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The Dilogarithm Function of a Real Argument

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Abstract. This paper is a user's guide to the dilogarithm function

$$Li_2(z) = -\int_0^z \frac{\log(1-z)}{z}\,dz$$

of a real argument. It is intended for those who are primarily interested in the values of the dilogarithm rather than in its functional relationships.

The paper is deliberately written in the style of the book *Computer Approximations* by Hart, Cheney et al.

a. Definition and Analytical Behavior. The dilogarithm function $Li_2(z)$ is defined [1] by

$$(1)\qquad Li_2(z) = -\int_0^z \frac{\log(1-z)}{z}\,dz.$$

The function is real-valued for real values of $z \leqslant 1$ and has a logarithmic branch point at $z = 1$. It is usual to assign a branch cut along the real line from 1 to ∞ and to assign the imaginary part $-i\pi\log(x)$ to $Li_2(x)$ for real values of $x > 1$. In what follows, we deal only with the real part of the function $Li_2(x)$ for real arguments x.

Li_2 is asymptotic to $\pi^2/3 - \frac{1}{2}\log^2(x)$ for large x and to $-\pi^2/6 - \frac{1}{2}\log^2(-x)$ for large negative x.

Li_2 has a maximum at $x = 2$ and the value there is $\pi^2/4$.

Li_2 has a zero at the origin and at $x = 12.5951703698450161286398965\ldots$ [4].

Li_2 has infinite slope at $x = 1$.

b. Fundamental Identities.

1. *Expansions.*

$$(2)\qquad Li_2(x) = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \cdots, \qquad -1 \leqslant x \leqslant 1,$$

$$(3)\qquad Li_2(e^{-z}) = \frac{\pi^2}{6} + z\left(\log(z) - 1 - \frac{z}{4} + \frac{B_2 z^2}{2\cdot 3\cdot 2!} - \frac{B_4 z^4}{4\cdot 5\cdot 4!} + \cdots\right),$$

where B_n are the Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, etc., with the notation of [2]. A comparable expression for $Li_2(-e^{-z})$ can be derived by the use of Eq. (11).

2. *Functional Relations.* There are a number of functional relationships, all stated and derived in [1], which serve to reduce the computation of the dilogarithm to the interval $0 \leqslant x \leqslant \frac{1}{2}$.

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$$(4) \qquad Li_2(x) = \frac{\pi^2}{6} - \log(x)\log(1-x) - Li_2(1-x), \quad \frac{1}{2} < x < 1,$$

$$(5) \qquad Li_2(x) = \frac{\pi^2}{6} - \log(x)\log(x-1) - \frac{1}{2}\log(x) + Li_2\left(1 - \frac{1}{x}\right), \quad 1 < x \leqslant 2,$$

$$(6) \qquad Li_2(x) = \frac{\pi^2}{3} - \frac{1}{2}\log^2(x) - Li_2\left(\frac{1}{x}\right), \quad 2 < x < \infty,$$

$$(7) \qquad Li_2(x) = -\frac{1}{2}\log^2(1-x) - Li_2\left(\frac{x}{x-1}\right), \quad -1 \leqslant x < 0,$$

$$(8) \qquad Li_2(x) = -\frac{\pi^2}{6} - \frac{1}{2}\log(1-x)[2 \cdot \log(-x) - \log(1-x)] + Li_2\left(\frac{1}{1-x}\right), \quad -\infty < x < -1.$$

The intervals stated are those which provide for reduction to the range [1/2, 1], but the relationships are valid over much wider intervals.

3. *Improved Convergence.* It is possible to improve the n^2 rate of convergence of series (2) by well-known techniques. If the expression

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

is squared, multiplied by x^n and the infinite sum expanded in terms of logarithms and dilogarithms, then we obtain a series with n^4 convergence

$$(9) \qquad Li_2(x) = \left(\frac{x}{x+1}\right)\left[3 + \sum_1^\infty \frac{x^n}{n^2(n+1)^2}\right] - 2 \cdot \left(\frac{x-1}{x+1}\right)\log(1-x).$$

The n^6 analog is

$$(10) \qquad (1 + 4x + x^2)Li_2(x) = 4x^2 \sum_1^\infty \frac{x^n}{[n(n+1)(n+2)]^2} + 4x + \frac{23}{4}x^2 + 3(1-x^2)\log(1-x),$$

and both of these series converge for $|x| \leqslant 1$.

One can go arbitrarily far in this direction, but the expressions get more complex and are of little use for machine computation. On the other hand, they certainly are convenient for hand computation. The n^6 version of this procedure is used in [3] but in that case it only succeeds in getting wrong answers faster, as the resulting expression has enormous internal cancellation for arguments near 0.

c. **Error Propagation.** For values of $x < 0$, both the absolute derivative df/dx and the relative derivative $x/f \cdot df/dx$ are less than 1 in magnitude and so in this interval both relative errors and absolute errors in the argument are damped.

In the interval $0 \leqslant x \leqslant 2$, both of the derivatives rise from 1 at the origin, become infinite at $x = 1$, and then decrease to zero at $x = 2$. Therefore, both relative and absolute accuracy are completely lost in the neighborhood of the point $x = 1$. The absolute error is amplified in the interval $0 \leqslant x \leqslant 1.28...$, and the relative error in the interval $0 \leqslant x \leqslant 1.13\ldots$; they are damped otherwise.

In the interval $x > 2$, the derivative goes from zero towards its asymptote of $-\log(x)/x$ and is less than 1 throughout the interval. Therefore, absolute errors are damped for $x > 2$. The same is true for relative error except that near the zero at $x = 12.595\ldots$, all relative accuracy is lost.

d. Design of Subroutines. Because the function has value 0 and slope 1 at the origin, it is reasonable to look for relative accuracy near the origin. For this reason, the error curves for $Li_2(x)/x$ were leveled.

The use of Eqs. (4) through (8) to bring arguments into the range [0, 1/2] causes little loss of precision. Neither (4) nor (5) is valid for $x = 1$, which must be treated separately.

The series (3) can be used near $x = 1$ and no problems arise, but internal cancellation will cause loss of significance if the series is used for arguments near zero. For arguments larger than about 2, it is more practical to use Eq. (6). The convergence of series (3) is quite good, but little is gained by its use because of the two logarithms that are required.

Equations (9) and (10) have improved convergence, but are not very useful for machine computation. There is some error amplification with either series because of partial cancellation, but far more important is the complete loss of accuracy due to internal cancellation for small values of x. In the algorithm published in [3], Eq. (10) was used and led to the loss of two decimal places of accuracy for arguments near .01. The problem would be solved if a routine were used which returned accurate values of $\log(1 + x)$ as a function of x.

It does not seem necessary or practical to provide any remedy for the loss of relative significance near the zero at $x = 12.595\ldots$, since normally a standard of absolute accuracy is what is wanted in this range; and, in any case, little improvement could be obtained without a separate entry point.

No alarm exits are required and no overflow can occur. Some loss of precision will be experienced if the polynomials in the tables are evaluated in the normal way because the coefficients are not a decreasing sequence. The loss is not great, and full accuracy can be obtained by converting them to continued fraction form by the methods described in [5].

e. Checking. Consistency checks can be derived from the relationship

$$Li_2(x) + Li_2(-x) = \frac{1}{2} Li_2(x^2), \tag{11}$$

and the series (3) can be used for arguments near 1. There are also some special values for which the value of Li_2 can be expressed in simple terms. These are all derived in [1].

$$Li_2(1) = \frac{\pi^2}{6}, \quad Li_2(-1) = \frac{-\pi^2}{12}, \quad Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),$$

$$Li_2(2) = \frac{\pi^2}{4}, \quad Li_2(\phi) = \frac{\pi^2}{10} - \log^2(\phi),$$

where $\phi = (\sqrt{5} - 1)/2$. From this last equation, additional special values can be obtained by the use of (4)–(8) for the arguments $-\phi$, $1 + \phi$, $1 - \phi$, $-1 - \phi$, $2 + \phi$.

In addition, a host of two-term expressions can be derived from the relationships given in [1], for example,

$$6Li_2(3) - 3Li_2(-3) = 2\pi^2.$$

f. Constants. The following constants are needed for constructing or testing the dilogarithm.

$$\frac{\pi^2}{6} = 1.64493\ 40668\ 48226\ 43647\ 24151\ 66646,$$

$$\phi = 0.61803\ 39887\ 49894\ 84820\ 45868\ 34366,$$

$$\log 2 = 0.69314\ 71805\ 59945\ 30941\ 72321\ 21458.$$

Tables of values of Li_2 are given to 30D in Appendix B for testing purposes.

g. A Priori Calculation. Series (2) and relations (4) through (8) are entirely satisfactory for a priori calculation, and the values in Appendix B were calculated in this way.

h. Index Tables. The code name for the dilogarithm is DILOG. Coefficients for rational approximations to $Li_2(x)$ are given in Appendix A. The coefficients are given to enough decimal places that each of them ends with some meaningless digits. Pn is the coefficient of x^n in the numerator and Qn is the coefficient of x^n is the denominator. The error criterion for these approximations is ***relative.***

Index for the Dilogarithm Function

$$\frac{Li_2(x)}{x} \approx \frac{P(x)}{Q(x)}$$

Range	Precision	N	M	Index
[0,1/2]	2.5	0	1	0001
	3.9	1	1	0002
	5.3	1	2	0003
	6.7	2	2	0004
	8.1	2	3	0005
	9.4	3	3	0006
	10.8	3	4	0007
	12.1	4	4	0008
	13.5	4	5	0009
	14.9	5	5	0010
	16.2	5	6	0011
	17.6	6	6	0012
	19.0	6	7	0013
	20.3	7	7	0014
	21.7	7	8	0015
	23.1	8	8	0016
	24.5	8	9	0017

Appendix A — Tables of Coefficients

DILOG 0001

P00	−.35218 119 e1
Q00	−.35315 131 e1
Q01	+.10000 000 e1

DILOG 0002

P00	−.20071 3706 e1
P01	+.50517 5398 e0
Q00	−.20068 8475 e1
Q01	+.10000 0000 e1

DILOG 0003

P00	+.11202 62486 e2
P01	−.60994 87077 e1
Q00	+.11202 67809 e2
Q01	−.89026 98834 e1
Q02	+.10000 00000 e1

DILOG 0004

P00	+.47255 72352 87 e1
P01	−.36933 25898 50 e1
P02	+.30733 84547 38 e0
Q00	+.47255 71321 36 e1
Q01	−.48746 41717 99 e1
Q02	+.10000 00000 00 e1

DILOG 0005

P00	−.35678 64461 4688 e2
P01	+.38344 98584 1892 e2
P02	−.82200 39545 6182 e1
Q00	−.35678 64492 1010 e2
Q01	+.47264 67970 1406 e2
Q02	−.16072 49108 7932 e2
Q03	+.10000 00000 0000 e1

DILOG 0006

P00	−.12094 27332 56777 e2
P01	+.15973 19286 84940 e2
P02	−.51710 43301 77459 e1
P03	+.21090 57593 73841 e0
Q00	−.12094 27332 09355 e2
Q01	+.18996 76051 13668 e2
Q02	−.85764 08638 60377 e1
Q03	+.10000 00000 00000 e1

DILOG 0007

P00	+.11457 07921 85155 e3
P01	−.18453 60987 11386 e3
P02	+.85371 10007 43272 e2
P03	−.10046 95943 82901 e2
Q00	+.11457 07921 86959 e3
Q01	−.21317 87970 97653 e3
Q02	+.12593 57220 61229 e3
Q03	−.25005 15993 62086 e2
Q04	+.10000 00000 00000 e1

DILOG 0008

P00	+.32595 30720 85431 65159 e2
P01	−.60686 62378 52880 01793 e2
P02	+.35267 27757 69340 34438 e2
P03	−.64953 30814 72632 88026 e1
P04	+.15545 34477 52578 84132 e0
Q00	+.32595 30720 85198 73078 e2
Q01	−.68835 45058 18661 12910 e2
Q02	+.48854 43919 87334 97920 e2
Q03	−.13097 76047 86466 37907 e2
Q04	+.10000 00000 00000 00000 e1

DILOG 0009

P00	−.37051 65037 10351 62272 e3
P01	+.79633 01479 42916 54315 e3
P02	−.57463 07648 13207 03737 e3
P03	+.15564 69546 41903 62861 e3
P04	−.11663 52700 60940 89867 e2
Q00	−.37051 65037 10362 39104 e3
Q01	+.88895 92738 73661 53621 e3
Q02	−.75570 20830 25256 65596 e3
Q03	+.26895 65072 25653 85462 e3
Q04	−.35675 07897 42226 89707 e2
Q05	+.10000 00000 00000 00000 e1

DILOG 0010

P00	−.91023 77491 81049 07308 e2
P01	+.21879 45728 57946 16534 e3
P02	−.18477 14011 91366 16282 e3
P03	+.63948 88550 52845 88473 e2
P04	−.77002 69013 11109 15982 e1
P05	+.12019 30729 97785 98757 e0
Q00	−.91023 77491 81047 87870 e2
Q01	+.24155 05165 87430 02759 e3
Q02	−.23504 52775 67011 75740 e3
Q03	+.10156 02444 82463 98449 e3
Q04	−.18430 14376 33000 95898 e2
Q05	+.10000 00000 00000 00000 e1

Appendix A Tables of Coefficients

DILOG 0011

P00	+.12050 99126 13416 71705 797 e4
P01	−.32407 60925 45394 74989 552 e4
P02	+.31928 97693 82412 89723 624 e4
P03	−.13920 67146 52669 65241 768 e4
P04	+.25212 66291 93634 06616 773 e3
P05	−.13120 03443 27163 41584 577 e2
Q00	+.12050 99126 13416 72354 682 e4
Q01	−.35420 35706 98751 66000 880 e4
Q02	+.39445 06717 66915 11608 434 e4
Q03	−.20599 52998 38311 16588 803 e4
Q04	+.50200 96220 27681 16987 420 e3
Q05	−.48063 99725 87360 84391 455 e2
Q06	+.10000 00000 00000 00000 000 e1

DILOG 0012

P00	+.26082 98491 62942 04666 308 e3
P01	−.76841 39388 14390 59816 780 e3
P02	+.85348 27535 32016 71957 160 e3
P03	−.43992 89135 68751 87758 195 e3
P04	+.10321 32694 37217 26909 276 e3
P05	−.88086 82055 75880 69558 540 e1
P06	+.09619 21508 34802 05029 510 e0
Q00	+.26082 98491 62942 04603 695 e3
Q01	−.83362 14011 05125 79870 690 e3
Q02	+.10329 07009 45683 41906 278 e4
Q03	−.62183 29313 81994 08960 331 e3
Q04	+.18557 22003 75370 61281 857 e3
Q05	−.24566 28970 55685 47746 454 e2
Q06	+.10000 00000 00000 00000 000 e1

DILOG 0013

P00	−.39376 56825 06549 39008 69818 e4
P01	+.12718 63208 12074 54040 54796 e5
P02	−.15915 09794 82176 85545 84827 e5
P03	+.96551 92830 34263 56777 23740 e4
P04	−.28875 69599 93662 44649 97881 e4
P05	+.37744 98030 27333 13134 91492 e3
P06	−.14449 66290 66206 05559 99854 e2
Q00	−.39376 56825 06549 39012 64089 e4
Q01	+.13703 04628 74738 27740 77041 e5
Q02	−.18903 34209 50788 87087 56612 e5
Q03	+.13104 57120 70715 80257 63505 e5
Q04	−.47622 75177 78801 06982 93108 e4
Q05	+.85467 15159 14565 42532 50347 e3
Q06	−.62157 70486 62897 94249 42839 e2
Q07	+.10000 00000 00000 00000 00000 e1

DILOG 0014

P00	−.76228 68752 68307 55565 29146 e3
P01	+.26593 22028 10696 76496 63031 e4
P02	−.36658 75881 69095 11133 52891 e4
P03	+.25242 59912 68957 86370 99977 e4
P04	−.90015 67040 61893 56220 23769 e3
P05	+.15414 86633 63843 26413 69576 e3
P06	−.98371 85185 51047 29849 34359 e1
P07	+.07901 99588 74990 35527 52300 e0
Q00	−.76228 68752 68307 55564 95060 e3
Q01	+.28498 93746 92404 45363 39191 e4
Q02	−.42936 50776 72548 33820 87482 e4
Q03	+.33286 60675 80586 94705 18154 e4
Q04	−.14028 76448 66007 43044 11522 e4
Q05	+.31054 86484 17207 55907 43310 e3
Q06	−.31501 28765 22552 48064 46416 e2
Q07	+.10000 00000 00000 00000 00000 e1

DILOG 0015

P00	+.12915 24527 68067 19048 33413 153 e5
P01	−.48707 48997 96284 78978 83707 474 e5
P02	+.73980 21817 32133 90851 28615 142 e5
P03	−.57734 44511 67653 13018 85864 690 e5
P04	+.24413 47509 26561 33778 64169 145 e5
P05	−.53824 75484 01822 64205 83399 478 e4
P06	+.53401 60722 15763 45243 03757 297 e3
P07	−.15675 76496 90785 42058 87741 061 e2
Q00	+.12915 24527 68067 19048 33648 966 e5
Q01	−.51936 30129 88301 58742 69853 176 e5
Q02	+.85529 26624 49424 06422 49934 724 e5
Q03	−.74153 26436 34868 83755 91676 737 e5
Q04	+.36177 94839 86391 68291 52183 411 e5
Q05	−.98145 94889 34009 78101 09998 809 e4
Q06	+.13603 99725 08465 88957 67875 526 e4
Q07	−.77945 49609 01779 21118 40945 980 e2
Q08	+.10000 00000 00000 00000 00000 000 e1

Appendix A Tables of Coefficients

DILOG 0016

P00	+.22632 49801 88246 51566 37263 082 e4
P01	−.91237 68147 22541 57332 40743 276 e4
P02	+.15028 97069 22976 16692 38432 998 e5
P03	−.12983 35516 47249 36918 14757 240 e5
P04	+.62678 10607 03798 93162 33309 136 e4
P05	−.16596 90181 28602 58740 78635 080 e4
P06	+.21776 88398 66709 61256 57883 458 e3
P07	−.10798 42257 41933 85334 23708 485 e2
P08	+.06625 37985 85450 08732 77232 000 e0
Q00	+.22632 49801 88246 51566 37244 265 e4
Q01	−.96895 80597 69603 20223 84242 575 e4
Q02	+.17199 89364 15124 61900 55387 137 e5
Q03	−.16348 16162 13100 36113 49349 316 e5
Q04	+.89588 20514 14481 32671 99807 015 e4
Q05	−.28332 11086 19335 65154 45871 761 e4
Q06	+.48737 77106 49642 09210 64865 660 e3
Q07	−.39232 99443 20329 20231 03784 706 e2
Q08	+.10000 00000 00000 00000 00000 000 e1

DILOG 0017

P00	−.42486 09415 77076 57842 58155 13413 e5
P01	+.18324 45317 79306 99272 26355 13651 e6
P02	−.32752 66692 97550 16252 22235 86638 e6
P03	+.31312 24406 90220 04497 04229 56820 e6
P04	−.17221 86978 19825 42758 08769 31586 e6
P05	+.54428 19292 61565 48293 77288 42213 e5
P06	−.92731 78839 56725 07722 62213 32049 e4
P07	+.72397 96495 23300 32103 75229 23595 e3
P08	−.16815 16155 14046 44958 09755 69676 e2
Q00	−.42486 09415 77076 57842 58156 60209 e5
Q01	+.19386 60553 18733 90718 32947 51241 e6
Q02	−.37127 25059 98599 45511 55176 56852 e6
Q03	+.38705 52752 61533 98115 08213 82566 e6
Q04	−.23814 72006 59810 43990 22174 46162 e6
Q05	+.87588 91008 97049 12543 31268 45284 e5
Q06	−.18567 76580 98068 96609 58836 46794 e5
Q07	+.20563 29290 98181 83699 97666 62708 e4
Q08	−.95414 18759 91525 51414 12346 86148 e2
Q09	+.10000 00000 00000 00000 00000 00000 e1

Appendix B **Table of Values**

x	dilog(x)	dilog(−x)
0.02	0.02010 08990 18693 19554 07554 12493	−.01990 08790 15136 83978 03429 51326
0.04	0.04040 72753 24338 30714 85438 84616	−.03960 69550 96577 74164 44971 90472
0.06	0.06092 48424 59884 97429 97989 79316	−.05912 32198 62624 05075 13265 18607
0.08	0.08165 95876 98643 26519 84798 78575	−.07845 44530 82443 20233 10428 59707
0.10	0.10261 77910 99391 13111 38373 69057	−.09760 52352 29321 58384 11033 41851
0.12	0.12380 60463 27036 04241 91590 27599	−.11657 99590 82834 61011 46645 73331
0.14	0.14523 12834 45341 06292 76916 19326	−.13538 28404 67294 84931 46218 22263
0.16	0.16690 07939 17664 07294 79383 65447	−.15401 79282 04449 01808 36404 34686
0.18	0.18882 22580 86085 85333 51615 95038	−.17248 91133 50156 30597 85670 89030
0.20	0.21100 37754 39704 77261 11850 96074	−.19080 01377 77535 61903 69131 53766
0.22	0.23345 38980 30463 79482 90300 55518	−.20895 46021 62683 75361 59640 17283
0.24	0.25618 16674 51142 89958 72563 42099	−.22695 59734 23426 03914 79135 51882
0.26	0.27919 66558 56947 81806 36320 58343	−.24480 75916 56568 19155 30583 87318
0.28	0.30250 90115 91760 42080 74882 03837	−.26251 26766 14689 63828 14481 10532
0.30	0.32612 95100 75476 06953 00356 94175	−.28007 43337 59582 90423 02169 72305
0.32	0.35006 96107 23578 47318 89214 61188	−.29749 55599 25939 29006 45124 21852
0.34	0.37434 15208 08806 40225 50754 18013	−.31477 92486 25755 01654 55329 16910
0.36	0.39895 82673 43333 70582 97277 79501	−.33192 81950 21138 43916 44077 01793
0.38	0.42393 37782 65871 39834 97496 33474	−.34894 51005 90698 17912 44188 38894
0.40	0.44928 29744 71281 66446 47334 02376	−.36583 25775 12449 62799 07642 19653
0.42	0.47502 18745 33404 08139 35289 30687	−.38259 31527 84163 30333 52296 65162
0.44	0.50116 77143 61562 43905 59007 77479	−.39922 92721 00266 84401 50984 49953
0.46	0.52773 90845 19805 88377 26427 90164	−.41574 33035 02780 22313 94403 76484
0.48	0.55475 60885 54961 36293 86023 45297	−.43213 75408 22291 05810 09356 72666
0.50	0.58224 05264 65012 50590 26563 20160	−.44841 42069 23646 20244 30644 05916
0.52	0.61021 61084 47686 30856 00642 63976	−.46457 54567 69831 73312 71348 55858
0.54	0.63870 87053 75584 14655 17892 92715	−.48062 33803 16422 46266 59134 64193
0.56	0.66774 66441 53605 33959 65122 80099	−.49656 00052 47992 27462 13152 34110
0.58	0.69736 10583 74981 58928 43392 30087	−.51238 72995 66977 36384 86383 22300
0.60	0.72758 63077 16333 38951 35362 96840	−.52810 71740 44666 53659 86724 07090
0.62	0.75846 04836 08688 03740 05632 51928	−.54372 14845 43247 82973 29346 86901
0.64	0.79002 60243 47134 53637 84003 64632	−.55923 20342 17161 51240 02766 04810
0.66	0.82233 04706 44328 18045 23915 05023	−.57464 05756 01389 52239 87398 06762
0.68	0.85542 74037 47997 71006 39836 28277	−.58994 88125 93744 76391 38948 24356
0.70	0.88937 76242 86038 73860 10062 74807	−.60515 84023 37705 28397 44268 87577
0.72	0.92425 06536 36126 09887 33867 69629	−.62027 09570 11863 53723 63675 23198
0.74	0.96012 66752 33516 95678 12087 11469	−.63528 80455 31625 61763 05047 10184
0.76	0.99709 90883 05810 39591 17436 45829	−.65021 11951 68395 78868 20618 25034
0.78	1.03527 79342 17013 40369 73927 23810	−.66504 18930 91114 57256 46970 19132
0.80	1.07479 46000 08248 35939 54519 22854	−.67978 15878 34681 09120 62517 40538
0.82	1.11580 84509 85135 48265 04017 91877	−.69443 16906 99479 62228 50710 72771
0.84	1.15851 64875 07309 75080 57252 17889	−.70899 35770 85944 03774 77578 98548
0.86	1.20316 79608 60418 34740 45474 20250	−.72346 85877 67829 76983 49251 49256
0.88	1.25008 75841 99261 87156 76547 59981	−.73785 80301 07619 24916 89438 27238
0.90	1.29971 47230 04958 72517 10604 94193	−.75216 31792 17261 62037 26927 13427
0.92	1.35267 51610 43712 62837 43299 77602	−.76638 52790 67239 37138 25679 85458
0.94	1.40992 83004 64020 36892 89545 59279	−.78052 55435 46761 87296 59639 82604
0.96	1.47312 58602 39736 72826 22694 85409	−.79458 51574 77707 41046 03670 48107
0.98	1.54579 97120 31465 60971 30517 62219	−.80856 52775 84769 91754 44933 33163
1.00	1.64493 40668 48226 43647 24151 66646	−.82246 70334 24113 21823 62075 83323

Appendix B

Table of Values

x	dilog(x)	dilog(−x)
1.1	1.96199 91013 05568 59305 31769 92766	−0.89083 80902 62282 60587 13094 28418
1.2	2.12916 94303 83959 65944 43055 69819	−0.95740 53085 58781 24819 80102 86629
1.3	2.24088 78398 53646 06643 46665 50977	−1.02228 40383 02275 31998 67189 86228
1.4	2.31907 30363 09661 14059 79490 62073	−1.08557 79563 47125 07555 48895 82744
1.5	2.37439 52702 72480 20067 74997 63072	−1.14738 06603 75570 75407 99766 33863
1.6	2.41313 11379 74625 25843 30895 29909	−1.20777 69923 69138 19407 37201 65256
1.7	2.43935 42708 85838 95641 68592 51849	−1.26684 41458 66363 44732 09055 17040
1.8	2.45587 64585 04301 73501 23015 43645	−1.32465 25988 23732 07003 59126 90929
1.9	2.46472 33024 89591 31709 61169 58476	−1.38126 69046 33378 82028 12297 72061
2.0	2.46740 11002 72339 65470 86227 49969	−1.43674 63668 83680 94636 29020 23894
2.1	2.46505 79753 80808 73317 16305 66323	−1.49114 56181 51680 57356 36375 68660
2.2	2.45858 66019 99741 85173 79866 78759	−1.54451 51190 48595 20388 57728 16970
2.3	2.44869 25157 43370 11182 36557 39529	−1.59690 15905 87111 27205 35146 39329
2.4	2.43594 10990 45935 57746 92034 45568	−1.64834 83904 74691 23187 57568 61321
2.5	2.42079 08065 65933 84391 36565 93893	−1.69889 58419 95014 17304 80912 84015
2.6	2.40361 72122 68632 40213 31734 51324	−1.74858 15225 97636 56164 80729 86990
2.7	2.38473 07615 37743 34716 85278 64660	−1.79744 05180 82914 58608 72293 48866
2.8	2.36439 01023 28581 97356 82109 13407	−1.84550 56472 76696 84533 46582 42627
2.9	2.34281 22472 93540 60311 74075 53046	−1.89280 76612 85357 11543 48493 26700
3.0	2.32018 04233 13098 39640 61944 73703	−1.93937 54207 66708 95307 72717 19178
3.1	2.29665 02065 78988 94807 82442 90375	−1.98523 60541 15665 58057 45252 75757
3.2	2.27235 43686 91871 25146 41087 80631	−2.03041 50990 21474 09647 38030 50602
3.3	2.24740 67413 67447 88102 05419 16266	−2.07493 66294 87449 59659 68002 06148
3.4	2.22190 53253 95980 12378 06175 33989	−2.11882 33700 99811 52150 48267 36057
3.5	2.19593 48115 75780 80425 28561 00843	−2.16209 67990 77975 09509 85494 95119
3.6	2.16956 86397 83855 06533 97895 79840	−2.20477 72414 25323 35187 45676 36039
3.7	2.14287 06921 26057 63222 12336 34587	−2.24688 39533 19760 86544 00554 57373
3.8	2.11589 66938 59746 74448 02919 20201	−2.28843 51987 31344 67761 41315 45051
3.9	2.08869 53792 15163 27085 18141 48904	−2.32944 83191 25246 75884 14383 12332
4.0	2.06130 94667 77317 41669 14414 52151	−2.36993 97969 98365 83198 55374 25350
4.1	2.03377 64796 21663 45664 82498 64539	−2.40992 53139 03926 70454 68915 67960
4.2	2.00612 94381 39823 82731 05445 07153	−2.44941 98035 37803 90388 57630 41577
4.3	1.97839 74478 83292 85112 04457 56021	−2.48843 75003 90962 47280 09002 97322
4.4	1.95060 62003 85219 56750 64384 45576	−2.52699 19844 12570 69479 29469 74302
4.5	1.92277 84014 93970 98602 95627 31318	−2.56509 62220 76555 18311 89302 08462
4.6	1.89493 41390 46013 16477 03739 51243	−2.60276 26041 99429 90729 48352 33582
4.7	1.86709 11995 54661 79686 66273 71950	−2.64000 29808 18128 37605 21632 41083
4.8	1.83926 53418 71258 90049 30678 76821	−2.67682 86934 02456 93301 38321 46069
4.9	1.81147 05343 92159 44793 84312 29342	−2.71325 06046 46956 25099 38005 77169
5.0	1.78371 91612 66630 62774 35597 34722	−2.74927 91260 60808 29002 55875 15376
5.1	1.75602 22021 52051 16424 42237 74355	−2.78492 42435 51450 23674 68378 90143
5.2	1.72838 93893 20815 63305 13293 18289	−2.82019 55411 77325 16212 90219 00941
5.3	1.70082 93453 14731 77579 55061 12117	−2.85510 22232 27345 39246 82011 72730
5.4	1.67334 97038 41301 06466 79958 59280	−2.88965 31347 68856 70483 29570 71014
5.5	1.64595 72161 91491 43588 90252 74286	−2.92385 67807 91902 40027 90726 90568
5.6	1.61865 78451 14125 28392 84508 08686	−2.95772 13440 65167 07672 23928 22094
5.7	1.59145 68477 94820 98091 31485 02729	−2.99125 47018 07933 64839 93087 10288
5.8	1.56435 88493 47146 27749 18691 70541	−3.02446 44412 72544 18575 53491 06472
5.9	1.53736 79080 21911 30309 64093 28589	−3.05735 78743 23068 93724 46768 36844
6.0	1.51048 75731 70602 17738 28544 27622	−3.08994 20510 88031 58128 30461 57063

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