THE DIMENSION OF ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

BY

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§ 1. Introduction

The starting point of this paper is the well known theorem of Dvoretzky [6] on the existence of almost spherical sections of convex bodies. Our interest is in getting good estimates for the dimension of the almost spherical sections. It turns out that it is possible to obtain quite sharp general estimates which in many interesting examples give actually the best possible results. The theorem of Dvoretzky is by now a very important tool in Banach space theory, in particular in the so called local theory of Banach spaces. It is not surprising therefore that sharp estimates on the dimension of the spherical sections have also many applications.

Though the main purpose of this paper is to present new results, we have tried to make several parts of the paper selfcontained by presenting also some proofs of known results. Most of these proofs are simpler or different from the proofs appearing already in the literature. An announcement of the main new results of this paper appeared in [10].

The basic idea of our approach is that used by the third named author [29] in proving Dvoretzky's theorem. We use the approach of [29] in a slightly simplified form and examine in detail the estimates which go into each step of the proof. Let up point out however, that for Dvoretzky's theorem itself there is known by now a shorter measure theoretic proof (cf. [32] and especially [9]) as well as functional analytic proofs (cf. [19], [35]).

The approach of [29] and of the present paper is based on a lemma of P. Levý [22] which is a simple consequence of the isoperimetric inequality for subsets of the unit sphere $S^{n-1}=\{(x_1, x_2, \ldots, x_n); \sum_{i=1}^n x_i^2=1\}$ in R^n. This inequality states that among all the closed subset of S^{n-1} having the same $(n-1)$-dimensional surface measure the caps are the sets which have the smallest boundary measure (the boundary measure itself will actually not appear explicitely in this paper; we use instead a quantity which enters in the usual

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definition of boundary measure). The isoperimetric inequality was as far as we know, proved first in the general case by E. Schmidt [33]. The original proof of Schmidt was very long and has been simplified (see e.g. Dinghas [5]). We present in the appendix to this paper a proof of the isoperimetric inequality which seems to be simpler than the other available proofs, though it is based on classical ideas (like symmetrization) which are discussed in detail e.g. in the book of Hadwiger [14].

In section 2 we show how to deduce from Levy's lemma the main general estimates on the dimension of spherical sections. In order to explain these results let us recall first the definition of the Banach-Mazur distance coefficient $d(X, Y)$ between isomorphic Banach spaces X and Y: $d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|$, T ranges over the isomorphisms from X onto $Y\}$. Our first general result on the dimension of almost spherical sections is the following. Let $(X, \|\ \|)$ be an n dimensional Banach space, let $|||\ |||$ be an inner product norm on X with $\|x\| \leqslant b|||x|||$, $x \in X$ and let M_r be the median of $r(x) = \|x\|$ on $\{x;\ |||x||| = 1\}$ with respect to the rotation invariant measure on this set. Then whenever $k \leqslant \eta n M_r^2/b^2$ there is a subspace Y of X with dim $Y = k$ and $d(Y, l_2^k) \leqslant 2$ (here η is a positive absolute constant). By using duality we deduce from this result a formula which is very useful in applications

$$k_1 k_2 \geqslant \eta \|P\|^2 n^2/d^2(X, l_2^n) \tag{1.1}$$

where k_1 and k_2 are such that there exist subspaces $Y \subset X$, $Z \subset X^*$ with $d(Y, l_2^{k_1}) \leqslant 2$, $d(Z, l_2^{k_2}) \leqslant 2$ and $\|P\|$ is the norm of a projection P which projects X onto Y or X^* onto Z (depending on whether $k_2 \geqslant k_1$ or $k_1 \geqslant k_2$). Another formula proved in section 2 connects the dimension of the almost Hilbertian subspaces of X to the projection constant of X^*.

Section 3 contains various applications and examples related to (1.1). We compute for example the exact (up to a multiplicative constant) dimension of the largest possible almost Hilbertian subspaces of the spaces l_p^n, $1 \leqslant p \leqslant \infty$ and C_p^n, $1 \leqslant p \leqslant \infty$. We show that (1.1) implies the following inequality concerning convex n-dimensional polytopes which are symmetric with respect to the origin. If Q is such a polytope having $2s$ extreme points and $2v$ faces, then $\log v \cdot \log s \geqslant \gamma n$ where γ is some absolute constant. We present examples of polytopes which show that this result (as well as other consequences of (1.1)) are asymptotically the best possible. We also present in section 3, as an application of (1.1) (mainly of the term $\|P\|$ appearing in it), an answer to a question of A. Pelczynski on absolutely summing operators.

In section 4 we examine the Dvoretzky Rogers lemma [7] and its relation to the discussion of section 2. We show first (following the reasoning in [29]) how to deduce from the Dvoretzky Rogers lemma the fact that every Banach space X of dimension n has an almost Hilbertian subspace of dimension $\eta \log n$. This of course is Dvoretzky's theorem

(Dvoretzky's original proof gave however the weaker estimate $\eta\,(\log n)^{1/2}$). In section 4 we present also a new version of the Dvoretzky Rogers lemma whose proof is very simple. This version gives some information which does not seem to follow from the known versions and proofs of the Dvoretzky Rogers lemma.

Section 5 is devoted to the study of the connection between the dimension of almost spherical sections and the notion of the cotype of a Banach space. We show that the cotype of an infinite-dimensional Banach space X can be determined up to an arbitrary $\varepsilon > 0$ by considering the dimension of almost Hilbertian subspaces of finite-dimensional subspaces of X. We also present in this section an example due to W. B. Johnson which clarifies the role of the ε.

The results of sections 2–5 are all in the following general direction: Given a Banach space X with $\dim X = n$ then for some suitable k (often surprisingly large) "most" k-dimensional subspaces of X are close to l_2^k. In section 6 we investigate the situation where for a given $k = k(n)$ all k-dimensional subspaces are close to l_2^k. It turns out that if $\log k(n)/\log n \geq \gamma$ for some positive γ independent of n then X itself must be close to an inner product space (i.e. l_2^n). This fact is proved by examining some constants which enter naturally in the study of the type or cotype of a Banach space. We present in section 6 also some consequences of this result. In particular, by combining this result with a result from section 2, we are able to solve the local version of the complemented subspaces problem: If all the subspaces of a finite dimensional space X are nicely complemented then X must be close to an inner product space.

In section 7 we indicate briefly some directions in which the approach used in section 2 can possibly provide additional interesting information.

As we mentioned already above, the paper ends with an appendix which contains a proof of the isoperimetric inequality.

In this paper we consider only Banach spaces over the real field. The results and their proofs are valid (with some obvious minor modifications) also in the complex case. There are several universal constants which enter into the estimates below. These constants are mostly denoted by the letters η, γ, c. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper.

§ 2. The basic estimates

As mentioned in the introduction our approach is based on an isoperimetric inequality for subsets of the usual Euclidean sphere. Before stating this inequality let us introduce some notation.

Let $S^{n-1}=\{x=(x_1, ..., x_n)\in R^n; \sum_{i=1}^n |x_i|^2=1\}$. The sphere S^{n-1} has a natural structure as a measure space as well as a metric space. We let μ_{n-1} denote the unique rotation invariant measure on S^{n-1} normalized by $\mu_{n-1}(S^{n-1})=1$. As the metric on S^{n-1} we take the usual geodesic distance, i.e. $d(x, y)$ is the angle between the lines joining x and y to the origin ($0\leqslant d(x, y)<\pi$ for all $x, y\in S^{n-1}$). The intersection of S^{n-1} with half spaces of R^n are called caps; these are exactly the balls with respect to the metric d, i.e. sets of the form $B(X_0, r)=\{x\in S^{n-1}, d(x_0, x)\leqslant r\}$. For a set $A\subset S^{n-1}$ and $\varepsilon>0$ we let A_ε be the set $\{x\in S^{n-1}; d(x, A)\leqslant\varepsilon\}$.

THEOREM 2.1 [33]. *Let A be a closed subset of S^{n-1} and let B be a cap of S^{n-1} so that $\mu_{n-1}(A)=\mu_{n-1}(B)$. Then for every $\varepsilon>0$, $\mu_{n-1}(A_\varepsilon)\geqslant\mu_{n-1}(B_\varepsilon)$.*

Observe that if B is the cap $B(x, r)$ then B_ε is the cap $B(x, r+\varepsilon)$. The requirement that $\mu_{n-1}(A)=\mu_{n-1}(B(x, r))$ determines r uniquely and therefore it determines also $\mu_{n-1}(B(x, r)_\varepsilon)$ for every $\varepsilon>0$. A proof of Theorem 2.1 will be presented in the appendix.

The usefulness of Theorem 2.1 stems from the fact that for a cap the quantity $\mu_{n-1}(B)$ is trivially evaluated, namely

$$\mu_{n-1}(B(x_0, r))=\gamma_n\int_{-\pi/2}^{r-\pi/2}\cos^{n-2}t\,dt \tag{2.1}$$

where

$$\gamma_n^{-1}=\int_{-\pi/2}^{\pi/2}\cos^{n-2}t\,dt=\Gamma\left(\frac{n-1}{2}\right)\Gamma(\tfrac{1}{2})/\Gamma\left(\frac{n}{2}\right). \tag{2.2}$$

The key to all the results in this section is that for large n, $\mu_{n-1}(B(x, \varepsilon+\pi/2))$ is very close to 1 already for small ε. Consequently, by Theorem 2.1, the same is true for $\mu_{n-1}(A_\varepsilon)$ whenever $\mu_{n-1}(A)=1/2$. The sets A which we shall use will be level sets of functions f defined on S^{n-1}. We state now the lemma of Levy which is the precise version of Theorem 2.1 which will be applied in the sequel.

LEMMA 2.2. *Let $f(x)$ be a continuous real valued function on S^{n-1}. Let M_f be the median of f, i.e. the unique number satisfying*

$$\mu_{n-1}\{x; f(x)\leqslant M_f\}\geqslant 1/2, \quad \mu_{n-1}\{x; f(x)\geqslant M_f\}\geqslant 1/2.$$

Let $A^f=\{x; f(x)=M_f\}$. Then for every $\varepsilon>0$

$$\mu_{n-1}((A^f)_\varepsilon)\geqslant 2\gamma_n\int_0^\varepsilon\cos^{n-2}t\,dt. \tag{2.3}$$

Proof. Let $A^f_+=\{x;\ f(x)\geqslant M_f\}$ and $A^f_-=\{x;\ f(x)\leqslant M_f\}$. Then $\mu_{n-1}(A^f_+)\geqslant 1/2$ and $\mu_{n-1}(A^f_-)\geqslant 1/2$. Observe that for every $\varepsilon>0$

$$(A^f_+)_\varepsilon \cap (A^f_-)_\varepsilon = (A^f)_\varepsilon.$$

Hence by Theorem 2.1 and (2.1)

$$1-\mu_{n-1}((A^f)_\varepsilon)\leqslant 2\left(1-\gamma_n\int_{-\pi/2}^{\varepsilon}\cos^{n-2}t\,dt\right)$$

and this implies (2.3). □

It is convenient to give (2.3) a simpler form. The function $f(t)=e^{t^2/2}\cos t$ decreases on $[0,\pi/2]$. Hence for $0\leqslant\varepsilon\leqslant\pi/2$

$$\int_\varepsilon^{\pi/2}\cos^n t\,dt\leqslant\int_\varepsilon^{\pi/2}e^{-nt^2/2}\,dt\leqslant e^{-n\varepsilon^2/2}\int_0^\infty e^{-nu^2/2}\,du=(\pi/2n)^{1/2}e^{-n\varepsilon^2/2}.$$

Also since

$$\Gamma\left(\frac{n}{2}+1\right)\leqslant\Gamma\left(\frac{n+1}{2}\right)^{1/2}\Gamma\left(\frac{n+3}{2}\right)^{1/2}\leqslant\left(\frac{(n+1)}{2}\right)^{1/2}\Gamma\left(\frac{n+1}{2}\right)$$

we get that

$$\int_0^{\pi/2}\cos^n t\,dt=\Gamma\left(\frac{n+1}{2}\right)\Gamma(\tfrac12)/2\Gamma\left(\frac{n}{2}+1\right)\geqslant\left(\frac{\pi}{2(n+1)}\right)^{1/2}.$$

Thus we deduce from (2.3) that for $\varepsilon\leqslant 1$

$$\mu_{n-1}((A^f)_\varepsilon)\geqslant 1-\left(\frac{n-1}{n-2}\right)^{1/2}e^{-(n-2)\varepsilon^2/2}\geqslant 1-4e^{-n\varepsilon^2/2}. \tag{2.4}$$

We apply now Lemma 2.2 to the function $r(x)=\|x\|$ defined on $S^{n-1}=\{x;\ |||x|||=1\}$ where $\|\cdot\|$ is a norm in an n dimensional Banach space X and $|||\cdot|||$ is an inner product norm on X so that

$$a|||x|||\leqslant\|x\|\leqslant b|||x|||,\quad x\in X \tag{2.5}$$

for suitable $0<a\leqslant b<\infty$. Let M_r be the median of $r(x)$ with respect to the rotation invariant measure μ_{n-1} on S^{n-1} and let $A=\{x;\ |||x|||=1,\ \|x\|=M_r\}$. Assume that $y\in A_\varepsilon$ for some $\varepsilon>0$; i.e. y is a point on S^{n-1} so that $d(x,y)\leqslant\varepsilon$ for some $x\in A$. Recall that $d(x,y)$ is the geodesic distance from x to y on S^{n-1} and this quantity is always larger than $|||x-y|||$. Hence $\|x-y\|\leqslant b|||x-y|||\leqslant b\varepsilon$ and thus

$$M_r-b\varepsilon\leqslant\|y\|\leqslant M_r+b\varepsilon,\quad y\in A_\varepsilon.$$

It follows now from Lemma 2.2 (or more precisely from (2.4)) that

$$1-\mu_{n-1}\{y;\ |||y|||=1,\quad \big|\,\|y\|-M_r\big| \leqslant b\varepsilon\} < 4e^{-n\varepsilon^2/2}. \tag{2.6}$$

From this inequality it is easy to deduce the following proposition.

PROPOSITION 2.3. *Let $(X, \|\ \|)$ be a Banach space of dimension n. Let $|||\cdot|||$ be an inner product norm on X so that (2.5) holds. Let M_r be the median of $r(x)=\|x\|$ on $\{x;\ |||x|||=1\}$ and let $0<\varepsilon\leqslant 1$. Let $\{y_i\}_{i=1}^m$ be any m points of norm 1 in l_2^n where $m\leqslant e^{n\varepsilon^2/2}/4$. Then there is an isometry U from l_2^n onto $(X, |||\cdot|||)$ so that*

$$M_r-b\varepsilon \leqslant \|Uy_i\| \leqslant M_r+b\varepsilon,\quad i=1,2,\dots,m. \tag{2.7}$$

Proof. Let U_0 by any isometry from l_2^n onto $(X, |||\ |||)$. Let σ be the normalized Haar measure on the space of all orthogonal transformations V of $(X, |||\ |||)$. By (2.6) we have for every unit vector y in l_2^n

$$\sigma\{V;\ M_r-b\varepsilon \leqslant \|VU_0y\| \leqslant M_r+b\varepsilon\} > 1-1/m.$$

Hence there is at least one orthogonal transformation V_0 of $(X, |||\ |||)$ so that (2.7) holds for $U=V_0U_0$. □

Remark. If we take as m a number which is smaller than $4^{-1}e^{n\varepsilon^2/2}$ say $\gamma e^{n\varepsilon^2/2}$ with a small γ then the proof of the proposition shows that for "most" choices of V_0 (or more precisely, for a set of V_0's of large σ measure) the operator $U=V_0U_0$ has the desired properties.

The most useful way to apply Proposition 2.3 is to choose the points $\{y_i\}_{i=1}^m$ so that they form a "δ net" in a subspace of l_2^n of a suitable dimension. In this case $\{Uy_i\}_{i=1}^m$ also form a δ net in a subspace of $(X, |||\cdot|||)$ of the same dimension. By the term "δ net" in a metric space (K, ϱ) we mean a subset K_0 of K so that for every $x\in K$ there is a $y\in K_0$ with $\varrho(x,y)\leqslant\delta$.

We need first two elementary lemmas, both well known.

LEMMA 2.4. *Let X be a k-dimensional Banach space and let $\delta>0$. Then $\{x\in X;\ \|x\|=1\}$ has a δ net of cardinality $\leqslant(1+2/\delta)^k$.*

Proof. Let $\{x_i\}_{i=1}^m$ be a maximal subset of the boundary of the unit ball B of X consisting of points whose mutual distances are larger than δ. The maximality of $\{x_i\}_{i=1}^m$ implies that this set is a δ net of $\{x\in X;\ \|x\|=1\}$. The sets $x_i+\frac{1}{2}\delta B$ are all disjoint and are contained in $(1+\delta/2)B$. By comparing volumes we get that $m(\delta/2)^k\leqslant(1+\delta/2)^k$ and this proves the lemma. □

LEMMA 2.5. *Let* $(X, \|\cdot\|)$ *be a Banach space and let* $|||\cdot|||$ *be an equivalent inner product norm on* X. *Let* $0<\delta, \varrho<1$ *and assume that for a* δ *net* K_0 *of the set* $\{x;\ |||x|||=1\}$ *we know that* $1-\varrho\leq\|x\|\leq 1+\varrho$ *for every* $x\in K_0$. *Then for every* $x\in X$

$$F(\varrho,\delta)^{-1}|||x|||\leq\|x\|\leq F(\varrho,\delta)\,|||x|||$$

for some function $F(\varrho,\delta)$ *(depending only on* ϱ *and* δ *but not on* X*) for which* $\lim_{\varrho,\delta\to 0} F(\varrho,\delta)=1$.

Proof. If $|||x_0|||=1$ we can represent x_0 as $x_0=x_1+\delta x_2+\delta^2x_3+\dots$ with $x_i\in K_0$ for every i. Hence $\|x_0\|\leq(1+\varrho)/(1-\delta)$ i.e. $\|x\|\leq(1+\varrho)|||x|||/(1-\delta)$ for every $x\in X$. We identify X with X^* via the inner product induced by $|||\ \ |||$ and let $\|x\|_*$ be the norm of x as an element of X^* (i.e. $\|x\|_*=\sup\{(x,z);\ \|z\|\leq 1\}$). For every $x\in K_0$ let y_x be an element in X so that $(y_x,x)=1$ and $\|y_x\|_*\leq 1/(1-\varrho)$. Then

$$1\leq|||y_x|||\leq(1+\varrho)/(1-\delta)(1-\varrho),\quad |||x+y_x|||\geq 2.$$

Hence

$$|||x-y_x|||^2\leq 2+2(1+\varrho)^2/(1-\delta)^2(1-\varrho)^2-4=\Delta^2(\varrho,\delta)$$

where $\lim_{\delta,\varrho\to 0}\Delta(\varrho,\delta)=0$.

Put $\tilde{y}_x=y_x/|||y_x|||$. Then $|||\tilde{y}_x-x|||\leq\Delta(\varrho,\delta)$. Hence the set $\{\tilde{y}_x;\ x\in K_0\}$ forms a $\delta+\Delta(\varrho,\delta)$ net in $\{y;\ |||y|||=1\}$. Since $\|\tilde{y}_x\|_*\leq 1/(1-\varrho)$ for every $x\in K_0$ we get from the first step of the proof that

$$\|y\|_*\leq|||y|||/(1-\varrho)(1-\delta-\Delta(\varrho,\delta)),\quad y\in X$$

and consequently

$$\|x\|\geq|||x|||(1-\varrho)(1-\delta-\Delta(\varrho,\delta)),\quad x\in X$$

and this concludes the proof. □

Summarizing the preceding results we get

THEOREM 2.6. *For every* $\tau>0$ *there is a constant* $\eta(\tau)>0$ *having the following property. Let* $(X,\|\ \|)$ *be an* n *dimensional Banach space and let* $|||\cdot|||$ *be an inner product norm on* X *so that* (2.5) *holds. Let* M_r *be the median of* $r(x)=\|x\|$ *on* $\{x;\ |||x|||=1\}$. *Then for* $k=[\eta(\tau)\cdot nM_r^2/b^2]$ *there is a* k*-dimensional subspace* Y *of* X *so that* $d(Y, l_2^k)\leq 1+\tau$.

Proof. Given $\tau>0$ choose $\delta>0$ and $\varrho>0$ so that $F^2(\varrho,\delta)\leq 1+\tau$ where F is the function appearing in Lemma 2.5. We claim that $\eta(\tau)=\varrho^2/4\log(3/\delta)$ has the desired property. Indeed, by Proposition 2.3 with $\varepsilon=\varrho M_r/b$ we can find, for every choice of $m=[\frac{1}{4}\exp(n\varrho^2M_r^2/2b^2)]$ unit vectors $\{y_i\}_{i=1}^m$ in l_2^n, an isometry U from l_2^n onto $(X, M_r^{-1}|||\cdot|||)$ so that $1-\varrho\leq\|Uy_i\|\leq 1+\varrho$ for every $1\leq i\leq m$. By Lemma 2.4 and the fact that by our choice of $\eta(\tau)$ and k

$$(1+2/\delta)^k\leq(3/\delta)^k=e^{k\log(3/\delta)}\leq m$$

we may choose the points $\{y_i\}_{i=1}^m$ so that they form a δ net in a k-dimensional subspace Y_0 of l_2^n. By Lemma 2.5 we get that $d(UY_0, l_2^k) \leqslant 1+\tau$.

Remark. The probabilistic nature of the proof (see the remark following Proposition 2.3) shows that if we replace the $\eta(\tau)$ by a smaller constant then we can achieve that "most" k-dimensional subspaces of X are of distance $\leqslant 1+\tau$ from l_2^k. "Most" means here a set of measure close to 1 (the closeness to 1 depends on the choice of $\eta(\tau)$) in the natural measure induced by $|||\cdot|||$ on the Grassmann manifold of k-dimensional subspaces of X. Since this measure depends on the choice of the inner product $|||\cdot|||$ it is not an intrinsic notion related to $(X, \|\cdot\|)$.

The usefulness of Theorem 2.6 depends very much on the possibility of choosing "good" inner product norms $|||\cdot|||$ on X and on the possibility to evaluate the median M_r. Let us recall the fact due to F. John [15] that for every Banach space X of dimension n, $d(X, l_2^n) \leqslant n^{1/2}$. More precisely, if E is the (unique) ellipsoid of minimal volume containing the unit ball B of X then $B \supset n^{-1/2}E$, i.e. $|||x||| \leqslant \|x\| \leqslant n^{1/2}|||x|||$ for every $x \in X$ where $|||\cdot|||$ is the norm whose unit ball is E.

For inner product norms such as this one (i.e. for which $b/a \leqslant n^{1/2}$ in (2.5)) the median M_r behaves like the mean $\int_{S^{n-1}} \|x\| d\mu_{n-1}(x)$, a quantity which in practice is much easier to compute. The reason for this is the high concentration of $r(x) = \|x\|$ near its median which is exhibited in (2.6).

LEMMA 2.7. *There is an absolute constant c so that whenever (2.5) holds with $b \leqslant n^{1/2}$ (where $n = \dim X$) then*

$$\left| \int_{S^{n-1}} \|x\| \, d\mu_{n-1}(x) - M_r \right| < c.$$

Proof. For every integer m we have by (2.6)

$$\mu_{n-1}\{x;\ |||x||| = 1; \quad m \leqslant \big| \|x\| - M_r \big| \leqslant m+1\} \leqslant 4e^{-m^2/2}$$

and thus $c = \sum_{m=1}^{\infty} 4(m+1)e^{-m^2/2}$ has the required property. □

It follows from Lemma 2.7 that whenever $b/a \leqslant n^{1/2}$ we have

$$\tfrac{1}{2} \leqslant M_r^{-1} \int_{S^{n-1}} \|x\| \, d\mu_{n-1}(x) \leqslant \gamma \tag{2.8}$$

for some absolute positive constant γ. The high concentration of $r(x)$ near its median shows also that for every $\varrho > 0$, $\int_{S^{n-1}} \|x\|^\varrho d\mu_{n-1}(x)$ behaves like M_r^ϱ, i.e. that

$$\gamma(\varrho)^{-1} \leqslant \int_{S^{n-1}} \|x\|^\varrho \, d\mu_{n-1}(x) \Big/ \left(\int_{S^{n-1}} \|x\| \, d\mu_{n-1}(x) \right)^\varrho \leqslant \gamma(\varrho) \tag{2.9}$$

for some positive constant $\gamma(\varrho)$, whenever $b/a \leqslant n^{1/2}$. The constant $\gamma(\varrho)$ is bounded on every compact set in $(0, \infty)$.

We shall present now some variants of Theorem 2.6 which do not require the computation of M_r. A completely trivial way to eliminate M_r is to observe that always $M_r \geqslant a$ and thus by choosing the inner product norm in (2.5) so that $b/a = d(X, l_2^n)$ we get that

$$k_\tau(X) \geqslant \eta(\tau)\, n/d^2(X, l_2^n) \tag{2.10}$$

where $k = k_\tau(X)$ is an integer so that X contains a subspace Y with $d(Y, l_2^k) \leqslant 1+\tau$.

We present now an improved version of (2.10) which involves the projection constant of X^*. Let us recall that the projection constant $\lambda(X)$ of a Banach space X is the infimum of all the numbers λ such that whenever X is a subspace of a Banach space Z there is a projection of norm $\leqslant \lambda$ from Z onto X. It is known (cf. [17], [12]) that $\lambda(X) \leqslant n^{1/2}$ for every X with $\dim X = n$. For a Banach space X we denote by $\pi_1(X)$ the number

$$\pi_1(X) = \sup\left\{\sum_{j=1}^{k} \|x_j\|; \quad \{x_j\}_{j=1}^{k} \in X,\ \sum_{j=1}^{k} |x^*(x_j)| \leqslant \|x^*\| \quad \text{all} \quad x^* \in X^*\right\}.$$

In terms of absolutely summing operators (which will be discussed briefly in the next section) $\pi_1(X)$ is the 1-absolutely summing norm of the identity of X. The number $\pi_1(X)$ is related to $\lambda(X)$ by the inequality $\lambda(X)\pi_1(X) \geqslant n$, and this inequality becomes an equality for spaces X which have "enough" isometries (e.g. if X has a basis whose symmetric constant is 1, cf. [12]).

THEOREM 2.8. *For every $\tau > 0$ there is a $\gamma(\tau) > 0$ so that for every Banach space X of dimension n the following formula holds*

$$k_\tau(X) \geqslant \gamma(\tau)\, n^2/d^2(X, l_2^n)\lambda^2(X^*). \tag{2.11}$$

As before, $k_\tau(X)$ is an integer such that X has a subspace whose distance from the Hilbert space of the same dimension is $\leqslant 1+\tau$.

Proof. We deduce (2.11) directly from Theorem 2.6 and an estimate of M_r in terms of $\pi_1(X^*)$ which is due to Rutovitz [31], which we reproduce now.

Let $|||\cdot|||$ be an inner product norm on X so that (2.5) holds with $b/a = d(X, l_2^n)$. We identify X with X^* via the inner product induced by $|||\cdot|||$ and as before (the proof of Lemma 2.5) let $\|x\|_*$ be the norm of x as an element in X^*. Clearly

$$b^{-1}|||x||| \leqslant \|x\|_* \leqslant a^{-1}|||x|||, \quad x \in X. \tag{2.12}$$

Let $\{x_j\}_{j=1}^k$ be elements in X so that

$$\sum_{j=1}^k \|x_j\|_* \geq \pi_1(X^*)/2 \tag{2.13}$$

$$\sum_{j=1}^k |(x, x_j)| \leq \|x\|, \quad x \in X. \tag{2.14}$$

By integrating (2.14) on $\{x, |||x||| = 1\}$ with respect to the normalized rotation invariant measure on this set we get

$$\int_{|||x|||=1} \|x\| \, d\mu_{n-1}(x) \geq \sum_{j=1}^k \int_{|||x|||=1} |(x, x_j)| \, d\mu_{n-1}(x) = \alpha_n \sum_{j=1}^k |||x_j||| \tag{2.15}$$

where

$$\alpha_n = \int_{\sum_{i=1}^n t_i^2 = 1} |t_1| \, d\mu_{n-1}(t).$$

A direct computation shows that $\alpha_n = 2\gamma_n/(n-1)$ where γ_n is the number defined in (2.2). Hence $\alpha_n \geq n^{-1/2}/2$. By (2.8), (2.12), (2.13) we deduce from (2.15) that for some absolute constant $\eta > 0$

$$M_r \geq \eta \cdot n^{1/2} \sum_{j=1}^k |||x_j||| \geq \eta a n^{-1/2} \sum_{j=1}^k \|x_j\|_* \geq \eta a n^{-1/2} \pi_1(X^*)/2 \geq \eta a n^{1/2}/2\lambda(X^*). \tag{2.16}$$

Substituting this estimate for M_r in the value of $k_\tau(X)$ appearing in the statement of Theorem 2.6 we get

$$k_\tau(X) \geq \eta(\tau)\eta^2 n^2 a^2/4b^2\lambda^2(X^*) = \eta(\tau)\eta^2 n^2/4d^2(X, l_2^n)\lambda^2(X^*),$$

and this proves the theorem. □

Remarks. That (2.11) is an improvement on (2.10) follows from the fact that $\lambda(X^*) \leq n^{1/2}$. For spaces which do not have "enough" isometries it is preferable to state (2.11) in the sharper form $k_\tau(X) \geq \gamma(\tau)\pi_1^2(X^*)/d^2(X, l_2^n)$. We mention also that the proof of Theorem 2.8 and (2.9) shows that for every $\varrho > 0$ there is a $\gamma(\tau, \varrho)$ so that

$$k_\tau(X) \geq \gamma(\tau, \varrho)\pi_\varrho^2(X^*)/d^2(X, l_2^n) \tag{2.17}$$

where $\pi_\varrho(X^*)$ is the ϱ absolutely summing norm of the identity of X^* (the estimate (2.17) becomes better as ϱ approaches 0).

Perhaps the most useful version of Theorem 2.6 is obtained by considering simultaneously $k_\tau(X)$ and $k_\tau(X^*)$.

THEOREM 2.9. *For every $\tau > 0$ there is a $\delta(\tau) > 0$ so that for every Banach space X of dimension n the following formula holds*

$$k_\tau(X)\,k_\tau(X^*) \geq \delta(\tau)\,n^2\|P\|^2/d^2(X, l_2^n). \tag{2.18}$$

Here $\|P\|$ is the norm of a projection P which acts either on X or on X^*. If $k_1=k_\tau(X)\leqslant k_2=k_\tau(X^*)$ then P is a projection from X onto a subspace Y for which $d(Y, l_2^{k_1})\leqslant 1+\tau$. If $k_2<k_1$ then P is a projection from X^* onto a subspace Z for which $d(Z, l_2^{k_2})\leqslant 1+\tau$.

Proof. Let $|||\cdot|||$ be an inner product norm on X so that $|||x|||\leqslant\|x\|\leqslant d|||x|||$ for every $x\in X$ where $d=d(X, l_2^n)$. By identifying X with X^* via the inner product induced by $|||\cdot|||$ get that $d^{-1}|||x|||\leqslant\|x\|_*\leqslant|||x|||$ for every $x\in X$ where $\|x\|_*$ denotes the norm of x when considered as a functional on $(X, \|\ \|)$. In other words $X^*=(X, \|\ \|_*)$. Let M_r and M_{r^*} be the medians of $r(x)=\|x\|$ and $r^*(x)=\|x\|_*$ on $\{x;\ |||x|||=1\}$. By Theorem 2.6 there are subspaces Y of X and Z of X^* so that $d(Y, l_2^{k_1})\leqslant 1+\tau$ and $d(Z, l_2^{k_2})\leqslant 1+\tau$ where $k_1\geqslant \eta(\tau)nM_r^2/d^2$ and $k_2\geqslant\eta(\tau)nM_{r^*}^2$. In order to prove (2.18) it suffices therefore to find a projection P so that $\|P\|\leqslant\gamma M_r M_{r^*}$ for some absolute constant γ.

We recall now that if $\eta(\tau)$ is chosen small enough we can ensure not only that $d(Y, l_2^{k_1})\leqslant 1+\tau$ but that for a set of isometries U of $(X, |||\cdot|||)$ whose Haar measure is $>1/2$ we have that $d(UY, l_2^{k_1})\geqslant 1+\tau$ (see the remark following Proposition 2.3). A similar remark holds for X^* and Z. Thus if e.g. $k_1\leqslant k_2$ there is no loss of generality to assume that $Y\subset Z$. We claim now that the orthogonal (with respect to $|||\cdot|||$) projection P from X onto Y satisfies $\|P\|\leqslant(1+\tau)^2M_rM_{r^*}$. Indeed, we have for every $y\in Y$

$$\|y\|\leqslant M_r(1+\tau)\,|||y|||,\quad \|y\|_*\leqslant M_{r^*}(1+\tau)\,|||y|||.$$

Hence for every $x\in X$

$$|||Px|||^2=(Px, Px)=(Px, x)\leqslant\|Px\|_*\|x\|\leqslant M_{r^*}(1+\tau)\,|||Px|||\,\|x\|$$

and thus

$$\|Px\|\leqslant M_r(1+\tau)\,|||Px|||\leqslant(1+\tau)^2M_rM_{r^*}\|x\|.\qquad\square$$

Remark. We would like once again to point out the measure theoretic nature of the proof and the result. If $\delta(\tau)$ is chosen small enough we get that most (in the sense of having large measure with respect to the natural measure induced by $|||\cdot|||$ on the Grassmanian) subspaces Y of X of dimension k_1 satisfy $d(Y, l_2^{k_1})\leqslant 1+\tau$ and similarly for subspaces of X^* of dimension k_2. Moreover if $k_1\leqslant k_2$ then on most subspaces Y of X of dimension k_1 there is a projection P from X onto Y whose norm may be used in the right hand side of (2.18).

To conclude this section we state explicitely two formulas which are weaker versions of (2.18). Since always $\|P\|\geqslant 1$ and $d(X, l_2^n)\leqslant n^{1/2}$ we get

$$k_\tau(X)\,k_\tau(X^*)\geqslant\delta(\tau)\,n \tag{2.19}$$

$$\max\,(k_\tau(X),\ k_\tau(X^*))\geqslant(\delta(\tau)\cdot n)^{1/2}. \tag{2.20}$$

It should be pointed out that in order to derive (2.19) (and thus (2.20)) we do not have to go through the construction of the projection P in the proof of Theorem 2.9. It is clear directly that we always have $M_r M_{r^*} \geq 1$. This is a consequence of the trivial inequality $1 = (x, x) \leq \|x\| \|x\|_* $ for every x with $|||x||| = 1$.

§ 3. Examples and applications

In this section we present several applications of the results of section 2 and especially of Theorem 2.9. We shall mostly consider subspaces Y with $d(Y, l_2^k) \leq 2$ and not the more general case of $1+\tau$. The passage from 2 to $1+\tau$ changes the dimension we get by a multiplicative constant which depends only on τ. This follows by applying (2.10) to a Banach space X with $d(X, l_2^n) \leq 2$.

Example 3.1. Let $X = l_p^n$, $1 \leq p \leq \infty$, $n = 1, 2, \ldots$ There is a constant c so that X contains a subspace Y with $\dim Y = k$ and $d(Y, l_2^k) \leq 2$ where

(i) $k = c \log n$, if $p = \infty$
(ii) $k = cn^{2/p}$, if $2 \leq p < \infty$
(iii) $k = cn$, if $1 \leq p \leq 2$.

In all three cases these are the best possible estimates.

Proof. Let $|||\cdot|||$ be the usual Euclidean norm in R^n, i.e. $|||(t_1, t_2, \ldots, t_n)||| = (\sum_{i=1}^n t_i^2)^{1/2}$. For $p \geq 2$ we have that $n^{(1/p)-(1/2)} |||x||| \leq \|x\| \leq |||x|||$ and in particular $d(l_p^n, l_2^n) \leq n^{(1/2)-(1/p)}$. Assertion (ii) is therefore a consequence of (2.10). In the other cases however, (2.10) does not give the desired result.

For $1 \leq p \leq 2$ we have $|||x||| \leq \|x\| \leq n^{(1/p)-(1/2)} |||x|||$ and it is quite easy to compute M_r. By (2.8) and (2.9) we have

$$M_r^p \sim \int_{|||x|||=1} \|x\|^p \, d\mu_{n-1}(x) = n \int_{\Sigma t_i^2 = 1} |t_1|^p \, d\mu_{n-1}(t)$$

where the symbol $\alpha_n \sim \beta_n$ means that α_n / β_n is bounded and bounded away from 0. Also, since $1 \leq p \leq 2$

$$n^{-1/2}/2 \leq \int |t_1| \, d\mu_{n-1}(t) \leq \left(\int |t_1|^p \, d\mu_{n-1}(t) \right)^{1/p} \leq \left(\int |t_1|^2 \, d\mu_{n-1}(t) \right)^{1/2} = n^{-1/2}$$

we get that $M_r \sim n^{(1/p)-(1/2)}$ and thus (iii) is a consequence of Theorem 2.6.

Also (i) can be verified by computing M_r and applying Theorem 2.6. This computation is somewhat more delicate and will be done in the next section in a somewhat more general context (cf Proposition 4.3). The simplest way to prove (i) is by a direct argument. Let $\{y_j\}_{j=1}^n$ be a $\frac{1}{2}$—net on the boundary of the unit ball of l_2^k. By Lemma 2.4 n can be taken $\leqslant 5^k$, i.e. $k \geqslant \log n/\log 5$. The operator T: $l_2^k \to l_\infty^n$ defined by $Tx((x, y_1), (x, y_2), ..., (x, y_n))$ satisfies $\frac{1}{2}\|x\| \leqslant \|Tx\| \leqslant \|x\|$ for every $x \in l_2^k$ and thus $d(l_2^k, Tl_2^k) \leqslant 2$.

Let us verify that (ii) and (iii) are the best possible estimates. For (iii) this is obvious. For (ii) the simplest argument is the following one (taken from [2]). Let $u_j = (t_{j,1}, t_{j,2}, ..., t_{j,n})$, $j = 1, 2, ..., k$ be vectors in l_p^n so that

$$\left(\sum_{j=1}^{k} \lambda_j^2\right)^{1/2} \leqslant \left\|\sum_{j=1}^{k} \lambda_j u_j\right\| \leqslant 2\left(\sum_{j=1}^{k} \lambda_j^2\right)^{1/2}$$

for any choice of scalars $\{\lambda_j\}_{j=1}^n$. Then if $\{r_j(s)\}_{j=1}^k$ denote the Rademacher functions on $[0, 1]$ we have for every $s \in [0, 1]$

$$k^{p/2} \leqslant \left\|\sum_{j=1}^{k} r_j(s)\, u_j\right\|^p = \sum_{i=1}^{k} \left|\sum_{j=1}^{k} r_j(s)\, t_{j,i}\right|^p.$$

By integrating with respect to s and using Khintchine's inequality we deduce that for some $c(p)$

$$k^{p/2} \leqslant c(p) \sum_{i=1}^{n} \left(\sum_{j=1}^{k} t_{j,i}^2\right)^{p/2}.$$

Since by our assumption on the $\{u_j\}_{j=1}^k$ we have that $(\sum_{j=1}^k t_{j,i}^2)^{1/2} \leqslant 2$ for $1 \leqslant i \leqslant n$ it follows that $k^{p/2} \leqslant 2^p c(p)\, n$ and thus (ii) cannot be improved.

Assertion (iii) is also a consequence of (2.11). Indeed since $\lambda(l_\infty^n) = 1$ we have that $\lambda(l_q^n) \leqslant d(l_q^n, l_\infty^n) \leqslant n^{1/q}$ for $2 \leqslant q < \infty$. Thus if $(1/p) + (1/q) = 1$ and $X = l_p^n$ with $1 \leqslant p \leqslant 2$ we get from (2.11) that $k_\tau(X) \geqslant \gamma(\tau)\, n^2/n = \gamma(\tau)\, n$.

It is perhaps also instructive to deduce (ii) and (iii) from (2.18). Let $X = l_p^n$ with $2 < p < \infty$. By (2.18)

$$k(X)\, k(X^*) \geqslant \delta n^2/d^2(l_p^n, l_2^n) = \delta n^{1+(2/p)}.$$

Since we verified directly that $k(X) \leqslant C(p)\, n^{2/p}$ and trivially $k(X^*) \leqslant n$ we deduce (ii) and (iii) (for $p > 1$). In this way we get however (iii) in a slightly weaker form in which c is replaced by a constant depending on p. The way to overcome this difficulty is to take into account also the term $\|P\|$ in (2.18) which we ignored above. Using this term we can deduce (iii) also for $p = 1$. Indeed since $\lambda(l_2^n) = (2n/\pi)^{1/2}$ (cf. [31], [12]) we get by taking $X = l_\infty^n$ in (2.18) that $\|P\| \geqslant (k_1/2\pi)^{1/2}$ and since $k_1 k_2 \geqslant \delta n^2 k_1/2\pi n$ it follows that $k_2 \geqslant \delta n/2\pi$. □

We have still to verify that (i) is the best possible estimate. This fact will be used several times in the sequel and we state it therefore explicitly (cf. [29]).

PROPOSITION 3.2. *Let T be an operator from l_2^k into l_∞^n so that $\|T\|\,\|T^{-1}\|\leqslant\lambda$. Then $\log n\geqslant C(\lambda)k$ where $C(\lambda)$ is a positive constant depending on λ.*

Proof. By the remark in the beginning of this section it suffices to prove the proposition for some $\lambda>1$ say for $\lambda=8/7$. We may assume that $\|T\|=1$. Let $\{e_i\}_{i=1}^n$ be the unit vector basis in $l_1^n=(l_\infty^n)^*$ and put $y_i=T^*e_i\in l_2^k$, $1\leqslant i\leqslant n$. The vectors y_i are all in the unit ball of l_2^k and for every $x\in l_2^k$ with $\|x\|=1$ there is an i so that $\min(\|x+y_i\|, \|x-y_i\|)\leqslant 1/2$. Indeed, since $\|T^{-1}\|\leqslant 8/7$ there is an i so that $|(x, y_i)|=|Tx(e_i)|\geqslant 7/8$ and hence for a suitable choice of the sign $\|x\pm y_i\|^2=\|x\|^2+\|y_i\|^2\pm 2(x, y_i)\leqslant 1/4$. The union of the balls in l_2^k with centers $\{\pm y_i\}_{i=1}^n$ and radius 3/2 cover, thus the ball with radius 2 centered at the origin. By comparing volumes we get that $2^k\leqslant 2n(3/2)^k$ and this proves the proposition. $\square$

In Example 3.1 we exhibited various ways in which the estimates of section 2 can be used in verifying (ii) and (iii). (Another probabilistic method for proving (ii) is presented in [2].) We shall now illustrate still another approach in which the results of section 2 can be applied by computing the dimensions of almost Hilbertian subspaces of the spaces C_p^n, $1\leqslant p\leqslant\infty$, $n=1, 2, 3, \ldots$ The Banach space C_p^n is the space $B(l_2^n)$ of all operators T from l_2^n into itself endowed with the norm $\|T\|_p=(\sum_{i=1}^n \lambda_i^p)^{1/p}$ where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of $|T|=(TT^*)^{1/2}$. For $p=\infty$ we get the usual operator norm, for $p=2$ we get the Hilbert-Schmidt norm. Recall that $(C_p^n)^*$ is isometric to C_q^n where $p^{-1}+q^{-1}=1$. Clearly $\dim C_p^n=n^2$.

Example 3.3. Let $X=C_p^n$, $1\leqslant p\leqslant\infty$, $n=1, 2, \ldots$ There is a constant c so that X contains a subspace Y with $\dim Y=k$ and $d(Y, l_2^k)\leqslant 2$ where

(i) $k=cn^{1+2/p}$ if $2\leqslant p\leqslant\infty$

(ii) $k=cn^2$ if $1\leqslant p\leqslant 2$.

These estimates are the best possible.

Proof. We settle first the case $p=\infty$. For every $1\leqslant p\leqslant\infty$ it is obvious that C_p^n has a subspace isometric to l_2^n. This proves (i) for $p=\infty$. In order to prove that (i) cannot be improved for $p=\infty$ it is enough by Proposition 3.2 to show that $d(C_\infty^n, Z)\leqslant 2$ for some subspace Z of $l_\infty^{2^{\gamma n}}$ for some constant γ. This follows from Lemma 2.4 and the fact that $\|T\|_\infty\geqslant\frac{1}{2}\max_{1\leqslant i,j\leqslant m}(Tx_i, x_j)$ for every $T\in B(l_2^n)$ if $\{x_i\}_{i=1}^m$ is a $\frac{1}{4}$—net in the boundary of the unit ball of l_2^n.

We pass now to the case $2\leqslant p<\infty$. Let $Y\subset C_p^n$ be such that $d(Y, l_2^k)\leqslant 2$ and let Y_0 be the same space as Y but with the norm induced from C_∞^n. Since for every $T\in B(l_2^n)$,

$\|T\|_\infty \leq \|T\|_p \leq n^{1/p}\|T\|_\infty$ we get that $d(Y_0, l_2^k) \leq 2n^{1/p}$. By (2.10) we get that Y_0 has a subspace Y_1 with $d(Y_1, l_2^s) \leq 2$ where $s \geq \eta k/4n^{2/p}$. By the case $p=\infty$ we deduce that $s \leq cn$ and hence

$$k \leq \eta_0 n^{1+2/p} \tag{3.1}$$

for some constant η_0. Observe also that for $p>2$ and $T \in B(l_2^n)$, $\|T\|_p \leq \|T\|_2 \leq n^{(1/2)-(1/p)}\|T\|_p$ and thus $d(C_p^n, l_2^{n^2}) \leq n^{(1/2)-(1/p)}$. It follows from (2.18) that

$$k(C_p^n)\, k(C_q^n) \geq \delta(n^2)^2/(n^{(1/2)-(1/p)})^2 = \delta n^{2+(1+2/p)}. \tag{3.2}$$

Assertions (i) and (ii) are immediate consequences of (3.1) and (3.2). Also, (3.1) asserts that (i) is the best possible estimate. Clearly (ii) is also the best possible. □

We pass now to some examples of spaces whose unit balls are polytopes. First we present a general result on polytopes.

THEOREM 3.4. *There is an absolute positive constant γ so that the following holds. For every convex polytope Q in R^n which is symmetric with respect to the origin (and has the origin as an interior point) we have*

$$\log s \log t \geq \gamma n, \tag{3.3}$$

where $2s$ is the number of vertices of Q and $2t$ is the number of $(n-1)$-dimensional faces of Q.

Proof. Let X be the Banach space whose unit ball is Q. Then X is isometric to a subspace of l_∞^t and X^* is a subspace of l_∞^s. Inequality (3.3) is thus an immediate consequence of (2.19) and Proposition 3.2. □

For furture reference we state explicitly the following consequence of (3.3).

$$\max(\log s, \log t) \geq (\gamma n)^{1/2} \tag{3.4}$$

Theorem 3.4 is a partial solution to the so called "lower bound problem" for symmetric convex polytopes: Give a lower bound for the number of $(n-1)$ dimensional faces for a symmetric polytope in R^n having $2s$ vertices. For s with $\log s \leq \gamma n$ formula (3.3) gives such a lower estimate and as we shall see below this estimate is asymptotically the best possible in this range. Let us note that for general (i.e. not necessarily symmetric) polytopes the lower bound problem was solved in precise terms by Barnette [1]. For non symmetric polytopes the situation is different from that expressed in (3.3). In the non symmetric case we have no logarithmic (or equivalently exponential) terms, this is seen by considering e.g. a simplex in R^n where $s=t=n+1$.

Let us point out that though Theorem 3.4 is a theorem in the combinatorial theory of convex polytopes its proof allows us to make some statements which relate combinatorial quantities to the global shape of the polytope. We note first that our proof shows e.g. that the following variant of (3.3) holds. Let Q_1 and Q_2 be two symmetric convex polytopes in R^n so that $Q_1 \subset Q_2 \subset 2Q_1$. Then (3.3) is valid if we take as $2s$ the number of vertices of Q_1 and as $2t$ the number of faces of Q_2. Note also that (3.3) was derived from (2.19) which in turn was obtained from (2.18) by replacing $d(X, l_2^n)$ by its upper bound $n^{1/2}$ and $\|P\|$ by its lower bound 1. Thus in case where Q is nearer to an ellipsoid than $n^{1/2}$ or if Q does not have nicely complemented almost spherical sections our proof gives a stronger estimate than (3.3).

Our next example is of a very simple polytope for which we have almost equality in (3.4). Its main purpose is to illustrate how the term $\|P\|$ in (2.18) can be used to solve a problem in the theory of p-absolutely summing operators.

Example 3.5. The space $X = (\overbrace{l_1^n \oplus l_1^n \oplus \ldots \oplus l_1^n}^{n})_\infty$ has a subspace Y of dimension $k \geqslant cn$ with $d(Y, l_2^k) \leqslant 2$ so that there is a projection P from X onto Y with $\|P\| \leqslant c(\log n)^{1/2}$.

Proof. The number of extreme points in the unit ball of X is $2s = (2n)^n$ and the number of extreme points of the unit ball of $X^* = (\overbrace{l_\infty^n \oplus l_\infty^n \oplus \ldots \oplus l_\infty^n}^{n})_1$ is $2t = n \cdot 2^n$. Observe that $\log t \sim n$ and $\log s \sim n \log n$ while $\dim X = n^2$ i.e. in this case (3.4) is close to being an equality.

We apply now (2.18) to this space X and obtain $k(X)\,k(X^*) \geqslant \delta n^2 \|P\|^2$. From the evaluation of s and t and Proposition 3.2 we deduce that $k(X) \leqslant \gamma n$ and $k(X^*) \leqslant \gamma n \log n$ and thus $\|P\| \leqslant \gamma (\log n)^{1/2}/\delta^{1/2}$. From this information it does not follow directly that $k(X) \geqslant \eta n$ for some $\eta > 0$ (and not only $k(X) \geqslant \eta n/\log n$). In order to show that $k(X) \geqslant \eta n$ we go back to Theorem 2.6 and evaluate M_r. The norm in X is given by $\|x\| = \sup_{1 \leqslant j \leqslant n}(\sum_{i=1}^n |x_{i,j}|)$. As an inner product norm $|||\cdot|||$ in X we take $|||x||| = (\sum_{i,j=1}^n |x_{i,j}|^2)^{1/2}$. Clearly $|||x||| \cdot n^{-1/2} \leqslant \|x\| \leqslant |||x|||n^{1/2}$ for every $x \in X$. Also

$$\int_{|||x|||=1} \|x\|\, d\mu_{n^2-1}(x) \geqslant \sum_{i=1}^n \int_{|||x|||} |x_{i,1}|\, d\mu_{n^2-1}(x) \geqslant n \cdot \frac{1}{2} \cdot \frac{1}{(n^2)^{1/2}} = \frac{1}{2},$$

and thus, by Theorem 2.6, $k(X) \geqslant \eta n^2/4(n^{1/2})^2 = \eta n/4$. This concludes the proof. □

Remark. We do not know whether there is a sequence k_n with $\lim_{n\to\infty} k_n = \infty$ and a constant c independent of n so that X contains a subspace Y with $\dim Y = k_n$, $d(Y, l_2^{k_n}) \leqslant 2$ and so that there is a projection P from X onto Y with $\|P\| \leqslant c$.

Example 3.5 enables the solution of a problem of A. Pelczynski concerning p-absolutely summing operators. Recall that a bounded linear operator $T: X \to Y$ is said to be p absolutely summing, $1 \leq p < \infty$ if there is a constant K so that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leq K \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{1/p}$$

for every choice of n and $\{x_i\}_{i=1}^n$ in X. It is easily verified and well known that if $p_1 < p_2$ then every p_1-absolutely summing operator is also p_2-absolutely summing. Thus the smallest family of operators among those defined above is that of 1-absolutely summing operators. A fundamental result of Grothendieck [13] (cf. also [23]) asserts that every bounded linear operator from l_1 into l_2 is 1-absolutely summing. It was proved in [23] that this result actually characterizes in a certain sense the pair (l_1, l_2) among all pairs of Banach spaces (e.g. if X has an unconditional basis and the pair $(X, Y) \neq (l_1, l_2)$ then there is a bounded operator from X to Y which is not 1-absolutely summing). The situation changes if we consider pairs (X, Y) such that every bounded linear operator from X to Y is 2-absolutely summing. Here there are much more examples, e.g. for every $1 \leq p \leq 2$ the pair (c_0, l_p) has this property (cf. [23]). Pelczynski conjectured that the class of all Banach spaces X such that every bounded $T: X \to l_2$ is 2-absolutely summing is closed under direct sum in the sense of l_1. It follows easily from Example 3.5 that this is false.

PROPOSITION 3.6. *There exists a bounded linear operator* $T: (c_0 \oplus c_0 \oplus \ldots)_1 \to l_2$ *which is not p-absolutely summing for any* $p < \infty$.

Proof. By Example 3.5 there is for every n a projection P_n from $X = (c_0 \oplus c_0 \oplus \ldots)_1$ onto a subspace Y_n so that $d(Y_n, l_2^n) \leq 2$ and $\|P_n\| \leq c(\log n)^{1/2}$. The space $Y = (\sum_{n=1}^{\infty} \oplus Y_{2^n})_2$ is isomorphic to l_2. The operator $T: X \to Y$ defined by

$$Tx = (P_2 x, 4^{-1} P_4 x, \ldots, n^{-2} P_{2^n} x, \ldots)$$

is bounded but it is not p-absolutely summing for any $p < \infty$ (apply the definition of a p-absolutely summing operator to a set $\{x_i\}_{i=1}^{2^n}$ which corresponds to an orthonormal basis in $l_2^{2^n}$ by the isomorphism from Y_{2^n} onto $l_2^{2^n}$).

We conclude this section by an example which shows that (3.4) (and therefore also (2.20)) are sharp estimates.

Example 3.7. Let $\{X_n\}_{n=1}^{\infty}$ be the sequence of Banach spaces defined by $X_1 = l_\infty^2$ and for $n \geq 1$

$$X_{2n} = (X_{2n-1} \oplus X_{2n-1})_1, \quad X_{2n+1} = (X_{2n} \oplus X_{2n})_\infty.$$

Then $\dim X_{2n}=2^{2n}$ and the unit ball of X_{2n} is a polytope with $2s_n$ vertices and $2t_n$ faces with $\log s_n$, $\log t_n \leqslant 3\cdot 2^n$.

Proof. A straightforward verification shows that $2s_n=2^{2^{n+1}-1}$ and $2t_n=2^{2^{n+1}+2^n-2}$. $\square$

Remarks. 1. Observe that by (2.18) and Proposition 3.2 there is a constant $\gamma>0$ so that for every n there is a subspace $Y_n\subset X_{2n}$ with $\dim Y_n=k_n\geqslant\gamma\cdot 2^n$, $d(Y_n, l_2^{k_n})\leqslant 2$ and a projection P_n from X_{2n} onto Y_n with $\|P_n\|\leqslant\gamma^{-1}$.

2. Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of integers so that $\sum_{n=1}^{\infty} 2^{-n}\log m_n<\infty$. If we change the definition of the $\{X_n\}_{n=1}^{\infty}$ in the example above by putting $X_{2n}=(\overbrace{X_{2n-1}\oplus X_{2n-1}\oplus\ldots\oplus X_{2n-1}}^{m_n})_1$ and keeping the definition of X_{2n+1} then we get that $\dim X_{2n}=2^n\cdot m_1 m_2\ldots m_n$, $\log s_n\sim 2^n$ and $\log t_n\sim m_1 m_2\ldots m_n$. This shows that (3.3) can become an equality (up to a constant independent of n) if $\log s$ is given in advance provided that $\log n=O(\log s)$ and $\log s=O(n/\log n)$.

§ 4. The Dvoretzky Rogers lemma

In this section we discuss the "Dvoretzky Rogers Lemma" and various variants of it. We also show how this lemma is used in proving Dvoretzky's theorem on the existence of almost spherical sections.

We begin by stating the Dvoretzky-Rogers Lemma in its original formulation (cf. [7]).

THEOREM 4.1. *Let $(X, \|\ \|)$ be an n dimensional Banach space and let E be the ellipsoid of maximal volume contained in the unit ball of X. Let $|||\ |||$ be the inner product norm induced on X by E. Then there exists an orthonormal basis $\{u_i\}_{i=1}^n$ of $(X, |||\ |||)$ and vectors $\{x_i\}_{i=1}^n$ in X so that $\|x_i\|=|||x_i|||=1$ and*

$$x_i=\sum_{j\leqslant i} a_{i,j}u_j,\quad a_{i,i}^2=1-\sum_{j<i} a_{i,j}^2\geqslant\frac{n-i+1}{n},\ 1\leqslant i\leqslant n. \tag{4.1}$$

This theorem is by now well known and its proof has been reproduced in many places (e.g. in Day's book) so we do not reproduce its proof here. We just mention that the proof is based on a variational argument.

Let us point out explicitly one immediate consequence of (4.1). We have for every i

$$\begin{aligned}\|a_{i,i}u_i\| &\geqslant \|x_i\|-\|x_i-a_{i,i}u_i\|\geqslant 1-|||x_i-a_{i,i}u_i|||\\ &=1-(1-a_{i,j}^2)^{1/2}=a_{i,j}^2/(1+(1-a_{i,j}^2)^{1/2})\end{aligned}$$

and thus $\|u_i\|\geqslant|a_{i,i}|/(1+(1-a_{i,j}^2)^{1/2})$. Consequently, by (4.1)

$$\|u_i\|\geqslant 1/2\quad\text{if } i-1\leqslant\tfrac{9}{25}n. \tag{4.2}$$

In other words there are $c \dim X$ vectors in X which are orthogonal with respect to $|||\ |||$ and on which the norms $|||\cdot|||$ and $||\ ||$ differ by the factor two at most. There is an obvious duality between the ellipsoid E of maximal volume in the unit ball of X and the ellipsoid E' of minimal volume containing the unit ball of X^*. Thus the result of Fritz John [15] which was mentioned already in section 2 shows that in the notation of Theorem 4.1

$$n^{-1/2}|||x||| \leqslant ||x|| \leqslant |||x||| \text{ for every } x \in X \tag{4.3}$$

Theorem 4.1 enables us to give a non trivial estimate of the median M_r of $r(x)=||x||$ on $\{x;\ |||x|||=1\}$ i.e. on the surface of E. For obtaining this estimate we need the following computational result.

LEMMA 4.2. *Let* $1 \leqslant m \leqslant n$ *be integers. Then*

$$\int_{S^{n-1}} \max_{1\leqslant i<m} |t_i|\, d\mu_{n-1}(t) \geqslant c(\log\ m/n)^{1/2} \tag{4.4}$$

where c *is an absolute constant,* $S^{n-1}=\{(t_1, \dots, t_n);\ \sum_{i=1}^n t_i^2=1\}$ *and* μ_{n-1} *is the normalized rotation invariant measure on* S^{n-1}.

Proof. Let ν be the measure on R^n whose density is given by $\exp(-\pi \sum_{i=1}^n t_i^2)$. For every continuous real valued function $f(t)$ on S^{n-1} let $\tilde{f}(t)$ be a function on $R^n \sim \{0\}$ defined by $\tilde{f}(t)=|||t|||f(t/|||t|||)$ where $|||t|||=(\sum_{i=1}^n t_i^2)^{1/2}$. It follows immediately from the uniqueness of a normalized rotation invariant measure on S^{n-1} that there is a constant λ_n so that

$$\int_{R^n} \tilde{f}(t)\, d\nu(t) = \lambda_n \int_{S^{n-1}} f(t)\, d\mu_{n-1}(t),\ f \in C(S^{n-1}). \tag{4.5}$$

By taking $f \equiv 1$ in (4.5) we deduce that

$$\begin{aligned}\lambda_n = \int_{R^n} |||t|||\, d\nu(t) &\leqslant \left(\int_{R^n} |||t|||^2\, d\nu(t)\right)^{1/2} = \left(\int_{R^n} \sum_{i=1}^n t_i^2 \exp\left(-\pi \sum_{i=1}^n t_i^2\right) dt_1 \dots dt_n\right)^{1/2} \\ &= \left(n \int_{-\infty}^{\infty} t^2 e^{-\pi t^2}\, dt\right)^{1/2} = (n/2\pi)^{1/2}.\end{aligned} \tag{4.6}$$

Observe next that for every $\varepsilon>0$, and $0<\delta<1/\pi$ we have $\nu\{t;\ \max_{1\leqslant i\leqslant m}|t_i| \leqslant (\delta \log m)^{1/2}\}<\varepsilon$ provided $m \geqslant m(\delta, \varepsilon)$. Indeed, put $\alpha=(\delta \log m)^{1/2}$, then

$$\int_\alpha^\infty e^{-\pi t^2}\, dt \geqslant \frac{1}{4\pi\alpha} e^{-\pi\alpha^2} = \frac{1}{4\pi}(\delta\ \log\ m)^{-1/2} m^{-\pi\delta} > \tfrac{1}{2}(1-\varepsilon^{1/m})$$

and thus

$$\nu\{t;\max_{1\leqslant i\leqslant m}|t_i|\leqslant\alpha\}=\left(\int_{-\alpha}^{\alpha}e^{-\pi t^2}\,dt\right)^m<(\varepsilon^{1/m})^m=\varepsilon$$

by using this observation for $\varepsilon=\delta=1/4$ say, (4.5) for the function $f(t)=\max_{1\leqslant i\leqslant m}|t_i|$ and (4.6) we deduce that

$$\int_{S^{n-1}}f(t)\,d\mu_{n-1}(t)=\lambda_n^{-1}\int_{R^n}f(t)\,d\nu(t)\geqslant\lambda_n^{-1}\cdot\tfrac{3}{4}\cdot(\tfrac{1}{4}\log m)^{1/2}\geqslant c(\log m/n)^{1/2}.\qquad\square$$

PROPOSITION 4.3. *With the notation preceding Lemma 4.2 we have $M_r\geqslant\eta(\log n/n)^{1/2}$ for some absolute constant $\eta>0$.*

Proof. By Lemma 2.7 it is enough to prove that

$$\int_{S^{n-1}}\left\|\sum_{i=1}^n t_i u_i\right\|d\mu_{n-1}(t)\geqslant\gamma(\log n/n)^{1/2}$$

for some constant γ. Let $\{r_i(s)\}_{i=1}^\infty$ denote the Rademacher functions on [0, 1]. Observe that for vectors $\{w_i\}_{i=1}^n$ in a Banach space $\int_0^1\|\sum_{i=1}^n r_i(s)w_i\|ds$ is just the average of $\|\sum_{i=1}^n\pm w_i\|$ over all choices of signs and is therefore $\geqslant\max_i\|w_i\|$. Consequently by (4.2) and (4.4) we get for $m=[4n/25-1]$

$$\int_{S^{n-1}}\left\|\sum_{i=1}^n t_i u_i\right\|d\mu_{n-1}(t)=\int_0^1\int_{S^{n-1}}\left\|\sum_{i=1}^n r_i(s)\,t_i u_i\right\|d\mu_{n-1}(t)\,ds$$

$$\geqslant\int_{S^{n-1}}\max_{1\leqslant i\leqslant n}\|t_i u_i\|\,d\mu_{n-1}(t)\geqslant\tfrac{1}{2}\int_{S^{n-1}}\max_{1\leqslant i\leqslant m}|t_i|\,d\mu_{n-1}(t)\geqslant\gamma(\log n/n)^{1/2}$$

$\square$

From Theorem 2.6, Proposition 4.3 and (4.3) we deduce immediately Dvoretzky's theorem (with an estimate first given in [29], which is better than the original estimate)

THEOREM 4.4. *There is an absolute constant c so that every Banach space X with* $\dim X=n$ *has a subspace Y with* $\dim Y=K\geqslant c\log n$ *and* $d(Y,l_2^k)\leqslant 2$.

It follows from Proposition 3.2 that this estimate cannot be improved in the general case. By combining Theorem 4.4 with 2.10 we get that if $\dim X=n$ then $X\supset Y$ with $\dim Y=k$, $d(Y,l_2^k)\leqslant 2$ and $k\geqslant c\max(\log n,n/d^2(X,l_2^n))$. It is natural to ask in view of this formula whether Theorem 4.4 can be improved in situations where we know that $d(X,l_2^n)=o(n^{1/2})$ (but of course $d(X,l_2^n)\geqslant c(n/\log n)^{1/2}$). The answer is negative as the following example shows.

Example 4.5. There is a Banach space X of dimension n so that $d(X, l_2^n) = (n/\log n)^{1/2}$ and so that X does not have subspaces close to a Hilbert space of dimension $\geq C \log n$.

Proof. Let X be R^n with the norm defined as follows

$$\|(t_1, t_2, \ldots, t_n)\| = \max\left((\log n/n)^{1/2}\left(\sum_{i=1}^n t_i^2\right)^{1/2},\quad |t_i|,\ i = 1, \ldots, n\right)$$

Put $|||(t_1, \ldots, t_n)||| = (\sum_{i=1}^n t_i^2)^{1/2}$. Then $(\log n/n)^{1/2}|||x||| \leq \|x\| \leq |||x|||$ for every $x \in X$ and thus $d(X, l_2^n) \leq (n/\log n)^{1/2}$ (actually we have an equality in this inequality). Let Y be a subspace of X and let $|||\ |||_0$ be an inner product norm on Y so that $|||y|||_0 \leq \|y\| \leq 2|||y|||_0$ for every $y \in Y$. It is an elementary fact that every ellipsoid in R^{2k} with the origin as center has a k-dimensional section through 0 which is a ball (cf. e.g. [6]). Thus by passing if necessary to a subspace of Y of half the dimension we may assume without loss of generality that $|||y|||_0 = a|||y|||$ for some $a > 0$ and every $y \in Y$.

We consider now separately two cases.

Case (*i*): $a > (\log n/n)^{1/2}$. In this case $\|y\| > (\log n/n)^{1/2}|||y|||$ and thus $\|y\| = \max_{1 \leq i \leq n} |t_i|$ for every $y = (t_1, t_2 \ldots, t_n) \in Y$. The fact that dim $Y \leq C \log n$ follows now from Proposition 3.2.

Case (*ii*): $a \leq (\log n/n)^{1/2}$. Let $y_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$, $1 \leq i \leq k$ be an orthonormal basis in $(Y, |||\ |||_0)$. Then $\sum_{j=1}^n u_{i,j}^2 = a^{-2}$ for every i. Also for every choice of $\{\lambda_i\}_{i=1}^k$ with $\sum_{i=1}^k \lambda_i^2 = 1$ we have

$$\max_j \left|\sum_{i=1}^k \lambda_i u_{i,j}\right| \leq \left\|\sum_{i=1}^k \lambda_i y_i\right\| \leq 2\left|\left|\left|\sum_{i=1}^k \lambda_i y_i\right|\right|\right|_0 = 2$$

and hence $\sum_{i=1}^k u_{i,j}^2 \leq 4$ for every j. Consequently

$$a^{-2}k = \sum_{i=1}^k \sum_{j=1}^n u_{i,j}^2 = \sum_{j=1}^n \sum_{i=1}^k u_{i,j}^2 \leq 4n$$

or $k \leq 4na^2 \leq 4 \log n$. □

From Theorem 4.1 we deduced (cf. (4.2)) the existence of many orthogonal vectors with respect to $|||\cdot|||$ in which the right hand inequality of (4.3) is close to being equality. In general this is not true for the left hand inequality of (4.3). More precisely, if $\lambda = \sup\{|||x|||;\ \|x\| = 1\}$ then in general we cannot find many othogonal points on the surface of the ellipsoid appearing in Theorem 4.1 for which $\|x\|$ is up to a small constant factor (say a constant independent of n) equal to $1/\lambda$. As we shall see in the next section it is useful for some purposes to have an inner product norm $|||\ |||_0$ with say $a|||x|||_0 \leq \|x\| \leq b|||x|||_0$ so that both sides of the inequality are up to a constant factor precise for a large set of orthonormal vectors with respect to $|||\cdot|||_0$. We shall prove now a version of the Dvoretzky Rogers lemma which ensures this. The proof we present is much simpler than the known proofs of Theorem 4.1.

THEOREM 4.6. *Let* $(X, \|\ \|)$ *be a Banach space of dimension* n *and let* $|||\cdot|||$ *be an inner product norm on* X *so that* $|||x||| \geq \|x\| \geq |||x|||/d$ *for every* $x \in X$ *and some* $d \leq n$. *Let* K *and* γ *be constants so that* $0<\gamma<1$ *and* $K>1$.

Then there exists a subspace Y *of* X *with* $\dim Y = m \geq n^{1+\log(1-\gamma)/\log K}$ *and constants* a *and* b *so that*

$$a|||y||| \leq \|y\| \leq b|||y||| \tag{4.7}$$

for every $y \in Y$ *and so that for every subspace* Y_0 *of* Y *with* $\dim Y_0 \geq (1-\gamma)m$ *there are vectors* y_1 *and* y_2 *in* Y_0 *satisfying*

$$|||y_1||| = |||y_2||| = 1, \quad \|y_1\| \geq b/K, \quad \|y_2\| \leq aK. \tag{4.8}$$

Before proving this result let us make some comments. The main point in this result is that it ensures for both inequalities of (4.7) the existence of many orthonormal vectors in which these inequalities are close to being sharp. Indeed, by an obvious inductive argument we get for $k=[\gamma m]$, k vectors $\{u_i\}_{i=1}^k$ in Y which are orthonormal with respect to $|||\ |||$ and for which $\|u_i\| \geq b/K$ as well as k vectors $\{v_i\}_{i=1}^k$ in Y which are orthonormal with respect to $|||\cdot|||$ and for which $\|v_i\| < aK$. Let us also remark that the d appearing in the statement need not be the distance from X to l_2^n. The fact that we allow as d a number as large as n enables us to apply Theorem 4.6 without using the result of Fritz John.

Proof. If we cannot take X itself as Y then there is a subspace X_1 of X with $\dim X_1 \geq (1-\gamma)n$ so that in X_1 we have either $|||x||| \geq \|x\| \geq K|||x|||/d$ or $|||x|||/K \geq \|x\| \geq |||x|||/d$ for every $x \in X$. If we cannot take X_1 as Y there is a subspace X_2 of X_1 of dimension $\geq (1-\gamma)^2 n$ so that on it for suitable $j_{1,2}, j_{2,2}$ (each being either 0 or 1 or 2 with $j_{1,2}+j_{2,2}=2$) we have

$$K^{-j_{1,2}}|||x||| \geq \|x\| \geq K^{j_{2,2}}|||x|||/d \quad x \in X_2.$$

We continue in an obvious way. If this process does not stop before the p'th step we get a subspace X_p of X with $\dim X_p \geq (1-\gamma)^p n$ and so that for suitable $j_{1,p}, j_{2,p} \in \{0, 1, 2, \ldots, p\}$ with $j_{1,p}+j_{2,p}=p$

$$K^{-j_{1,p}}|||x||| \geq \|x\| \geq K^{j_{2,p}}|||x|||/d, \quad x \in X_p.$$

Consequently $K^p \leq d \leq n$, i.e. $p \leq \log n/\log K$. Hence $\dim Y = (1-\gamma)^p n \geq (1-\gamma)^{\log n/\log K} \cdot n$. This concludes the proof. $\square$

Another version of Theorem 4.1 appears in the paper of Larman and Mani [21]. Larman and Mani apply a variational argument which is very similar to that used by Dvoretzky and Rogers (but somewhat more involved) to an ellipsoid which determines the Banach-Mazur distance $d(X, l_2^n) = d$, rather than the ellipsoid of maximal value contained in the unit ball of X. Using this ellipsoid they are able to find 2 sequences of k orthogonal points like the points $\{u_1\}_{i=1}^k$ and $\{v_i\}_{i=1}^k$ described after the statement of Theorem 4.6. In Larman and Mani's proof we get $k \sim cd$. This gives relatively little information when d is small but very useful information if d is close to its maximum i.e. $n^{1/2}$.

Usually an ellipsoid which determines the distance from l_2^n is very far from the ellipsoid of maximal volume (this is a source of many difficulties which occur if we try to apply the results of this paper to various open problems). Let us mention a simple and well known example which exhibits this difference. Let X be the Banach space obtained by taking in R^n the norm whose unit ball B is the convex hull of $B_0 = \{(t_1, t_2, ..., t_n), \sum_{i=1}^n t_i^2 = 1\}$ and the points $\pm(1, 1, 1, 1, ...)$. Then B_0 is the ellipsoid of maximal volume in B (it is even the ellipsoid of maximal volume in the unit cube $Q = \{(t_1, ..., t_n); \max_i |t_i| \leq 1\}$. We have $B_0 \subset B \subset n^{1/2} B_0$ and $n^{1/2}$ cannot be replaced by a smaller number. On the other hand the set B is obtained by rotating a 2 dimensional figure (namely the convex hull of $\pm(n^{1/2}, 0)$ and $\{(t_1, t_2); t_1^2 + t_2^2 \leq 1\}$) and thus $d(X, l_2^n) \leq 2^{1/2}$.

§ 5. Connection with the notion of cotype

The results proved in the previous sections have an interesting connection with the notion of the cotype of a Banach space. Let us first recall the definitions of the notions of type and cotype.

A Banach space X is said to be of type p (resp. cotype q) if there is a constant $\alpha < \infty$ (resp $\beta > 0$) so that for every integer m and every choice of $\{x_i\}_{i=1}^m$ in X we have

$$\int_0^1 \left\| \sum_{i=1}^m r_i(s)\, x_i \right\| ds \leq \alpha \left(\sum_{i=1}^m \|x_i\|^p \right)^{1/p} \tag{5.1}$$

respectively

$$\int_0^1 \left\| \sum_{i=1}^m r_i(s)\, x_i \right\| ds \geq \beta \left(\sum_{i=1}^m \|x_i\|^q \right)^{1/q} \tag{5.2}$$

where $\{r_i\}_{i=1}^\infty$ denote as before the Rademacher functions on $[0, 1]$. Let us recall that from the classical Khintchine inequality it follows that necessarily for every X, $1 \leq p \leq 2 \leq q \leq \infty$. The constants α resp. β are called the p-type and q-cotype constants of X. We mention also a result of Kahane [18] which asserts that there is a constant $\gamma(r)$ for every $r > 1$ so that for every Banach space X every choice of m and $\{x_i\}_{i=1}^m$

$$\left(\int_0^1 \left\|\sum_{i=1}^m r_i(s)\, x_i\right\|^r\right)^{1/r} \leqslant \gamma(r) \int_0^1 \left\|\sum_{i=1}^m r_i(s)\, x_i\right\| ds. \tag{5.3}$$

Since obviously $(\int_0^1 \|\sum_{i=1}^m r_i(s)x_i\|^r)^{1/r} \geqslant \int_0^1 \|\sum_{i=1}^m r_i(s)x_i\| ds$ we see that of functions $f(s)$ of the form $\|\sum_{i=1}^m r_i(s)x_i\|$ all the L_r norms are equivalent and thus we could use on the left hand side of (5.1) and (5.2) and L_r norm (with of course a suitable change in α and β).

In the preceding section (the proof of Proposition 4.3) we used already integrals of the form appearing in (5.1) and (5.2) for computing medians. For spaces in which we know the type and the cotype this computation gives naturally more information.

PROPOSITION 5.1. *Let $(X, \|\ \|)$ be an n dimensional space whose p-type constant is α and whose q-cotype constant is β for some $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. Let $|||\cdot|||$ be an inner product norm on X so that $a|||x||| \leqslant \|x\| \leqslant b|||x|||$ for every $x \in X$ (with $b/a \leqslant n^{1/2}$). Let $\{u_i\}_{i=1}^n$ be an orthonormal basis of $(X, |||\ |||)$ so that $\|u_1\| \geqslant \|u_2\| \geqslant \ldots \geqslant \|u_n\|$. Let M_r be the median of $r(x) = \|x\|$ on $\{x;\ |||x||| = 1\}$. Then*

$$c\alpha\left(\sum_{i=1}^n \|u_i\|^{2p/(2-p)}\right)^{(2-p)/2p} \geqslant M_r \geqslant c^{-1}\beta \max_{1 \leqslant m \leqslant n} \{\|u_m\| m^{1/q} n^{-1/2}\}, \tag{5.3}$$

where c is an absolute constant.

Proof. We apply Lemma 2.7 which allows us to replace M_r by the mean of $\|x\|$ (it is here that the assumption $b/a \leqslant n^{1/2}$ is used). We have

$$\begin{aligned}
\int_{S^{n-1}} \|x\|\, d\mu_{n-1}(x) &= \int_{S^{n-1}} \|\sum t_i u_i\|\, d\mu_{n-1}(t) = \int_{S^{n-1}} \int_0^1 \left\|\sum_{i=1}^n r_i(s)\, t_i\, u_i\right\| ds\, d\mu_{n-1}(t) \\
&\leqslant \alpha \int_{S^{n-1}} \left(\sum_{i=1}^n \|t_i u_i\|^p\right)^{1/p} d\mu_{n-1}(t) \\
&\leqslant \alpha\left(\sum_{i=1}^n \|u_i\|^{2p/(2-p)}\right)^{(2-p)/2p} \int_{S^{n-1}} \left(\sum_{i=1}^n |t_i|^2\right)^{1/2} d\mu_{n-1}(t) \\
&= \alpha\left(\sum_{i+1}^n \|u_i\|^{2p/(2-p)}\right)^{(2-p)/2p}.
\end{aligned}$$

This proves the left hand inequality of (5.3). To prove the right hand inequality we use the following estimate for every $1 \leqslant m \leqslant n$

$$\begin{aligned}
\int_{S^{n-1}} \left(\sum_{i=1}^n \|t_i u_i\|^q\right)^{1/q} d\mu_{n-1}(t) &\geqslant \|u_m\| \int_{S^{n-1}} \left(\sum_{i=1}^m |t_i|^q\right)^{1/q} d\mu_{n-1}(t) \\
&\geqslant \|u_m\| m^{1/q-1/2} \int_{S^{n-1}} \left(\sum_{i=1}^m |t_i|^2\right)^{1/2} d\mu_{n-1}(t) \geqslant \tfrac{1}{2} \|u_m\| m^{1/q} n^{-1/2}. \qquad \square
\end{aligned}$$

In general it is convenient to use different orthonormal bases in the two inequalities of (5.3). For the right hand inequality we use $\{u_i\}_{i=1}^n$ for which $\|u_i\|$ is as large as possible (preferably with $\|u_i\|$ close to b) while for the left hand inequality we use $\{u_i\}_{i=1}^k$ for which $\|u_i\|$ is a small as possible (preferably $\|u_i\|$ close to a). The right hand inequality is more convenient to use since for it, it is enough to have a good estimate on $\|u_i\|$ only for a large set of indices i and not for all $1 \leqslant i \leqslant n$. In particular if we take as $|||\ |||$ the norm induced by the ellipsoid of maximum volume in the unit ball of X we have $b=1$ and by (4.2) we may choose the $\{u_i\}_{i=1}^n$ so that $\|u_i\| \geqslant 1/2$ for $1 \leqslant i \leqslant 9n/25$. Hence we get that in this case $M_r \geqslant c\beta n^{1/q-1/2}$. Thus we deduce from Theorem 2.6 the following result

THEOREM 5.2. *Let X be a Banach space of cotype $2 \leqslant q < \infty$ and let β be the q-cotype constant of X. Then for every subspace $X_0 \subset X$ with* $\dim X_0 = n$ *there is a subspace $Y \subset X_0$ so that $d(Y, l_2^k) \leqslant 2$ and $k \geqslant \eta\beta^2 n^{2/q}$ where η is an absolute constant.*

Since every uniformly convex space is of cotype q for some $q < \infty$ (cf. [27]) it follows that in the presence of uniform convexity we always get Hilbertian subspace of X_0 of dimension $\geqslant (\dim X_0)^\delta$ for some $\delta > 0$.

The space $L_q(0, 1)$ is of cotype $\max(2, q)$ and the same is true for the space C_q of operators on l_2 with the norm $\|T\|_q = (\operatorname{trace}(TT^*)^{q/2})^{1/q}$, cf. [34]. By applying Theorem 5.2 to the spaces $l_q^n \subset L_q(0, 1)$ we recover exactly the results proved in Example 3.1 (for $1 \leqslant q < \infty$). This shows that the result of Theorem 5.2 is sharp. By applying Theorem 5.2 to the spaces $C_q^n \subset C_q$ we get the same result as that in Example 3.3 for $1 \leqslant q < 2$ while for $q > 2$ Theorem 5.2 does not give the best possible result which is presented in Example 3.3. Theorem 5.2 gives however much more information than Examples 3.1 and 3.3 (at least for $1 \leqslant q \leqslant 2$) since it applies not only to the spaces l_q^n and C_q^n but to *any* finite dimensional subspace of $L^q(0, 1)$ respectively C_q.

Theorem 5.2 can actually be used to characterize the cotype of a Banach space (up to $\varepsilon > 0$). A result of Krivine and Maurey and Pisier (cf. [28]) states that if X is a Banach space and if $q_0 = \inf\{q;\ X \text{ is of cotype } q\}$ then for every n there is a subspace Y_n of X so that $d(Y_n, l_{q_0}^n) \leqslant 2$. Note that by Example 3.1 these spaces Y_n do not contain subspaces close to Euclidean spaces of dimension larger than cn^{2/q_0}. Consequently if X is a Banach space which satisfies the conclusion of Theorem 5.2 then X is of cotype $q+\varepsilon$ for every $\varepsilon > 0$. The ε cannot be dropped in this statement. This is shown by the following example due to W. B. Johnson.

Example 5.3. Let $q \geqslant 2, \varepsilon > 0$ and a sequence of integers $\{a_n\}_{n=1}^\infty$ be given so that $\lim_n a_n = \infty$. Then there is a Banach space X of type 2 which is not of cotype q and yet has the following property. For every subspace $Y \subset X$ with $\dim Y = n$ there is a subspace $Z \subset Y$ with $\dim Z \geqslant n - a_n$ and a subspace $\tilde{Z}$ of l_q so that $d(Z, \tilde{Z}) \leqslant 1 + \varepsilon$.

This example which was kindly communicated to us by W. B. Johnson is a slight modification of another example of Johnson's which appeared in [16]. Observe that the a_n may tend to ∞ as slowly as we wish. Thus if $q = 2$ and e.g. $a_n = [\log n]$ every subspace Y of X with $\dim Y = n$ has an almost Hilbertian subspace of codimension $\leqslant \log n$. Also if $q > 2$ we get from Theorem 5.2 that, if $a_n \leqslant n/2$ say, the space $\tilde{Z}$ and therefore also Y has an almost Hilbertian subspace of dimension $\geqslant \eta n^{2/q}$.

Proof. We call a finite collection $\{E_j\}_{j=1}^n$ of subsets of the positive integers allowable if the E_j are mutually disjoint and the smallest integer in $\bigcup_{j=1}^n E_j$ is $\geqslant n+1$. Let X_0 be the linear space of all sequences of scalars which are eventually 0. For every subset E of the integers and every $x \in X$ we denote by Ex the vector defined by $Ex(i) = x(i)$ if $i \in E$ and $Ex(i) = 0$ if $i \notin E$. Let $\delta > 0$ and $\eta > 1$ be such $(1+\delta)\eta \leqslant 1 + \varepsilon$.

We define now inductively a sequence of norms $\{\|\cdot\|_k\}_{k=1}^\infty$ on X_0 as follows: $\|x\|_1 = \|x\|_{c_0} = \max_i |x(i)|$, and for $k \geqslant 1$

$$\|x\|_{k+1} = \max\left(\|x\|_k, \eta^{-1} \sup\left(\left(\sum_{j=1}^n \|E_j x\|_k^q\right)^{1/q}, \{E_j\}_{j=1}^n \text{ allowable, } n = 1, 2, \ldots\right)\right).$$

It is easy to see that $\|x\|_k \leqslant (\sum_{i=1}^\infty |x(i)|^q)^{1/q}$ for every k and thus $\lim_k \|x\|_k = \|x\|$ exists for every $x \in X_0$. The completion of X_0 with respect to this limiting norm is denoted by $\tilde{X} = \tilde{X}(q, \eta)$.

It is easily verified that the unit vectors $\{e_i\}_{i=1}^\infty$ form an unconditional basis of $\tilde{X}$ and

$$\|x\| = \max\left(\|x\|_{c_0}, \eta^{-1} \sup\left(\left(\sum_{j=1}^n \|E_j x\|^q\right)^{1/q}, \{E_j\}_{j=1}^n \text{ allowable}\right)\right). \tag{5.4}$$

Also, if $\{x_j\}_{j=1}^m$ are elements in $\tilde{X}$ with disjoint support (with respect to the unit vector basis) we get that

$$\left\|\sum_{j=1}^m x_j\right\| \leqslant \left(\sum_{j=1}^m \|x_j\|^q\right)^{1/q}. \tag{5.5}$$

The example presented in [16] is the one obtained by taking $q = 1$ and $\eta = 2$. The main technical argument in [16] is the proof of the fact that with these choices of q and η, $\tilde{X}(q, \eta)$ does not contain a subspace isomorphic to l_1. The same argument (with some obvious minor modifications) shows that for a general $1 \leqslant q \leqslant \infty$ and $\eta > 1$ the space $\tilde{X}(q, \eta)$ does not have a subspace isomorphic to l_q. If follows from this that $\tilde{X}(q, \eta)$ has no infinite-dimensional subspace of cotype q. Indeed, suppose that W is an infinite-dimensional subspace of X of cotype q. Then by the standard gliding hump argument we get a sequence $\{w_j\}_{j=1}^{\infty}$ of vectors of norm 1 in W which are almost disjointly supported with respect to the unit vector basis. Since W is of cotype q there is a $\beta > 0$ so that $\|\sum_{j=1}^{\infty} \lambda_j w_j\| \geqslant \beta(\sum_{j=1}^{\infty} |\lambda_j|^q)^{1/q}$ for every choice of scalars $\{\lambda_j\}_{j=1}^{\infty}$. By combining this inequality with (5.5) we deduce that $\{w_j\}_{j=1}^{\infty}$ is equivalent to the unit vector basis of l_q and this, as mentioned above, is impossible.

We quote now a result due essentially to Pelczynski and Rosenthal [30] (cf. also [16]) which we shall need in the sequel. For every integer n and $\delta > 0$ there is an integer $N = N(n, \delta)$ having the following property. If W is a space with an unconditional basis (whose unconditionality constant is 1) and Y is a subspace of W with $\dim Y = n$ then there is a subspace $\tilde{Y} \subset W$ spanned by N vectors with mutually disjoint supports and an operator T: $Y \to \tilde{Y}$ so that $\|y\| \leqslant \|Ty\| \leqslant (1+\delta)\|y\|$ for every $y \in Y$.

Using this function $N(n, \delta)$ we define in terms of the given sequence $\{a_n\}_{n=1}^{\infty}$ and the δ we have chosen above a sequence k_i of positive integers so that $k_i \geqslant \max \{N(n, \delta);$ $a(n) \leqslant i\}$. We claim that the subspace X of $\tilde{Y}$ which is spanned by the sequence $\{e_{k_i}\}_{i=1}^{\infty}$ has the required properties.

To prove this fix n and let

$$V = \{x \in X \subset \tilde{X};\ x(k_i) = 0, \quad i = 1, 2, \ldots, a_n\}.$$

Observe that if $\{y_j\}_{j=1}^{m}$ are non zero vectors in V which have mutually disjoint supports and $m \leqslant N(n, \delta) \leqslant k_{a_n}$ then the sequence $\{\operatorname{supp} y_j\}_{j=1}^{\infty}$ of the supports of the y_j's is allowable and thus by (5.4) and (5.5)

$$\left(\sum_{j=1}^{m} \|\lambda_j y_j\|^q\right)^{1/q} \geqslant \left\|\sum_{j=1}^{m} \lambda_j y_j\right\| \geqslant \eta^{-1}\left(\sum_{j=1}^{m} \|\lambda_j y_j\|^q\right)^{1/q},$$

λ_j scalars.

Hence $d(\operatorname{span} \{y_j\}_{j=1}^{m}, l_q^m) \leqslant \eta$.

Let now $Y \subset X$ with $\dim Y = n$ and let $Z = Y \cap V$. Clearly $\dim Z \geqslant n - a_n$. The discussion above shows that Z is $(1+\delta)$-isomorphic to a subspace of V spanned by $N(n, \delta)$ disjointly supported vectors which, as we have just seen, is η isomorphic to l_q^N.

That X is not of cotype q was already verified above. We omit the proof of the fact that X is of type 2 since this fact is not needed in the present context. □

In order to complete the discussion of the relation between the cotype of a space and the dimension of almost spherical sections we present another example.

Example 5.4. For every $q \geqslant 2$ there is a Banach space X of type 2 and of cotype $q+\varepsilon$ for every $\varepsilon > 0$ yet there is no constant c so that every $Y \subset X$ with $\dim Y = n$ has a subspace Z with $k = \dim Z \geqslant cn^{2/q}$ and $d(Z, l_2^k) \leqslant 2$.

Proof. As X we take the Orlicz sequence space l_M where $M(t)=t^q/|\log t|$ near $t=0$. The fact that X is of cotype $q+\varepsilon$ for every $\varepsilon>0$ and of type 2 follows from the following general result (cf. [25] and [11]). An Orlicz sequence space l_M for which $M(2t)/M(t)\leqslant\gamma$ for $t\in(0, 1]$ is of type p, $p\leqslant 2$ (respectively of cotype q, $q\geqslant 2$) if and only if

$$M(uv)/M(v)\leqslant Ku^q \quad \text{resp.} \quad M(uv)/M(v)\geqslant K^{-1}u^q$$

for all u, $v\in(0, 1]$ and some $K<\infty$.

As for the dimension of almost Hilbertian subspaces, let X_n be the span of the first n unit vectors in l_M and let $Y_n\subset X_n$ with $\dim Y_n=k_n$ and $d(Y_n, l_2^{k_n})\leqslant 2$. Since l_M is of type 2 it follows from a result of Maurey [26] that there is a projection P_n from l_M (and thus from X_n) onto Y_n with $\|P_n\|\leqslant\gamma$ with γ independent of n. By comparing projection constants we get

$$\tfrac{1}{2}(2k_n/\pi)^{1/2}\leqslant \tfrac{1}{2}\lambda(l_2^{k_n})\leqslant\lambda(Y_n)\leqslant\|P_n\|\lambda(X_n)\leqslant\gamma\, d(X_n, l_\infty^n)\leqslant\gamma/M^{-1}\left(\frac{1}{n}\right).$$

A short computation shows that this implies that $k_n\leqslant c(n/\log n)^{2/q}$, for some constant c independent of n. □

Remark. The estimate we just obtained for k_n is precise. For every subspace Y of l_M where $M(t)=t^q/|\log t|$ $(q\geqslant 2)$ with $\dim Y=n$ there is a subspace $Z\subset Y$ with $k=\dim Z\geqslant\eta(n/\log n)^{2/q}$ so that $d(Z, l_2^k)\leqslant 2$. This follows from a generalization of theorem 5.2 which involves the notion of (Gaussian) cotype f where f is a non negative nonincreasing function on $[0, (\pi/2)^{1/2}]$. A Banach space X is said to be of cotype f (with f-cotype constant c) if for every finite sequence $\{x_i\}_{i=1}^n\subset X$ with

$$I=\int_\Omega\left\|\sum_{i=1}^n g_i(\omega)\,x_i\right\|dP(\omega)\neq 0$$

one has $\sum_{i=1}^n f(\|x_i\|/I)\leqslant c$ ($g_i(\omega)$ are independent normalized Gaussian variables in the probability space (Ω, P)).

Now in the setting of Proposition 5.1 we have

$$J=\int_\Omega\left\|\sum_{i=1}^n g_i(\omega)\,u_i\right\|dP(\omega)=c_n\int_{S^{n-1}}\|x\|\,d\mu_{n-1}(x)$$

where c_n is a constant depending only on n. More precisely the proof of Lemma 4.2 shows that c_n equals $\int_\Omega(\sum_{i=1}^n g_i^2(\omega))^{1/2}dP(\omega)$ and hence $c_n\geqslant cn^{1/2}$ where $c>0$ is an absolute constant. Since, as in the proof of Theorem 5.2

$$\sum_{i=1}^n f(\|x_i\|/J)\geqslant(9/25)\cdot f(1/2J)$$

we obtain that

$$\int_{S^{n-1}} \|x\|\, d\mu_{n-1}(x) = J/c_n \geqslant \gamma n^{-1/2}/f^{-1}(\gamma/n) \tag{5.6}$$

for some constant γ.

From (5.6) we get a useful estimate for M_r and hence for $k(X)$ whenever we know f. In the case of Orlicz sequence spaces there is an explicit formula for f (cf. [25] and [11])

$$f(t) = \inf\left\{\frac{t^2 M(uv)}{u^2 M(v)};\quad t < u \leqslant 1,\quad 0 < v \leqslant 1\right\} \tag{5.7}$$

for $t \in (0, 1)$. The estimate for k_n mentioned in the beginning of this remark follows from (5.6) and (5.7) after some elementary computations.

§ 6. Spaces all whose subspaces of a given dimension are almost Hilbertian

In the previous sections we investigated for a given Banach space X with $\dim X = n$ the question for what integer k does X have k dimensional subspaces which are close to being Hilbertian. Our proofs usually proved not only the existence of such subspaces but that "most" subspaces of a given dimension are almost Hilbertian. In this section we consider the problem under what conditions can all the k dimensional subspaces of X be close to Hilbertian. It will turn out that if k is large enough (compared to n) then X itself must be close to being Hilbertian.

Convenient tools for handling the problem mentioned in the preceding paragraph are again the notions of type and cotype which were introduced in the previous section. In this section we shall only be interested in type 2 and cotype 2 but we shall examine in more detail the constants associated with these two notions.

Let X be a Banach space and let N be an integer. We let $\alpha_n(X)$ resp. $\beta_n(X)$ be the smallest numbers such that the following inequality

$$\beta_n(X)^{-1}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2} \leqslant \left(\int_0^1 \left\|\sum_{i=1}^{n} r_i(t)\, x_i\right\|^2 dt\right)^{1/2} \leqslant \alpha_n(X)\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2} \tag{6.1}$$

holds for every choice of $\{x_i\}_{i=1}^n$ in X.

It is clear that the sequences $\alpha_n(X)$ and $\beta_n(X)$ are monotonely increasing with n and that X is of type 2 (resp. cotype 2) if and only if $\sup_n \alpha_n(X) < \infty$ (resp. $\sup_n \beta_n(X) < \infty$). It is also clear that if X is a Hilbert space then $\alpha_n(X) = \beta_n(X) = 1$ for every n. These equations characterize spaces which are isometric to Hilbert space (indeed, the equation $\alpha_2(X) = \beta_2(X) = 1$ is precisely the parallelogram identity). Spaces X which are isomorphic to Hilbert space are characterized by the relation $\sup_n \alpha_n(X) < \infty$ and $\sup_n \beta_n(X) < \infty$. This is a result of Kwapien [20]. We shall state (and apply) in the sequel Kwapien's result in a slightly stronger form.

The quantities $\alpha_n(X)$ and $\beta_n(X)$ were investigated by Maurey and Pisier [28]. We shall need here the following simple facts concerning these constants.

$$\alpha_{nm}(X) \leqslant \alpha_n(X)\,\alpha_m(X), \quad \beta_{nm}(X) \leqslant \beta_n(X)\beta_m(X), \quad m, n = 1, 2, \ldots \tag{6.2}$$

$$\beta_n(X) \leqslant \alpha_n(X^*). \tag{6.3}$$

Let us recall the proof of these inequalities. The first inequality in (6.2) follows from

$$\begin{aligned}\int_0^1 \left\| \sum_{i=1}^{nm} r_i(t)\,x_i \right\|^2 dt &= \int_0^1 \int_0^1 \Bigg\| r_1(s)\left(\sum_{i=1}^{n} r_i(t)\,x_i\right) \\ &\quad + r_2(s)\left(\sum_{i=n+1}^{2n} r_i(t)\,x_i\right) + \ldots + r_m(s)\left(\sum_{i=nm-n+1}^{nm} r_i(t)\,x_i\right)\Bigg\|^2 dt\,ds \\ &\leqslant \alpha_m(X)^2 \int_0^1 \sum_{j=1}^{m} \left\| \sum_{i=(j-1)n+1}^{jn} r_i(t)x_i \right\|^2 dt \leqslant \alpha_m(X)^2\,\alpha_n(X)^2 \sum_{i=1}^{nm} \|x_i\|^2.\end{aligned}$$

The proof of the second inequality in (6.2) is the same. Inequality (6.3) is proved as follows. Pick any $\{x_i\}_{i=1}^n \in X$ and let $\{x_i^*\}_{i=1}^n \in X^*$ be such that $\|x_i\|^2 = \|x_i^*\|^2 = x_i^*(x_i)$ for every i. Then

$$\begin{aligned}\sum_{i=1}^{n} \|x_i\|^2 = \sum_{i=1}^{n} x_i^*(x_i) &= \int_0^1 \left(\sum_{i=1}^{n} r_i(t)\,x_i^*\right)\left(\sum_{i=1}^{n} r_i(t)\,x_i\right) dt \\ &\leqslant \left(\int_0^1 \left\|\sum_{i=1}^{n} r_i(t)\,x_i^*\right\|^2 dt\right)^{1/2} \left(\int_0^1 \left\|\sum_{i=1}^{n} r_i(t)\,x_i\right\|^2 dt\right)^{1/2} \\ &\leqslant \alpha_n(X^*)\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \left(\int_0^1 \left\|\sum_{i=1}^{n} r_i(t)\,x_i\right\|^2 dt\right)^{1/2}.\end{aligned}$$

As in the remark at the end of the preceding section it will be useful for us to introduce variants of $\alpha_n(X)$ and $\beta_n(X)$ obtained by replacing the Rademacher functions by independent random variables $g_i(\omega)$ on some probability space (Ω, P) each having a normalized (i.e. mean 0 and variance 1) Gaussian distribution. We let for every integer n, $\tilde{\alpha}_n(X)$ and $\tilde{\beta}_n(X)$ be the smallest constants for which

$$\tilde{\beta}_n(X)^{-1}\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \leqslant \left(\int_\Omega \left\|\sum_{i=1}^{n} g_i(w)\,x_i\right\|^2 dP(\omega)\right)^{1/2} \leqslant \tilde{\alpha}_n(X)\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \tag{6.4}$$

for every choice of $\{x_i\}_{i=1}^n \subset X$.

We shall use the following relation between $\alpha_n(X)$ and $\tilde{\alpha}_n(X)$ (resp. $\beta_n(X)$ and $\tilde{\beta}_n(X)$)

$$\tilde{\alpha}_n(X) \leqslant \alpha_n(X), \quad \tilde{\beta}_n(X) \leqslant \beta_n(X), \quad n = 1, 2, \ldots \tag{6.5}$$

Indeed, by the symmetry of the g_i's we get

$$\int_\Omega \left\| \sum_{i=1}^n g_i(\omega)\, x_i \right\|^2 dP(\omega) = \int_0^1 \int_\Omega \left\| \sum_{i=1}^n r_i(t)\, g_i(\omega)\, x_i \right\|^2 dP(\omega)\, dt$$
$$\leqslant \alpha_n(X)^2 \int_\Omega \sum_{i=1}^n |g_i(\omega)|^2 \|x_i\|^2 dP(\omega) = \alpha_n(X)^2 \sum_{i=1}^n \|x_i\|^2$$

and analogus inequality holds for the $\beta_n(X)$. We mention without proof some other results on the relation between these constants which can be found in [28] but which we shall not use here. For every Banach space X and every n, $\alpha_n(X) \leqslant (\pi/2)^{1/2}\, \tilde{\alpha}_n(X)$. On the other hand there is a constant c so that $\beta_n(X) \leqslant c\tilde{\beta}_n(X)$ for all n if and only if X is of cotype q for some $q < \infty$. The constants $\tilde{\alpha}_n(X)$ and $\tilde{\beta}_n(X)$ are in general not submultiplicative in the sense of (6.2). From the results we quoted it follows that $\tilde{\alpha}_n(X)$ is always equivalent to a submultiplicative sequence (namely $\alpha_n(X)$). This is false for $\tilde{\beta}_n(X)$. For example if $X = c_0$ it is not hard to verify (this is related to Lemma 4.2) that $\tilde{\beta}_n(c_0) \sim (n/\log n)^{1/2}$. The importance of $\tilde{\alpha}_n(X)$ and $\tilde{\beta}_n(X)$ in our context stems from the following lemma:

LEMMA 6.1. *Let X be a Banach space with* $\dim X = n$. *Then* $\tilde{\alpha}_M(X) = \tilde{\alpha}_{n(n+1)/2+1}(X)$ *and* $\tilde{\beta}_M(X) = \tilde{\beta}_{n(n+1)/2+1}(X)$ *for every* $m > n(n+1)/2+1$.

Proof. Let $m > n(n+1)/2+1$ and let $\{x_i\}_{i=1}^m$ be any m elements n X. Then $Q(x^*) = \sum_{i=1}^m x^*(x_i)^2$ is a quadratic form on X^*. Since the dimension of the space of quadratic forms on X^* is $n(n+1)/2$ it follows from Caratheodory's theorem that $Q(x^*)$ can be represented as a convex combination of fewer than m quadratic forms of the set $\{mx^*(x_i)^2\}_{i=1}^m$. Hence there exist $\lambda_i \geqslant 0$ with $\sum_{i=1}^m \lambda_i = 1$ so that at least one λ_i (say λ_1) vanishes and $Q(x^*) = m \sum_{i=1}^m \lambda_i x^*(x_i)^2$. Let $\lambda = \max_{1 \leqslant i \leqslant m} \lambda_i$ (say that $\lambda = \lambda_2$) and put $y_i = (\lambda_i/\lambda)^{1/2} x_i$ and $z_i = (1 - \lambda_i/\lambda)^{1/2} x_i$, $1 \leqslant i \leqslant m$. Then

$$y_1 = 0, \quad z_2 = 0, \tag{6.6}$$

$$\sum_{i=1}^m \|x_i\|^2 = \sum_{i=1}^m \|y_i\|^2 + \sum_{i=1}^m \|z_i\|^2. \tag{6.7}$$

Also $Q(x^*) = \lambda m \sum_{i=1}^m x^*(y_i)^2$ and thus

$$\int_\Omega \left\| \sum_{i=1}^m g_i(\omega)\, y_i \right\|^2 dP(\omega) = (\lambda m)^{-1} \int_\Omega \left\| \sum_{i=1}^m g_i(\omega)\, x_i \right\|^2 dP(\omega). \tag{6.8}$$

(Here we used the basic property of normal variables namely that the value of $\int_\Omega \|\sum_{i=1}^m g_i(\omega)u_i\|^2 dP(\omega)$ depends only on the quadratic form $\sum_{i=1}^m x^*(u_i)^2$.) Similarly

$$\int_\Omega \left\| \sum_{i=1}^m g_i(\omega) z_i \right\|^2 dP(\omega) = (1-(\lambda m)^{-1}) \int_\Omega \left\| \sum_{i=1}^m g_i(\omega) x_i \right\|^2 dP(\omega). \tag{6.9}$$

Since by (6.6) the sequences $\{z_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ consist of at most $m-1$ non-zero vectors we get from (6.7), (6.8) and (6.9) that $\tilde{\alpha}_m(X) \leq \tilde{\alpha}_{m-1}(X)$ and $\tilde{\beta}_m(X) \leq \tilde{\beta}_{m-1}(X)$ and this concludes the proof. □

Remark. By using a variant of Caratheodory's theorem valid for cones a similar argument shows that we can actually replace $n(n+1)/2+1$ by $n(n+1)/2$.

In order to apply Lemma 6.1 we have to use the theorem of Kwapien [20], which was mentioned already above, in the following form. Let X be a Banach space. Then there is a Hilbert space H so that $d(X, H) \leq \sup_n \tilde{\alpha}_n(X)\tilde{\beta}_n(X)$ (This result has of course a meaningful contents only if $\sup_n \tilde{\alpha}_n(X)\tilde{\beta}_n(X) < \infty$).

THEOREM 6.2. *Let X be an n dimensional Banach space and let $1 < k \leq n$. Then*

$$d(X, l_2^n) \leq (\alpha_k(X)\beta_k(X))^{1+2\log n/\log k}. \tag{6.10}$$

In particular if $d(Y, l_2^k) \leq c$ for every k dimensional subspace Y of X then

$$d(X, l_2^n) \leq c^{2(1+2\log n/\log k)}. \tag{6.11}$$

Proof. Let $r = [1+2\log n/\log k]$. Then $k^r \geq n^2 \geq n(n+1)/2+1$. By (6.2) and (6.5)

$$\tilde{\alpha}_{k^r}(X) \leq \alpha_{k^r}(X) \leq \alpha_k(X)^r,\ \tilde{\beta}_{k^r}(X) \leq \beta_{k^r}(X) \leq \beta_k(X)^r.$$

By Lemma 6.1 and Kwapien's theorem we have that $d(X, l_2^n) \leq \tilde{\alpha}_{k^r}(X)\tilde{\beta}_{k^r}(X)$ and this proves the first assertion of the theorem. The second assertion follows from the following obvious remark

$$\alpha_k(X) = \sup\{\alpha_k(Y),\ Y \subset X, \dim X = k\} \leq \sup\{d(Y, l_2^k),\ Y \subset X, \dim Y = k\}$$

and the analogous remark for $\beta_k(X)$. □

Another consequence of Lemma 6.1 is the following

THEOREM 6.3. *Let X be a Banach space of type $p \leq 2$ and cotype $q \geq 2$. Then there is a constant γ (depending only on the type p and cotype q constants of X) so that for every $Y \subset X$ with* $\dim Y = n$

$$d(Y, l_2^n) \leq \gamma n^{2(1/p-1/q)}. \tag{6.12}$$

Proof. By the definition of type p and the result of Kahane (cf (5.3)) it follows that for some $\eta(p)$

$$\int_0^1 \left\| \sum_{i=1}^{n^2} r_i(t)\, x_i \right\|^2 dt \leqslant \eta(p) \left(\sum_{i=1}^{n^2} \|x_i\|^p \right)^{1/2} \leqslant \eta(p)\, n^{2(1/p-1/2)} \left(\sum_{i=1}^{n^2} \|x_i\|^2 \right)^{1/2}.$$

Hence $\tilde{\alpha}_{n^2}(Y) \leqslant \alpha_{n^2}(Y) \leqslant \eta(p)\, n^{2(1/p-1/2)}$. Similarly $\tilde{\beta}_{n^2}(Y) \leqslant \eta(q)\, n^{2(1/2-1/q)}$. An application of Kwapien's theorem gives the desired results. □

We do not know whether the factor 2 appearing in the exponent of n in (6.12) is really necessary. Our proof would show that it could be dropped if we knew that $\tilde{\alpha}_{n^2}(X) \leqslant c\tilde{\alpha}_n(X)$ for some absolute constant c and all X with $\dim X = n$. In the case of $X = L_p(0, 1)$ or $X = C_p$ (6.12) takes the form $d(Y, l_2^n) \leqslant \gamma n^{2|1/p-1/2|}$ for every $Y \subset X$ with $\dim Y = n$. We do not know even in this special case whether the factor 2 is necessary (without the factor 2 we would certainly get the best possible result since $d(l_p^n, l_2^n) = n^{|1/p-1/2|}$).(1)

Observe that Theorem 6.3 asserts in particular that the second statement of Theorem 6.2 is in a sense the best possible. Indeed let $k(n)$ be a function of n so that $\lim_n \log k(n)/\log n = 0$. Choose for every n a $p_n > 2$ so that $k(n)^{2(1/2-1/p_n)} = 2$, and let $X_n = l_{p_n}^n$. Then by (6.12), $d(Y, l_2^{k(n)}) \leqslant 2\gamma$ for every $Y \subset X_n$ with $\dim Y = k(n)$ while $d(X_n, l_2^n) = n^{1/2-1/p_n} = 2^{\log n/2 \log k(n)} \to \infty$. Theorem 6.2 states of course that we cannot find such X_n if $\lim_n \inf \log k(n)/\log n > 0$.

We present now a variant of Theorem 6.3 where the factor 2 in the exponent in (6.12) is eliminated in the expense of replacing Y by a suitable subspace of Y of relatively high dimension.

PROPOSITION 6.4. *Let X be a Banach space of type $p \leqslant 2$ and cotype $q \geqslant 2$. Then there is a constant η (depending only on the type p and cotype q constants of X) so that for every $Y \subset X$ with $\dim Y = n$ there is a subspace $Z \subset Y$, $k = \dim Z \geqslant \frac{1}{2} n^{1-(\log\log n)^2/\log n}$ so that $d(Z, l_2^k) \leqslant \eta k^{1/p-1/q+(3/\log\log k)}$.*

Proof. We apply Theorem 4.6 to a given subspace Y of X of dimension n with $K = n^{1/\log\log n}$ and $\gamma = 1 - 1/\log n$. We obtain a subspace U of Y with $\dim U = m \geqslant n^{1-(\log\log n)^2/\log n}$ and an inner product norm $|||\cdot|||$ on U so that $a|||y||| \leqslant \|y\| \leqslant b|||y|||$ for every $y \in U$ and suitable constants a and b and so that for every subspace $U_0 \subset U$ with $\dim U_0 \geqslant m/\log n$ there are vectors y_1 and y_2 in U_0 with

$$|||y_1||| = |||y_2||| = 1, \quad \|y_1\| \geqslant b/K \quad \text{and} \quad \|y_2\| \leqslant aK.$$

In particular, we can find $k \geqslant (1 - 1/\log n)m$ vectors $\{u_j\}_{j=1}^k$ in U so that the $\{u_j\}_{j=1}^k$ are orthonormal with respect to $|||\cdot|||$ and satisfy $\|u_j\| \leqslant aK$, $j = 1, \ldots, k$. Let $Z = \operatorname{span}\{u_j\}_{j=1}^k$.

(1) (Added in proof). D. R. Lewis proved recently that if $Y \subset L_p(0,1)$ with $\dim Y = n$ and $1 < p < \infty$ then $d(Y, l_2^n) \leqslant n^{|1/p-1/2|}$.

Clearly dim $Z=k\geqslant m/2$. By the construction of Z there are $h\geqslant(1-2/\log n)m\geqslant m/2$ vectors $\{v_i\}_{i=1}^h$ in Z so that the $\{v_i\}_{i=1}^h$ are orthogonal with respect to $|||\cdot|||$ and satisfy $\|v_i\|\geqslant b/K$ for every i. We shall use these vectors $\{u_j\}_{j=1}^k$ and $\{v_i\}_{i=1}^h$ in order to estimate $M=\int\|z\|\,d\mu_{k-1}(z)$, where the integration is taken over $\{z;\, z\in Z,\ |||z|||=1\}$ with the usual rotation invariant measure μ_{k-1}. By the proof of Proposition 5.1 applied to the $\{u_j\}_{j=1}^k$ we get that

$$M\leqslant\alpha\left(\sum_{j=1}^{k}\|u_j\|^{2p/(p-2)}\right)^{(2-p)/2p}\leqslant\alpha aKk^{1/p-1/2} \tag{6.13}$$

where α is the p-type constant of X. Similarly if β is q-cotype constant of X we get by the proof of the same proposition (this time using the $\{v_i\}_{i=1}^h$) that

$$M\geqslant\tfrac{1}{2}\beta bK^{-1}h^{1/q}k^{-1/2}\geqslant\beta bK^{-1}k^{1/2-1/q}2^{-1-1/q}. \tag{6.14}$$

By combining (6.13) and (6.14) we get that

$$d(Z,l_2^k)\leqslant b/a\leqslant\eta(\alpha,\beta)\,K^2k^{1/p-1/q}\leqslant\eta(\alpha,\beta)\,k^{1/p-1/q+(3/\log\log k)}$$

and this concludes the proof. □

Proposition 6.4 is of interest only in the case where p is close to 2 and q is relatively large. If $q=2$ Proposition 6.4 is of course much weaker than Theorem 5.2.

We return to Lemma 6.1 and present some additional applications of it. This lemma can be used in getting new information concerning the so called three space problem which was treated in [8]. This problem is the following: Let X be a Banach space which has a subspace Y so that X/Y and Y are both isomorphic to a Hilbert space. How far can X itself be from a Hilbert space?

Theorem 6.5. *Let X be a Banach space having a subspace Y so that X/Y and Y are isomorphic to a Hilbert space. Then there is a constant c so that for every $Z\subset X$ with* dim $Z=n$, $d(Z,l_2^n)\leqslant c(\log n)^2$.

Proof. As we shall show below (Proposition 6.6) it is enough to prove the theorem in case where Y and X/Y are both isometric to a Hilbert space. In [8] the following generalization of (6.2) is proved

$$\alpha_{nm}(X)\leqslant\alpha_n(Y)\alpha_m(X)+\alpha_n(Y)\alpha_m(X/Y)+\alpha_n(X)\alpha_m(X/Y),\quad n,m=1,2,\ldots \tag{6.15}$$

In our case we have $\alpha_n(Y)=\alpha_n(X/Y)=1$ for every n and thus we get from (6.15) that $\alpha_{m^2}(X)\leqslant 2\alpha_m(X)+1$ for every m. Since $\alpha_2(X)\leqslant d(X,l_2^2)\leqslant 2^{1/2}$ we deduce by easy induction that $\alpha_{2^{2^k}}(X)\leqslant 2^k(2^{1/2}+1)-1$ for every k or

$$\alpha_n(X)\leqslant(2^{3/2}+2)\log_2 n,\quad n=3,6,\ldots, \tag{6.16}$$

By (6.3) we deduce from (6.16) that also $\beta_n(X) \leqslant (2^{3/2}+2)\log_2 n$. Hence by Kwapien's theorem and Lemma 6.1 it follows that for every $Z \subset X$ with $\dim Z = n$

$$d(Z, l_2^n) \leqslant \tilde{\alpha}_{n^2}(X)\tilde{\beta}_{n^2}(X) \leqslant \alpha_{n^2}(X)\beta_{n^2}(X) \leqslant 16(3+2^{3/2})(\log_2 n)^2. \tag{6.17}$$

PROPOSITION 6.6. *Let $X \supset Y$, Y_0 and Z_0 be Banach spaces such that Y is isomorphic to Y_0 and X/Y is isomorphic to Z_0. Then there is a Banach space $X_0 \supset Y_0$ so that X_0/Y_0 is isometric to Z_0 and*

$$d(X, X_0) \leqslant d(Y, Y_0)\, d(X/Y, Z_0).$$

Proof. Another way to formulate this proposition is the following. Given Banach spaces X, Y_0 and Z_0 and bounded linear operators $T\colon Y_0 \to X$ and $S\colon X \to Z_0$ so that $ST = 0$, $\|Ty\| \geqslant \|y\|$, $y \in Y_0$ and $\|Sx\| \geqslant \inf_{y \in Y_0} \|x + Ty\|$ for all $x \in X$. Then there is a norm $\|\ \|_0$ on X so that

$$\|T\|^{-1}\|x\| \leqslant \|x\|_0 \leqslant \|S\|\,\|x\| \quad x \in X$$

and so that in $\|\cdot\|_0$, T becomes an isometry and S a quotient map. It is easily checked that such a norm $\|\cdot\|_0$ is obtained by taking as its unit ball the set of all x of the form

$$x = Ty + x_1, \quad \|y\|_{Y_0} + \max(\|x_1\|, \|Sx_1\|) < 1.$$

Remark. It follows from Theorem 6.5 in particular that if $X \supset Y$ with $n = \dim X$ and Y and X/Y both inner product spaces then $d(X, l_2^n) \leqslant c(\log_2 n)^2$ for $c = 16(3+2^{3/2})$. In [8] an example is constructed of such an X for which $d(X, l_2^n) \geqslant \eta(\log n)^{1/2}$ for some $\eta > 0$. Hence up to the exponent of the $\log n$ Theorem 6.5 is the best possible. Observe also that in the setting of Theorem 6.5 the distance from X to a Hilbert space is essentially the same as the smallest norm of a projection from X onto Y.

We conclude this section with the solution to the local version of the complemented subspaces problem. In [24] it was proved that if X is an infinite dimensional Banach space so that every closed subspace of X is complemented then X is isomorphic to Hilbert space. The first step in this proof was the simple observation (made in [3]) that it is enough to prove that there is a function $\lambda \to f(\lambda)$ so that if on any closed subspace of X there is a projection of norm $\leqslant \lambda$ then the distance of X from a Hilbert space is $\leqslant f(\lambda)$. In this formulation this problem makes sense also without the assumption that $\dim X = \infty$. The proof given in [24] does not suffice for proving this case. In order to clarify this point let us review briefly the argument of [24]. It went as follows. We start with an arbitrary finite dimensional subspace $Z \subset X$ with $\dim Z = k$ say, and put $d(Z, l_2^k) = d$. By the assumption there is a projection Q from X onto Z with $\|Q\| \leqslant \lambda$. The space $(I-Q)X$ is infinite-dimensional and hence by Dvoretzky's theorem there is a $Z_0 \subset (I-Q)X$ such that $d(Z_0, l_2^k) \leqslant 2$. Let T be an operator from Z onto Z_0 with $\|x\|/2d \leqslant \|Tx\| \leqslant \|x\|$ for every $x \in Z$. By using the assumption there is a projection P from X onto $\{x+2^6\lambda^2 Tx, x\in Z\}$ with norm $\leqslant \lambda$. The computation done in [24] shows that this implies that

$$d = d(Z, l_2^k) \leqslant \lambda^4 \cdot 2^9. \tag{6.18}$$

The point where this argument breaks down in case $\dim X < \infty$ is the choice of Z_0. This difficulty can however by overcome if we use the results of section 2 as well as Theorem 6.2.

THEOREM 6.7. *There is a function $\lambda \to f(\lambda)$ so that if X is a Banach space with* $\dim X = n$ *such that for every $Y \subset X$ there is a projection of norm $\leqslant \lambda$ from X onto Y then $d(X, l_2^n) \leqslant f(\lambda)$. One can take $f(\lambda) = c\lambda^{32}$ for a suitable constant c.*

Proof. If every subspace of X is λ-complemented then every subspace of X^* is $(1+\lambda)$ complemented. Indeed, if $Y \subset X^*$ and P is a projection from X onto $Y^\perp \subset X$ then $I_{X^*} - P^*$ is a projection from X^* onto Y. By (2.19) we have $k(X)k(X^*) \geqslant \delta n$ for some $\delta > 0$. Hence there is no loss of generality to assume that every subspace of X is $(\lambda+1)$-complemented and there is a subspace Y in X with $\dim Y = k \geqslant (\delta n)^{1/2}$ and $d(Y, l_2^k) \leqslant 2$. Let Z be any subspace of X with $\dim Z = m < k/2$. Let Q be a projection from X onto Z with $\|Q\| \leqslant \lambda + 1$. We take now as Z_0 an m dimensional subspace of $Y \cap \operatorname{kern} Q$, and proceed as in [24]. It follows from (6.18) that $d(Z, l_2^m) \leqslant 2^9(\lambda+1)^4$. Consequently if $m = [\frac{1}{2}(\delta n)^{1/2}]$ then $\alpha_m(X) \leqslant 2^9(\lambda+1)^4$ and $\beta_m(X) \leqslant 2^9(\lambda+1)^4$. Consequently by Theorem 6.2 $d(X, l_2^n) \leqslant c\lambda^{32}$ for a suitable constant c. □

Remark. By repeating the same argument using this time (2.18) instead of (2.19) and using in (2.18) the estimate we just obtained for $d(X, l_2^n)$ we actually get that $d(X, l_2^n) \leqslant c_1\lambda^{24}$ provided that $n > c_2\lambda^{96}$.

§ 7. Additional results

The approach we presented in section 2 for proving the existence of almost spherical sections was based on Proposition 2.3. This proposition allows us to choose any m points on the surface of the unit ball in l_2^n. The choice we made in section 2 was always that of a δ-net in a subspace of a suitable dimension. It is however quite likely that other choices of the points in Proposition 2.3 will lead to some useful consequences. We present here one result obtained by making a somewhat different choice of the points appearing in Proposition 2.3.

Let $\{e_j\}_{j=1}^n$ be an orthonormal basis in l_2^n, let $k<n$ be an integer and let $\delta>0$. We choose the points $\{y_i\}_{i=1}^m$ so that they form a δ-net in all the subspaces of l_2^n spanned by choosing k distinct elements out of $\{e_j\}_{j=1}^n$. In view of Lemma 2.4 an estimate for the number m of points needed is given by

$$(1+2/\delta)^k\binom{n}{k} \leqslant (1+2/\delta)^k n^k/k! \leqslant (c(\delta)\, n/k)^k$$

for some constant $c(\delta)$. Comparing this estimate with the one allowed by Proposition 2.3 we get that we can make the desired choice if

$$k \log n/k \leqslant \gamma(\delta)\, nM_r^2/b^2. \tag{7.1}$$

Consequently we get the following result.

THEOREM 7.1. (*i*) *There is an absolute constant $\eta>0$ so that the following is true. Let X be a Banach space of dimension n and let $|||\cdot|||$ be an inner product norm so that* (2.5) *holds. Let $k=[\eta n M_r^2/b^2 \log n]$. Then there exists a basis $\{x_j\}_{j=1}^n$ of X so that*

$$\left(\sum_{j=1}^n \lambda_j^2\right)^{1/2} \leqslant \left\|\sum_{j=1}^n \lambda_j x_j\right\| \leqslant 2\left(\sum_{j=1}^n \lambda_j^2\right)^{1/2} \tag{7.2}$$

for every choice of scalars $\{\lambda_j\}_{j=1}^n$ for which at most k scalars are different from 0.

(ii) *Let X be a Banach space of cotype* 2. *Then there is a constant $\gamma>0$* (*depending only on the cotype* 2 *constant of X*) *and a basis $\{x_j\}_{j=1}^n$ of X so that* (7.2) *holds for every choice of scalars $\{\lambda_j\}_{j=1}^n$ so that at most γn of the scalars are different from* 0.

To derive (ii) we use (7.1) and the estimate on M_r deduced in section 5 (for the $|||\cdot|||$ appearing in the Dvoretzky Rogers theorem). Let us note that in the case of statement (ii) we get by the triangle inequality that the right hand inequality in (7.2) holds for arbitrary $\{\lambda_j\}_{j=1}^n$ provided we replace 2 by $2\gamma^{-1/2}$.

One unfortunate fact about the basis $\{x_j\}_{j=1}^n$ appearing in (7.2) is that it is usually a "bad" basis from the Banach space point of view since its basis constant may be large. For example if $X=l_1^n$ we may apply assertion (ii) of Theorem 7.1. The basis $\{x_j\}_{j=1}^n$ we get will always have a basis constant $\geqslant cn^{1/2}$ since every projection from l_1^n on a subspace close to l_2^k has norm $\geqslant \gamma_0 k^{1/2}$.

Another general remark which may be useful in some applications is the following. Since the approach in section 2 is probabilistic in nature we can ensure the existence of subspaces which are almost Hilbertian in several given norms simultaneously. More precisely, given τ and s there is an $\eta(\tau, s)>0$ so that if $\{\|\ \|_j\}_{j=1}^s$ are norms on R^n and $|||\cdot|||$ is an inner product norm on R^n then there is a subspace $Y\subset R^n$ with dim $Y=k$ so that

$d((Y, \|\cdot\|_j), l_2^k) \leqslant 1+\tau$ for $j=1, \ldots, s$, where $k=[\eta(\tau, s) \min_j nM_{r_j}^2/b_j^2]$ (and M_{r_j} and b_j have the obvious meaning). A similar but somewhat more complicated statement can be made if instead of keeping s fixed we allow also s to grow with n.

§ 8. Appendix—A proof of the isoperimetric inequality

We present in this appendix a complete proof of Theorem 2.1. We start by introducing some notations. For a subset $A \subset S^{n-1}$ we let $r(A)$ be the radius of A defined by $\min\{r; \exists x \in S^{n-1}, A \subset B(x, r)\}$. It is obvious that the minimum always exists. The geodesic metric d on S^{n-1} induces a natural metric δ (the Hausdorff metric) on the space of closed non-empty subsets of S^{n-1} by putting $\delta(A, B) = \min\{r; A \subset B_r, B \subset A_r\}$. The set of closed subsets of S^{n-1} forms a compact metric space in this metric δ. The function $R(A)$ is clearly a continuous function of A, the function $\mu_{n-1}(A)$ is however only upper semicontinuous i.e. $\delta(A^k, A) \to 0$ implies $\mu_{n-1}(A) \geqslant \lim_k \sup \mu_{n-1}(A^k)$.

The main tool in the proof of Theorem 2.1 will be the notion of spherical symmetrization. Let $Z \subset S^{n-1}$ be a closed set and let γ be a half circle on S^{n-1} joining the pair of antipodal points x_0 and $-x_0$. For each $y \in \gamma$ let H^y be the hyperplane in R^n which is ortogonal to the line joining x_0 with $-x_0$ and which contains y. The intersection $H^y \cap S^{n-1}$ is an $(n-2)$-sphere which we denote by $S^{n-2, y}$ (if $y = x_0$ or $y = -x_0$ then $H^y \cap S^{n-1}$ is just a single point). We let $\mu_{n-2,y}$ be the unique normalized rotation invariant measure on $S^{n-2,y}$. For $y \in \gamma$ we let $A^y = A \cap H^y$ and let B^y be a cap in $S^{n-2,y}$ with center y such that $\mu_{n-2,y}(B^y) = \mu_{n-2,y}(A^y)$ (if $\mu_{n-2,y}(A^y) = 0$ we let B^y be either the empty set or the set consisting of the point y only depending on whether A^y is empty or not; we do the same also in case $y = \pm x_0$). The set $B = \bigcup_{y \in \gamma} B^y$ is called the symmetrization of A with respect to γ and is denoted by $\sigma_\gamma(A)$. It follows from Fubini's theorem that $\mu_{n-1}(B) = \mu_{n-1}(A)$. It is also easily seen that the function $y \to \mu_{n-2,y}(A^y) = \mu_{n-2,y}(B^y)$ is upper semi continuous and hence B is a closed subset of S^{n-1}.

The proof of Theorem 2.1 is by induction on n. The theorem is trivial for $n=1$. For general n it is an immediate consequence of the following three lemmas.

LEMMA 8.1. *Let $A \subset S^{n-1}$ be a closed set and let $M(A) = \{C \subset S^{n-1}$; C closed, $\mu_{n-1}(C) = \mu_{n-1}(A)$, $\mu_{n-1}(C_\varepsilon) \leqslant \mu_{n-1}(A_\varepsilon)$ for every $\varepsilon > 0\}$. Then there is a $B \in M(A)$ with minimal radius, i.e.* $\min\{r(C); C \subset M(A)\}$ *exists.*

LEMMA 8.2. *Let $A \subset S^{n-1}$ be a closed set. Then for every half circle γ, $\sigma_\gamma(A) \in M(A)$.*

LEMMA 8.3. *Let $B \subset S^{n-1}$ be a closed set which is not a cap. Then there exist a finite family of half circles $\{\gamma_i\}_{i=1}^n$ so that $r(\sigma_{\gamma_n}(\sigma_{\gamma_{n-1}} \ldots \sigma_{\gamma_1}(B))) < r(B)$.*

The induction hypothesis will be used in the proof of Lemma 8.2 (the proof of this lemma for $n=n_0$ assumes the validity of Theorem 2.1 for $n=n_0-1$).

It is clear from the definition of $M(A)$ that $B\in M(A)$ and $C\in M(B)$ implies $C\in M(A)$. Thus Lemmas 8.2 and 8.3 imply that an element of minimal radius in $M(A)$ must be a cap. By Lemma 8.1 such an element always exists i.e. $M(A)$ must contain a cap. This is exactly the assertion of Theorem 2.1.

Proof of Lemma 8.1. Since the function $B\to r(B)$ is continuous it is enough to show that $M(A)$ is a closed subset in the space of all closed subsets of S^{n-1}. Assume that $B^k\in M(A)$, $k=1, \ldots$, and $\delta(B^k, B)\to 0$, and let $\varepsilon\geqslant 0$. For every $\eta>0$ we have for large enough k that $B\subset B^k_\eta$ and thus $B_\varepsilon\subset B^k_{\varepsilon+\eta}$. Consequently

$$\mu_{n-1}(B_\varepsilon)\leqslant \mu_{n-1}(B^k_{\varepsilon+\eta})\leqslant \mu_{n-1}(A_{\varepsilon+\eta}).$$

It follows that

$$\mu_{n-1}(B_\varepsilon)\leqslant \inf_\eta \mu_{n-1}(A_{\varepsilon+\eta})=\mu_{n-1}\Big(\bigcap_{\eta>0} A_{\varepsilon+\eta}\Big)=\mu_{n-1}(A_\varepsilon).$$

In particular for $\varepsilon=0$ we have $\mu_{n-1}(B)\leqslant\mu_{n-1}(A)$. On the other hand $\mu_{n-1}(B)\geqslant \lim_k \sup \mu_{n-1}(B^k)=\mu_{n-1}(A)$. This shows that $B\in M(A)$. □

Proof of Lemma 8.2. Let A be a closed subset of S^{n-1} and let γ be a half circle on S^{n-1} joining x_0 with $-x_0$. Let u be the midpoint of γ. We identify the sphere $S^{n-2,u}$ with S^{n-2} (this identification preserves the usual $n-2$ dimensional measure as well as the distances). For every $y\in\gamma$ $(y\neq\pm x_0)$ we define a mapping τ_y from $S^{n-2,y}$ onto $S^{n-2,u}=S^{n-2}$ by projecting along the meridians (i.e. for $x\in S^{n-2,y}$, $\tau_y(x)$ is the point on $S^{n-2,u}$ which belongs to the half circle joining x_0, x, $-x_0$). It is obvious that there is a function f such that whenever $x_1\in S^{n-2,y_1}$, $x_2\in S^{n-2,y_2}$ then

$$d(x_1, x_2)=f(y_1, y_2, d(\tau_{y_1}(x_1), \tau_{y_2}(x_2))).$$

Hence there is for every y_1, $y_2\in\gamma$, $\varepsilon>0$ (with $d(y_1,y_2)\leqslant\varepsilon$) an $\eta(y_1, y_2, \varepsilon)\geqslant 0$ so that for every subset C of S^{n-2,y_1}, $C_\varepsilon\cap S^{n-2,y_2}=\tau_{y_2}^{-1}((\tau_{y_1}C)_{\eta(y_1,y_2,\varepsilon)})$. (If $d(y_1,y_2)>\varepsilon$ then clearly $C_\varepsilon\cap S^{n-2,y_2}=\phi$). Hence for every $\varepsilon>0$, $y\in\gamma$ $(y\neq\pm x_0)$ we have

$$\tau_y((A_\varepsilon)^y)=\bigcup_{\substack{z\in\gamma\\ d(z,y)\leqslant\varepsilon}}(\tau_z A^z)_{\eta(z,y,\varepsilon)}. \tag{8.1}$$

Let $B=\sigma_\gamma(A)$. By applying (8.1) to B we get

$$\tau_y((B_\varepsilon)^y)=\bigcup_{\substack{z\in\gamma\\ d(z,y)\leqslant\varepsilon}}(\tau_z B^z)_{\eta(z,y,\varepsilon)}. \tag{8.2}$$

For every $z \in \gamma$ the set $\tau_z B^z$ is by definition a cap in S^{n-1} for which $\mu_{n-2}(\tau_z B^z) = \mu_{n-2}(\tau_z A^z)$. By the induction hypothesis (i.e. Theorem 2.1 for $n-1$) we get that

$$\mu_{n-2}((\tau_z B^z)_{\eta(z,y,\varepsilon)}) \leqslant \mu_{n-2}((\tau_z A^z)_{\eta(z,y,\varepsilon)}) \tag{8.3}$$

for all admissible y, z, ε. Since all the sets appearing in the right hand side of (8.2) are caps with the same center (namely u) we get from (8.1), (8.2) and (8.3).

$$\begin{aligned}\mu_{n-2}(\tau_y(B_\varepsilon)^y) &= \sup_{\substack{z\in\gamma \\ d(z,y)\leqslant\varepsilon}} \mu_{n-2}((\tau_z B^z)_{\eta(z,y,\varepsilon)}) \\ &\leqslant \sup_{\substack{z\in\gamma \\ d(z,y)\leqslant\varepsilon}} \mu_{n-2}((\tau_z A^z)_{\eta(z,y,\varepsilon)}) \leqslant \mu_{n-2}(\tau_y(A_\varepsilon)^y).\end{aligned} \tag{8.4}$$

In other words we have for every $y \neq \pm x_0$ in γ that

$$\mu_{n-2,y}((B_\varepsilon)^y) \leqslant \mu_{n-2,y}((A_\varepsilon)^y). \tag{8.5}$$

By Fubini's theorem this implies that $\mu_{n-1}(B_\varepsilon) \leqslant \mu_{n-1}(A_\varepsilon)$ and this concludes the proof of the lemma. □

Proof of Lemma 8.3. Let $B \subset S^{n-1}$ be a closed set which is not a cap and let $r = r(B)$. Then there exists a $u \in S^{n-1}$ such that $B \subset B(u, r)$. We shall perform now some spherical symmetrizations on B. All these symmetrizations will be with respect to half circles γ whose middle point is u. All these symmetrizations leave $B(u, r)$ invariant. We perform first one symmetrization of B with respect to any half circle γ_1 with center u. Since B is not a cap $\sigma_{\gamma_1}(B)$ is not all of $B(u, r)$ and thus it does not contain the entire boundary $\partial B(u, r)$ of $B(u, r)$.

We make now the following two simple observations

(i) Let C be a closed subset of $B(u, r)$ and $x \in \partial B(u, r) \sim C$ then for every half circle γ whose center is u, $\sigma_\gamma(C)$ does not contain x.

(ii) Let C be a closed subset of $B(u, r)$ and G a relatively open subset of $\partial B(u, r)$ which is disjoint from C. Then for every $x \in \partial B(u, r)$ there is a relatively open set G_x of $\partial B(u, r)$ and a half circle γ_x with center u so that $G_x \cap \sigma_{\gamma_x}(C) = \phi$. (Both G_x and γ_x depend only on x and G but not on C).

By using (i) and (ii) (taking as G an open subset of $\partial B(u, r)$ which does not belong to $\sigma_{\gamma_1}(B)$) and the compactness of $\partial B(u, r)$ we deduce that there are $\{x_i\}_{i=2}^n$ in $\partial B(u, r)$ so that $\sigma_{\gamma_{x_n}}(\sigma_{\gamma_{x_{n-1}}} \dots \sigma_{\gamma_{x_2}}(\sigma_{\gamma_1}(B))\dots)$ is disjoint from $\partial B(u, r)$ and thus has a radius smaller than r. This concludes the proof of Lemma 8.3 and thus of Theorem 2.1. □

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