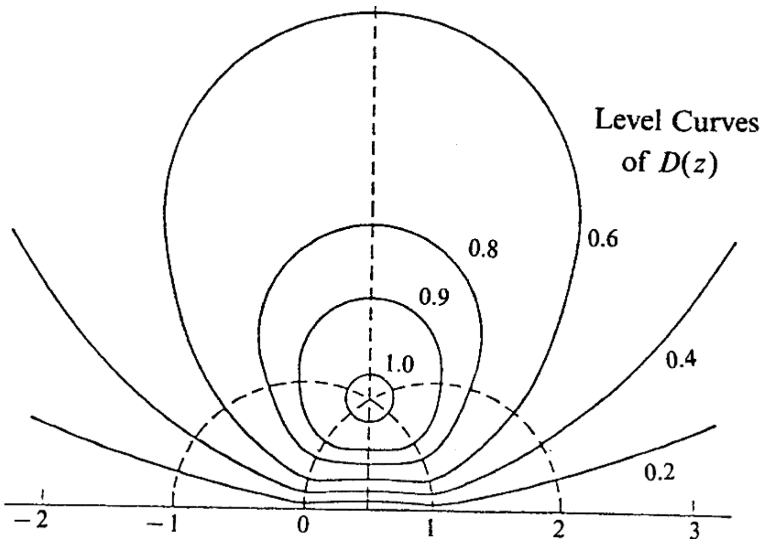
D

The function $\mathrm{Li}_2(z)$, extended as above to $\mathbb{C} \smallsetminus [1, \infty)$, jumps by $2\pi i \log|z|$ as z crosses the cut. Thus the function $\mathrm{Li}_2(z) + i \arg(1-z) \log|z|$, where arg denotes the branch of the argument lying between $-\pi$ and π, is continuous. Surprisingly, its imaginary part

$$D(z) \;=\; \Im(\mathrm{Li}_2(z)) \;+\; \arg(1-z)\log|z|$$

is not only continuous, but satisfies

(I) $D(z)$ is real analytic on $\mathbb{C}$ except at the two points 0 and 1, where it is continuous but not differentiable (it has singularities of type $r \log r$ there).



The above graph shows the behaviour of $D(z)$. We have plotted the level curves $D(z) \;=\; 0,\, 0.2,\, 0.4,\, 0.6,\, 0.8,\, 0.9,\, 1.0$ in the upper half-plane. The values in the lower half-plane are obtained from $D(\bar{z}) = -D(z)$. The maximum of D is $1.0149\ldots$, attained at the point $(1+i\sqrt{3})/2$.

The function $D(z)$, which was discovered by D. Wigner and S. Bloch (cf. [1]), has many other beautiful properties. In particular:

(II) $D(z)$, which is a real-valued function on $\mathbb{C}$, can be expressed in terms of a function of a single real variable, namely

$$D(z) = \frac{1}{2}\left[D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-1/z}{1-1/\bar{z}}\right) + D\left(\frac{1/(1-z)}{1/(1-\bar{z})}\right)\right] \qquad (2)$$

which expresses $D(z)$ for arbitrary complex z in terms of the function

$$D(e^{i\theta}) = \Im[\mathrm{Li}_2(e^{i\theta})] = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} .$$

(Note that the real part of Li_2 on the unit circle is elementary: $\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi-\theta)}{4}$ for $0 \le \theta \le 2\pi$.) Formula (2) is due to Kummer.

(III) All of the functional equations satisfied by $\mathrm{Li}_2(z)$ lose the elementary correction terms (constants and products of logarithms) when expressed in terms of $D(z)$. In particular, one has the 6-fold symmetry

$$\begin{aligned} D(z) &= D\left(1-\frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) \\ &= -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{-z}{1-z}\right) \end{aligned} \tag{3}$$

and the five-term relation

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0, \tag{4}$$

while replacing Li_2 by D in the many-term relation (1) makes the constant $C(f)$ disappear.

The functional equations become even cleaner if we think of D as being a function not of a single complex number but of the cross-ratio of four such numbers, i.e., if we define

$$\widetilde{D}(z_0, z_1, z_2, z_3) = D\left(\frac{z_0-z_2}{z_0-z_3}\,\frac{z_1-z_3}{z_1-z_2}\right) \quad (z_0, z_1, z_2, z_3 \in \mathbb{C}). \tag{5}$$

Then the symmetry properties (3) say that $\widetilde{D}$ is invariant under even and anti-invariant under odd permutations of its four variables, the five-term relation (4) takes on the attractive form

$$\sum_{i=0}^{4} (-1)^i\, \widetilde{D}(z_0, \ldots, \widehat{z_i}, \ldots, z_4) = 0 \quad (z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})) \tag{6}$$

(we will see the geometric interpretation of this later), and the multi-variable formula (1) generalizes to the following beautiful formula:

$$\sum_{\substack{z_1 \in f^{-1}(a_1) \\ z_2 \in f^{-1}(a_2) \\ z_3 \in f^{-1}(a_3)}} \widetilde{D}(z_0, z_1, z_2, z_3) \;=\; n\, \widetilde{D}(a_0, a_1, a_2, a_3) \quad (z_0, a_1, a_2, a_3 \in \mathbb{P}^1(\mathbb{C}))\,,$$

where $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a function of degree n and $a_0 = f(z_0)$. (Equation (1) is the special case when f is a polynomial, so $f^{-1}(\infty)$ is ∞ with multiplicity n.)

Finally, we mention that a real-analytic function on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$, built up out of the polylogarithms in the same way as $D(z)$ was constructed from the dilogarithm, has been defined by Ramakrishnan [6]. His function (slightly modified) is given by

$$D_m(z) = \Re\left(i^{m+1}\left[\sum_{k=1}^{m} \frac{(-\log|z|)^{m-k}}{(m-k)!}\mathrm{Li}_k(z) - \frac{(-\log|z|)^m}{2m!}\right]\right)$$

(so $D_1(z) = \log|z^{1/2} - z^{-1/2}|$, $D_2(z) = D(z)$) and satisfies

$$\begin{aligned} D_m\left(\frac{1}{z}\right) &= (-1)^{m-1} D_m(z)\,, \\ \frac{\partial}{\partial z} D_m(z) &= \frac{i}{2z}\left(D_{m-1}(z) + \frac{i}{2}\frac{(-i\log|z|)^{m-1}}{(m-1)!}\frac{1+z}{1-z}\right). \end{aligned}$$

However, it does not seem to have analogues of the properties (II) and (III): for example, it is apparently impossible to express $D_3(z)$ for arbitrary complex z in terms of only the function $D_3(e^{i\theta}) = \sum_{n=1}^{\infty}(\cos n\theta)/n^3$, and passing from Li_3 to D_3 removes many but not all of the numerous lower-order terms in the various functional equations of the trilogarithm, e.g.:

$$\begin{aligned} &D_3(x) + D_3(1-x) + D_3\left(\frac{x}{x-1}\right) \\ &\qquad = D_3(1) + \frac{1}{12}\log\bigl|x(1-x)\bigr| \log\left|\frac{x}{(1-x)^2}\right| \log\left|\frac{x^2}{1-x}\right|, \\ &D_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) + D_3(xy) + D_3\left(\frac{x}{y}\right) - 2D_3\left(\frac{x(1-y)}{y(1-x)}\right) - 2D_3\left(\frac{1-y}{1-x}\right) \\ &\quad -2D_3\left(\frac{x(1-y)}{x-1}\right) - 2D_3\left(\frac{y(1-x)}{y-1}\right) - 2D_3(x) - 2D_3(y) \\ &\qquad = 2D_3(1) - \frac{1}{4}\log|xy| \log\left|\frac{x}{y}\right| \log\left|\frac{x(1-y)^2}{y(1-x)^2}\right|. \end{aligned}$$

Nevertheless, these higher Bloch-Wigner functions do occur. In studying the so-called "Heegner points" on modular curves, B. Gross and I had to study for $n = 2, 3, \ldots$ "higher weight Green's functions" for $\mathfrak{H}/\Gamma$ ($\mathfrak{H}$ = complex upper half-plane, $\Gamma = SL_2(\mathbb{Z})$ or a congruence subgroup). These are functions $G_n(z_1, z_2) = G_n^{\mathfrak{H}/\Gamma}(z_1, z_2)$ defined on $\mathfrak{H}/\Gamma \times \mathfrak{H}/\Gamma$, real analytic in both variables except for a logarithmic singularity along the diagonal $z_1 = z_2$, and satisfying $\Delta_{z_1} G_n = \Delta_{z_2} G_n = n(n-1)G_n$, where $\Delta_z = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the hyperbolic Laplace operator with respect to $z = x + iy \in \mathfrak{H}$. They are obtained as

$$G_n^{\mathfrak{H}/\Gamma}(z_1,z_2) = \sum_{\gamma\in\Gamma} G_n^{\mathfrak{H}}(z_1,\gamma z_2)$$

where $G_n^{\mathfrak{H}}$ is defined analogously to $G_n^{\mathfrak{H}/\Gamma}$ but with $\mathfrak{H}/\Gamma$ replaced by $\mathfrak{H}$. The functions $G_n^{\mathfrak{H}}$ $(n = 2, 3, \dots)$ are elementary, e.g.,

$$G_2^{\mathfrak{H}}(z_1,z_2) = \left(1 + \frac{|z_1-z_2|^2}{2y_1y_2}\right)\log\frac{|z_1-z_2|^2}{|z_1-\bar z_2|^2} \; + \; 2\,.$$

In between $G_n^{\mathfrak{H}}$ and $G_n^{\mathfrak{H}/\Gamma}$ are the functions $G_n^{\mathfrak{H}/\mathbb{Z}} = \sum_{r\in\mathbb{Z}} G_n^{\mathfrak{H}}(z_1, z_2 + r)$. It turns out [10] that they are expressible in terms of the D_m $(m = 1, 3, \dots, 2n-1)$, e.g.,

$$\begin{aligned} G_2^{\mathfrak{H}/\mathbb{Z}}(z_1,z_2) &= \frac{1}{4\pi^2 y_1y_2}\left(D_3(e^{2\pi i(z_1-z_2)}) + D_3(e^{2\pi i(z_1-\bar z_2)})\right) \\ &\quad + \frac{y_1^2+y_2^2}{2y_1y_2}\left(D_1(e^{2\pi i(z_1-z_2)}) + D_1(e^{2\pi i(z_1-\bar z_2)})\right)\,. \end{aligned}$$

I do not know the reason for this connection.

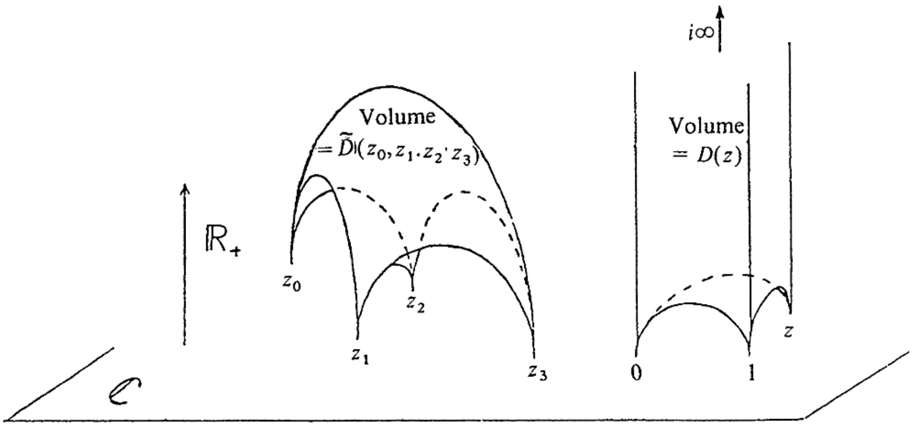
4 Volumes of hyperbolic 3-manifolds ...

The dilogarithm occurs in connection with measurement of volumes in euclidean, spherical, and hyperbolic geometry. We will be concerned with the last of these. Let $\mathfrak{H}_3$ be the Lobachevsky space (space of non-euclidean solid geometry). We will use the half-space model, in which $\mathfrak{H}_3$ is represented by $\mathbb{C}\times\mathbb{R}_+$ with the standard hyperbolic metric in which the geodesics are either vertical lines or semicircles in vertical planes with endpoints in $\mathbb{C}\times\{0\}$ and the geodesic planes are either vertical planes or else hemispheres with boundary in $\mathbb{C}\times\{0\}$. An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\partial\mathfrak{H}_3 = \mathbb{C}\cup\{\infty\} = \mathbb{P}^1(\mathbb{C})$. Let Δ be such a tetrahedron. Although the vertices are at infinity, the (hyperbolic) volume is finite. It is given by

$$\mathrm{Vol}(\Delta) = \widetilde{D}(z_0,z_1,z_2,z_3), \tag{7}$$

where $z_0,\dots,z_3 \in \mathbb{C}$ are the vertices of Δ and $\widetilde{D}$ is the function defined in (5). In the special case that three of the vertices of Δ are ∞, 0, and 1, equation (7) reduces to the formula (due essentially to Lobachevsky)

$$\mathrm{Vol}(\Delta) = D(z). \tag{8}$$



In fact, equations (7) and (8) are equivalent, since any 4-tuple of points $z_0, \ldots, z_3$ can be brought into the form $\{\infty, 0, 1, z\}$ by the action of some element of $SL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$, and the group $SL_2(\mathbb{C})$ acts on $\mathfrak{H}_3$ by isometries.

The (anti-)symmetry properties of $\widetilde{D}$ under permutations of the z_i are obvious from the geometric interpretation (7), since renumbering the vertices leaves Δ unchanged but may reverse its orientation. Formula (6) is also an immediate consequence of (7), since the five tetrahedra spanned by four at a time of $z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})$, counted positively or negatively as in (6), add up algebraically to the zero 3-cycle.

The reason that we are interested in hyperbolic tetrahedra is that these are the building blocks of hyperbolic 3-manifolds, which in turn (according to Thurston) are the key objects for understanding three-dimensional geometry and topology. A hyperbolic 3-manifold is a 3-dimensional riemannian manifold M which is locally modelled on (i.e., isometric to portions of) hyperbolic 3-space $\mathfrak{H}_3$; equivalently, it has constant negative curvature -1. We are interested in complete oriented hyperbolic 3-manifolds that have finite volume (they are then either compact or have finitely many "cusps" diffeomorphic to $S^1 \times S^1 \times \mathbb{R}_+$). Such a manifold can obviously be triangulated into small geodesic simplices which will be hyperbolic tetrahedra. Less obvious is that (possibly after removing from M a finite number of closed geodesics) there is always a triangulation into *ideal* tetrahedra (the part of such a tetrahedron going out towards a vertex at infinity will then either tend to a cusp of M or else spiral in around one of the deleted curves). Let these tetrahedra be numbered $\Delta_1, \ldots, \Delta_n$ and assume (after an isometry of $\mathfrak{H}_3$ if necessary) that the vertices of Δ_ν are at ∞, 0, 1 and z_ν. Then

$$\mathrm{Vol}(M) \;=\; \sum_{\nu=1}^{n} \mathrm{Vol}(\Delta_\nu) \;=\; \sum_{\nu=1}^{n} D(z_\nu)\,. \tag{9}$$

Of course, the numbers z_ν are not uniquely determined by Δ_ν since they depend on the order in which the vertices were sent to $\{\infty, 0, 1, z_\nu\}$, but the non-uniqueness consists (since everything is oriented) only in replacing z_ν by $1 - 1/z_\nu$ or $1/(1 - z_\nu)$ and hence does not affect the value of $D(z_\nu)$.

One of the objects of interest in the study of hyperbolic 3-manifolds is the "volume spectrum"

$$\mathbf{Vol} \;=\; \{\mathrm{Vol}(M) \mid M \text{ a hyperbolic 3-manifold}\} \subset \mathbb{R}_+\,.$$

From the work of Jørgensen and Thurston one knows that **Vol** is a countable and well-ordered subset of $\mathbb{R}_+$ (i.e., every subset has a smallest element), and its exact nature is of considerable interest both in topology and number theory. Equation (9) as it stands says nothing about this set since any real number can be written as a finite sum of values $D(z)$, $z \in \mathbb{C}$. However, the parameters z_ν of the tetrahedra triangulating a complete hyperbolic 3-manifold satisfy an extra relation, namely

$$\sum_{\nu=1}^{n} z_\nu \wedge (1 - z_\nu) \;=\; 0, \tag{10}$$

where the sum is taken in the abelian group $\Lambda^2\mathbb{C}^\times$ (the set of all formal linear combinations $x \wedge y$, $x, y \in \mathbb{C}^\times$, subject to the relations $x \wedge x = 0$ and $(x_1x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$). (This follows from assertions in [3] or from Corollary 2.4 of [5] applied to suitable x and y.) Now (9) *does* give information about **Vol** because the set of numbers $\sum_{\nu=1}^{n} D(z_\nu)$ with z_ν satisfying (10) is countable. This fact was proved by Bloch [1]. To make a more precise statement, we introduce the *Bloch group*. Consider the abelian group of formal sums $[z_1] + \cdots + [z_n]$ with $z_1, \ldots, z_n \in \mathbb{C}^\times \smallsetminus \{1\}$ satisfying (10). As one easily checks, it contains the elements

$$[x] + [\frac{1}{x}], \quad [x] + [1 - x], \quad [x] + [y] + [\frac{1 - x}{1 - xy}] + [1 - xy] + [\frac{1 - y}{1 - xy}] \tag{11}$$

for all x and y in $\mathbb{C}^\times - \{1\}$ with $xy \neq 1$, corresponding to the symmetry properties and five-term relation satisfied by $D(\,\cdot\,)$. The Bloch group is defined as

$$\mathcal{B}_\mathbb{C} = \{[z_1] + \cdots + [z_n] \text{ satisfying (10)}\}/(\text{subgroup generated by the elements (11)}) \tag{12}$$

(this is slightly different from the usual definition). The definition of the Bloch group in terms of the relations satisfied by $D(\,\cdot\,)$ makes it obvious that D extends to a linear map $D : \mathcal{B}_\mathbb{C} \to \mathbb{R}$ by $[z_1] + \cdots + [z_n] \mapsto D(z_1) + \cdots + D(z_n)$, and Bloch's result (related to Mostow rigidity) says that the set $D(\mathcal{B}_\mathbb{C})$ coincides with $D(\mathcal{B}_{\overline{\mathbb{Q}}})$, where $\mathcal{B}_{\overline{\mathbb{Q}}}$ is defined by (12) but with the z_ν lying in $\overline{\mathbb{Q}}^\times \smallsetminus \{1\}$. Thus $D(\mathcal{B}_\mathbb{C})$ is countable, and (9) and (10) imply that **Vol** is contained in this countable set. The structure of $\mathcal{B}_{\overline{\mathbb{Q}}}$, which is very subtle, will be discussed below.

We give an example of a non-trivial element of the Bloch group. For convenience, set $\alpha = \dfrac{1-\sqrt{-7}}{2}$, $\beta = \dfrac{-1-\sqrt{-7}}{2}$. Then

$$\begin{aligned} &2\cdot\Big(\frac{1+\sqrt{-7}}{2}\Big)\wedge\Big(\frac{1-\sqrt{-7}}{2}\Big) + \Big(\frac{-1+\sqrt{-7}}{4}\Big)\wedge\Big(\frac{5-\sqrt{-7}}{4}\Big) \\ &= 2\cdot(-\beta)\wedge\alpha + \Big(\frac{1}{\beta}\Big)\wedge\Big(\frac{\alpha^2}{\beta}\Big) = \beta^2\wedge\alpha - \beta\wedge\alpha^2 = 2\cdot\beta\wedge\alpha - 2\cdot\beta\wedge\alpha = 0\,, \end{aligned}$$

so

$$2\Big[\frac{1+\sqrt{-7}}{2}\Big] + \Big[\frac{-1+\sqrt{-7}}{4}\Big] \in \mathcal{B}_{\mathbb{C}}\,. \tag{13}$$

This example should make it clear why non-trivial elements of $\mathcal{B}_{\mathbb{C}}$ can only arise from algebraic numbers: the key relations $1+\beta=\alpha$ and $1-\beta^{-1}=\alpha^2/\beta$ in the calculation above forced α and β to be algebraic.

5 ... and values of Dedekind zeta functions

Let F be an algebraic number field, say of degree N over $\mathbb{Q}$. Among its most important invariants are the discriminant d, the numbers r_1 and r_2 of real and imaginary archimedean valuations, and the Dedekind zeta-function $\zeta_F(s)$. For the non-number-theorist we recall the (approximate) definitions. The field F can be represented as $\mathbb{Q}(\alpha)$ where α is a root of an irreducible monic polynomial $f\in\mathbb{Z}[x]$ of degree N. The discriminant of f is an integer d_f and d is given by $c^{-2}d_f$ for some natural number c with $c^2\,|\,d_f$. The polynomial f, which is irreducible over $\mathbb{Q}$, in general becomes reducible over $\mathbb{R}$, where it splits into r_1 linear and r_2 quadratic factors (thus $r_1\ge 0$, $r_2\ge 0$, $r_1+2r_2=N$). It also in general becomes reducible when it is reduced modulo a prime p, but if $p\nmid d_f$ then its irreducible factors modulo p are all distinct, say $r_{1,p}$ linear factors, $r_{2,p}$ quadratic ones, etc. (so $r_{1,p}+2r_{2,p}+\cdots=N$). Then $\zeta_F(s)$ is the Dirichlet series given by an Euler product $\prod_p Z_p(p^{-s})^{-1}$ where $Z_p(t)$ for $p\nmid d_f$ is the monic polynomial $(1-t)^{r_{1,p}}(1-t^2)^{r_{2,p}}\cdots$ of degree N and $Z_p(t)$ for $p\,|\,d_f$ is a certain monic polynomial of degree $\le N$. Thus (r_1,r_2) and $\zeta_F(s)$ encode the information about the behaviour of f (and hence F) over the real and p-adic numbers, respectively.

As an example, let F be an imaginary quadratic field $\mathbb{Q}(\sqrt{-a})$ with $a\ge 1$ squarefree. Here $N=2$, $d=-a$ or $-4a$, $r_1=0$, $r_2=1$. The Dedekind zeta function has the form $\sum_{n\ge1} r(n)n^{-s}$ where $r(n)$ counts representations of n by certain quadratic forms of discriminant d; it can also be represented as the product of the Riemann zeta function $\zeta(s)=\zeta_{\mathbb{Q}}(s)$ with an L-series $L(s)=\sum_{n\ge1}\big(\frac{d}{n}\big)\,n^{-s}$ where $\big(\frac{d}{n}\big)$ is a symbol taking the values ±1 or 0 and which is periodic of period $|d|$ in n. Thus for $a=7$

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \frac{1}{2} \sum_{(x,y)\neq(0,0)} \frac{1}{(x^2+xy+2y^2)^s}$$
$$= \Big(\sum_{n=1}^{\infty} n^{-s}\Big) \Big(\sum_{n=1}^{\infty} (\frac{-7}{n}) n^{-s}\Big)$$

where $(\frac{-7}{n})$ is $+1$ for $n \equiv 1,\,2,\,4 \pmod 7$, -1 for $n \equiv 3,\,5,\,6 \pmod 7$, and 0 for $n \equiv 0 \pmod 7$.

One of the questions of interest is the evaluation of the Dedekind zeta function at suitable integer arguments. For the Riemann zeta function we have the special values cited at the beginning of this paper. More generally, if F is totally real (i.e., $r_1 = N$, $r_2 = 0$), then a theorem of Siegel and Klingen implies that $\zeta_F(m)$ for $m = 2, 4, \dots$ equals $\pi^{mN}/\sqrt{|d|}$ times a rational number. If $r_2 > 0$, then no such simple result holds. However, in the case $F = \mathbb{Q}(\sqrt{-a})$, by using the representation $\zeta_F(s) = \zeta(s)L(s)$ and the formula $\zeta(2) = \pi^2/6$ and writing the periodic function $(\frac{d}{n})$ as a finite linear combination of terms $e^{2\pi i k n/|d|}$ we obtain

$$\zeta_F(2) = \frac{\pi^2}{6\sqrt{|d|}} \sum_{n=1}^{|d|-1} (\frac{d}{n})\, D(e^{2\pi i n/|d|}) \qquad (F \text{ imaginary quadratic}),$$

e.g.,

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3\sqrt{7}} \Big(D\big(e^{2\pi i/7}\big) + D\big(e^{4\pi i/7}\big) - D\big(e^{6\pi i/7}\big)\Big)\,.$$

Thus the values of $\zeta_F(2)$ for imaginary quadratic fields can be expressed in closed form in terms of values of the Bloch-Wigner function $D(z)$ at algebraic arguments z.

By using the ideas of the last section we can prove a much stronger statement. Let $\mathcal{O}$ denote the ring of integers of F (this is the $\mathbb{Z}$-lattice in $\mathbb{C}$ spanned by 1 and $\sqrt{-a}$ or $(1+\sqrt{-a})/2$, depending whether $d = -4a$ or $d = -a$). Then the group $\Gamma = SL_2(\mathcal{O})$ is a discrete subgroup of $SL_2(\mathbb{C})$ and therefore acts on hyperbolic space $\mathfrak{H}_3$ by isometries. A classical result of Humbert gives the volume of the quotient space $\mathfrak{H}_3/\Gamma$ as $|d|^{3/2}\,\zeta_F(2)/4\pi^2$. On the other hand, $\mathfrak{H}_3/\Gamma$ (or, more precisely, a certain covering of it of low degree) can be triangulated into ideal tetrahedra with vertices belonging to $\mathbb{P}^1(F) \subset \mathbb{P}^1(\mathbb{C})$, and this leads to a representation

$$\zeta_F(2) = \frac{\pi^2}{3|d|^{3/2}} \sum_{\nu} n_\nu\, D(z_\nu)$$

with n_ν in $\mathbb{Z}$ and z_ν *in F itself* rather than in the much larger field $\mathbb{Q}(e^{2\pi i n/|d|})$ ([8], Theorem 3). For instance, in our example $F = \mathbb{Q}(\sqrt{-7})$ we find

$$\zeta_F(2) = \frac{4\pi^2}{21\sqrt{7}} \Big(2D(\frac{1+\sqrt{-7}}{2}) + D(\frac{-1+\sqrt{-7}}{4})\Big)\,.$$

This equation together with the fact that $\zeta_F(2) = 1.89484144897\cdots \neq 0$ implies that the element (13) has infinite order in $\mathcal{B}_{\mathbb{C}}$.

In [8], it was pointed out that the same kind of argument works for all number fields, not just imaginary quadratic ones. If $r_2 = 1$ but $N > 2$ then one can again associate to F (in many different ways) a discrete subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that $\mathrm{Vol}(\mathfrak{H}_3/\Gamma)$ is a rational multiple of $|d|^{1/2}\zeta_F(2)/\pi^{2N-2}$. This manifold $\mathfrak{H}_3/\Gamma$ is now compact, so the decomposition into ideal tetrahedra is a little less obvious than in the case of imaginary quadratic F, but by decomposing into non-ideal tetrahedra (tetrahedra with vertices in the interior of $\mathfrak{H}_3$) and writing these as differences of ideal ones, it was shown that the volume is an integral linear combination of values of $D(z)$ with z of degree at most 4 over F. For F completely arbitrary there is still a similar statement, except that now one gets discrete groups Γ acting on $\mathfrak{H}_3^{r_2}$; the final result ([8], Theorem 1) is that $|d|^{1/2} \times \zeta_F(2)/\pi^{2(r_1+r_2)}$ is a rational linear combination of r_2-fold products $D(z^{(1)})\cdots D(z^{(r_2)})$ with each $z^{(i)}$ of degree ≤ 4 over F (more precisely, over the i^{th} complex embedding $F^{(i)}$ of F, i.e. over the subfield $\mathbb{Q}(\alpha^{(i)})$ of $\mathbb{C}$, where $\alpha^{(i)}$ is one of the two roots of the i^{th} quadratic factor of $f(x)$ over $\mathbb{R}$).

But in fact much more is true: the $z^{(i)}$ can be chosen in $F^{(i)}$ itself (rather than of degree 4 over this field), and the phrase "rational linear combination of r_2-fold products" can be replaced by "rational multiple of an $r_2 \times r_2$ determinant." We will not attempt to give more than a very sketchy account of why this is true, lumping together work of Wigner, Bloch, Dupont, Sah, Levine, Merkuriev, Suslin, ... for the purpose (references are [1], [3], and the survey paper [7]). This work connects the Bloch group defined in the last section with the algebraic K-theory of the underlying field; specifically, the group[2] $\mathcal{B}_F$ is equal, at least after tensoring it with $\mathbb{Q}$, to a certain quotient $K_3^{\mathrm{ind}}(F)$ of $K_3(F)$. The exact definition of $K_3^{\mathrm{ind}}(F)$ is not relevant here. What is relevant is that this group has been studied by Borel [2], who showed that it is isomorphic (modulo torsion) to $\mathbb{Z}^{r_2}$ and that there is a canonical homomorphism, the "regulator mapping," from it into $\mathbb{R}^{r_2}$ such that the co-volume of the image is a non-zero rational multiple of $|d|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2}$; moreover, it is known that under the identification of $K_3^{\mathrm{ind}}(F)$ with $\mathcal{B}_F$ this mapping corresponds to the composition $\mathcal{B}_F \to (\mathcal{B}_{\mathbb{C}})^{r_2} \xrightarrow{D} \mathbb{R}^{r_2}$, where the first arrow comes from using the r_2 embeddings $F \subset \mathbb{C}$ $(\alpha \mapsto \alpha^{(i)})$. Putting all this together gives the following beautiful picture. The group $\mathcal{B}_F/\{\text{torsion}\}$ is isomorphic to $\mathbb{Z}^{r_2}$. Let $\xi_1, \ldots, \xi_{r_2}$ be any r_2 linearly independent elements of it, and form the matrix with entries $D(\xi_j^{(i)})$, $(i, j = 1, \ldots, r_2)$. Then the determinant of this matrix is a non-zero rational multiple of $|d|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2}$. If instead of taking any r_2 linearly independent elements we choose the ξ_j to

[2] It should be mentioned that the definition of $\mathcal{B}_F$ which we gave for $F = \mathbb{C}$ or $\overline{\mathbb{Q}}$ must be modified slightly when F is a number field because $F^\times$ is no longer divisible; however, this is a minor point, affecting only the torsion in the Bloch group, and will be ignored here.

be a basis of $\mathcal{B}_F/\{\text{torsion}\}$, then this rational multiple (chosen positively) is an invariant of F, independent of the choice of ξ_j. This rational multiple is then conjecturally related to the quotient of the order of $K_3(F)_{\text{torsion}}$ by the order of the finite group $K_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ denotes the ring of integers of F (Lichtenbaum conjectures).

This all sounds very abstract, but is in fact not. There is a reasonably efficient algorithm to produce many elements in $\mathcal{B}_F$ for any number field F. If we do this, for instance, for F an imaginary quadratic field, and compute $D(\xi)$ for each element $\xi \in \mathcal{B}_F$ which we find, then after a while we are at least morally certain of having identified the lattice $D(\mathcal{B}_F) \subset \mathbb{R}$ exactly (after finding k elements at random, we have only about one chance in 2^k of having landed in the same non-trivial sublattice each time). By the results just quoted, this lattice is generated by a number of the form $\kappa|d|^{3/2}\zeta_F(2)/\pi^2$ with κ rational, and the conjecture referred to above says that κ should have the form $3/2T$ where T is the order of the finite group $K_2(\mathcal{O}_F)$, at least for $d < -4$ (in this case the order of $K_3(F)_{\text{torsion}}$ is always 24). Calculations done by H. Gangl in Bonn for several hundred imaginary quadratic fields support this; the κ he found all have the form $3/2T$ for some integer T and this integer agrees with the order of $K_2(\mathcal{O}_F)$ in the few cases where the latter is known. **E**
Here is a small excerpt from his tables:

$\|d\|$	7	8	11	15	19	20	23	24	31	35	39	40	$\cdots$	303	472	479	491	555	583
T	2	1	1	2	1	1	2	1	2	2	6	1	$\cdots$	22	5	14	13	28	34

(the omitted values contain only the primes 2 and 3; 3 occurs whenever $d \equiv 3 \bmod 9$ and there is also some regularity in the powers of 2 occurring). Thus one of the many virtues of the mysterious dilogarithm is that it gives, at least conjecturally, an effective way of calculating the orders of certain groups in algebraic K-theory!

To conclude, we mention that Borel's work connects not only $K_3^{\text{ind}}(F)$ and $\zeta_F(2)$ but more generally $K_{2m-1}^{\text{ind}}(F)$ and $\zeta_F(m)$ for any integer $m > 1$. No elementary description of the higher K-groups analogous to the description of K_3 in terms of $\mathcal{B}$ is known, but one can at least speculate that these groups and their regulator mappings may be related to the higher polylogarithms and that, more specifically, the value of $\zeta_F(m)$ is always a simple multiple of a determinant ($r_2 \times r_2$ or $(r_1 + r_2) \times (r_1 + r_2)$ depending whether m is even or odd) whose entries are linear combinations of values of the Bloch-Wigner-Ramakrishnan function $D_m(z)$ with arguments $z \in F$. As the simplest case, one can guess that for a *real* quadratic field F the value of $\zeta_F(3)/\zeta(3)$ $(= L(3)$, where $L(s)$ is the Dirichlet L-function of a real quadratic character of period d) is equal to $d^{5/2}$ times a simple rational linear combination of differences $D_3(x) - D_3(x')$ with $x \in F$, where x' denotes the conjugate of x over $\mathbb{Q}$. Here is one (numerical) example of this:

$$2^{-5}5^{5/2}\zeta_{\mathbb{Q}(\sqrt{5})}(3)/\zeta(3) \;=\; D_3\Big(\frac{1+\sqrt{5}}{2}\Big) - D_3\Big(\frac{1-\sqrt{5}}{2}\Big) - \frac{1}{3}\big[D_3(2+\sqrt{5}) - D_3(2-\sqrt{5})\big]$$

(both sides are equal approximately to 1.493317411778544726). I have found
F many other examples, but the general picture is not yet clear.

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Notes on Chapter I.

A The comment about "too little-known" is now no longer applicable, since the dilogarithm has become very popular in both mathematics and mathematical physics, due to its appearance in algebraic K-theory on the one hand and in conformal field theory on the other. Today one needs no apology for devoting a paper to this function.

B From the point of view of the modern theory, the arguments of the dilogarithm occurring in these eight formulas are easy to recognize: they are the totally real algebraic numbers x (off the cut) for which x and $1-x$, if non-zero, belong to the same rank 1 subgroup of $\overline{\mathbb{Q}}^{\times}$, or equivalently, for which $[x]$ is a torsion element of the Bloch group. The same values reappear in connection with Nahm's conjecture in the case of rank 1 (see §3 of Chapter II).

C Wojtkowiak proved the general theorem that any functional equation of the form $\sum_{j=1}^{J} c_j \mathrm{Li}_2(\phi_j(z)) = C$ with constants $c_1, \ldots, c_J$ and C and rational flunctions $\phi_1(z), \ldots, \phi_J(z)$ is a consequence of the five-term equation. (It is not known whether this is true with "rational" replaced by "algebraic".) The proof is given in §2 of Chapter II.

D As well as the Bloch-Wigner function treated in this section, there are several other modifications of the "naked" dilogarithm $\mathrm{Li}_2(z)$ which have nice properties. These are discussed in §1 of Chapter II.

E Now much more information about the actual order of $K_2(\mathcal{O}_F)$ is available, thanks to the work of Browkin, Gangl, Belabas and others. Cf. [7], [3] of the bibliography to Chapter II.

F The statement "the general picture is not yet clear" no longer holds, since after writing it I found hundreds of further numerical examples of identities between special values of polylogarithms and of Dedekind zeta functions and was able to formulate a fairly precise conjecture describing when such identities occur. A statement of this conjecture and a description of the known results can be found in §4 of Chapter II and in the literature cited there.

G This paper is still in preparation!

Chapter II. Further aspects of the dilogarithm

As explained in the preface to this paper, in this chapter we give a more detailed discussion of some of the topics treated in Chapter I and describe some of the developments of the intervening seventeen years. In Section 1 we discuss six further functions which are related to the classical dilogarithm Li_2 and the Bloch-Wigner function D: the Rogers dilogarithm, the "enhanced" dilogarithm, the double logarithm, the quantum dilogarithm, the p-adic dilogarith, and the finite dilogarithm. Section 2 treats the functional equations of the dilogarithm function in more detail than was done in Chapter I and describes a general method for producing such functional equations, as well as presenting Wojtkowiak's proof of the fact that all functional equations of the dilogarithm whose arguments are rational functions of one variable are consequences of the 5-term relation. In §3 we discuss Nahm's conjecture relating certain theta-series-like q-series with modular properties to torsion elements in the Bloch group (as explained in more detail in his paper in this volume) and show how to get some information about this conjecture by using the asymptotic properties of these q-series. The last section contains a brief description of the (mostly conjectural) theory of the relationships between special values of higher polylogarithm functions and special values of Dedekind zeta functions of fields, a topic which was brought up at the very end of Chapter I but which had not been fully developed at the time when that chapter was written.

1 Variants of the dilogarithm function

As explained in Chapter I, one of the disadvantages of the classical dilogarithm function $\mathrm{Li}_2(z)$ is that, although it has a holomorphic extension beyond the region of convergence $|z| < 1$ of the defining power series $\sum_{n=1}^{\infty} z^n/n^2$, this extension is many-valued. This complicates all aspects of the analysis of the dilogarithm function. One way to circumvent the difficulty, discussed in detail in §§3–4 of Chapter I, is to introduce the Bloch-Wigner dilogarithm function $D(z) = \Im\big[\mathrm{Li}_2(z) - \log|z|\,\mathrm{Li}_1(z)\big]$, which extends from the original region of definition $0 < |z| < 1$ to a continuous function $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ which is (real-) analytic on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$; this function has an appealing interpretation as the volume of an ideal hyperbolic tetrahedron and satisfies "clean" functional equations which do not involve products of ordinary logarithms.

It turns out, however, that there are other natural dilogarithm functions besides Li_2 and D which have interesting properties. In this section we shall discuss six of these: the Rogers dilogarithm L, which is similar to D but is defined on $\mathbb{P}^1(\mathbb{R})$ (where D vanishes); the "enhanced" dilogarithm $\widehat{D}$, which takes values in $\mathbb{C}/\pi^2\mathbb{Q}$ and is in some sense a combination of the Rogers and Bloch-Wigner dilogarithms, but is only defined on the Bloch group of

$\mathbb{C}$ rather than for individual complex arguments; the double logarithm $\mathrm{Li}_{1,1}$, the simplest of the multiple polylogarithms, which has two arguments but can be expressed in terms of ordinary dilogarithms; the quantum dilogarithm of Faddeev and Kashaev, which will play a role in the discussion of Nahm's conjecture in §3; and, very briefly, the p-adic and the modulo p analogues of the dilogarithm.

A. The Rogers dilogarithm. This function is defined in the interval $(0,1)$ by

$$L(x) \;=\; \mathrm{Li}_2(x) \,+\, \frac{1}{2}\log(x)\log(1-x) \qquad \text{if } 0<x<1$$

and then extended to the rest of $\mathbb{R}$ by setting $L(0)=0$, $L(1)=\pi^2/6$, and

$$L(x) \;=\; \begin{cases} 2\,L(1) \;-\; L(1/x) & \text{if } x>1, \\ -\,L(x/(x-1)) & \text{if } x<0. \end{cases}$$

The resulting function is then a monotone increasing continuous real-valued function on $\mathbb{R}$ and is (real-)analytic except at 0 and 1, where its derivative becomes infinite. At infinity it is not continuous, since one has

$$\lim_{x\to+\infty} L(x) \;=\; 2L(1) \;=\; \frac{\pi^2}{3}\,, \qquad \lim_{x\to-\infty} L(x) \;=\; -L(1) \;=\; -\frac{\pi^2}{6}\,,$$

but it *is* continuous if we consider it modulo $\pi^2/2$. Moreover, the new function $\overline{L}(x) := L(x) \pmod{\pi^2/2}$ from $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{R}/\frac{\pi^2}{2}\mathbb{Z}$, just like its complex analogue $D(z)$, satisfies "clean" functional equations with no logarithm terms, in particular the reflection properties

$$\overline{L}(x)+\overline{L}(1-x) \;=\; \overline{L}(1), \qquad \overline{L}(x)+\overline{L}(1/x) \;=\; -\overline{L}(1)$$

and the 5-term functional equation

$$\overline{L}(x)+\overline{L}(y)+\overline{L}\Big(\frac{1-x}{1-xy}\Big)+\overline{L}(1-xy)+\overline{L}\Big(\frac{1-y}{1-xy}\Big) \;=\; 0.$$

If we replace $\overline{L}$ by L in the left-hand sides of these three equations, then their right-hand sides must be replaced by piecewise continuous functions whose values depend on the positions of the arguments: $L(1)$ in the first equation, $2L(1)$ for $x>0$ or $-L(1)$ for $x<0$ in the second, and $-3L(1)$ for $x,\,y<0$, $xy>1$ and $+3L(1)$ otherwise in the third. The proofs of these and all other functional equations result from the elementary formula

$$L'(x) \;=\; -\,\frac{1}{2x}\,\log(1-x) \;-\; \frac{1}{2(1-x)}\,\log(x)\,.$$

The special values of the dilogarithm function listed in §1 of Chapter I become simpler when expressed in terms of the Rogers dilogarithm (e.g. one has simply $L(1/2) = \pi^2/12$, $L((3-\sqrt{5})/2) = \pi^2/15$ instead of the previous expressions involving $\log^2(2)$ and $\log^2((1+\sqrt{5})/2))$ and the same holds also for more complicated identities involving several values of the dilogarithm at algebraic arguments. Such identities, of which several will be discussed in §2, reflect the fact that the corresponding linear combination of arguments represents a torsion element in the Bloch group of $\overline{\mathbb{Q}}$. They play a role in quantum field theory, where the constants appearing on the right-hand sides of the identities, renormalized by dividing by $L(1)$, occur as central charges of certain rational conformal field theories.

B. The enhanced dilogarithm. The Bloch-Wigner dilogarithm is the imaginary part of $\mathrm{Li}_2(z)$ (corrected by a multiple of $\log|z|\,\mathrm{Li}_1(z)$ to make its analytic properties better) and hence vanishes on $\mathbb{P}^1(\mathbb{R})$, while the Rogers dilogarithm is the restriction of $\mathrm{Li}_2(z)$ (corrected by a multiple of $\log|z|\,\mathrm{Li}_1(z)$ to make its analytic properties better) to $\mathbb{P}^1(\mathbb{R})$ and takes its values most naturally in the circle group $\mathbb{R}/(\pi^2/2)\mathbb{Z}$. It is reasonable to ask whether there is then a function $\widehat{D}(z)$ with values in $\mathbb{C}/(\pi^2/2)\mathbb{Z}$ or at least $\mathbb{C}/\pi^2\mathbb{Q}$ whose imaginary part is $D(z)$ and whose restriction to $\mathbb{P}^1(\mathbb{R})$ is $L(z)$. In fact such a function does not exist if we demand that the argument belongs to $\mathbb{P}^1(\mathbb{C})$, but it *does* exist if we either pass to a suitable infinite covering of $\mathbb{P}^1(\mathbb{C})\smallsetminus\{0,1,\infty\}$ or else consider only combinations $\sum n_j\widehat{D}(z_j)$ where $\sum n_j[z_j]$ belongs to the Bloch group $\mathcal{B}_{\mathbb{C}}$ defined in §4 of Chapter I. This extended function, which following [47] we call the "enhanced dilogarithm," plays an important role in W. Nahm's article in this volume and is discussed in some detail there, so we will be relatively brief here.

We begin with the extension of Li_2 and D to covers of punctured projective space. Let $X = \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0,1,\infty\}$. Its fundamental group is the free group on two generators and its universal cover $\widetilde{X}$ is isomorphic to the complex upper half-plane $\mathfrak{H}$ (with covering map given by the classical Legendre modular function $\lambda : \mathfrak{H}/\Gamma(2) \to X$), and naturally Li_2 becomes a single-valued holomorphic function on this universal cover, but we do not have to go this far if we only want the values of $\mathrm{Li}_2(z)$ modulo $\pi^2\mathbb{Q}$. Instead, it suffices to take the universal abelian cover $\widehat{X}$: the abelianization of $\pi_1(X)$ is $\mathbb{Z}\oplus\mathbb{Z}$, with generators corresponding to the monodromy around 0 and 1 and hence to the multi-valuedness of $\log(z)$ and $\log(1-z)$, so $\widehat{X}$ is given by choosing branches of these two logarithms, i.e.,

$$\widehat{X} \;=\; \bigl\{(u,v)\in\mathbb{C}^2 \mid e^u+e^v=1\bigr\},$$

with covering map $\pi : \widehat{X} \to X$ given by $(u,v) \mapsto z = e^u = 1-e^v$. It is on this space that we will define $\widehat{D}$.

Actually, to get a $\mathbb{C}/4\pi^2\mathbb{Z}$-valued version of Li_2, we do not even need to pass to $\widehat{X}$, but only to the smaller abelian cover corresponding to choosing a branch of $\log(1-z)$ only, i.e., the cover $X' = \mathbb{C} - 2\pi i\mathbb{Z}$, with covering map

$X' \to X$ given by $v \mapsto 1 - e^v$. Indeed, from the formula $\mathrm{Li}_2'(z) = \frac{1}{z} \log \frac{1}{1-z}$ we see that the function

$$F(v) = \mathrm{Li}_2(1 - e^v) \qquad\qquad (v \in X')$$

has derivative given by $F'(v) = \dfrac{-v}{1 - e^{-v}}$, which is a one-valued meromorphic function on $\mathbb{C}$ with simple poles at $v \in 2\pi i\mathbb{Z}$ whose residues all belong to $2\pi i\mathbb{Z}$. It follows that F itself is a single-valued function on X' with values in $\mathbb{C}/(2\pi i)^2\mathbb{Z}$. This function satisfies $F(v + 2\pi is) = F(v) - 2\pi is \log(1 - e^v)$. We now define $\widehat{D}$ on $\widehat{X}$ by

$$\widehat{D}(\hat{z}) \;=\; F(v) \,+\, \frac{uv}{2} \qquad \text{for } \hat{z} = (u, v) \in \widehat{X}\,.$$

This is a holomorphic function from $\widehat{X}$ to $\mathbb{C}/(2\pi i)^2\mathbb{Z}$ whose behavior under the covering transformations of $\widehat{X} \to X$ is given by

$$\widehat{D}((u + 2\pi ir,\, v + 2\pi is)) \;=\; \widehat{D}((u, v)) \,+\, \pi i(rv - su) \,+\, 2\pi^2 rs \qquad (r,\, s \in \mathbb{Z})$$

and whose relation to the Bloch-Wigner function $D(z)$ is given by

$$\Im(\widehat{D}(\hat{z})) \;=\; D(z) \,+\, \frac{1}{2}\,\Im(\bar{u}v) \qquad (\hat{z} = (u, v), \quad \pi(\hat{z}) = z). \qquad (1)$$

(For more details, see [47], pp. 578–579, where the definition of $\widehat{D}$ is given somewhat differently.)

Now let $\xi = \sum n_j [z_j]$ be an element of the Bloch group $\mathcal{B}_{\mathbb{C}}$. This means, first of all, that the numbers $z_j \in X$ and $n_j \in \mathbb{Z}$ satisfy $\sum n_j (z_j) \wedge (1 - z_j) = 0$ in $\Lambda^2(\mathbb{C}^*)$ and, secondly, that ξ is considered only up to the addition of five-term relations $[x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1 - xy] + \left[\frac{1-y}{1-xy}\right]$ with $x,\, y \in X$, $xy \neq 1$. If we lift each z_j to some $\hat{z}_j = (u_j, v_j)$ in $\widehat{X}$, then the relation $\sum n_j (z_j) \wedge (1 - z_j) = 0$ in $\Lambda^2(\mathbb{C}^*)$ says that the sum $\sum n_j\, (u_j) \wedge (v_j) \in \Lambda^2(\mathbb{C})$ has the form $2\pi i \wedge A$ for some $A \in \mathbb{C}$ depending on the liftings $\hat{z}_j = (u_j, v_j)$. If we change these lifts to $\hat{z}_j' = (u_j', v_j')$ with $u_j' = u_j + 2\pi i r_j$, $v_j' = v_j + 2\pi i s_j$ with r_j and s_j in $\mathbb{Z}$, then A changes to $A' = A + \sum n_j (r_j v_j - s_j u_j + 2\pi i r_j s_j)$, so the formula given above for the behavior of $\widehat{D}$ under covering transformations of $\widehat{X}$ implies that the expression ("enhanced dilogarithm")

$$D^{\mathrm{enh}}(\xi) \;=\; \sum_j n_j\, \widehat{D}(\hat{z}_j) \;-\; \pi i A \;\in\; \mathbb{C}/\pi^2\mathbb{Q}$$

is independent of the choice of lifts $\hat{z}_j$. (This independence is true only modulo $\pi^2\mathbb{Q}$, and not in general modulo $2\pi^2\mathbb{Z}$, as asserted in [47], because the group generated by the z_j and $1 - z_j$ may contain torsion of arbitrary order.) Any 5-term relation is in the kernel of D^{enh}, so D^{enh} does indeed give a well-defined map from the Bloch group $\mathcal{B}_{\mathbb{C}}$ to $\mathbb{C}/\pi^2\mathbb{Q}$. (The treatment in [30] is more precise since Nahm works with an extension $\widehat{\mathcal{B}}_{\mathbb{C}}$ of $\mathcal{B}_{\mathbb{C}}$ on which the value of D^{enh} makes sense modulo $2\pi^2\mathbb{Z}$.) Furthermore, the relation $\sum n_j(z_j) \wedge (1-z_j) = 0$ implies that $\sum \Im(\bar{u}_j v_j)$ belongs to $\pi^2\mathbb{Q}$, so formula (1) implies that $\Im D^{\mathrm{enh}}(\xi) = D(\xi)$ for any $\xi \in \mathcal{B}_{\mathbb{C}}$.

In §4 of Chapter I we explained the relation of D to hyperbolic volumes. In particular, if M is any oriented compact hyperbolic 3-manifold (or complete hyperbolic 3-manifold with cusps), and if we triangulate M into oriented ideal hyperbolic tetrahedra Δ_j, then the expression $\xi_M = \sum [z_j]$, where z_j is the cross-ratio of the vertices of Δ_j, lies in $\mathcal{B}_{\mathbb{C}}$ and the interpretation of $D(z_j)$ as $\mathrm{Vol}(\Delta_j)$ implies that the imaginary part of $D^{\mathrm{enh}}(\xi_M)$ is the hyperbolic volume of M. The corresponding interpretation of the real part $\Re D^{\mathrm{enh}}(\xi_M) \in \mathbb{R}/\pi^2\mathbb{Q}$ is that it is equal (up to a normalizing factor) to the Chern-Simons invariant of M. For further discussion of this, see Neumann [33] as well as [32] and [42].

We refer the reader to §7 of [47] for an interesting number-theoretic application of the enhanced dilogarithm related to a conjectural formula which is both a generalization of the classical Kronecker limit formula and a refinement of (a special case of) the Gross-Stark conjecture on special values of Artin L-functions. Very roughly, if $\mathcal{A}$ is an ideal class of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, then the value at $s = 2$ of the partial zeta function $\zeta_{K,\mathcal{A}}(s) = \sum_{\mathfrak{a}\in\mathcal{A}} N(\mathfrak{a})^{-s}$ is known by results of Deninger [13] and Levin [25] to be of the form $\pi^2 d^{-3/2} D(\xi_{K,\mathcal{A}})$ for some $\xi_{K,\mathcal{A}} \in \mathcal{B}_{\overline{\mathbb{Q}}}$, and in [47] an "enhanced" partial zeta value $\zeta^{\mathrm{enh}}_{K,\mathcal{A}}(2) \in \mathbb{C}/\pi^2 d^{1/2}\mathbb{Q}$ is defined for which the formula $\zeta^{\mathrm{enh}}_{K,\mathcal{A}}(2) = \pi^2 d^{-3/2} D^{\mathrm{enh}}(\xi_{K,\mathcal{A}})$ can be conjectured and tested numerically in many examples.

C. The double logarithm. In recent years there has been a resurgence of interest in the "multiple zeta values"

$$\zeta(k_1, \ldots, k_m) \;=\; \sum_{\substack{n_1,\ldots,n_m\in\mathbb{Z} \\ 0<n_1<\cdots<n_m}} \frac{1}{n_1^{k_1}\,\cdots\, n_m^{k_m}}$$

originally defined (for $m = 2$) by Euler; these numbers turn up in particular in connection with quantum invariants of knots and with the calculation of certain Feynman diagram integrals. The multiple zeta values are simply the specializations to $x_1 = \cdots = x_m = 1$ of the multiple polylogarithm functions

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1, \ldots, x_m) \;=\; \sum_{\substack{n_1,\ldots,n_m\in\mathbb{Z} \\ 0<n_1<\cdots<n_m}} \frac{x_1^{n_1}\,\cdots\, x_m^{n_m}}{n_1^{k_1}\,\cdots\, n_m^{k_m}}$$

which for $m = 1$ is the ordinary polylogarithm function.

In the hierarchy of multiple polylogarithm functions, the key invariant is the total weight $k_1 + \cdots + k_m$. The only multiple polylogarithm of weight 1 is the ordinary logarithm $\mathrm{Li}_1(x) = -\log(1-x)$, but there are two multiple polylogarithms of weight 2, the dilogarithm function $\mathrm{Li}_2(x)$ and the *double logarithm function* [21]

$$\mathrm{Li}_{1,1}(x,y) \;=\; \sum_{0<m<n} \frac{x^m y^n}{m\,n} \qquad (x,\, y \in \mathbb{C},\;\; |y|<1,\;\; |xy|<1).$$

The remarkable fact here is that the function $\mathrm{Li}_{1,1}$, which has two arguments and hence is a priori a more complicated type of object than the one-argument function Li_2, can in fact be expressed in terms only of the latter:

Proposition 1. *For* $x,\, y \in \mathbb{C}$ *with* $|xy|<1,\ |y|<1$ *we have*

$$\mathrm{Li}_{1,1}(x,y) \;=\; \mathrm{Li}_2\Big(\frac{xy-y}{1-y}\Big) \;-\; \mathrm{Li}_2\Big(\frac{-y}{1-y}\Big) \;-\; \mathrm{Li}_2(xy)\,. \tag{2}$$

Before proving this identity, we mention some equivalent formulas and consequences. First of all, the double logarithm function satisfies the identity—the simplest case of the “shuffle relations” satisfied by all multiple zeta values and multiple polylogarithms—

$$\mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(xy) \;=\; \mathrm{Li}_1(x)\,\mathrm{Li}_1(y)\,, \tag{3}$$

which is an immediate consequence of the fact that any pair of positive integers (m,n) must satisfy exactly one of the three conditions $0<m<n$, $0<n<m$, or $0<m=n$. Combining this with (2) and interchanging the roles of x and y, we can rewrite (2) in the equivalent form

$$\mathrm{Li}_{1,1}(x,y) \;=\; \mathrm{Li}_1(x)\,\mathrm{Li}_1(y) \;+\; \mathrm{Li}_2\Big(\frac{-x}{1-x}\Big) \;-\; \mathrm{Li}_2\Big(\frac{xy-x}{1-x}\Big)\,, \tag{4}$$

which is slightly less pretty than (2) in that it involves products of logarithms as well as dilogarithms, but has the advantage of containing only two rather than three dilogarithms. And if we use (4) to express both $\mathrm{Li}_{1,1}(x,y)$ and $\mathrm{Li}_{1,1}(y,x)$ in (3) in terms of dilogarithms, we obtain what is perhaps the most natural proof of the five-term relation.

We now give the proof of Proposition 1 (in the form (4) or (2)). In fact, just for fun we give *three* proofs.

(i) We have

$$\begin{aligned}\frac{\partial}{\partial y}\,\mathrm{Li}_{1,1}(x,y) \;&=\; \sum_{0<m<n} \frac{x^m}{m}\, y^{n-1} \;=\; \sum_{m=1}^{\infty} \frac{x^m}{m}\,\frac{y^m}{1-y}\\ &=\; \frac{1}{1-y}\,\log\frac{1}{1-xy}\,.\end{aligned}$$

The derivative with respect to y of the right-hand side of (4) (or of (2)) has the same value and both sides of (4) (or of (2)) vanish at $y=0$.

(ii) We have

$$\begin{aligned}\frac{\partial}{\partial x}\,\mathrm{Li}_{1,1}(x,y) \;&=\; \sum_{0<m<n} x^{m-1}\,\frac{y^n}{n} \;=\; \sum_{n=1}^{\infty} \frac{1-x^{n-1}}{1-x}\,\frac{y^n}{n}\\ &=\; \frac{1}{1-x}\,\log\frac{1}{1-y} \;-\; \frac{1}{x(1-x)}\,\log\frac{1}{1-xy}\,.\end{aligned}$$

The derivative with respect to x of the right-hand side of (4) (or of (2)) has the same value and both sides of (4) (or of (2)) vanish at $x = 0$.

(iii) Write the right-hand side of (4) as

$$\sum_{m,\,n\geq 1} \frac{x^m}{m}\,\frac{y^n}{n} \quad + \quad \sum_{k=1}^{\infty}\Bigl(\frac{-x}{1-x}\Bigr)^k\,\frac{1-(1-y)^k}{k^2}$$

$$= \sum_{m,\,n\geq 1} x^m y^n \left[\frac{1}{mn} - \sum_{n\leq k\leq m} \frac{(-1)^{k-1}}{k^2}\binom{m-1}{k-1}\binom{k}{n}\right]$$

and then verify as an elementary combinatorial exercise that the expression in square brackets, which clearly equals $\dfrac{1}{mn}$ if $m < n$, vanishes if $m \geq n$.

It seems very surprising that the beautiful identities (2) and (4) are not better known.

D. The quantum dilogarithm The "quantum dilogarithm," studied by Faddeev-Kashaev [15], Kirillov [22] and other authors, is the function of two variables defined by the series

$$\mathrm{Li}_2(x;q) \;=\; \sum_{n=1}^{\infty} \frac{x^n}{n\,(1-q^n)}\,. \tag{5}$$

It is a q-deformation of the ordinary dilogarithm in the sense that

$$\lim_{\varepsilon\to 0}\bigl(\varepsilon\,\mathrm{Li}_2(x;e^{-\varepsilon})\bigr) \;=\; \mathrm{Li}_2(x) \qquad\qquad (|x|<1)\,; \tag{6}$$

indeed, using the expansion $\frac{1}{1-e^{-t}} \;=\; \frac{1}{t}+\frac{1}{2}+\frac{t}{12}-\frac{t^3}{720}+\cdots$ we obtain a complete asymptotic expansion

$$\mathrm{Li}_2(x;e^{-\varepsilon}) \;=\; \mathrm{Li}_2(x)\,\varepsilon^{-1} + \frac{1}{2}\log\Bigl(\frac{1}{1-x}\Bigr) + \frac{x}{1-x}\,\frac{\varepsilon}{12} - \frac{x+x^2}{(1-x)^3}\,\frac{\varepsilon^3}{720} + \cdots$$

as $\varepsilon \to 0$ with x fixed, $|x| < 1$.

The function (5) belongs to the world of "q-series." These series, about which there is a very extensive literature—with the letter "q" having been the traditional choice long before it was realized that there was any connection with the "q" of "quantum"—are functions of a formal (or small complex) variable q which are given by convergent infinite series whose terms are rational functions of q with rational coefficients. For instance, the q-hypergeometric functions, a very important subclass which includes some classical modular forms and related functions like Ramanujan's "mock theta functions" (which have occurred in connection with quantum invariants of 3-manifolds [24]) are given by series whose nth term has the form $\prod_{i=1}^n R(q,q^i)$ for some rational function $R(x,y)$ of two variables. The classical aspects of q-series are those having to do with the behavior as q tends to 0 and typically are concerned with proving identities $F(q) = G(q)$ between two given q-series considered as elements of $\mathbb{Q}[[q]]$. This is usually done either by purely combinatorial arguments (such as interpreting the coefficients of q^n in $F(q)$ and $G(q)$ as the numbers of partitions of n of two different types and then giving a bijection between these) or else via algebraic tricks such as introducing an extra parameter x and showing that both $F(x,q)$ and $G(x,q)$ satisfy the same functional equations under $x \mapsto qx$ (as in the proof of Proposition 2 below). The quantum aspects, on the other hand, are the ones that emerge when one studies the asymptotic behavior of the q-series as q tends to 1 (or more generally to a root of unity) rather than to 0, an example being the asymptotic expansion of $\mathrm{Li}_2(x; e^{-\varepsilon})$ as $\varepsilon \to 0$ given above.

In the study of q-hypergeometric functions and other q-series, an important role is played by the q-analogues

$$(q)_n \;:=\; \prod_{m=1}^{n} (1-q^m)\,, \qquad (x;\,q)_n \;:=\; \prod_{i=0}^{n-1} (1-q^i x)$$

of the classical factorial function and Pochhammer symbol, respectively. One can also allow $n = \infty$ and set

$$(q)_\infty \;:=\; \prod_{m=1}^{\infty} (1\,-\,q^m)\,, \qquad (x;\,q)_\infty \;:=\; \prod_{i=0}^{\infty}(1\,-\,q^i\,x),$$

the q-analogues of the classical gamma function; the function $(q)_\infty$ is up to a factor $q^{1/24}$ the modular form $\eta(\tau)$ (Dedekind eta-function), where $q = e^{2\pi i\tau}$. Observe that the finite products can be expressed in terms of the infinite ones by $(q)_n = (q)_\infty/(q^{n+1};q)_\infty$ and $(x;q)_n = (x;q)_\infty/(q^n x;q)_\infty$. Following a much-practised abuse of notation we will consider q as given and omit it from the notations, writing simply $(x)_n$ and $(x)_\infty$ instead of $(x;q)_n$ and $(x;q)_\infty$. This causes no confusion with the notations $(q)_n$ and $(q)_\infty$ since $(q)_n = (q;q)_n$ and $(q)_\infty = (q;q)_\infty$, but is an abuse of notation because, for instance, $(q)_2$ means $(1-q)(1-q^2)$ but $(x)_2$ means $(1-x)(1-qx)$ rather than $(1-x)(1-x^2)$.

The first surprise is now that the quantum dilogarithm $\mathrm{Li}_2(x;q)$ is essentially equivalent to the q-gamma function $(x)_\infty$! This is the third part of the following simple (and well-known) result which gives the expansions of the functions $(x)_\infty$, $1/(x)_\infty$ and $\log(x)_\infty$ as power series in x. All three formulas will play a role in §3 in connection with Nahm's conjecture.

Proposition 2. *For $x,\, q \in \mathbb{C}$ with $|x| < 1,\ |q| < 1$ we have the power series expansions*

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\binom{n}{2}}}{(q)_n}\, x^n\,, \tag{7}$$

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n}\,, \tag{8}$$

$$-\ \log(x;q)_\infty = \mathrm{Li}_2(x;q)\,. \tag{9}$$

Proof. All three of these identities can be proved in essentially the same way. To emphasize this, we present the three proofs simultaneously. Since $(x)_\infty$ is obviously a power series in x with constant term 1, we can write

$$(x)_\infty = \sum_{n=0}^{\infty} a_n\, x^n\,, \qquad \frac{1}{(x)_\infty} = \sum_{n=0}^{\infty} b_n\, x^n\,, \qquad -\log(x)_\infty = \sum_{n=1}^{\infty} c_n\, x^n$$

for some coefficients a_n, b_n and c_n depending on q, $a_0 = b_0 = 1$. Combining each of these expansions with the functional equation $(x)_\infty = (1-x)\,(qx)_\infty$ and comparing the coefficients of x^n on both sides, we find

$$(1-q^n)\,a_n = -q^{n-1}\,a_{n-1}\,, \quad (1-q^n)\,b_n = b_{n-1}\,, \quad (1-q^n)\,c_n = \frac{1}{n}\,,$$

from which the desired formulas

$$a_n = \frac{(-1)^n\, q^{\binom{n}{2}}}{(q)_n}\,, \qquad b_n = \frac{1}{(q)_n} \qquad c_n = \frac{1}{n}\cdot\frac{1}{1-q^n} \tag{10}$$

follow immediately or by induction. Note that the third identity of the proposition can also be proved directly, without using the functional equation of $(x)_\infty$, by the calculation

$$-\log(x)_\infty \;=\; \sum_{i=0}^{\infty} -\log(1-q^i x) \;=\; \sum_{i=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\,q^{in}\,x^n \;=\; \sum_{n=1}^{\infty}\frac{x^n}{n\,(1-q^n)}\,,$$

i.e., $\mathrm{Li}_2(x;q) = \sum_{i=0}^{\infty}\mathrm{Li}_1(q^i x)$.

The second surprise is the discovery by Faddeev and Kashaev [15] that the q-dilogarithm satisfies a non-commutative 5-term functional equation which degenerates in the limit $q \to 1$ to the classical 5-term functional equation of the classical dilogarithm. We content ourselves with stating and proving the first statement only, referring the reader for the second statement (which involves the use of the Baker-Campbell-Hausdorff formula) to the original paper, or to the more recent survey paper by Zudilin [48].

Proposition 3 ([38], [15], [22]). *Let u and v be non-commuting variables satisfying the commutation relation*

$$u\,v \;=\; q\,v\,u\,. \tag{11}$$

Then

$$(v)_\infty\,(u)_\infty \;=\; (u)_\infty\,(-vu)_\infty\,(v)_\infty\,. \tag{12}$$

Proof. Expanding each factor $(x)_\infty$ in (12) by equation (7) and observing that $v^n u^m = q^{-mn} u^m v^n$ and $(vu)^s = q^{-\binom{s+1}{2}} u^s v^s$, we find that (12) is equivalent to the generating series identity

$$\sum_{m,\,n\geq 0} q^{-mn}\, a_m\, a_n\, u^m\, v^n \;=\; \sum_{r,\,s,\,t\geq 0} (-1)^s q^{-\binom{s+1}{2}}\, a_r\, a_s\, a_t\, u^{r+s}\, v^{s+t}$$

with a_n as in (10) or, comparing coefficients of like monomials, to the combinatorial identity

$$\sum_{\substack{r,\,s,\,t\geq 0\\ r+s=m,\,s+t=n}} \frac{q^{rt}}{(q)_r\,(q)_s\,(q)_t} \;=\; \frac{1}{(q)_m\,(q)_n} \qquad (m,\,n\geq 0)\,. \tag{13}$$

(Amusingly, if we write (12) in the equivalent form $(v)_\infty^{-1}(-vu)_\infty^{-1}(u)_\infty^{-1} = (u)_\infty^{-1}(v)_\infty^{-1}$ and expand each term $(x)_\infty^{-1}$ using (8) instead of (7), then the combinatorial identity to be proved turns out to exactly the same formula (13), but with q replaced by q^{-1}.) Identity (13) can be proved either using generating functions (now commutative!) by multiplying both sides by $x^m y^n$, summing over $m,\,n\geq 0$, and applying (8) and the easy identity $\sum_{r=0}^\infty \frac{(y)_r}{(q)_r}x^r = \frac{(xy)_\infty}{(x)_\infty}$, or else by using the standard recursion property $\left[{m+1 \atop s}\right] = q^s\left[{m \atop s}\right] + \left[{m \atop s-1}\right]$ of the q-binomial coefficient $\left[{m \atop s}\right] = \frac{(q)_m}{(q)_s(q)_{m-s}}$ to show that the numbers $C_{m,n} := \sum_s \left[{m \atop s}\right] q^{(m-s)(n-s)}\frac{(q)_n}{(q)_{n-s}}$ satisfy $C_{m+1,n} = q^n C_{m,n} + (1-q^n)C_{m,n-1}$ and hence by induction $C_{m,n}=1$ for all $m,\,n\geq 0$.

E. The p-adic dilogarithm and the dianalog. The next dilogarithm variant we mention is the p-adic dilogarithm, studied by R. Coleman and other authors. We fix a prime number p and define

$$\mathrm{Li}_2^{(p)}(x) \;=\; \sum_{n>0,\;p\nmid n} \frac{x^n}{n^2}\,. \tag{14}$$

This function can be written as $\mathrm{Li}_2(x) - p^{-2}\mathrm{Li}_2(x^p)$, so in the complex domain it is simply a combination of ordinary dilogarithms and of no independent interest, but because we have omitted the terms in (14) with p's in the denominator, the power series converges p-adically for all p-adic numbers x with valuation $|x|_p < 1$. The function $\mathrm{Li}_2^{(p)}(x)$, and the corresponding higher p-adic polylogarithms $\mathrm{Li}_m^{(p)}(x)$, have good properties of analytic continuation and are related to p-adic L-functions [11]. Furthermore, the p-adic dilogarithm and p-adic polylogarithms have modified versions analogous to the Bloch-Wigner dilogarithm and Bloch-Wigner-Ramakrishnan polylogarithms which satisfy the same "clean" functional equations as their complex counterparts [41].

Finally, instead of working over the p-adic numbers we can work over the finite field $\mathbb{F}_p$ and consider the finite sum

$$\pounds_2(x) \;=\; \pounds_2^{(p)}(x) \;=\; \sum_{0<n<p} \frac{x^n}{n^2}\,, \tag{15}$$

a polynomial with coefficients in $\mathbb{F}_p$. The corresponding analogue $\pounds_1(x) = \sum_{n=1}^{p-1} x^n/n$ of the 1-logarithm was first proposed (under the name "The $1\frac{1}{2}$-logarithm") by M. Kontsevich in a note in the informal Festschrift prepared on the occasion of F. Hirzebruch's retirement as director of the Max Planck Institute for Mathematics in Bonn [23]. Kontsevich showed that this function, as a function from $\mathbb{F}_p$ to $\mathbb{F}_p$, satisfies the 4-term functional equation

$$\pounds_1(x+y) \;=\; \pounds_1(y) \,+\, (1-y)\pounds_1\Bigl(\frac{x}{1-y}\Bigr) \,+\, y\,\pounds_1\Bigl(-\frac{x}{y}\Bigr) \tag{16}$$

(the mod p analogue of the "fundamental equation of information theory" satisfied by the classical entropy function $-x\log x - (1-x)\log(1-x)$), midway between the 3-term functional equation of $\log(xy) - \log(x) - \log(y) = 0$ of the classical logarithm function and the 5-term functional equation of the classical dilogarithm, and also that $\pounds_1$ is characterized uniquely by this functional equation together with the two one-variable functional equations $\pounds_1(x) = \pounds_1(1-x)$ and $x\pounds_1(1/x) = -\pounds_1(x)$. Kontsevich's question whether the higher polylogarithm analogues $\pounds_m(x) = \sum_{n=1}^{p-1} x^n/n^m$ satisfied similar equations was taken up and answered positively by Ph. Elbaz-Vincent and H. Gangl [14]. They called these functions "polyanalogs", an amalgam of the words "analogue," "polylog," and "pollyanna" (an American term suggesting exaggerated or unwarranted optimism). Presumably the correct term for the case $m = 2$ would then be "dianalog", which has a pleasing British flavo(u)r.

The main property of the dianalog and its higher-order generalizations, generalizing the identities found by Kontsevich for $m = 1$, is that if we consider them as functions from $\mathbb{F}_p$ to $\mathbb{F}_p$ (rather than as polynomials with coefficients in $\mathbb{F}_p$) they satisfy functional equations which are reminiscent of, but of a somewhat different type than, the functional equations of the classical polylogarithms. In particular, Elbaz-Vincent and Gangl proved that the dianalog function satisfies several functional equations: the easy symmetry property $\pounds_2(x) = x\pounds_2(1/x)$, the somewhat less obvious three-term relation $\pounds_2(1-x) - \pounds_2(x) + x\pounds_2(1-1/x) = 0$, and the "Kummer-Spence analogue"

$$\begin{aligned}
&\pounds_2(xy) \,+\, y\,\pounds_2\Bigl(\frac{x}{y}\Bigr) \,-\, (1+y)\,\pounds_2(x) \,-\, (1+x)\,\pounds_2(y) \\
&\quad -(1-y)\left[\pounds_2\left(\frac{y(x-1)}{1-y}\right) - \pounds_2\left(\frac{1-x}{1-y}\right)\right] \\
&\quad -x(1-y)\left[\pounds_2\left(\frac{y(1-x)}{x(1-y)}\right) - \pounds_2\left(\frac{x-1}{x(1-y)}\right)\right] \;=\; 0\,,
\end{aligned}$$

each of which is the analogue of a classical functional equation of the trilogarithm. There is also a 22-term functional equation based on Cathelineau's differential version of the trilogarithm. Moreover, in each of the functional equations for $\pounds_1$ and $\pounds_2$, if one replaces the polynomial factors preceding the polyanalogs (for instance, the factors $1-y$ and y preceding $\pounds_1(x/(1-y))$ and $\pounds_1(-x/y)$ in (16)) by their pth powers, then the functional equation becomes true as an identity between polynomials in $\mathbb{F}_p[x,y]$ and not merely as an equality between functions from $\mathbb{F}_p \times \mathbb{F}_p$ to $\mathbb{F}_p$. Finally, by passing via the p-adics and using a recent result of Besser [4] expressing the polyanalogs (now again considered as functions rather than polynomials) as the mod p reductions of certain derivatives of modified p-adic polylogarithm functions, the authors show how functional equations of the mth classical complex polylogarithm induce by a process of differentiation corresponding functional equations of the $(m-1)$-st polyanalog. The whole story is intimately related to Cathelineau's theory of infinitesimal polylogarithms (infinitesimal or Lie version of the Bloch group), which is yet another and even more subtle manifestation of the world of polylogarithms.

We do not give any further details, referring the reader to the original papers [8] and [14].

2 Dilogarithm identities

In Chapter I of this paper we discussed both functional equations of the dilogarithm function and numerical identities involving the values of dilogarithms at algebraic arguments. Here we discuss both topics in more detail. In subsection **A** we give the algebraic characterization of arbitrary functional equations of the dilogarithm and prove Wojtkowiak's theorem that all functional equations whose arguments are rational functions of one variable are consequences of the five-term functional equation. In subsections **B** and **C** we discuss specific examples of identities of the form $\sum D(\alpha_i) = 0$ or $\sum L(\alpha_i) \in \mathbb{Q}\,\pi^2$, where the α_i are complex or real algebraic numbers, respectively, and describe a general method for producing such examples.

A key role in all these considerations is played by the five-term relation. We recall its statement from Chapter I. A sequence of $\{x_i\}_{i\in\mathbb{Z}}$ of real or complex numbers such that $1 - x_i = x_{i-1}x_{i+1}$ for all i automatically satisfies $x_{i+5} = x_i$. By a *5-cycle* we mean any (cyclically ordered) 5-tuple of numbers obtained in this way. Equivalently, a 5-cycle can be defined as the set of cross-ratios of the five (appropriately ordered) sub-4-tuples of a set of 5 distinct points in the projective line, or—in a different ordering—simply as any set of the form $\bigl(x,\, y,\, \dfrac{1-x}{1-xy},\, 1-xy,\, \dfrac{1-y}{1-xy}\bigr)$ with $x,\, y \notin \{0,1,\infty\}$, $xy \neq 1$. The five-term equation in its various guises says that the sum of the values of the Bloch-Wigner dilogarithm D at arguments belonging to a 5-cycle of complex numbers, or of the Rogers dilogarithm $L \pmod{\pi^2/2}$ at the numbers of a real 5-cycle, vanishes. This fact and related algebraic properties of 5-cycles turn up in a surprising number of contexts in quite different parts of mathematics: in the theory of webs (Bol's counterexample and correction to a theorem of Blaschke, later generalized by Chern and Griffiths [9]), in the study of the torsion in the group of birational transformations (Cremona transformations) of $\mathbb{P}^2(\mathbb{C})$ [1], in the study of the 1-dimensional Schrödinger equation for the potential $|x|^3$ [39], and in the study of the symmetry properties of the Apéry-Beukers-type integrals leading to the best currently known irrationality measures for π^2 [34]. However, we will not discuss these connections here, restricting ourselves only to the aspects directly related to the dilogarithm.

A. Functional equations of the dilogarithm. By a "functional equation of the dilogarithm" we mean any collection of integers n_i and rational or algebraic functions $x_i(t)$ of one or several variables such that $\sum n_i \mathrm{Li}_2(x_i(t))$ is a finite combination of products of two logarithms, or such that $\sum n_i D(x_i(t))$ (resp. $\sum n_i L(x_i(t))$ if all the $x_i(t)$ are real) is constant (resp. locally constant). A number of examples were given in Section 2 of Chapter I, with only the statement "All of the functional equations of Li_2 are easily proved by differentiation" by way of proof. That is a true, but somewhat ad hoc, statement, since it does not give an algebraic way to recognize or characterize functional equations of the dilogarithm. It is, however, easy to give such a criterion ([45], Prop. 1 of §7): it is necessary and sufficient that $\sum n_i(x_i(t)) \wedge (1 - x_i(t))$ be independent of t, i.e., that the element $\xi = \sum n_i[x_i(t)]$ of the group ring of the function field in which the x_i lie be in the kernel of the boundary map $\partial : [x] \mapsto (x) \wedge (1 - x)$ used to define the Bloch group (cf. §4 of Chapter I or §4 below). For convenience, we ignore 2-torsion.

Let us check this criterion for each of the functional equations given in §2 of Chapter I. For the one-variable functional equations corresponding to $\xi = [x] + [1 - x]$ or $\xi = [x] + [1/x]$ the statement $\partial(\xi) = 0$ is trivial. For the five-term equation we can either verify directly that the 5-term expression

$$V(x, y) \;=\; [x] + [y] + \Big[\frac{1 - x}{1 - xy}\Big] + [1 - xy] + \Big[\frac{1 - y}{1 - xy}\Big] \qquad (17)$$

is annihilated by ∂ or else use the more symmetric description of the five-term relation as $\xi = \sum_{i \pmod 5} [x_i]$ with $1 - x_i = x_{i-1}x_{i+1}$ and then calculate

$$\partial(\xi) \;=\; \sum_i (x_i) \wedge (x_{i-1}x_{i+1}) \;=\; \sum_i \big((x_i) \wedge (x_{i-1}) \;-\; (x_{i+1}) \wedge (x_i)\big) \;=\; 0\,.$$

Similarly, the six-term relation of Kummer and Newman corresponds to

$$\xi \;=\; 2\sum_i [x_i] \;-\; \sum_i [-x_{i-1}x_{i+1}/x_i]\,,$$

where $\{x_i\}_{i \in \mathbb{Z}/3\mathbb{Z}}$ is a cyclically numbered triple of numbers with $\sum x_i^{-1} = 1$, and here we find

$$\begin{aligned}\partial(\xi) \;&=\; \sum_i \big(2\,(x_i) \wedge (1 - x_i) \;-\; (-x_{i-1}x_{i+1}/x_i) \wedge ((1 - x_{i-1})(1 - x_{i+1}))\big) \\ &=\; \sum_j \big(2(x_j) - (-x_j x_{j-1}/x_{j+1}) - (-x_{j+1}x_j/x_{j-1})\big) \wedge (1 - x_j) \;=\; 0\,.\end{aligned}$$

Finally, the "strange many-variable equation" given in eq. (1) of Chapter I corresponds to the expression

$$\xi \;=\; [z] \;-\; \sum_{i=1}^{n}\sum_{j=1}^{n}[x_i/a_j]\,,$$

where $x_1,\dots,x_n$ and $a_1,\dots,a_n$ are the roots (counted with multiplicity) of $f(x)=z$ and $f(a)=1$, respectively, for some polynomial f of degree n without constant term. Then from the two identities $C\prod_i(t-x_i)=f(t)-z$ and $C\prod_j(t-a_j)=f(t)-1$, where $C\neq 0$ is the coefficient of t^n in $f(t)$, we find (modulo 2-torsion)

$$\begin{aligned}
\partial(\xi) \;&=\; (z)\wedge(1-z)-\sum_{i=1}^{n}(x_i)\wedge\Bigl(\prod_{j=1}^{n}\frac{a_j-x_i}{a_j}\Bigr)+\sum_{j=1}^{n}(a_j)\wedge\Bigl(\prod_{i=1}^{n}(a_j-x_i)\Bigr)\\
&=\; (z)\wedge(1-z)-\sum_{i=1}^{n}(x_i)\wedge(1-z)+\sum_{j=1}^{n}(a_j)\wedge\bigl(\frac{1-z}{C}\bigr)\\
&=\; (z)\wedge(1-z)-\Bigl(\frac{(-1)^{n-1}z}{C}\Bigr)\wedge(1-z)\;+\;\Bigl(\frac{(-1)^{n-1}}{C}\Bigr)\wedge\bigl(\frac{1-z}{C}\bigr)\\
&=\; 0\,.
\end{aligned}$$

The corresponding calculation for the yet more general functional equation given in the first line after equation (6) of Chapter I is left to the reader.

As already mentioned, the criterion $\sum n_i\bigl(x_i(t)\bigr)\wedge\bigl(1-x_i(t)\bigr)=0$ for a functional equation of the dilogarithm can be reformulated as saying that the element $\xi=\sum n_i[x_i(t)]$ belongs to the Bloch group of the corresponding function field. It is then reasonable to ask whether it in fact must be zero in this Bloch group, i.e., whether ξ is necessarily equal (modulo $\mathbb{Z}[\mathbb{C}]$) to a linear combination of five-term relations. This is conjectured, but not known to be true, when the $x_i(t)$ are allowed to be algebraic functions or rational functions of more than one variable. But in the case of rational functions of one variable, an elementary proof was found by Wojtkowiak. We reproduce his argument here in a slightly modified form.

Proposition 4 [40]. *(i) Any rational function of one variable is equivalent modulo the five-term relation to a linear combination of linear functions.*
(ii) Any functional equation of the dilogarithm with rational functions of one variable as arguments is a consequence of the five-term relation.
(Part (ii) is to be interpreted up to constants, i.e. the five-term relation suffices to give all relations $\sum_i D(x_i(t))=C$ but not necessarily to determine C.)

Proof. (i) Let $f(t)$ be an element of the field $\mathbb{C}(t)$ of rational functions in one variable. We want to show that the element $[f]\in\mathbb{Z}[\mathbb{C}(t)]$ is equivalent modulo five-term relations to a $\mathbb{Z}$-linear combination of elements of the form $[a_it+b_i]$. We do this by induction on the degree. Write $f(t)$ as $A(t)/B(t)$, where $A(t)$ and

$B(t)$ are polynomials of degree $\le n$, not both constant. Since we are working modulo the five-term relation, we can replace f by $1/f$ or $1/(1-f)$ if necessary to ensure that both $A(t)$ and $C(t) := B(t) - A(t)$ are non-constant. Choose a root a of $A(t)$ and a root c of $C(t)$ and set $g(t) = \dfrac{c-a}{t-a}$, $A^*(t) = g(t)A(t)$ and $D(t) = B(t) - A^*(t)$, so that $\deg(A^*) \le n-1$, $\deg(D) \le n$, and $D(c) = 0$. Then modulo the five-term relation, we have

$$\begin{aligned}[f] \;&\equiv\; -[g] + [fg] - \Big[\frac{1-f}{1-fg}\Big] + \Big[1 - \frac{1-g}{1-fg}\Big] \\ &\equiv\; -\left[\frac{c-a}{t-a}\right] + \left[\frac{A^*(t)}{B(t)}\right] - \left[\frac{C(t)/(t-c)}{D(t)/(t-c)}\right] + \left[\frac{(c-a)C(t)/(t-c)}{(t-a)D(t)/(t-c)}\right].\end{aligned}$$

Here each rational function appearing on the right has a numerator of degree $\le n-1$ and a denominator of degree $\le n$. Moreover, if B has degree $\le n-1$, then all terms on the right have both numerator and denominator of degree $\le n-1$. Therefore we have reduced any rational function with numerator and denominator of degrees $\le (n,n)$ to a combination of rational functions with numerator and denominator of degrees $\le (n-1,n)$, and any rational function with numerator and denominator of degrees $\le (n-1,n)$ to a combination of rational functions with numerator and denominator of degrees $\le (n-1,n-1)$. Iterating this procedure, we can keep reducing the degrees until we get to (0,1), i.e. (after inversion), until only linear functions appear. (Note that the number of applications of the five-term relation needed to reduce a rational function to a linear combination of linear ones grows exponentially with the degree n; more precisely, it is at most $(1+\sqrt{2})^{2n}/4$.)

(ii) By what we have just proved, any element $\xi = \xi(t) \in \mathbb{Z}[\mathbb{C}(t)]$ can be written modulo the 5-term relation as $\xi_0 + \sum n_i[\ell_i(t)]$, where $\xi_0 \in \mathbb{Z}[\mathbb{C}]$, $n_i \in \mathbb{Z}$ and the ℓ_i are non-constant linear functions of t. We can write each $\ell_i(t)$ as $(t-c_i)/(c_i'-c_i)$ with $c_i,\, c_i' \in \mathbb{C}$ distinct and (since we may replace $[\ell_i(t)]$ by $-[1-\ell_i(t)]$ modulo the five-term relation) $0 \le \arg(c_i'-c_i) < \pi$. The derivative of $D(\ell_i(t))$ is proportional to $(t-c_i)^{-1}\log|t-c_i'| - (t-c_i')^{-1}\log|t-c_i|$, and since these functions are linearly dependent for different i (as one sees by looking at their singularities), we deduce that $D(\xi(t))$ is constant if and only if $n_i = 0$ for all i, i.e., if $\xi(t) \equiv \xi_0$ modulo the five-term relation, as claimed. This proof also shows that every element of $\mathbb{Z}[\mathbb{C}(t)]/(\text{5-term relation})$ has a *unique* representative of the form $\xi_0 + \sum n_i[a_it+b_i]$ with $0 \le \arg(a_i) < \pi$.

B. Relations among special values of the dilogarithm. Relations among values of $D(\alpha)$, or among values of $L(\alpha)$ modulo π^2, correspond to torsion in the Bloch group. (More precisely, an element of $\mathcal{B}[\overline{\mathbb{Q}}]$ is torsion if and only if its Bloch-Wigner dilogarithm in all complex embeddings vanishes, in which case its Rogers dilogarithm in all real embeddings is a rational multiple of π^2.) Such relations are of interest in various contexts in combinatorics (asymptotics of certain q-hypergeometric series, like mock theta functions at roots of unity) [24], [49] and mathematical physics (models in rational conformal field theory, where the value of the Rogers dilogarithm divided by π^2 corresponds to the central charge) [17], [30]. If one has found a conjectural relation of this sort (which can be done empirically, e.g. by computing lots of values of the Bloch-Wigner or Rogers dilogarithm at algebraic arguments and searching for $\mathbb{Z}$-relations among them by the LLL algorithm), then one can always verify its correctness by finding an explicit expression of some multiple of it as a linear combination of five-term relations. We will describe a few examples and a general construction.

(a) Recall that the five-term relation is $\sum[x_i]$, where $\{x_i\}$ is a cyclically ordered 5-tuple of numbers satisfying $1-x_i = x_{i-1}x_{i+1}$. The simplest example is when $x_i = \alpha$ for all i, where α is one of the two roots of the quadratic equation $1-\alpha = \alpha^2$, i.e., $\alpha = (-1 \pm \sqrt{5})/2$. It follows that the element $[\alpha]$ of $\mathbb{Z}[\mathbb{Q}(\sqrt{5})]$ is killed by 5 in the Bloch group for each of these two numbers. That they are really 5-torsion and not trivial follows from the fact that the corresponding values $L((-1 \pm \sqrt{5})/2) = \pm\pi^2/10$ of the Rogers dilogarithm have the denominator 5 when divided by $\pi^2/6$.

(b) We give a less trivial example which will be used in §3 in connection with Nahm's conjecture. Set $\varepsilon = \sqrt{\alpha}$, with $\alpha = (-1+\sqrt{5})/2$ the inverse golden ratio as in Example (a). The field $F = \mathbb{Q}(\varepsilon)$ has two real and two (conjugate) complex embeddings. Define $\xi \in \mathbb{Z}[F]$ by

$$Q_1 \;=\; \varepsilon\,, \qquad Q_2 \;=\; \frac{1}{1+\varepsilon}\,, \qquad \xi = [Q_1] \,+\, [Q_2]\,. \tag{18}$$

From the identities

$$1 - Q_1 \;=\; Q_1^4\,Q_2\,, \qquad 1 - Q_2 \;=\; Q_1\,Q_2 \tag{19}$$

we obtain

$$\partial(\xi) \;=\; (Q_1)\wedge(4(Q_1)+(Q_2)) \,+\, (Q_2)\wedge((Q_1)+(Q_2)) \;=\; 0\,, \tag{20}$$

so ξ belongs to the Bloch group of F. To see that it is torsion, we use the relation (17) with $x = y = \varepsilon$:

$$V(\varepsilon,\,\varepsilon) \;=\; 2\,[\varepsilon] \,+\, 2\,\Big[\frac{1-\varepsilon}{1-\varepsilon^2}\Big] \,+\, \big[1-\varepsilon^2\big] \;=\; 2\,\xi \,+\, [1-\alpha]\,, \tag{21}$$

and since we have already seen in (a) that $[\alpha]$ and hence $[1-\alpha]$ are 5-torsion elements, this shows that $10\,\xi = 0$ in the Bloch group. To see that it really has this denominator, we calculate that $L(\xi)/L(1) = 13/10$ (numerically, but then exactly since we have just shown that $10\,L(\xi)/L(1)$ must be an integer).

(c) In example (a), the torsion element in the Bloch group had the form $[x]$ for a single number $x = (-1 \pm \sqrt{5})/2$. Another such example, even more obvious, is given by $x = 1/2$, for which $x \wedge (1-x) = x \wedge x = 0$ and $L(x) = \pi^2/12$. We claim that the only torsion elements of the Bloch group of $\mathbb{C}$ of the form $[x]$ with $x \in \mathbb{C} \smallsetminus \{0,1\}$ are these examples and the ones deduced from them using $[1/x] \equiv -[x]$ and $[1-x] \equiv -[x]$, i.e., the nine numbers $x = 1/2$, -1, 2, $(\pm 1 \pm \sqrt{5})/2$ and $(3 \pm \sqrt{5})/2$. (Compare this list with the special values of the dilogarithm given in §1 of Chapter I.) Indeed, for the element $[x]$ to belong to the Bloch group of $\mathbb{C}$ after tensoring with $\mathbb{Q}$), we must have that $x \wedge (1-x) = 0$ up to torsion, i.e. $x = \alpha t^p$, $1 - x = \beta t^q$ for some non-zero complex number t, integers p and q, and roots of unity α and β. Moreover, x must be totally real if $[x]$ is to be torsion in $\mathcal{B}_{\mathbb{C}}$, since this condition implies that $D(x^\sigma) = 0$ for all conjugates x^σ and D is non-zero for non-real arguments (cf. the picture in §3 of Chapter I), so we must in fact have $x = \pm t$, $1 - x = \pm t^q$ with t totally real. Replacing x by one of the six numbers x, $1-x$, $1/x$, $1-1/x$, $1/(1-x)$ or $x/(x-1)$, we may assume that $0 < 1 - x \le x < 1$ and hence that $x = t^p$, $1 - x = t^q$ with $0 < t < 1$ and $q \ge p \ge 1$. But it is easily checked that the only equations of the form $t^p + t^q = 1$ with $q \ge p \ge 1$ which have only real roots are $t + t = 1$ and $t + t^2 = 1$, corresponding to $x = 1/2$ and $x = (\sqrt{5} - 1)/2$, as claimed.

Remark about torsion. In examples (a) and (b), we showed that the elements ξ under consideration were torsion in the Bloch group by writing some multiple of them as a combination of five-term relations, and that they were non-trivial by computing the Rogers dilogarithm $L(\xi)$ and checking that $L(\xi)/L(1)$ had a non-trivial denominator. This method would not work if ξ belonged to a number field F having no real embeddings, but in that case we could use instead the enhanced dilogarithm of §1**B** (with respect to a fixed embedding of F into $\mathbb{C}$) and check numerically that its value was torsion but not zero. This would also work, of course, for a field having both real and non-real embeddings , e.g. for example (b) and the embedding given by $\varepsilon = \sqrt{(-1-\sqrt{5})/2}$. Note, however, that in contrast to the real case, the statement that a torsion element of the Bloch group is non-trivial is not absolute, but depends on the number field, because it can happen that an element which is non-trivial torsion in the Bloch group of one number field becomes trivial in the Bloch group of a larger field containing more roots of unity. A simple example where this happens is given by $\xi = [-1] \in \mathcal{B}_{\mathbb{Q}}$, which is non-trivial in $\mathcal{B}_{\mathbb{R}}$ because the number $L(-1)/L(1) = -1/2$ is non-integral, but which is trivial in $\mathcal{B}_{\mathbb{Q}(i)}$ because applying the duplication relation $[x^2] \equiv 2[x] + 2[-x]$ (an easy consequence of the five-term relation) to $x = i$ gives $[-1] \equiv 2[i] + 2[1/i] \equiv 0$. As a less trivial example, we saw in example (a) that the inverse golden ratio α is 10-torsion in $\mathcal{B}_F$ for $F = \mathbb{Q}(\sqrt{5})$, but if we pass to the field $F = \mathbb{Q}(\zeta)$, where ζ is a 5th root of unity, then it becomes zero because modulo the relations $[x] \equiv -[1-x]$ and $[x] \equiv -[x/(x-1)]$ we have

$$V(-\zeta,\,1{+}\zeta) = [-\zeta] + [1{+}\zeta] + [-\zeta^2] + [-\zeta^2/(-\zeta^2{-}1)] + [\zeta^3{+}\zeta^2{+}1] \;\equiv\; [1/\alpha]\,.$$

In fact, a theorem of Merkur'ev and Suslin [27] implies that this phenomenon *always* happens: every torsion element in the Bloch group of a number field becomes trivial in the Bloch group of a larger number field containing sufficiently many roots of unity.

C. Dilogarithm identities from triangulated 3-manifolds. Finally, we describe a simple method for producing examples of torsion elements in Bloch groups, using combinatorial triangulated 3-manifolds. (See also [18].) For this purpose, it is convenient to think of L or D as functions of real or complex oriented 3-simplices. By an *oriented n-simplex* in $\mathbb{P}^1(\mathbb{C})$ we mean an $(n+1)$-tuple of points in $\mathbb{P}^1(\mathbb{C})$ together with an ordering up to even permutations; more precisely, such a simplex has the form $[x_0,\dots,x_n]$ with $x_j \in \mathbb{P}^1(\mathbb{C})$ and with the convention that $[x_{\pi(0)},\dots,x_{\pi(n)}] = \operatorname{sgn}(\pi)\,[x_0,\dots,x_n]$ for $\pi \in \mathfrak{S}_{n+1}$. Let $\mathfrak{C}_n$ denote the free abelian group on oriented n-simplices. There are boundary maps $\partial : \mathfrak{C}_n \to \mathfrak{C}_{n-1}$ defined by the usual formula $\partial([x_0,\dots,x_n]) = \sum_{i=0}^n (-1)^i [x_0,\dots,\widehat{x_i},\dots,x_n]$, and the sequence (with $\varepsilon([x]) = 1$)

$$\cdots \longrightarrow \mathfrak{C}_4 \xrightarrow{\partial} \mathfrak{C}_3 \xrightarrow{\partial} \mathfrak{C}_2 \xrightarrow{\partial} \mathfrak{C}_1 \xrightarrow{\partial} \mathfrak{C}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{22}$$

is exact. The function $D : \mathbb{C} \to \mathbb{R}$ defines a function $\widetilde{D} : \mathfrak{C}_3 \to \mathbb{R}$ which associates to a 3-simplex in $\mathbb{P}^1(\mathbb{C})$ the value of D on the cross-ratio of its vertices, $\widetilde{D}([a,b,c,d]) = D\big(\frac{a-d}{a-c}\frac{b-c}{b-d}\big)$, as in Chapter I. This is well-defined (i.e., transforms under the action of $\pi \in \mathfrak{S}_4$ by $\operatorname{sgn}(\pi)$) and is 0 on $\partial(\mathfrak{C}_4)$ by the five-term relation, since the element $V(x,y)$ in (17) is simply the boundary of the 4-simplex $[\infty, 0, x, 1, y^{-1}]$ and any 4-simplex is equivalent to such a one under the action of $PGL_2(\mathbb{C})$ on $\mathfrak{C}_3$. (Notice that $\widetilde{D}$ is invariant under the action of $PGL_2(\mathbb{C})$ on $\mathfrak{C}_3$, since the cross-ratio is.) Because of the exactness of (22), we can say equivalently that $\widetilde{D}$ vanishes on $\operatorname{Ker}(\mathfrak{C}_3 \xrightarrow{\partial} \mathfrak{C}_2)$ or that it factors through ∂: if we define a map $\approx{D} : \mathfrak{C}_2 \to \mathbb{R}$ by $\overset{\approx}{D}([a,b,c]) = -\widetilde{D}([a,b,c,\infty])$ (here "∞" could be replaced by any other fixed base-point $x_0 \in \mathbb{P}^1(\mathbb{C})$), then for every oriented 3-simplex $[a,b,c,d] \in \mathfrak{C}_3$ we have

$$\begin{aligned}
\overset{\approx}{D}\big(\partial([a,b,c,d])\big) &= \overset{\approx}{D}\big(-[a,b,c] + [a,b,d] - [a,c,d] + [b,c,d]\big) \\
&= \widetilde{D}\big([a,b,c,\infty] - [a,b,d,\infty] + [a,c,d,\infty] - [b,c,d,\infty]\big) \\
&= \widetilde{D}\big([a,b,c,d] \;-\; \partial([a,b,c,d,\infty])\big) \\
&= \widetilde{D}([a,b,c,d])
\end{aligned}$$

and hence $\widetilde{D} = \overset{\approx}{D} \circ \partial$.

We can think of an element ξ of $\operatorname{Ker}(\mathfrak{C}_3 \xrightarrow{\partial} \mathfrak{C}_2)$ as a closed, triangulated, oriented near-3-manifold M, smooth except possibly at its vertices (it is a union of oriented tetrahedra glued to each other along their faces, and is hence automatically smooth on the interior of its 3-, 2- or 1-simplices, while at a vertex its topology is that of a cone on some compact oriented surface), together with a map ϕ from the vertices of M to $\mathbb{P}^1(\mathbb{C})$. Any such element $\xi =$

$\sum[\sigma_i]$ gives an identity $\widetilde{D}(M,\phi) := \sum \widetilde{D}(\sigma_i) = 0$ among values of the Bloch-Wigner dilogarithm. This identity can be written explicitly as a combination of five-term equations by the calculation above (just replace each simplex $\sigma = [a,b,c,d]$ of M by $[\sigma,\infty] := [a,b,c,d,\infty]$). We can also perform the same construction over $\mathbb{R}$, starting from a triangulated near-3-manifold M and a map ϕ from its 0-skeleton to $\mathbb{P}^1(\mathbb{R})$; then the element $\widetilde{L}(M,\phi)$ defined as the sum over the 3-simplices σ of M of the value of L at the cross-ratio of the images of the vertices of σ under ϕ will be an integral multiple of $\pi^2/2$ by virtue of the functional equation of the Rogers dilogarithm.

Here is an example. Let M be the join of an m-gon and an n-gon, where m and n are two positive integers. If we write these two polygons as $\{x_j\}_{j \pmod m}$ and $\{y_k\}_{k \pmod n}$, then this means that M is the union of 3-simplices $[x_j, x_{j+1}, y_k, y_{k+1}]$ ($j \in \mathbb{Z}/m\mathbb{Z}$, $k \in \mathbb{Z}/n\mathbb{Z}$). Then

$$\partial(M) \;=\; \sum_{j,\,k}\bigl([x_{j+1},y_k,y_{k+1}] - [x_j,y_k,y_{k+1}] + [x_j,x_{j+1},y_{k+1}] - [x_j,x_{j+1},y_k]\bigr)$$

vanishes because the first two terms in the parentheses cancel when we sum over all j with k fixed and the last two when we sum over all k with j fixed. If we map the $m+n$ vertices of M to points x_j, $y_k \in \mathbb{C}$ (which we simply denote by the same letters, omitting the map ϕ), then we get a functional equation

$$\sum_{j \pmod m,\ k \pmod n} D\left(\frac{y_k - y_{k+1}}{y_k - x_j}\,\frac{x_j - x_{j+1}}{y_{k+1} - x_{j+1}}\right) \;=\; 0\,,$$

valid for any complex numbers x_j, y_k. Specializing to $x_j = e^{2iX_j}$, $y_k = e^{2iY_k}$ with X_j, $Y_k \in \mathbb{R}$ gives real cross-ratios and a Rogers dilogarithm identity

$$\frac{2}{\pi^2}\sum_{j \pmod m,\ k \pmod n} L\left(\frac{\sin(Y_k - Y_{k+1})}{\sin(Y_k - X_j)}\,\frac{\sin(X_j - X_{j+1})}{\sin(Y_{k+1} - X_{j+1})}\right) \quad\in\; \mathbb{Z}\,,$$

where the value of the resulting integer depends on the ordering of the points X_j and Y_k on the circle $\mathbb{R}/\pi\mathbb{Z}$. In particular, if they are equally spaced we find

$$\frac{1}{\pi^2}\sum_{\substack{j \pmod m\\ k \pmod n}} L\left(\frac{\sin\bigl(\frac{\pi}{n}\bigr)\,\sin\bigl(\frac{\pi}{m}\bigr)}{\sin\bigl(\frac{k\pi}{n} - \frac{j\pi}{m} + t\bigr)\,\sin\bigl(\frac{(k+1)\pi}{n} - \frac{(j+1)\pi}{m} + t\bigr)}\right)$$
$$= \min(m,n) \,-\, 1.$$

If also $m = n$, then each term occurs n times, so the equation reduces to

$$\frac{1}{\pi^2}\sum_{k \pmod n} L\left(\frac{\sin^2\bigl(\frac{\pi}{n}\bigr)}{\sin^2\bigl(\frac{k\pi}{n} + t\bigr)}\right) \;=\; 1 - \frac{1}{n} \qquad (n \in \mathbb{N},\ t \in \mathbb{R})\,.$$

Finally, specializing a fourth time to $t = 0$ and using $L(+\infty) = \pi^2/3 = 2L(1)$, we find

$$L(1)^{-1} \sum_{0<k<n} L\left(\frac{\sin^2(\pi/n)}{\sin^2(k\pi/n)}\right) \;=\; 4 - \frac{6}{n}\,,$$

an equation well known in the physics literature.

3 Dilogarithms and modular functions

A fascinating and almost completely unsolved problem is to understand the overlap between the classes of q-hypergeometric functions and modular forms or functions, the prototypical case being given by the famous Rogers-Ramanujan identities. Nahm's conjecture gives us a first glimpse of an answer, which surprisingly involves dilogarithms and the Bloch group. In subsections **A** and **B** we describe the conjecture and some of the examples which motivate it, while subsection **C** contains an asymptotic analysis of the q-series involved and the proof that the conjecture is true in the simplest case.

A. q-hypergeometric series and Nahm's conjecture. Consider the two power series $G(q)$ and $H(q)$ by

$$G(q) \;=\; \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}\,, \qquad H(q) \;=\; \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} \qquad\qquad (|q|<1),$$

where $(q)_n$ as in §1 denotes the product $(1-q)(1-q^2)\cdots(1-q^n)$. The classical Rogers-Ramanujan identities—discovered by Rogers in 1897, rediscovered by Ramanujan in 1915 and then given a third proof by both authors jointly and many further proofs in subsequent years—says that these two series have product developments

$$G(q) \;=\; \prod_{n\,\equiv\,\pm1 \;(\mathrm{mod}\;5)} \frac{1}{1-q^n}\,, \qquad H(q) \;=\; \prod_{n\equiv\,\pm2\;(\mathrm{mod}\;5)} \frac{1}{1-q^n}\,.$$

The important thing here is not so much that these functions have product expansions as that, up to rational powers of q, they are both modular functions. Indeed, by the Jacobi triple product formula, we can rewrite the identities as

$$G(q) \;=\; \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n\, q^{(5n^2+n)/2}, \quad H(q) \;=\; \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n\, q^{(5n^2+3n)/2}$$

or, even more intelligently, as

$$q^{-1/60}\, G(q) \;=\; \frac{\theta_{5,1}(z)}{\eta(z)}\,, \qquad q^{11/60}\, H(q) \;=\; \frac{\theta_{5,2}(z)}{\eta(z)}\,, \qquad\qquad (23)$$

where $q = e^{2\pi iz}$ with $z \in \mathfrak{H}$ (upper half-plane) and

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \quad \theta_{5,j}(z) = \sum_{n \equiv 2j-1 \pmod{10}} (-1)^{[n/10]} \, q^{n^2/40} \,.$$

The point is that $\eta(z)$, $\theta_{5,1}(z)$ and $\theta_{5,2}(z)$ are all modular forms of weight $1/2$ and therefore that the functions on the right-hand side of (23) are modular functions, i.e., they are invariant under $z \mapsto \frac{az+b}{cz+d}$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belonging to some subgroup of finite index of $SL(2,\mathbb{Z})$. Indeed, if we combine them into a single vector-valued function

$$g(z) = \begin{pmatrix} q^{-1/60} G(q) \\ q^{11/60} H(q) \end{pmatrix} \qquad \bigl(z \in \mathfrak{H},\ q^{\alpha} := e^{2\pi i \alpha z}\bigr),$$

then we have transformation formulas with respect to the full modular group:

$$g(z+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11} \end{pmatrix} g(z), \quad g\Bigl(-\frac{1}{z}\Bigr) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} g(z) \tag{24}$$

(with $\zeta_N := e^{2\pi i/N}$) and hence $g(\gamma(z)) = \rho(\gamma) g(z)$ for all $\gamma \in SL(2,\mathbb{Z})$ and some representation $\rho :\ SL(2,\mathbb{Z}) \to GL(2,\mathbb{C})$.

The functions $G(q)$ and $H(q)$ are special examples of what are called *q-hypergeometric series*, i.e., series of the form $\sum_{n=0}^{\infty} A_n(q)$ where $A_0(q)$ is a rational function and $A_n(q) = R(q,q^n) A_{n-1}(q)$ for all $n \ge 1$ for some rational function $R(x,y)$ with $\lim\limits_{x\to 0} \lim\limits_{y \to 0} R(x,y) = 0$. (For G and H one has $A_0 = 1$ and $R(x,y) = x^{-1}y^2/(1-y)$ or $y^2/(1-y)$, respectively.) There are only a handful of examples known of q-hypergeometric series which are also modular, and, as already mentioned in the introduction to this section, the problem of describing when this happens in general is an important and fascinating question, but totally out of reach for the moment. A remarkable conjecture of Werner Nahm, discussed in more detail in his paper [30] in this volume as well as in his earlier articles [28], [29], relates the answer to this question in a very special case to dilogarithms and Bloch groups on the one hand and to rational conformal field theory on the other.

Nahm's conjecture actually concerns certain r-fold hypergeometric series (defined as above but with n running over $(\mathbb{Z}_{\ge 0})^r$ rather than just $\mathbb{Z}_{\ge 0}$). Let A be a positive definite symmetric $r \times r$ matrix, B a vector of length r, and C a scalar, all three with rational coefficients. We define a function $f_{A,B,C}(z)$ by the r-fold q-hypergeometric series

$$f_{A,B,C}(z) \;=\; \sum_{n=(n_1,\dots,n_r) \in\ (\mathbb{Z}_{\ge 0})^r} \frac{q^{\frac{1}{2} n^t A n + B^t n + C}}{(q)_{n_1} \cdots (q)_{n_r}} \qquad (z \in \mathfrak{H})$$

and ask when $f_{A,B,C}$ is a modular function. Nahm's conjecture does not answer this question completely, but predicts which A can occur. If $A = (a_{ij})$

is any positive definite symmetric $r \times r$ matrix with rational entries, we can consider the system

$$1 - Q_i \;=\; \prod_{j=1}^{r} Q_j^{a_{ij}} \qquad\qquad (i = 1, \dots, r) \tag{25}$$

of r equations in r unknowns, which we can write in abbreviated notation as $1 - Q = Q^A$, $Q = (Q_1, \dots, Q_r)$. We suppose first that A has integral coefficients, so that the equations in (25) are polynomial. Since there are as many equations as unknowns, we expect that the solutions form a 0-dimensional variety, i.e., there are only finitely many solutions and (hence) all lie in $\overline{\mathbb{Q}}$, but in any case, the system certainly has solutions in $\overline{\mathbb{Q}}^r$. For any such a solution $Q = (Q_1, \dots, Q_r)$ we can consider the element

$$\xi_Q \;=\; [Q_1] + \cdots + [Q_r] \quad \in \; \mathbb{Z}[F]\,,$$

where F is the number field $\mathbb{Q}(Q_1, \dots, Q_r)$. Then in $\Lambda^2(F^\times)$ we find

$$\begin{aligned} \partial(\xi_Q) \;&=\; \sum_{i=1}^{r} (Q_i) \wedge (1 - Q_i) \;=\; \sum_{i=1}^{r} (Q_i) \wedge \bigl(\prod_{j=1}^{r} Q_j^{a_{ij}}\bigr) \\ &=\; \sum_{i=1}^{r} (Q_i) \wedge \Bigl(\sum_{j=1}^{r} a_{ij}(Q_j)\Bigr) \;=\; \sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}\,(Q_i) \wedge (Q_j)\,, \end{aligned}$$

and this is 0 since a_{ij} is symmetric and $(Q_i) \wedge (Q_j)$ antisymmetric in i and j. Hence ξ_A belongs to the Bloch group of F. If A is not integral, then we have to be careful about the choice of determinations of the rational powers $Q_j^{a_{ij}}$ in (25). We require them to be consistent, i.e., we must have $Q_i = e^{u_i}$, $1 - Q_i = e^{v_i}$ for some vectors u, $v \in \mathbb{C}^r$ such that $v = Au$. This defines the minimal number field F in which the equations (25) make sense. For instance, if $A = \left(\begin{smallmatrix} 8/3 & 1/3 \\ 1/3 & 2/3 \end{smallmatrix}\right)$, we set $(Q_1, Q_2) = (\alpha, \alpha\beta^3)$ where α, $\beta \in \overline{\mathbb{Q}}$ are solutions of the system $1 - \alpha = \alpha^3\beta$, $1 - \alpha\beta^3 = \alpha\beta^2$; then $F = \mathbb{Q}(\alpha, \beta)$, and $\xi_Q = [\alpha] + [\alpha\beta^3]$ is an element of $\mathcal{B}(F)$. Nahm's conjecture is then

Conjecture. Let A be a positive definite symmetric $r \times r$ matrix with rational coefficients. Then the following are equivalent:

(i) The element ξ_Q is a torsion element of $\mathcal{B}(F)$ for every solution Q of (25).
(ii) There exist $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$ such that $f_{A,B,C}(z)$ is a modular function.

The main motivation for this conjecture comes from physics, and in fact one expects that all the modular functions $f_{A,B,C}$ which are obtained this way are characters of rational conformal field theories. (We will not discuss these aspects at all, referring the reader for this to Nahm's paper.) A further expectation, again predicted by the physics, is that if a matrix A satisfies the conditions of the conjecture, then the collection of modular functions occurring in statement (ii) span a vector space which is invariant under the action of

$SL(2,\mathbb{Z})$ (bosonic case) or at least $\Gamma(2)$ (fermionic case), even though the individual functions $f_{A,B,C}$ will in general have level greater than 2. For instance, in the Rogers-Ramanujan identities given above, each of the two functions (23) is modular (up to multiplication by a root of unity) only on the modular group $\Gamma_0(5)$ of level 5 (and on a yet much smaller group if we do not allow scalar multiples), but the vector space which they span is invariant under all of $SL(2,\mathbb{Z})$ by eq. (24). From a purely mathematical point of view, the motivation for the conjecture comes from the asymptotic analysis, discussed in subsection **B**, and from the known examples, some of which we now describe.

B. Examples and discussion. In this subsection we describe a number of examples which give numerical support for Nahm's conjecture and which show that two plausible alternative versions of the conjecture—one with a stronger and one with a weaker hypothesis on the matrix A—are not tenable.

(a) *Rank one examples.* If $r = 1$, then the parameters A, B and C are simply rational numbers and exactly seven cases are known where $f_{A,B,C}(z)$ is a modular function, given by the following table:

Table 1. The modular functions $f_{A,B,C}$ for $r = 1$

A	B	C	$f_{A,B,C}(z)$
2	0	$-1/60$	$\theta_{5,1}(z)/\eta(z)$
	1	$11/60$	$\theta_{5,2}(z)/\eta(z)$
1	0	$-1/48$	$\eta(z)^2/\eta(z/2)\eta(2z)$
	$1/2$	$1/24$	$\eta(2z)/\eta(z)$
	$-1/2$	$1/24$	$2\eta(2z)/\eta(z)$
$1/2$	0	$-1/40$	$\theta_{5,1}(z/4)/\theta_8(z)$
	$1/2$	$1/40$	$\theta_{5,2}(z/4)/\theta_8(z)$

with $\theta_{5,j}(z)$ and $\eta(z)$ as in (23) and $\theta_8(z) = \sum_{n>0}\left(\frac{8}{n}\right)q^{n^2/8} = \eta(z)\eta(4z)/\eta(2z)$. The first two entries in this table are just the Rogers-Ramanujan identities with which we began the discussion, and the product $\eta(z)f_{A,B,C}(z)$ in all seven cases is a unary theta series (i.e., a function $\sum \varepsilon(n)q^{\lambda n^2}$ with $\varepsilon(n)$ an even periodic function and λ a positive rational number). We will see in subsection **B** that these are the only triples $(A,B,C) \in \mathbb{Q}_+ \times \mathbb{Q}^2$ for which $f_{A,B,C}$ is modular. On the other hand, the element ξ_A when $r = 1$ consists of a single element $[Q_1]$, where $1 - Q_1 = Q_1^A$, so by the discussion in example (c) of §2**B** we know that the only values of $A > 0$ for which condition (i) of the conjecture is satisfied are $A = 1/2$, 1 or 2 (corresponding to $Q_1 = (-1+\sqrt{5})/2$, $1/2$ and $(3-\sqrt{5})/2$, respectively). Thus Nahm's conjecture holds for $r = 1$.

(b) *Totally real, or torsion in the Bloch group?* In the above examples, the element Q_1 was a totally real algebraic number. It is reasonable to ask whether the requirement for modularity is really condition (i) of the conjecture or merely the more elementary (but stronger) condition that equation (25) has only real solutions. To see that (i) really is the right condition, we consider the matrix $A = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. In this case (25) specializes to the system of equations (19) studied in example (b) of §2**B**, and by the discussion given there we know that the corresponding element $\xi_A \in \mathcal{B}(\mathbb{Q}(\xi_A))$ is torsion (of order 10) but is not totally real. So if (i) rather than total reality is the correct condition in the conjecture, then there should be pairs $(B, C) \in \mathbb{Q}^2 \times \mathbb{Q}$ for which $f_{A,B,C}$ is modular. After some experimentation, using a method which will be explained briefly at the end of subsection **C**, we find that there are indeed at least two such pairs, the corresponding identities being

$$f_{\left(\begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right), \frac{1}{120}}(z) = \frac{\theta_{5,1}(2z)}{\eta(z)}, \quad f_{\left(\begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1/2 \end{smallmatrix}\right), \frac{49}{120}}(z) = \frac{\theta_{5,2}(2z)}{\eta(z)},$$

where $\theta_{5,1}(z)$ and $\theta_{5,2}(z)$ are the same theta series as those occurring in (23). (These equations were not proved, but only verified to a high order in the power series in q.)

(c) *Must all solutions be torsion?* If A is any positive definite matrix, even with real coefficients, then the system of equations (25) has exactly one solution with all Q_i real and between 0 and 1. (To see this, one shows by induction on r the more general assertion that the system $1-Q_i = \lambda_i \prod_j Q_j^{a_{ij}}$ has a unique solution in $(0,1)^r$ for any real numbers $\lambda_1, \ldots, \lambda_r > 0$.) Denote this solution by $Q^0 = (Q_1^0, \ldots, Q_r^0)$. If the coefficients of A are rational, the Q_i^0 are real algebraic numbers (though not necessarily totally real—see (b)) and we obtain a specific element $\xi_A = \xi_{Q^0} \in \mathcal{B}(\overline{\mathbb{Q}} \cap \mathbb{R})$. If condition (i) of the conjecture is satisfied, then this must be a torsion element and hence the corresponding Rogers dilogarithm value $L(\xi_A) = \sum L(Q_i^0)$ must be a rational multiple of π^2. This criterion is numerically effective (one can find Q^0 numerically to high precision by an iterative procedure and then test $L(\xi_A)/\pi^2$ for rationality) and is the one used for the computer searches described in (d) and (e) below. One can reasonably ask whether it is in fact sufficient, i.e., whether it is sufficient in (i) to assume only that ξ_A is torsion. An example showing that this is not the case—we will see many others in (d) and (e)—is given by the matrix $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$. Here Q^0 is equal to $\left(\phi^{-1}\psi,\, \phi^4 - \phi^3\psi\right)$, with $\phi = (\sqrt{5}+1)/2$ and $\psi = (1+\sqrt{2\sqrt{5}-1})/2$, and this is torsion, as we can see numerically from the dilogarithm values ($L(\xi) = \frac{8}{5}L(1)$ for both ξ_{Q^0} and its real conjugate and $D(\xi) = 0$ for both non-real conjugates of ξ_{Q^0}) and could verify algebraically as in section 2**B**. But the equations $1-Q_1 = Q_1^8 Q_2^5$, $1-Q_2 = Q_1^5 Q_2^4$ have another Galois orbit of four solutions where (Q_1, Q_2) belong to a different quartic field and where $D(\xi) \neq 0$, so condition (i) of the conjecture is not satisfied. Here a computer search finds no B and C making $f_{A,B,C}$ modular.

(d) *Rank two examples.* An extensive search for positive definite matrices $A \in M_2(\mathbb{Q})$ for which $L(\xi_A)/L(1) \in \mathbb{Q}$ (specifically, a search over $A = \dfrac{1}{m}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ with integers a, b, c, m less than or equal to 100) found three infinite families

$$A = \begin{pmatrix} \lambda\alpha & 1-\alpha \\ 1-\alpha & \lambda^{-1}\alpha \end{pmatrix}, \quad \xi_A = \bigl(x,\, x^\lambda = 1-x\bigr), \quad L(\xi_A) = L(1),$$

$$A = \begin{pmatrix} \alpha & 2-\alpha \\ 2-\alpha & \alpha \end{pmatrix}, \quad \xi_A = \bigl(\tfrac{\sqrt{5}-1}{2},\, \tfrac{\sqrt{5}-1}{2}\bigr), \quad L(\xi_A) = \tfrac{6}{5}\,L(1),$$

$$A = \begin{pmatrix} \alpha & \frac{1}{2}-\alpha \\ \frac{1}{2}-\alpha & \alpha \end{pmatrix}, \quad \xi_A = \bigl(\tfrac{3-\sqrt{5}}{2},\, \tfrac{3-\sqrt{5}}{2}\bigr), \quad L(\xi_A) = \tfrac{4}{5}\,L(1),$$

and 22 individual solutions (excluding split ones), namely the matrices

A	$\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&2\\2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&3\\3&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}8&3\\3&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right)$
$L(\xi_A)/L(1)$	$5/4$	$13/10$	$10/7$	$3/2$	$3/2$	$8/5$

A	$\left(\begin{smallmatrix}11&9\\9&8\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}24&19\\19&16\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}2&1\\1&3/2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}5/2&2\\2&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}8/3&1/3\\1/3&2/3\end{smallmatrix}\right)$
$L(\xi_A)/L(1)$	$17/10$	$9/5$	$9/7$	$7/5$	$8/7$

and their inverses with $L(\xi_{A^{-1}})/L(1) = 2 - L(\xi_A)/L(1)$. (All of these examples except for $\left(\begin{smallmatrix}24&19\\19&16\end{smallmatrix}\right)$ were already given in 1995 by Nahm's student M. Terhoeven in his thesis ([37], pp. 48–49), based on a search in the smaller domain with "100" replaced by "11".) Of these, the only ones that satisfy the stronger condition that all solutions of (25) are torsion are

$$\begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}, \quad \begin{pmatrix} 2&1\\1&1 \end{pmatrix}, \quad \begin{pmatrix} 4&1\\1&1 \end{pmatrix}, \quad \begin{pmatrix} 4&2\\2&2 \end{pmatrix}, \quad \begin{pmatrix} 2&1\\1&3/2 \end{pmatrix}, \quad \begin{pmatrix} 4/3&2/3\\2/3&4/3 \end{pmatrix}$$

and their inverses, and indeed for each of these we find several values of B, C for which the function $f_{A,B,C}$ is (or appears to be) modular, while for the others we never find any. The list of these values is given in Table 2. The formulas for the corresponding modular forms for $A = \left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$ were given in (b), and the ones for $A = \left(\begin{smallmatrix}\alpha&1-\alpha\\1-\alpha&\alpha\end{smallmatrix}\right)$ are given by the formula

$$f_{\left(\begin{smallmatrix}\alpha&1-\alpha\\1-\alpha&\alpha\end{smallmatrix}\right),\left(\begin{smallmatrix}\alpha\nu\\-\alpha\nu\end{smallmatrix}\right),\frac{\alpha}{2}\nu^2-\frac{1}{24}}(z) = \frac{1}{\eta(z)} \sum_{n\in\mathbb{Z}+\nu} q^{\alpha n^2/2} \qquad (\forall \nu \in \mathbb{Q}), \tag{26}$$

which is easily proved using the identity

$$\sum_{\substack{m,n\ge 0\\ m-n=r}} \frac{q^{mn}}{(q)_m (q)_n} = \frac{1}{(q)_\infty} \qquad \text{for any } r \in \mathbb{Z} \tag{27}$$

(itself an easy consequence of eq. (7) and the Jacobi triple product formula). For reasons of space we do not give the other modular forms explicitly in Table 2, but only the numbers $c = c(A)$ and $K = K(A,B)$ defined—if $f_{A,B,C}$ is modular—by

$$f_{A,B,C}\bigl(e^{-\varepsilon}\bigr) = K\, e^{c\pi^2/6\varepsilon} + \mathrm{O}\bigl(e^{c'\pi^2/6\varepsilon}\bigr) \qquad (\varepsilon \to 0,\ c' < c)\,. \tag{28}$$

Table 2. Rank 2 examples for Nahm's conjecture. Here $\alpha_k = \sin(\pi k/8)$, $\beta_k = 2/\sqrt{5}\sin(\pi k/5)$ and $\gamma_k = 2/\sqrt{7}\sin(\pi k/7)$.

A	$\begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}, \quad c = 1$	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad c = \frac{3}{4}$					$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad c = \frac{5}{4}$				
B	$\begin{pmatrix} \alpha\nu \\ -\alpha\nu \end{pmatrix} \quad (\nu \in \mathbb{Q})$	$\begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -3/2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
C	$\alpha\nu^2/2 - 1/24$	$1/8$	$-1/32$	0	$1/8$	$7/32$	$25/24$	$-5/96$	$1/6$	$1/24$	$19/24$
K	$1/\sqrt{\alpha}$	1	α_3	$1/\sqrt{2}$	$1/2$	α_1	1	α_3	$1/\sqrt{2}$	$1/2$	α_1

A	$\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, \quad c = \frac{7}{10}$		$\begin{pmatrix} 1/3 & -1/3 \\ -1/3 & 4/3 \end{pmatrix}, \quad c = \frac{13}{10}$		$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}, \quad c = \frac{4}{7}$			$\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad c = \frac{10}{7}$		
B	$\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/6 \\ 2/3 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$
C	$1/120$	$49/120$	$3/40$	$1/120$	$-1/42$	$5/42$	$17/42$	$-5/84$	$1/21$	$1/84$
K	$\beta_1/\sqrt{2}$	$\beta_2/\sqrt{2}$	$\beta_1\sqrt{3/2}$	$\beta_2\sqrt{3/2}$	γ_3	γ_2	γ_1	$2\gamma_3$	$2\gamma_2$	$2\gamma_1$

A	$\begin{pmatrix} 3/2 & 1 \\ 1 & 2 \end{pmatrix}, \quad c = \frac{5}{7}$			$\begin{pmatrix} 1 & -1/2 \\ -1/2 & 3/4 \end{pmatrix}, \quad c = \frac{9}{7}$			$\begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}, \quad c = \frac{4}{5}$			$\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad c = \frac{6}{5}$		
B	$\begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 \\ 1/4 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$	$\begin{pmatrix} -1/3 \\ -2/3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
C	$1/168$	$-5/168$	$25/168$	$1/28$	$-3/56$	$1/56$	$1/30$	$1/30$	$-1/30$	$1/20$	$1/20$	$-1/20$
K	$\gamma_3\sqrt{2}$	$\gamma_2\sqrt{2}$	$\gamma_1\sqrt{2}$	$2\gamma_3$	$2\gamma_2$	$2\gamma_1$	$\beta_1\sqrt{3}$	$\beta_1\sqrt{3}$	$\beta_2\sqrt{3}$	$2\beta_1$	$2\beta_1$	$2\beta_2$

According to the analysis in **C** below, these numbers—of which c corresponds in conformal field theory to the effective central charge—are given by

$$c = \sum_{i=1}^{r} \left(1 - \frac{L(Q_i^0)}{L(1)}\right) = r - \frac{L(\xi_A)}{L(1)}, \quad K = \frac{1}{\sqrt{\det \widetilde{A}}} \prod_{i=1}^{r} \frac{(Q_i^0)^{b_i}}{\sqrt{1-Q_i^0}}, \tag{29}$$

where $Q^0 = (Q_1^0, \ldots, Q_r^0) \in (0,1)^r$ as above, $B = (b_1, \ldots, b_r)$, and

$$\widetilde{A} \;=\; A + \operatorname{diag}\left(\frac{Q_1^0}{1-Q_1^0}, \ldots, \frac{Q_r^0}{1-Q_r^0}\right). \tag{30}$$

(e) *Rank three examples.* We conducted similar experiments for 3×3 matrices, restricting the search to matrices with coefficients which are integral and ≤ 10. In this range we found over 100 matrices A satisfying $L(\xi_A)/L(1) \in \mathbb{Q}$, of which about one-third satisfied the stronger condition (i) of the conjecture. These consisted of members of a three-parameter infinite family

$$A = \alpha \begin{pmatrix} h^2 & h & -h \\ h & 1 & -1 \\ -h & -1 & 1 \end{pmatrix} + \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (\alpha \in \mathbb{Q},\ h \in \mathbb{Z},\ A_1 \in \{\tfrac{1}{2}, 1, 2\}) \tag{31}$$

together with eight sporadic solutions, and in all cases the computer found pairs (B, C) making $f_{A,B,C}$ apparently modular. The pairs found for the eight sporadic solutions (and for one member of the family (31) which had extra solutions) are given in Table 3, while those for the family (31) are given by

$$B = \alpha\nu \begin{pmatrix} h \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \frac{\alpha\nu^2}{2} - \frac{1}{24} + C_1 \tag{32}$$

where $\nu \in \mathbb{Q}$ and (A_1, B_1, C_1) is one of the 7 rank one solutions given in Table 1. Let us check that the matrices A in (31) indeed satisfy Nahm's criterion (i) and that the functions $f_{A,B,C)}$ for (B, C) as in (32) are indeed modular. For the first, we note that for this A Nahm's equations (25) take the form $1-Q_1 = Q_1^{A_1} R^h$, $1-Q_2 = Q_3 R$, $1-Q_3 = Q_2 R^{-1}$ with $R = (Q_1^h Q_2/Q_3)^\alpha$. The last two give $(1-Q_2)(1-Q_3) = Q_2 Q_3$ or $Q_2 + Q_3 = 1$, which implies first of all that $[Q_2] + [Q_3]$ is torsion in the Bloch group and secondly that $R = 1$ and hence (since h is integral!) that $1-Q_1 = Q_1^{A_1}$, which because of the choice of A_1 implies that $[Q_1]$ is also torsion. For the modularity, we note that $f_{A,B,C}$ for (A, B, C) as in (31) and (32) is equal to $\sum q^{Q(l,m,n)}/(q)_l (q)_m (q)_n$ where the sum is over all l, m, $n \geq 0$ and $Q(l, m, n)$ is the quadratic form

$$Q(l,m,n) \;=\; \frac{\alpha}{2}\,(hl + m - n - \nu)^2 \;+\; mn \;+\; \frac{A_1}{2}\,l^2 \;+\; B_1 l + C_1 \;-\; \frac{1}{24}\,.$$

The identity (27) then gives

Table 3. Rank 3 examples for Nahm's conjecture. In the first line α and ν are rational and h is integral. In the third line ν is rational.

A	$\begin{pmatrix} \alpha h^2+1 & \alpha h & -\alpha h \\ \alpha h & \alpha & 1-\alpha \\ -\alpha h & 1-\alpha & \alpha \end{pmatrix}, \quad c=\frac{3}{2}$		
B	$\begin{pmatrix} \alpha\nu h \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$	$\begin{pmatrix} \alpha\nu h+1/2 \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$	$\begin{pmatrix} \alpha\nu h-1/2 \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$
C	$\alpha\nu^2/2-1/16$	$\alpha\nu^2/2$	$\alpha\nu^2/2$

A	$\begin{pmatrix} \alpha h^2+2 & \alpha h & -\alpha h \\ \alpha h & \alpha & 1-\alpha \\ -\alpha h & 1-\alpha & \alpha \end{pmatrix}, \quad c=\frac{7}{5}$	
B	$\begin{pmatrix} \alpha\nu h \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$	$\begin{pmatrix} \alpha\nu h+1 \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$
C	$\alpha\nu^2/2-7/120$	$\alpha\nu^2/2+17/20$

A	$\begin{pmatrix} \alpha h^2+1/2 & \alpha h & -\alpha h \\ \alpha h & \alpha & 1-\alpha \\ -\alpha h & 1-\alpha & \alpha \end{pmatrix}, \quad c=\frac{8}{5}$	
B	$\begin{pmatrix} \alpha\nu h \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$	$\begin{pmatrix} \alpha\nu h+1/2 \\ \alpha\nu \\ -\alpha\nu \end{pmatrix}$
C	$\alpha\nu^2/2-1/15$	$\alpha\nu^2/2-1/60$

A	$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad c=\frac{3}{2}$				
B	$\begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$
C	$1/8$	$1/48$	$-1/24$	$1/8$	$3/16$

A	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad c=\frac{8}{7}$					
B	$\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}$
C	$-1/84$	$-1/84$	$5/84$	$5/84$	$17/84$	$17/84$

A	$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad c=\frac{9}{10}$					
B	$\begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3/2 \\ 1 \\ 1/2 \end{pmatrix}$
C	$1/40$	$-3/80$	$1/40$	$9/40$	$13/80$	$9/40$

A	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad c=1$						
B	$\begin{pmatrix} 0 \\ 2\nu \\ -2\nu \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1/2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1/2 \\ -1/2 \end{pmatrix}$
C	$\nu^2-1/24$	$7/48$	$7/48$	$5/24$	$5/24$	$-1/48$	$1/48$

A	$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad c=1$					
B	$\begin{pmatrix} -1 \\ -1 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$
C	$1/12$	$1/48$	$-1/24$	0	$1/3$	$11/24$

A	$\begin{pmatrix} 6 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad c=\frac{2}{3}$		
B	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$
C	$-1/30$	$1/12$	$11/36$

A	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad c=\frac{4}{5}$							
B	$\begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$
C	0	0	$-1/30$	$11/120$	$1/6$	$11/120$	$1/6$	$11/30$

A	$\begin{pmatrix} 4 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad c=\frac{3}{2}$				
B	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$
C	$-1/16$	$1/24$	$5/48$	$3/8$	$7/16$

A	$\begin{pmatrix} 8 & 4 & 1 \\ 4 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad c=\frac{9}{10}$	
B	$\begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1/2 \\ 1/2 \end{pmatrix}$
C	$1/40$	$9/40$

$$f_{A,B,C}(z) \;=\; \frac{1}{\eta(z)}\sum_{l\ge 0}\biggl(\sum_{r\in\mathbb{Z}} q^{\alpha(r+hl-\nu)^2/2}\biggr)\, q^{A_1l^2/2+B_1l+C_1}\,,$$

and (since h is integral!) we can now shift r by $-hl$ in the inner sum to see that $f_{A,B,C}$ is the product of f_{A_1,B_1,C_1} and the function (26). We also remark that one of the eight sporadic values of A also has an infinite family of (B,C) making $f_{A,B,C}$ modular (checked only numerically):

$$f_{\left(\begin{smallmatrix}2&1&1\\1&2&0\\1&0&2\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\2\nu\\-2\nu\end{smallmatrix}\right),\nu^2-\frac{1}{24}}(z) \;=\; \frac{1}{\eta(z)}\sum_{n\in\mathbb{Z}+\nu} q^{n^2} \qquad (\forall\nu\in\mathbb{Q}). \tag{33}$$

(f) *Duality.* Finally, we remark that the conditions on A in the conjecture are invariant under $A\mapsto A^{-1}$. For (i) this is clear, since if Q is a solution of (25) for some A then the vector $Q^\star=(1-Q_1,\dots,1-Q_r)$ is a solution for $A^\star=A^{-1}$ and $\xi_{Q^\star}$ equals $-\xi_Q$ modulo torsion in $\mathcal{B}(\overline{\mathbb{Q}})$. On the modular side, the analysis in **C** (specifically, eq. (38) below) suggests, though it does not prove, that if $f_{A,B,C}$ is modular then $f_{A^\star,B^\star,C^\star}$ is also modular and has an asymptotic expression as in (28) with (c,K) replaced by $(c^\star,K^\star)$, where

$$(A^\star,B^\star,C^\star,c^\star,K^\star) \;=\; \bigl(A^{-1},\,A^{-1}B,\,\tfrac12 B^tA^{-1}B-\tfrac{r}{24}-C,\,r-c,\,K\sqrt{\det A}\bigr).$$

These formulas, which one can verify in the examples in Table 2, were given by Nahm in [28], p. 663–4 and [29], p. 164. On the conformal field theory side, the involution $A\leftrightarrow A^{-1}$ is related to a duality found by Goddard-Kent-Olive [19] and to the so-called level-rank duality.

C. Asymptotic calculations. In this subsection we study the asymptotic behavior of $f_{A,B,C}(z)$ as $z\to 0$ ($q\to 1$), concentrating for simplicity on the case $r=1$, and use it to verify that the table of rank one solutions given in (a) of subsection **B** is complete. The analysis could in principle be carried out for larger values of r, but this would require a considerable effort and it is not clear whether one could use the method to complete the classification even in the next case $r=2$.

We consider real variables $A>0$, B and C and write $F_{A,B,C}(q)$ instead of $f_{A,B,C}(z)$, so that the definition (for $r=1$) becomes

$$F_{A,B,C}(q) \;=\; \sum_{n=0}^{\infty}\frac{q^{\frac12 An^2+Bn+C}}{(q)_n} \qquad (A,\,B,\,C\in\mathbb{R},\;\; A>0). \tag{34}$$

Since $F_{A,B,C}(q)=q^CF_{A,B,0}(q)$, it is enough to study the case $C=0$, in which case we omit the index C from the notation.

Proposition 5. *For any A, $B\in\mathbb{R}$, $A>0$, we have the asymptotic expansion*

$$\log F_{A,B}\bigl(e^{-\varepsilon}\bigr) \;\sim\; \sum_{j=-1}^{\infty} c_j(A,B)\,\varepsilon^j \qquad \bigl(\varepsilon\searrow 0\bigr), \tag{35}$$

with coefficients $c_j(A,B) \in \mathbb{R}$. The first two coefficients are given by

$$c_{-1}(A,B) \;=\; L(1) - L(Q)\,, \qquad c_0(A,B) \;=\; B\,\log Q \;-\; \tfrac{1}{2}\,\log \Delta\,, \tag{36}$$

where Q is the unique positive solution of $Q + Q^A = 1$, $L(x)$ is the Rogers dilogarithm, and $\Delta = A + Q - AQ$, while $c_j(A,B)$ for $j \ge 1$ is a polynomial of degree $j+1$ in B with coefficients in $\mathbb{Q}[Q,\, A,\, \Delta^{-1}]$, e.g.

$$\begin{aligned} c_1(A,B) \;=\; & \frac{1-Q}{2\Delta}\,B^2 \;-\; \frac{Q(1-Q)(1-A)}{2\Delta^2}B \\ & \qquad - \frac{(1-Q)((1-Q^2)A^3 - 3QA^2 + 3Q(1+Q)A - 2Q^2)}{24\Delta^3}\,. \end{aligned}$$

Before proving the proposition, we say a little bit more about the coefficients c_j. The polynomial $c_j = c_j(A,B)$ belongs to $\Delta^{-3j}\mathbb{Q}[A,Q,B\Delta]$. Its leading term is $\alpha_j B^{j+1}/(j+1)$ where $\sum \alpha_j x^j$ is the Taylor expansion of $\log Q(x)$, $Q(x) + e^{-x}Q(x)^A = 1$, e.g.

$$c_2 = -\frac{Q(1-Q)}{6\Delta^3}\,B^3 + \cdots\,, \quad c_3 = \frac{Q(1-Q)(3Q-(1+Q)\Delta)}{24\Delta^5}\,B^4 + \cdots\,.$$

We omit the other coefficients, giving only the constant term of c_2:

$$\begin{aligned} c_2(A,0) \;=\; & \frac{Q(1-Q)(A-1)}{48\,\Delta^6}\,\bigl[-4A(A+1)(A-1)^2Q^3 \\ & + 3A(A-1)(A-2)(2A+1)Q^2 \;+\; 2A^2(5A-4)Q - A^3(2A-1)\bigr]\,. \end{aligned}$$

We will describe three different approaches for obtaining the asymptotics of $\log F_{A,B}(e^{-\varepsilon})$ as $\varepsilon \to 0$. The first, based on a functional equation for $F_{A,B}$, is short and elementary, but gives the coefficients $c_j(A,B)$ only up to a term depending on A. The second is also elementary—essentially based on the Euler-Maclaurin summation formula—and gives the entire asymptotic expansion, but requires more work. The third, which is the one used by Nahm and his collaborators, is based on Cauchy's theorem together with formula (8) for the quantum dilogarithm, and also leads to a full expansion; this method also has a variant, using (7) rather than (8) for the quantum dilogarithm, which has apparently not been noticed before. It seemed worth presenting all three approaches, at least briefly, since the information they give is somewhat different and since each of them is applicable (at least in principle) to the general case of Nahm's conjecture and to many other problems of this type.

(a) For the first approach we set

$$\frac{F_{A,B}\bigl(e^{-\varepsilon}\bigr)}{F_{A,0}\bigl(e^{-\varepsilon}\bigr)} \;=\; Q^B\,H_{A,B}(\varepsilon)$$

where $H_{A,B}(\varepsilon) = \sum_{n=0}^{\infty} h_n(A,B)\,\varepsilon^n$ is a power series in ε satisfying $H_{A,B}(0) \equiv 1$, $H_{A,0}(\varepsilon) \equiv 1$. Knowing $H_{A,B}(\varepsilon)$ is tantamount to knowing $c_j(A,B) - c_j(A,0)$ for all $j \geq 1$. The functional equation

$$F_{A,B}(q) \;-\; F_{A,B+1}(q) \;=\; \sum_{n=1}^{\infty} \frac{q^{\frac{1}{2}An^2+Bn}}{(q)_{n-1}} \;=\; q^{\frac{1}{2}A+B}\, F_{A,B+A}(q) \qquad (37)$$

of $F_{A,B}(q)$ gives a functional equation

$$H_{A,B}(\varepsilon) \;=\; Q\, H_{A,B+1}(\varepsilon) \;+\; (1-Q)\, e^{-(\frac{1}{2}A+B)\varepsilon}\, H_{A,B+A}(\varepsilon)$$

for $H_{A,B}(\varepsilon)$. Write $H_{A,B}(\varepsilon) = \sum_{n=0}^{\infty} h_n(A,B)\,\varepsilon^n$ where $h_0 \equiv 1$ and $h_n(A,B)$ is a polynomial of degree $2n$ in B without constant term for $n \geq 1$. Substituting this into the functional equation and replacing B by $B-A$, we obtain

$$h_n(A,B-A) \;=\; Q\, h_n(A,B+1-A) \;+\; (1-Q) \sum_{s=0}^{n} \frac{(A/2-B)^s}{s!}\, h_{n-s}(A,B)\,,$$

and this equation gives the coefficient $h_{n,m}$ of B^m in $h_n(A,B)$ recursively in terms of earlier coefficients $h_{n',m'}$ with $n' < n$ or with $n' = n$, $m' < m$.

Remark. The functional equation (37) remains true formally if one replaces $F_{A,B}(q)$ by $F^{\star}_{A,B}(q) = q^{B^2/2A} F_{1/A,B/A}(q^{-1})$, so the formal power series $H_{A,B}(\varepsilon)$ satisfies a second functional equation $H_{1/A,B/A}(-\varepsilon) = e^{\varepsilon B^2/2A}$ $H_{A,B}(\varepsilon)$. (Note that the change $(A,B) \mapsto (1/A, B/A)$ changes Q to $1-Q = Q^A$ and does not change B^2/A or Q^A.) It follows that the coefficients $c_j(A,B)$ in (35) satisfy the functional equations

$$c_j(A,B) \;-\; (-1)^j\, c_j(1/A,B/A) \;=\; \gamma_j(A) \;+\; \begin{cases} B^2/2A & \text{if } j=1, \\ 0 & \text{otherwise} \end{cases}$$

for some constants $\gamma_j = \gamma_j(A)$ independent of B. Using the explicit formulas for c_j which will be computed below, we find that $\gamma_{-1} = L(1)$, $\gamma_0 = -\frac{1}{2}\log A$, $\gamma_1 = -\frac{1}{24}$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$. This suggests the **conjecture** that $\gamma_j = 0$ for all $j \geq 2$. Assuming this, we have the formal functional equation

$$F^{\star}_{A,B}\bigl(e^{-\varepsilon}\bigr) \;=\; \sqrt{A}\,\exp\bigl(-\frac{\pi^2}{6\varepsilon} + \frac{\varepsilon}{24} + \mathrm{O}\bigl(\varepsilon^N\bigr)\bigr)\, F_{A,B}\bigl(e^{-\varepsilon}\bigr) \qquad (\forall N > 0). \quad (38)$$

(b) The second method is based on the asymptotics of the individual terms in the series (34). Denote the nth term in this series by u_n. Then the ratio $u_n/u_{n-1} = q^{An+B-A/2}/(1-q^n)$ is small for q small and large for q very near 1, and becomes equal to 1 when q^n is near to the unique root $Q \in (0,1)$ of the equation $Q + Q^A = 1$. The terms that contribute are therefore those of the form $q^n = Qq^{-\nu}$ with $\nu \in \nu_0 + \mathbb{Z}$ of order much less than n, where ν_0 denotes the fractional part of $-\log(Q)/\log(q)$. To know the size of u_n in this range we first have to know the behavior of $(q)_n$ as $q \to 1$ and $n \to \infty$ with q^n tending to a fixed number Q. It is given by:

Lemma. *For fixed $Q \in (0,1)$ and $q = e^{-\varepsilon} \to 1$ with $q^n = Qq^{-\nu}$ with $n \to \infty$, $\nu = o(n)$, we have*

$$\log\Big(\frac{1}{(q)_n}\Big) \sim \Big[\frac{\pi^2}{6} - \mathrm{Li}_2(Q)\Big]\,\varepsilon^{-1} - \Big[(\nu - \frac{1}{2})\,\log\big(\frac{1}{1-Q}\big) + \frac{1}{2}\log\big(\frac{2\pi}{\varepsilon}\big)\Big]$$
$$-\Big[\frac{1}{24} + \frac{1}{2}\big(\nu^2 - \nu + \frac{1}{6}\big)\,\frac{Q}{1-Q}\Big]\,\varepsilon - \sum_{r=3}^{\infty} B_r(\nu)\,\mathrm{Li}_{2-r}(Q)\,\frac{\varepsilon^{r-1}}{r!}$$

as $\varepsilon \to 0$, where $B_r(\nu)$ denotes the rth Bernoulli polynomial and $\mathrm{Li}_{-j}(Q)$ *is the negative-index polylogarithm* $\mathrm{Li}_{-j}(Q) = \sum_{m=1}^{\infty} m^j Q^m = \big(Q\frac{d}{dQ}\big)^j \frac{1}{1-Q}$.

Proof. The Euler-Maclaurin formula or the modularity of $\eta(z)$ gives

$$\log\Big(\frac{1}{(q)_\infty}\Big) = \frac{\pi^2}{6\varepsilon} - \frac{1}{2}\log\big(\frac{2\pi}{\varepsilon}\big) - \frac{\varepsilon}{24} + \mathrm{O}\big(\varepsilon^N\big)$$

for all N as $\varepsilon \to 0$. On the other hand, we have

$$\begin{aligned}\log\Big(\frac{(q)_n}{(q)_\infty}\Big) &= \sum_{s=1}^{\infty}\log\Big(\frac{1}{1-q^{n+s}}\Big) = \sum_{s=1}^{\infty}\log\Big(\frac{1}{1-Qq^{s-\nu}}\Big)\\ &= \sum_{s=1}^{\infty}\sum_{k=1}^{\infty}\frac{Q^k}{k}\,q^{k(s-\nu)} = \sum_{k=1}^{\infty}\frac{Q^k}{k}\,\frac{e^{\nu k\varepsilon}}{e^{k\varepsilon}-1}\\ &= \sum_{k=1}^{\infty}\frac{Q^k}{k}\sum_{r=0}^{\infty}\frac{B_r(\nu)}{r!}\,(k\varepsilon)^{r-1} = \sum_{r=0}^{\infty}\frac{B_r(\nu)}{r!}\,\mathrm{Li}_{2-r}(Q)\,\varepsilon^{r-1}\,.\end{aligned}$$

Subtracting these two formulas gives the desired result. □

With the same conventions ($q = e^{-\varepsilon}$, $q^n = Qq^{-\nu}$), we also have

$$\begin{aligned}\log\big(q^{\frac{1}{2}An^2+Bn+C}\big) &= -\frac{A\varepsilon}{2}\Big(\frac{\log(1/Q)}{\varepsilon} - \nu\Big)^2 - B\varepsilon\Big(\frac{\log(1/Q)}{\varepsilon} - \nu\Big)\\ &= -\frac{\log(Q)\log(1-Q)}{2\varepsilon} + \big(B - \nu A\big)\log Q + \big(-\frac{A\nu^2}{2} + B\nu\big)\,\varepsilon\,,\end{aligned}$$

where we have used $A\log(Q) = \log(1-Q)$. Together with the lemma this gives

$$\log\Big(\frac{q^{\frac{1}{2}An^2+Bn+C}}{(q)_n}\Big) = \frac{L(1) - L(Q)}{\varepsilon} - \frac{1}{2}\log\frac{2\pi}{\varepsilon} + \log\frac{Q^B}{\sqrt{1-Q}}$$
$$-\Big(\frac{\Delta}{1-Q}\,\frac{\nu^2}{2} - \big(B + \frac{1}{2}\frac{Q}{1-Q}\big)\nu + \frac{1}{24}\frac{1+Q}{1-Q}\Big)\,\varepsilon - \sum_{r=3}^{\infty} B_r(\nu)\,\mathrm{Li}_{2-r}(Q)\,\frac{\varepsilon^{r-1}}{r!}$$

with $\Delta = A + Q - AQ$ as before. If we write this as $\log\varphi(\nu)$ where φ is a smooth function of rapid decay, then $F_{A,B}(q)$ equals $\sum_{\nu\equiv\nu_0 \pmod 1}\varphi(\nu)$, which by the Poisson summation formula can be written as $\sum_{r\in\mathbb{Z}}\widetilde{\varphi}(r)e^{2\pi i r\nu_0}$ where $\widetilde{\varphi}$ is the Fourier transform of φ. The smoothness of φ implies that all terms except $r=0$ give contributions which are $\mathrm{O}(\varepsilon^N)$ for all $N>0$ as $\varepsilon\to 0$. We therefore obtain the asymptotic expansion

$$F_{A,B}\bigl(e^{-\varepsilon}\bigr) \;=\; \frac{Q^B}{\sqrt{\Delta}}\cdot\exp\biggl(\frac{L(1)-L(Q)}{\varepsilon} \;-\; \frac{1+Q}{1-Q}\,\frac{\varepsilon}{24}\biggr)$$
$$\cdot\,\mathbf{I}\biggl[\exp\biggl(\bigl(B+\frac{1}{2}\frac{Q}{1-Q}\bigr)u\varepsilon-\sum_{r=3}^{\infty}B_r(u)\,\mathrm{Li}_{2-r}(Q)\,\frac{\varepsilon^{r-1}}{r!}\biggr)\biggr]\biggl(\frac{1-Q}{\Delta\varepsilon}\biggr),$$

where $\mathbf{I}[\;]$ is the functional defined formally by the integral transform

$$\mathbf{I}\bigl[H(u)\bigr](t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}e^{-u^2/2t}\,H(u)\,du\,, \tag{39}$$

and explicitly at the level of power series by

$$\mathbf{I}\biggl[\sum_{n=0}^{\infty}c_n\,u^n\biggr](t)\quad\sim\quad\sum_{n=0}^{\infty}(2n-1)!!\,c_{2n}\,t^n\,, \tag{40}$$

where $(2n-1)!! = (2n)!/2^n n!$ as usual. It is not immediately obvious why this expansion makes sense, since the argument of the functional grows like $1/\varepsilon$ as $\varepsilon\to 0$, but the power series to which it is applied has the property that the coefficient of u^n is $\mathrm{O}(\varepsilon^{2n/3})$ for every $n\ge 0$, and since the functional $\mathbf{I}[\;]$ has the scaling property $\mathbf{I}\bigl[H(\lambda u)\bigr](t) = \mathbf{I}\bigl[H(u)\bigr](\lambda^2 t)$, the final expression does indeed involve only positive powers of ε. (Choose $\lambda=\varepsilon^c$ with $\frac{1}{2}<c<\frac{2}{3}$.)

(c) The third method, which is based on a clever application of Cauchy's theorem going back originally to an old paper of Meinardus [26], was first used in this context by Nahm, Recknagel and Terhoeven [31] and is also the one used in Nahm's paper [30] in this volume, so that we will only indicate the main idea of the method here. We write $F_{a,b}(q)$ as

$$F_{A,B}(q) \;=\; C_{x^0}\biggl[\biggl(\sum_{n=-\infty}^{\infty}q^{\frac{1}{2}An^2+Bn}\,x^{-n}\biggr)\biggl(\sum_{n=0}^{\infty}\frac{1}{(q)_n}\,x^n\biggr)\biggr]$$

where $C_{x^0}[\,\cdot\,]$ denotes the constant term of a Laurent series. The first factor is a theta series with a well-understood asymptotic behavior as $q\to 1$ and the second equals $(x;q)_\infty^{-1}$ by equation (8) and hence also has known asymptotics. Specifically, the Poisson summation formula (or the Jacobi transformation formula for theta series) shows that

$$\theta_{A,B}(z,u) \;:=\; \sum_{n=-\infty}^{\infty}q^{\frac{1}{2}An^2+Bn}\,x^{-n} \;=\; \sqrt{\frac{i}{Az}}\;\sum_{n=-\infty}^{\infty}\mathbf{e}\bigl(-\frac{(u-Bz+n)^2}{2Az}\bigr)\,,$$

where $q = \mathbf{e}(z) := e^{2\pi i z}$, $x = \mathbf{e}(u)$. Hence we find, for any $u_0 \in \mathbb{C}$,

$$\begin{aligned} F_{A,B}(q) &= \int_{u_0}^{u_0+1} \theta_{A,B}(z,u)\, \bigl(\mathbf{e}(u);\, q\bigr)_\infty^{-1}\, du \\ &= \sqrt{\frac{i}{Az}} \int_{\Im(u)=\Im(u_0)} \mathbf{e}\Bigl(-\frac{(u-Bz)^2}{2Az}\Bigr)\, \bigl(\mathbf{e}(u);\, q\bigr)_\infty^{-1}\, du \\ &= \frac{1}{\sqrt{2\pi A\varepsilon}} \int_{\Im(t)=\text{const}} \exp\Bigl(-\frac{t^2}{2A\varepsilon} + \mathrm{Li}_2(q^B e^{it};\, q)\Bigr)\, dt\,. \end{aligned}$$

where in the last line we have substituted $u = Bz + t/2\pi$. The derivative with respect to t of the argument of exp equals $[A\log(1-e^{it}) - it]/iA\varepsilon + \mathrm{O}(1)$, which vanishes at $it = \log(1-Q) + \mathrm{O}(\varepsilon)$, where Q is the solution between 0 and 1 (or, of course, any other solution) of Nahm's equation $1-Q = Q^A$. Now moving the path of integration to a neighbourhood of this point and applying the method of stationary phase (saddle point method), we obtain the desired asymptotic expansion for $F_{A,B}\bigl(e^{-\varepsilon}\bigr)$. This method has the further advantage, as explained in Nahm's paper, that the contributions from the further saddle points could in principle be used to describe all the terms of the q-expansion of $f_{A,B,C}(-1/z)$ in the cases when $f_{A,B,C}$ is a modular function.

As already mentioned, one can also do an almost exactly similar calculation using the power series expansion (7) rather than (8), at least if $A > 1$: we now write

$$\begin{aligned} F_{A,B}(q) &= \mathbf{C}_{x^0}\Biggl[\Biggl(\sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{A-1}{2}n^2 + (B+\frac12)n}\, x^{-n}\Biggr)\Biggl(\sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\frac{n^2-n}{2}}}{(q)_n}\, x^n\Biggr)\Biggr] \\ &= \int_{u_0}^{u_0+1} \sqrt{\frac{i}{(A-1)z}} \sum_{n\in\mathbb{Z}} \mathbf{e}\Biggl(\frac{(u-(B+\frac12)z+n+\frac12)^2}{2(A-1)z}\Biggr)\, \bigl(\mathbf{e}(u); q\bigr)_\infty\, du \\ &= \frac{1}{\sqrt{2\pi(A-1)\varepsilon}} \int_{\Im(t)=\text{const}} \exp\Biggl(-\frac{t^2}{2(A-1)\varepsilon} - \mathrm{Li}_2(-q^{B+\frac12} e^{it};\, q)\Biggr)\, dt\,, \end{aligned}$$

and again an asymptotic expansion could now be obtained by the saddle point method. I have not carried out the details.

This completes our discussion and proof of Proposition 5. We remark that each of the methods described applies also to $r > 1$, and we again find an expansion of $\log F_{A,B}(q)$ of the form (35), with leading coefficient given by

$$c_{-1}(A,B) \;=\; \sum_{i=1}^{r}\bigl(L(1) - L(Q_i)\bigr) \;=\; rL(1) \,-\, L(\xi_A) \;=\; c(A)L(1)\,.$$

In particular, the modularity of $f_{A,B,C}(z)$ for any $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$ requires that $L(\xi_A)$ be a rational multiple of π^2 and hence suggests very strongly that ξ_A has to be a torsion element of $\mathcal{B}(\mathbb{Q}(\xi_A))$, as required by Nahm's conjecture.

The higher coefficients are given by a formula essentially the same as in the case $r=1$, except that the previous expression of the form $\mathbf{I}\big[H(u)\big]\big(\frac{1-Q}{\Delta\varepsilon}\big)$ with $\mathbf{I}[\;]$ defined by equation (30) or (31) must now be replaced by its multi-dimensional generalization

$$\frac{\sqrt{\det\widetilde{A}}}{(2\pi)^{r/2}}\int_{\mathbb{R}^r}\exp\bigl(-\frac{\varepsilon}{2}u\widetilde{A}u^t\bigr)\,H(u)\,du\quad\sim\quad\sum_{n=0}^{\infty}\frac{(2\varepsilon)^{-n}}{n!}\,\Delta_{\widetilde{A}}^{n}H(0),$$

where $\widetilde{A}$ is given by (30) and $\Delta_{\widetilde{A}}$ denotes the Laplacian with respect to the quadratic form $u\widetilde{A}u^t$.

We now use the proposition to classify the modular $f_{A,B,C}$ when $r=1$.

Theorem. *The only $(A,B,C)\in\mathbb{Q}_+\times\mathbb{Q}\times\mathbb{Q}$ for which $f_{A,B,C}(z)$ is a modular form are those given in Table 1.*

The basic idea of the proof is that, if $f(z)=F(q)$ is a modular form of weight k on some group Γ, then the function $g(z)=z^{-k}f(-1/z)$ is also a modular form (on the group $S\Gamma S$, where $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$) and hence has an expansion as $z\to 0$ of the form $a_0q^{n_0}+a_1q^{n_1}+\cdots$ for some rational exponents $n_0<n_1<\cdots$ and non-zero algebraic coefficients a_0, a_1, ... In particular we have $F\bigl(e^{-\varepsilon}\bigr)=f(i\varepsilon/2\pi)=(2\pi i/\varepsilon)^k\,a_0\,e^{-4\pi^2n_0/\varepsilon}+\mathrm{O}\bigl(\varepsilon^{-k}e^{-4\pi^2n_1/\varepsilon}\bigr)$ and hence $\log F\bigl(e^{-\varepsilon}\bigr)=-4\pi^2n_0\varepsilon^{-1}-k\log\varepsilon+c+\mathrm{O}(\varepsilon^N)$ for all $N>0$ as ε tends to 0, where $c=\log((2\pi i)^ka_0)$. Comparing this with the expansion of $\log F_{A,B,C}\bigl(e^{-\varepsilon}\bigr)=\log F_{A,B}\bigl(e^{-\varepsilon}\bigr)-C\varepsilon$ as given in equation (35), we see that if $f_{A,B,C}(z)$ is a modular form of weight k then

(i) the weight k must be 0 (i.e. $f_{A,B,C}(z)$ must be a modular *function*);
(ii) the number $c_{-1}(A,B)$ must be π^2 times a rational number;
(iii) the number $c_0(A,B)$ must be the logarithm of an algebraic number;
(iv) the number $c_1(A,B)$ must be a rational number (namely, C); and
(v) the numbers $c_j(A,B)$ must vanish for all $j\ge 2$.

The conditions (ii)–(iv) are useless for numerical work, since the rationals lie dense in $\mathbb{R}$, but condition (v) gives infinitely many constraints on the two real numbers A and B and can be used to determine their possible values. This approach was already used by Terhoeven in 1994 to prove a weaker version of the theorem, namely, that the only values of $A\in\mathbb{Q}^+$ for which $f_{A,0,C}$ is modular for some $C\in\mathbb{Q}$ are 1/2, 1 and 2. (See [36], where, however, the details of the calculation are not given.) To do this, Terhoeven computed $c_j(A,B)$ up to $j=2$ using the method (c) above, as developed in his earlier joint paper [31]. If $f_{A,0,C}$ is to be modular for some C, then the number $c_2(A,0)$ must vanish. We have $c_2(A,0)=P(A,Q)/(A+Q-AQ)^6$ for a certain polynomial $P(A,Q)$ with rational coefficients (given after Proposition 5), where Q is the root of $Q+Q^A=1$ in (0,1). By looking at the graph of the function $\Phi(Q)=P\bigl(\frac{\log(1-Q)}{\log Q},\,Q\bigr)$ on the interval (0,1), we find numerically that it has a simple zero at $Q_1=(3-\sqrt{5})/2$, a double zero at $Q_2=1/2$, a simple zero at $Q_3=$

$(\sqrt{5}-1)/2$, and no other zeros. Since $P(Q_1, \frac{1}{2}) = P(Q_2, 1) = P(Q_3, 2) = 0$ and since $\Phi(Q) = \Phi(1-Q)$, we know that there really are zeros of the specified multiplicities at Q_1, Q_2 and Q_3. It follows that they are the only zeros in this interval and hence that $f_{A,0,C}$ can only be modular for the three values given in Table 1.

To prove the full theorem we proceed in the same way, but since we now have two variables A and B to determine we must use at least two coefficients $c_j(A,B)$. The functions $c_2(A,B)$ and $c_3(A,B)$ are polynomials in B of degree 3 and 4, respectively, whose coefficients are known elements of $\mathbb{Q}(A,Q)$. For fixed $A \in \mathbb{R}^+$ the condition that the two polynomials $c_2(A,\,\cdot\,)$ and $c_3(A,\,\cdot\,)$ have a common zero is that their resultant vanishes. From the explicit formulas for c_2 and c_3 we find that this resultant has the form $R(A,Q)/(A+Q-AQ)^{39}$ where $R(A,Q)$ is a (very complicated) polynomial in A and Q with rational coefficients. Now a graphical representation of the function $\Psi(Q) = R\big(\frac{\log(1-Q)}{\log Q)}, Q\big)$ on the interval (0,1) (which we can unfortunately not show here since this function varies by many orders of magnitude in this range and hence has to be looked at at several different scales on appropriate subintervals) shows that it has *nine* zeros (counting multiplicities), namely:

$$\begin{aligned}
&\text{a simple zero at } Q_0 &&= 0.19600353454518444708574616009357\cdots\\
&\text{a double zero at } Q_1 &&= 0.3819660\cdots = (3-\sqrt{5})/2\\
&\text{a triple zero at } Q_2 &&= 0.5000000\cdots = 1/2\\
&\text{a double zero at } Q_3 &&= 0.6180339\cdots = (\sqrt{5}-1)/2\\
&\text{a simple zero at } Q_4 &&= 0.80399646545481555291425\cdots = 1-Q_0
\end{aligned}$$

and no other zeros. Since $\Psi(1-Q) = -\Psi(Q)$ and since we know from Table 1 that $\Psi(Q)$ has to have *at least* a double zero at $Q = Q_1$ and at $Q = Q_3$ and a triple one at $Q = Q_2$, we deduce that the numerically found zeros at these places are really there and correspond to the known cases of modularity. As to the new value Q_0 and $Q_4 = 1 - Q_0$ and the corresponding A-values $A_0 = 0.1338717368167616096950604067073585\cdots$ and $A_4 = A_0^{-1}$, we find that indeed $c_2(A_0,B)$ and $c_3(A_0,B)$ have a (simple) common root at $B = B_0 = -0.397053221675466369\ldots$, as they must since their resultant vanishes. But for the pair (A_0,B_0) we find numerically

$$c_{-1}(A_0,B_0)/\pi^2 \;=\; 0.12772794682938816298878 98\cdots$$

which is (apparently) not a rational number,

$$\exp(c_0(A_0,B_0)) \;=\; 3.4660299497719132664077586\cdots$$

which is (apparently) not algebraic,

$$c_1(A_0,B_0) \;=\; 0.4917635587907976876492549\cdots$$

which is (apparently) not rational, and finally

$$c_4(A_0,B_0) \;=\; 0.0175273497972616555765902\cdots$$

which is (definitely) not zero. It follows that the function $f_{A_0,B_0,c_1(A_0,B_0)}(z)$ is not modular and thus that the list given in Table 1 is complete. $\square$

This analysis was quite tedious and would become prohibitively so for $r \geq 2$ (although the method applies in principle) because we would need explicit formulas for many more of the coefficients $c_j(A,B)$. An alternative approach might be to refine Proposition 5 by showing that

$$\log F_{A,B}\big(\zeta\, e^{-\varepsilon}\big) \;\sim\; \sum_{j=-1}^{\infty} c_{j,\zeta}(A,B)\,\varepsilon^{j} \qquad \big(\varepsilon \searrow 0\big)$$

for every root of unity ζ. If $f_{A,B,C}(z)$ is modular for some $C \in \mathbb{Q}$, then the logarithm of $F_{A,B,C}\big(\zeta\, e^{-\varepsilon}\big) = \zeta^{C} e^{-C\varepsilon} F_{A,B}\big(\zeta\, e^{-\varepsilon}\big)$ has vanishing coefficient of ε^j in its expansion at $\varepsilon = 0$ for every $j \geq 1$. In particular we must have $c_{1,\zeta}(A,B) = C$ for all roots of unity ζ with the *same* constant C, so we get infinitely many constraints on A and B without having to calculate any of the Taylor expansions further than their $\mathrm{O}(\varepsilon)$ term. I have not yet tried to carry this out, but it seems to hold out good prospects for an easier proof of the $r = 1$ case and perhaps a reasonable attack on the higher rank cases of the conjecture as well.

Finally, we remark that Proposition 5 (or its generalization to $r > 1$) can be used to search efficiently for values of B for a given A for which $f_{A,B,C}$ may be modular for some C. Indeed, since the function $\phi(\varepsilon) = \log(F_{A,B}(e^{-\varepsilon})$ must have a terminating asymptotic expansion of the form $c_{-1}\varepsilon^{-1} + c_0 + c_1\varepsilon$ with an error term that is exponentially small in $1/\varepsilon$, we can simply check whether four successive values of $n\phi(\alpha/n)$ (say, those with $\alpha = 1$ and $n = 20$, 21, 22, 23) are approximated to high precision by a quadratic polynomial in n (i.e., whether the third difference of this 4-tuple is extremely small). In fact, since we know c_{-1} by (30), we can look instead at three sucessive values of $n\phi(\alpha/n) - c_{-1}n^2/\alpha$ and check whether they lie on a line (i.e., whether their second difference vanishes) to high precision. This can be done very rapidly and therefore we can search through a large collection of candidate vectors $B \in \mathbb{Q}^r$ for those which can correspond to some modular $f_{A,B,C}$. This is the method which was used to find the modular solutions for $r = 2$ amd 3 given in (d) and (e) of subsection **B**.

4 Higher polylogarithms

In Chapter I we already introduced the higher polylogarithm functions $\mathrm{Li}_m(z)$, their one-valued modifications $D_m(z)$, and the idea that there might be relations among special values of Dedekind zeta functions at integral arguments and the values of polylogarithms at algebraic arguments, analogous to those existing for the dilogarithm. But at the time when that text was written, I knew only a few sporadic results, and the chapter ends with the sentence "the general picture is not yet clear." After it was written I did a lot of numerical calculations with special values of higher polylogarithm functions and was able to formulate a general conjecture which has now been proved in a small number and numerically verified in a large number of cases. Since there are already several expositions of these conjectures ([45], [46], Chapter 1 of [47]), I give only a sketch here, the more so because higher polylogarithms seem so far to play much less of a role in mathematical physics than the dilogarithm. (See, however, [35].) Nevertheless, the higher polylogarithm story is very pretty and it seemed a pity to omit it entirely.

We describe the conjectures in Section A and some supporting examples in Section B. It turns out that the theory works better if one replaces the function $D_m(z)$ defined in Chapter I by the slightly different function

$$\mathcal{L}_m(z) \;=\; \Re_m\biggl(\sum_{k=0}^{m-1} B_k\,\frac{(\log|z|^2)^k}{k!}\,\mathrm{Li}_{m-k}(z)\biggr),$$

where $\Re_m$ denotes $\Re$ for m odd and $\Im$ for m even and B_k is the kth Bernoulli number. Thus $\mathcal{L}_2(z) = D_2(z) = D(z)$, but $\mathcal{L}_3(z) = D_3(z) + \frac{1}{6}\log^2|z|\,\log\bigl|\frac{1-z}{\sqrt{z}}\bigr|$. Like D_m, $\mathcal{L}_m$ is a single-valued real-analytic function on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0,1,\infty\}$, but it is has the advantages of being continuous at the three singular points, of having clean functional equations (for instance, the logarithmic terms in the two functional equations for D_3 given in §3 of Chapter I are absent when these are written in terms of $\mathcal{L}_3$), and of leading to a simpler formulation of the conjectures relating the polylogarithms to zeta values and to algebraic K-theory.

A. Higher Bloch groups and higher K-groups. In Chapter I we defined the Bloch group $\mathcal{B}(F)$ of a field F (where we have changed the notation from the previous $\mathcal{B}_F$, because we will have higher Bloch groups $\mathcal{B}_m(F)$ as well) as $\mathcal{A}(F)/\mathcal{C}(F)$, where $\mathcal{A}(F)$ is the set of all elements $\xi = \sum n_i[x_i] \in \mathbb{Z}[F]$ satisfying $\partial(\xi) := \sum n_i(x_i) \wedge (1-x_i) = 0$ and $\mathcal{C}(F)$ the subgroup generated by the five-term relations $V(x,y)$ and by its degenerations $[x] + [1/x]$ and $[x]+[1-x]$. Because the elements of $\mathcal{C}(\mathbb{C})$ are in the kernel of the Bloch-Wigner dilogarithm D we have a map $D : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$. The key statement is that for a number field F of degree r_1+2r_2 over $\mathbb{Q}$ (r_1 real and $2r_2$ complex embeddings), the group $\mathcal{B}(F)$ has rank r_2 and the map $\mathcal{L}_F : \mathcal{B}(F)/(\text{torsion}) \to \mathbb{R}^{r_2}$ induced by the values of D on the various conjugates of $\xi \in \mathcal{B}(F)$ (there are only r_2 essentially different such values because $D(\bar{x}) = -D(x)$) is an isomorphism onto a lattice whose covolume is essentially equal to $\zeta_F(2)$.

The conjectural picture for the mth polylogarithm, $m \ge 3$, is that we can introduce similarly defined higher Bloch groups $\mathcal{B}_m(F) = \mathcal{A}_m(F)/\mathcal{C}_m(F)$, where $\mathcal{A}_m(F)$ is a suitably defined subgroup of $\mathbb{Z}[F]$ and $\mathcal{C}_m(F) \subseteq \mathcal{A}_m(F)$ is the subgroup corresponding to the functional equations of the higher polylogarithm function $\mathcal{L}_m$, in such a way that the rank of $\mathcal{B}_m(F)$ is equal to r_2 or $r_1 + r_2$ (depending whether m is even or odd) and that the map $\mathcal{L}_{m,F} : \mathcal{B}_m(F)/(\text{torsion}) \to \mathbb{R}^{(r_1+)r_2}$ induced by the values of $\mathcal{L}_m$ on the various conjugates of $\xi \in \mathbb{Z}[F]$ is an isomorphism onto a lattice of finite covolume. (Note that $\mathcal{L}_m(\bar{x}) = (-1)^{m-1}\mathcal{L}_m(x)$, which is why there are only r_2 essentially different values of $\mathcal{L}_m$ if m is even, but r_1+r_2 if m is odd.) Again, this implies the existence of numerous $\mathbb{Q}$-linear relations among the values of

$\mathcal{L}_m(x)$, $x \in F$. Moreover, the covolume of the lattice $\mathrm{Im}(\mathcal{L}_{m,F})$ is supposed to be a simple multiple of $\zeta_F(m)$, so that one also obtains (conjectural) formulas for the values of the Dedekind zeta functions of arbitrary number fields at $s = m$ in terms of the mth polylogarithm function with algebraic arguments.

Furthermore, the whole picture is supposed to correspond to the higher K-groups of F, as already mentioned at the end of Chapter I. For each $i \geq 0$ one has K-groups $K_i(F)$ and $K_i(\mathcal{O}_F)$ which for $i \geq 2$ become isomorphic after tensoring with $\mathbb{Q}$. The group $K_i(\mathcal{O}_F)$ is finitely generated and its rank was determined by Borel [5]: it is 0 for i even and is r_2 or $r_1 + r_2$ (depending as before whether m is even or odd) if $i = 2m - 1$, $m \geq 2$. Moreover, Borel showed that there is a natural "regulator" map $R_{m,F} : K_{2m-1}(F) \to \mathbb{R}^{(r_1+)r_2}$ which gives an isomorphism of $K_{2m-1}(F)/(\text{torsion})$ onto a sublattice whose covolume is a simple multiple of $\zeta_F(m)$. The motivation for the polylogarithm conjectures as stated above is that we expect there to be an isomorphism (at least after tensoring with $\mathbb{Q}$) between $K_{2m-1}(F)$ and $\mathcal{B}_m(F)$ such that the Borel regulator map $R_{m,F}$ and the polylogarithm map $\mathcal{L}_{m,F}$ correspond. This is not known at all in general, but de Jeu [12] and Beilinson–Deligne [2] defined a map compatible with $R_{m,F}$ and $\mathcal{L}_{m,F}$ from (a version of) $\mathcal{B}_m(F)$ to $K_{2m-1}(F)$, and for $m = 3$ Goncharov [20] proved the surjectivity of this map, establishing in this special case my conjecture that $\zeta_F(3)$ for any number field F can be expressed in terms of polylogarithms.

In the rest of this subsection, we describe the inductive definition of the groups $\mathcal{B}_m(F)$ in a way which is algorithmically workable, though theoretically unfounded.

The first case is $m = 3$. A first candidate for $\mathcal{A}_3(F)$ is $\mathrm{Ker}(\partial_3)$, where $\partial_3 : \mathbb{Z}[F] \to F^\times \otimes \Lambda^2 F^\times$ is the map sending $[x]$ to $(x) \otimes ((x) \wedge (1 - x))$ if $x \neq 0, 1$ (and to 0 if $x = 0$ or 1). If this definition and the above hypothetical statements are correct, we deduce—even without knowing the definitions of $\mathcal{C}_3(F)$ and $\mathcal{B}_3(F)$—that any $r_1 + r_2 + 1$ values $\mathcal{L}_3(\sigma(x))$, where $\sigma : F \to \mathbb{C}$ is a fixed embedding and x ranges over values in F, should be $\mathbb{Q}$-linearly dependent. Numerical evidence supported this for totally real fields. For instance, if $F = \mathbb{Q}(\sqrt{5})$ then the three one-term elements $[1]$, $[(1+\sqrt{5})/2]$ and $[(1-\sqrt{5})/2]$ belong to $\mathrm{Ker}(\partial_3)$ and we find (numerically, but here provably) that $\mathcal{L}_3\bigl(\frac{1+\sqrt{5}}{2}\bigr) + \mathcal{L}_3\bigl(\frac{1-\sqrt{5}}{2}\bigr) = \frac{1}{5}\,\mathcal{L}_3(1)$. For $r_2 > 0$, however, the correct $\mathcal{A}_3(F)$ turns out to be a subgroup of $\mathrm{Ker}(\partial_3)$, as we now explain.

Let ϕ be any homomorphism from $F^\times$ to $\mathbb{Z}$. If $\xi = \sum n_i[x_i]$ belongs to $\mathrm{Ker}(\partial_3)$, then the element $\iota_\phi(\xi) = \sum n_i \phi(x_i)[x_i]$ of $\mathbb{Z}[F]$ belongs to $\mathrm{Ker}(\partial)$, as is easily checked, so $\iota_\phi(\xi) \in \mathcal{A}_2(F)$. The additional requirement for ξ to belong to $\mathcal{A}_3(F)$ is then that $\iota_\phi(\xi)$, for every ϕ, should belong to the subgroup $\mathcal{C}_2(F) = \mathcal{C}(F)$ of $\mathcal{A}_2(F) = \mathcal{A}(F)$. Since we know that elements of the quotient $\mathcal{B}_2(F) = \mathcal{A}_2(F)/\mathcal{C}_2(F)$ are detected (up to torsion) by the values of D on the various conjugates, we can check this condition numerically by calculating $D(\sigma(\iota_\phi(\xi)))$ for all embeddings $\sigma : F \to \mathbb{C}$. This gives at least an algorithmic way to get $\mathcal{A}_3(F)$.

The construction of $\mathcal{C}_3(F)$, and of the higher $\mathcal{A}_m$ and $\mathcal{C}_m$ groups, now proceeds by induction on m, assuming at each stage that the conjectural picture as described above holds. We would like to define each $\mathcal{C}_m(F)$ as the free abelian group generated by the specialization to F of all functional equations of $\mathcal{L}_m$, but we cannot do this because these functional equations are not known except for $m = 2$. (Goncharov [20] has given a functional equation for $m = 3$ which is conjectured, but not known, to be generic, and Gangl [16] has given isolated functional equations up to $m = 7$, but for higher m nothing is known.) Instead we proceed as follows. We define $\mathcal{A}_3(F)$ as above and let $\mathcal{L}_{3,F} : \mathcal{A}_3(F) \to \mathbb{R}^{r_1+r_2}$ be the map sending $\sum n_i[x_i]$ to $\left\{\sum n_i \mathcal{L}_3(\sigma(x_i))\right\}_\sigma$. If the conjectures are correct, then the image of this map should be a lattice of full rank $r_1 + r_2$ and the kernel (at least up to torsion) should be $\mathcal{C}_3(F)$. The first statement can be checked empirically by computing $\mathcal{L}_{3,F}(\xi)$ numerically for a large number of elements ξ of $\mathcal{A}_3(F)$. If all is well—and in practice it is—we soon find that these vectors, to high precision, are all elements of a certain lattice (of full rank) $\mathbb{L}_3 \subset \mathbb{R}^{r_1+r_2}$. This confirms the conjecture and at the same time allows us to define $\mathcal{C}_3(F)$ as the set of $\xi \in \mathcal{A}_3(F)$ whose image under $\mathcal{L}_{3,F}$ is the zero vector of $\mathbb{L}_3$, something which can be verified numerically because $\mathbb{L}_3 \subset \mathbb{R}^{r_1+r_2}$ is discrete. The definition of the higher groups $\mathcal{A}_m(F)$ and $\mathcal{C}_m(F)$ now proceeds the same way. Once we know $\mathcal{C}_{m-1}(F)$, we define $\mathcal{A}_m(F)$ as the set of $\xi \in \mathbb{Z}[F]$ for which $\iota_\phi(\xi)$ belongs to $\mathcal{C}_{m-1}(F)$ for every homomorphism $\phi : F^\times \to \mathbb{Z}$. (This is automatically a subgroup of $\mathrm{Ker}(\partial_m)$, where $\partial_m : \mathbb{Z}[F] \to \mathrm{Sym}^{m-2}(F^\times) \otimes \Lambda^2(F^\times)$ takes $[x]$ to $(x)^{m-1} \otimes ((x) \wedge (1-x))$, so in practise we start by finding elements of $\mathrm{Ker}(\partial_m)$ and only check the condition $\iota_\phi(\xi) \in \mathcal{C}_{m-1}(F)$ for these. Note also that for a given $\xi = \sum n_i[x_i]$ the condition on $\iota_\phi(\xi)$ only has to be checked for finitely many maps ϕ, namely for a basis of the dual group of the group generated by the x_i.) We then define $\mathcal{L}_{m,F} : \mathcal{A}_m(F) \to \mathbb{R}^{(r_1+)r_2}$ as before and compute the images of many elements of $\mathcal{A}_m(F)$ under $\mathcal{L}_{m,F}$ to high precision. If the image vectors do not lie in some lattice $\mathbb{L}_m \subset \mathbb{R}^{(r_1+)r_2}$ within the precision of the calculation, then the conjectural picture is wrong and we stop. If they do—and in practise this always happens—then we define $\mathcal{C}_m(F)$ as the kernel of the map $\mathcal{L}_{m,F}$ from $\mathcal{A}_m(F)$ to the discrete group $\mathbb{L}_m \approx \mathbb{Z}^{(r_1+)r_2}$, and the mth Bloch group $\mathcal{B}_m(F)$ as the quotient $\mathcal{A}_m(F)/\mathcal{C}_m(F)$.

B. Examples. We start with a numerical example for $m = 3$ showing that the condition $\partial_3(\xi) = 0$ is not enough to ensure $\xi \in \mathcal{A}_3(F)$ when F is not totally real. Let θ be the real root of $\theta^3 - \theta - 1 = 0$. The field $F = \mathbb{Q}(\theta)$ has $r_1 = r_2 = 1$. The six numbers $x_0 = 1$, $x_1 = \theta$, $x_2 = -\theta$, $x_3 = \theta^3$, $x_4 = -\theta^4$ and $x_5 = \theta^5$ have the property that x_i and $1 - x_i$ belong to the group generated by -1 and θ. (So does θ^2, but its polylogarithms of all orders are related to those of θ and $-\theta$ by the "distribution" property of polylogarithms, so we have omitted it.) Hence $\partial_3(x_i) = 0$ (up to torsion) for each i, but numerically we find that 3 of the 6 vectors $\mathcal{L}_{3,F}(x_i) = (\mathcal{L}_3(x_i), \mathcal{L}_3(x_i')) \in \mathbb{R}^2$ (where $'$ denotes one of the non-real embeddings of F into $\mathbb{C}$) are linearly independent, rather than only $r_1 + r_2 =$

2. If we observe that $D(x_i') = \ell_i D(\theta')$ with $(\ell_0,\dots,\ell_5) = (0,1,-1,2,1,-1)$ and that the values of $\phi(x_i)$ for any $\phi : F^\times \to \mathbb{Z}$ are proportional to the integers $(m_0,\dots,m_5) = (0,1,1,3,4,5)$, then we find that the condition for an integral linear combination $\xi = \sum n_i[x_i]$ of the $[x_i]$'s to belong to $\mathcal{A}_3(F)$ is that $\sum_i \ell_i m_i n_i = 0$. These elements form an abelian group of rank 5 whose image in $\mathbb{R}^2$ under the map $\mathcal{L}_{3,F} : [x] \mapsto (\mathcal{L}_3(x), \mathcal{L}_3(x'))$ now does turn out numerically to be a lattice $\mathbb{L}_3$ of rank 2, as predicted, with the covolume of $\mathbb{L}_3$ being related to $\zeta_F(3)$ in the expected way. Moreover, since we now have a rank 3 group of elements mapping to 0 in $\mathbb{L}_3$, and since the space of relevant maps ϕ is only 1-dimensional, we can continue the inductive process for two more steps with the same elements x_i, obtaining finally two elements $[x_0]$ and $[x_5] - 5([x_4] - [x_3] + 46[x_2] + 57[x_1] + [x_0])$ whose images under the map $\mathcal{L}_{5,F} : [x] \mapsto (\mathcal{L}_5(x), \mathcal{L}_5(x'))$ from $\mathbb{Z}[F]$ to $\mathbb{R}^2$ are $\zeta(5)\binom{1}{1}$ and (to high precision) $2^{-7}\pi^{-5}23^{9/2}\zeta_F(5)\zeta(5)^{-1}\binom{-2}{1}$, respectively.

In general, in order to construct elements in the higher Bloch groups of a number field F, we have to find as many elements x_i of F as possible such that all of the numbers x_i and $1-x_i$ belong to a subgroup $G \subset F^\times$ of small rank. For instance, if we start with $F = \mathbb{Q}$ and let G be the rank 2 subgroup of $\mathbb{Q}^\times$ generated by -1, 2 and 3, then there are exactly 18 elements $x \in \mathbb{Q}$ for which both x and $1-x$ belong to G, namely the numbers 2, 3, 4 and 9 and their images under the group of order 6 generated by $x \mapsto 1/x$ and $x \mapsto 1-x$. If we want to make elements of the 3rd Bloch group, we need only find combinations $\xi = \sum n_i[x_i]$ for which $\sum n_i(x_i) \otimes ((x_i) \wedge (1-x_i))$ vanishes: there is no further condition of vanishing in the previous Bloch group $\mathcal{B}_2(\mathbb{Q})$ because it is zero (up to torsion). If all x_i belong to the set of 18 elements described above, then each element $(x_i) \otimes ((x_i) \wedge (1-x_i))$ is a linear combination of only two elements $(2) \otimes ((2) \wedge (3))$ and $(3) \otimes ((2) \wedge (3))$ of $\mathbb{Q}^\times \otimes \Lambda^2(\mathbb{Q}^\times)$. We therefore get many non-trivial elements of $\mathcal{B}_3(\mathbb{Q})$, a typical one being $[8/9] - 3[3/4] - 6[2/3]$, and since $\mathcal{B}_3(\mathbb{Q})$ is supposed to be of rank 1, each of them should map under $\mathcal{L}_3$ to a rational multiple of $\zeta(3)$, something that can be checked numerically. (For the element just given, for instance, we find $\mathcal{L}_3(8/9) - 3\mathcal{L}_3(3/4) - 6\mathcal{L}_3(2/3) = -\frac{91}{12}\zeta(3)$.) To get interesting examples for $F = \mathbb{Q}$ and higher values of m, we have to allow more prime factors in x_i and $1-x_i$. On p. 386 of [46] one can find a numerical relation over $\mathbb{Z}$ (conjecturally the only one, and with very large coefficients) among 29 values $\mathcal{L}_7(x_i)$ where $x_i \in \mathbb{Q}$ has only the prime factors 2 and 3 and $1-x_i$ only the prime factors 2, 3, 5 and 7.

To get examples for even higher values of m, it is advantageous to go to higher number fields and choose all x_i to belong to a group $\langle -1, \alpha\rangle$ of rank 1, where α is chosen to be an algebraic number of very small height, so that there are as many multiplicative relations as possible among the numbers $1 \pm \alpha^n$ ("ladder"). The algebraic number of conjecturally smallest positive height (according to the famous Lehmer conjecture) is the root of the 10th degree equation $\alpha^{10}+\alpha^9-\alpha^7-\alpha^6-\alpha^5-\alpha^4-\alpha^3+\alpha+1 = 0$. This example was studied in [10], where 71 multiplicatively independent multiplicative relations were found among the numbers α and $1-\alpha^n$ $(n \in \mathbb{N})$, the largest n occurring in such a relation being 360, and these were used to detect numerical polylogarithm relations up to $m = 16$. Subsequently, Bailey and Broadhurst [6] noticed that there was one further multiplicative relation (this one with $n = 630$), and used it to find a relation among values of polylogarithms of order 17, the present world record.

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