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THE DIMENSION OF A CUT LOCUS ON A SMOOTH RIEMANNIAN MANIFOLD

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Abstract. In this note we prove that the Hausdorff dimension of a cut locus on a smooth Riemannian manifold is an integer.

It is very difficult to investigate the structure of a cut locus (cf. [1] for the definition) on a complete Riemannian manifold. The difficulty lies in the non-differentiability of a cut locus. This means that one cannot describe the structure of a cut locus in a smooth category. For example, it is not always triangulable (cf. [3]). In this note, the structure of a cut locus will be described in terms of the Hausdorff dimension (cf. [2], [9] for the definition of the Hausdorff dimension), that is, the aim of this note is to determine the Hausdorff dimension of the cut locus of a point on a complete, connected C^∞-Riemannian manifold. The cut locus on a 2-dimensional Riemannian manifold has been investigated in detail by many reseachers. Actually it is already known that the Hausdorff dimension of a cut locus on a smooth 2-dimensional Riemannian manifold is 0 or 1 (cf. [4], [5]). On the other hand, the Hausdorff dimension of a cut locus on a Riemannian manifold is not always an integer, if the order of differentiability of the Riemannian metric is low. In fact, for each integer $k \geq 2$, the first author, constructed in [6] an $n(k)$-dimensional sphere $S^{n(k)}$ with a C^k-Riemannian metric which admits a cut locus whose Hausdorff dimension is greater than 1, and less than 2 (cf. [5]). In this note we prove that the Hausdorff dimension of a cut locus on a C^∞-Riemannian manifold is an integer. More precisely, we prove the following theorem.

MAIN THEOREM. *Let M be a complete, connected smooth Riemannian manifold of dimension n, and C_p the cut locus of a point p on M. Then for each cut point q of p, there exists a postive number δ_0 and a non-negative integer $k \leq n-1$ such that for any positive $\delta \leq \delta_0$, the Hausdorff dimension of $B(q, \delta) \cap C_p$ is k. Here $B(q, \delta)$ denotes the open ball centered at q with radius δ.*

REMARK. The topological dimension is not greater than the Hausdorff dimension for a metric space. Since $C_p \cap B(q, \delta)$ contains a submanifold with the same dimension as the Hausdorff dimension of $C_p \cap B(q, \delta)$, both dimensions coincide.

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For some basic tools in Riemannian geometry refer to [1], [8].

Let M be a complete, connected, smooth (C^∞-) Riemannian manifold of dimension n and let S_pM denote the unit sphere of all unit tangent vectors to M at p. For each $v \in S_pM$, let $\rho(v)$ denote the distance from p to the cut point of it along a unit speed geodesic $\gamma_v : [0, \infty) \to M$ emanating from p with $\gamma_v'(0) = v$. If there is no cut point of p along γ_v, define $\rho(v) = \infty$. Then it is well-known that the function $\rho : S_pM \to [0, \infty]$ is continuous. The cut locus of p will be denoted by C_p. For each $v \in S_pM$, let $\lambda(v)$ denote the distance function on the tangent space T_pM of M at p between the zero vector to its first tangent conjugate point along γ_v. If there is no conjugate point of p along γ_v, define $\lambda(v) = \infty$. It follows from the proof of the Morse index theorem (cf. [8]) that the function $\lambda : S_pM \to [0, \infty]$ is continuous. Note also that $\rho \le \lambda$ on S_pM. We define two maps e_λ and e_ρ on $\{v \in S_pM \mid \lambda(v) < \infty\}$, $\{v \in S_pM \mid \rho(v) < \infty\}$ respectively by

$$e_\lambda(v) := \exp_p(\lambda(v)v)\,, \quad e_\rho(v) := \exp_p(\rho(v)v)\,,$$

where $\exp_p$ denotes the exponential map on T_pM. If a cut point q of p is conjugate to p along some minimal geodesic joining p to q, is called a *conjugate cut point*. Otherwise q is called a *non-conjugate cut point*. If a cut point q of p is non-conjugate and if there exist exactly two minimal geodesics joining p to q, then the cut point q will be called a *normal cut point*. It follows from the implicit function theorem that the set of all normal cut points forms a smooth hypersurface of M. For each $v \in S_pM$ with $\lambda(v) < +\infty$, let $N(v)$ denote the kernel of the map $(d\exp_p)_{\lambda(v)v}$ and denote its dimension by $\nu(v)$, which is called the conjugate multiplicity of the conjugate point $e_\lambda(v)$ along γ_v. It follows from a property of Jacobi fields that $N(v)$ can be identified with a linear subspace of the tangent space of S_pM at v. It follows from the implicit function theorem that if the function ν is constant on an open sebset U in S_pM, then λ is smooth on U.

LEMMA 1. *Suppose that* ν *is constant on an open subset* U *in* S_pM. *If* $\lambda(v_0) = \rho(v_0) < \infty$ *at a point* v_0 *in* U, *then any vector of* $N(v_0)$ *is mapped to the zero vector by the differential* de_λ *of* e_λ.

PROOF. Let w be any element of $N(v_0)$. Choose a smooth curve $v : (-1, 1) \to S_pM$ with $v(0) = v_0$, $v'(0) = w$ such that $v'(t) \in N(v(t))$ for each $t \in (-1, 1)$. Suppose that $de_\lambda(w)$ is non-zero. Since we get

$$de_\lambda(v'(t)) = \gamma'_{v(t)}(\lambda(v(t)))(\lambda \circ v)'(t)\,,$$

we may assume that $(\lambda \circ v)'(t)$ is negative on $[0, \delta]$ for some positive $\delta < 1$. The length $l(\delta)$ of the subarc $e_\lambda \circ v \mid [0, \delta]$ is

$$l(\delta) = \lambda(v_0) - \lambda(v(\delta))\,. \tag{1}$$

By the triangle inequality we have

$$l(\delta) + \lambda(v(\delta)) \ge d(p, e_\lambda(v_0)) = \rho(v_0) = \lambda(v_0)\,. \tag{2}$$

The equation (1) implies that the equality holds in (2). This is impossible, because $\lambda(v(\delta)) \geq \rho(v(\delta))$. Therefore $de_\lambda(N(v_0))=0$. □

If Q_p denotes the set of all conjugate cut points, then we have:

LEMMA 2. *The Hausdorff dimension of Q_p is not greater than $n-2$.*

PROOF. It follows from the Morse-Sard-Federer theorem [9] that the Hausdorff dimension of the set

$$\{\exp_p(w) \mid w \in T_pM, \operatorname{rank}(d\exp_p)_w \leq n-2\}$$

is not greater than $n-2$. Thus the Hausdorff dimension of the set

$$\{e_p(v) \in Q_p \mid v \in S_pM, \nu(v) \geq 2\}$$

is not greater than $n-2$. Thus it sufficies to prove that the Hausdorff dimension of the set $A_1 := \{e_p(v) \in Q_p \mid \nu(v)=1\}$ is not greater than $n-2$. By the proof of the Morse index theorem (cf. [8]), the function ν is locally constant around a neighborhood of each $v \in A_1$. Thus λ is smooth around each $v \in S_pM$ with $v \in A_1$. It follows from Lemma 1 that A_1 is a subset of

$$\{e_\lambda(v) \mid v \in S_pM, \nu(v)=1, \dim de_\lambda(T_vS_pM) \leq n-2\}.$$

Therefore by the Morse-Sard-Federer theorem, the Hausdorff dimension of A_1 is not greater than $n-2$. □

If L_p denotes the set of non-conjugate cut points which are not normal, then we have:

LEMMA 3. *The Hausdorff dimension of L_p is not greater than $n-2$. Thus the Hausdorff dimension of the cut locus of p is not greater than $n-1$.*

PROOF. Let q be any element of L_p. Let $v_1, \ldots, v_k$ be all the elements of $e_p^{-1}(q)$. It follows from the implicit function theorem that for each pair of two vectors v_i, v_j $(i<j)$ in $e_p^{-1}(q)$ there exist hypersurfaces W_i, W_j, $H_{i,j}$ containing $\rho(v_i)v_i$, $\rho(v_j)v_j$, q respectively such that for each $x \in H_{i,j}$ there exist vectors $w_i \in W_i$, $w_j \in W_j$ of the same length with $\exp_p w_i = \exp_p w_j = x$ (cf. [7]). Let v_i, v_j, v_k $(i<j<k)$ be any distinct three vectors in $e_p^{-1}(q)$. Since the tangent spaces of $H_{i,j}$ and $H_{j,k}$ at q are distinct, we may assume that the intersection $H_{i,j,k} := H_{i,j} \cap H_{j,k}$ forms an $(n-2)$-dimensional submanifold containing q, by taking smaller hypersurfaces $H_{i,j}$, $H_{j,k}$. If we set $H_q = \bigcup_{i<j<k} H_{i,j,k}$, then any cut point of L_p sufficiently close to q is an element of H_q. Moreover, the Hausdorff dimension of H_q is $n-2$. Therefore for each point $q \in L_p$ we can choose a subset H_q $(\ni q)$ of Hausdorff dimension $n-2$ such that $L_p \cap H_q$ is relatively open in L_p. Since M satisfies the second countability axiom, L_p is covered by at most a countable number of H_{q_i}, $q_i \in L_p$. Thus implies that the Hausdorff dimension of L_p is at most $n-2$. As was observed above, $C_p \setminus (L_p \cup Q_p)$ is a countable disjoint union of smooth

hypersurfaces of M. In particular its Hausdorff dimension is $n-1$. Thus the latter claim is clear from Lemma 2. □

LEMMA 4. *If $v_0 \in S_pM$ satisfies $\rho(v_0) < \lambda(v_0)$, then there exists a sequence of $\{v_j\}$ of elements in S_pM convergent to v_0 such that $e_\rho(v_j)$ is a normal cut point for each j.*

PROOF. Since the functions ρ, λ are continuous, there exists a relatively open neighborhood U around v_0 in S_pM on which $\rho < \lambda$. Since $d\exp_p$ has maximal rank at each $v \in U$, we have

$$\dim_{\mathrm{H}}(e_\rho|_U)^{-1}(Q_p \cup L_p) = \dim_{\mathrm{H}}(Q_p \cup L_p)$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. Thus by Lemmas 2 and 3,

$$\dim_{\mathrm{H}}(e_\rho|_U)^{-1}(Q_p \cup L_p) \le n-2 . \tag{3}$$

This inequalty implies that the set $U \setminus e_\rho^{-1}(Q_p \cup L_p)$ is open and dense in U, since $\dim_{\mathrm{H}} U = n-1$. Therefore if we get a sequence $\{v_i\}$ of points in $U \setminus e_\rho^{-1}(Q_p \cup L_p)$ convergent to v_0, then the sequence $\{e_\rho(v_i)\}$ of normal cut points converges to q. □

REMARK. The inequality (3) is a generalization of Lemmas 2.1 and 3.1 in [10].

PROOF OF THE MAIN THEOREM. Let q be any cut point of p. Suppose that there exists a sequence $\{v_j\}$ of tangent vectors in S_pM with $\lim_{j\to\infty} e_\rho(v_j) = q$ such that $\rho(v_j) < \lambda(v_j)$ for each j. By Lemma 4 for any positive ε

$$\dim_{\mathrm{H}} B(q, \varepsilon) \cap C_p \ge n-1 .$$

On the other hand, $\dim_{\mathrm{H}} C_p \le n-1$. Thus $\dim_{\mathrm{H}}(B(q, \varepsilon) \cap C_p) = n-1$ for any positive ε. Suppose that the cut point q does not admit a sequence $\{v_j\}$ as above. Then there exists a neighborhood W around $e_\rho^{-1}(q)$ in S_pM such that $\rho(w) = \lambda(w)$ for any $w \in W$. For each $v \in e_\rho^{-1}(q)$, we define a positive integer $k(v)$ by

$$k(v) := \liminf_{w \to v} \nu(w) .$$

Thus we may take a sufficiently small neighborhood $U(v)$ ($\subset W$) around v in S_pM such that $\min \nu|_{U(v)} = k(v)$. Since $e_\rho^{-1}(q)$ is compact, we may choose finitely many neighborhoods $U(v_1), \dots, U(v_l)$ from $U(v)$, $v \in e_\rho^{-1}(q)$, which cover $e_\rho^{-1}(q)$. Set $U_i := U(v_i)$, $k_i := k(v_i)$ for simplicity. Without loss of generality we may assume that

$$k_1 = \min\{k_i \mid 1 \le i \le l\} .$$

For each i, let

$$W_i := (\nu|_{U_i})^{-1}(k_1) .$$

If W_i is not empty, i.e. $k_1 = k_i$, then it follows from the Morse index theorem that W_i is an open subset of U_i. Therefore λ is smooth on $\bigcup_{i=1}^{l} W_i$. Since $\lambda = \rho$ on $\bigcup_{i=1}^{l} U_i \subset W$, it follows from the Morse-Sard-Federer theorem and Lemma 1 that

$$\dim_{\mathrm{H}} e_\rho\left(\bigcup_{i=1}^{l} W_i\right) \leq n-(k_1+1)\,, \quad \dim_{\mathrm{H}} e_\rho\left(\bigcup_{i=1}^{l} U_i \setminus W_i\right) \leq n-(k_1+1)\,.$$

Therefore we get

$$\dim_{\mathrm{H}} e_\rho\left(\bigcup_{i=1}^{l} U_i\right) \leq n-(k_1+1)\,. \tag{4}$$

Let δ_0 be a sufficiently small positive number satisfying

$$C_p \cap B(q, \delta_0) \subset e_\rho\left(\bigcup_{i=1}^{l} U_i\right).$$

By (4) we have

$$\dim_{\mathrm{H}} C_p \cap B(q, \delta_0) \leq n-(k_1+1)\,. \tag{5}$$

Let $\delta \in (0, \delta_0]$ be fixed. Since v_1 is an element of the closure of W_1, there exists an open subset $\tilde{W} \subset W_1$ such that $e_\rho(\tilde{W}) \subset C_p \cap B(q, \delta)$. By Theorem 3.3 in [11], $e_\rho(\tilde{W}) = e_\lambda(\tilde{W})$ is a submanifold of dimension $n-(k_1+1)$. Thus

$$\dim_{\mathrm{H}} C_p \cap B(q, \delta) \geq n-(k_1+1) \tag{6}$$

for any $\delta \in (0, \delta_0]$. Combining (5) and (6), we conclude the proof of the main theorem.

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