The Dimension of Random Ordered Sets

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ABSTRACT

Let $P = (X, <)$ be a finite ordered set and let $|P|$ denote the cardinality of the universe X. Also let $\Delta(P)$ denote the maximum degree of P, i.e., the maximum number of points comparable to any one point of P. Füredi and Kahn used probabilistic methods to show that the dimension of P satisfies $\dim(P) \le c_1\Delta(P)\log|P|$ and $\dim(P) \le c_2\Delta(P)\log^2\Delta(P)$ where c_1 and c_2 are positive absolute constants. In this article, we consider the probability space $\Omega(n, p)$ of bipartite ordered sets having n minimal elements and n maximal elements, where the events that any minimal element is less than any maximal element are independently distributed and each has probability $p = p(n)$. We show that for every $\epsilon > 0$, there exist $\delta, c > 0$ so that: (1) if $(\log^{1+\epsilon} n)/n < p \le 1/\log n$, then $\dim(P) > \delta pn \log pn$ for almost all $P \in \Omega(n, p)$; and (2) if $1/\log n \le p < 1 - n^{-1+\epsilon}$, then $\dim(P) > n - cn/p \log n$ for almost all $P \in \Omega(n, p)$. The first inequality is best possible up to the value of the constant δ when $p > (\log^{2+\epsilon} n)/n$. As to the accuracy of the second inequality, we have the trivial upper bound $\dim(P) \le n$ for all $P \in \Omega(n, p)$. We then develop a nontrivial upper bound which holds for almost all $P \in \Omega(n, p)$, when $p \ge 1/\log n$. This upper bound has the same form as the lower bound when p is constant. We also study the space $\mathscr{L}(n)$ of all labelled ordered sets on n points and show that there exist positive constants c_1, c_2 so that $n/4 - c_1 n/\log n < \dim(P) < n/4 - c_2 n/\log n$ for almost all $P \in \mathscr{L}(n)$.

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1. INTRODUCTION

For an ordered set $P = (X, <)$, the *cardinality* of P, denoted $|P|$, is the cardinality of its universe X. The *maximum degree* of P is $\Delta(P) = \max_{x \in P} |\{y \in X: y < x$ or $x < y\}|$. A *realizer* $\Sigma = \{L_1, \ldots, L_t\}$ of P is a collection of linear orders on X such that $x < y$ in P iff $x < y$ in L, for every $L \in \Sigma$. The *dimension* of P, denoted $\dim(P)$, is the least t such that P has a realizer of cardinality t. The ordered set S_n, consisting of the 1-element and $n-1$-element subsets of a n-element set ordered by inclusion, is so important to the dimension theory of ordered sets that it is called the *standard example*. The reader should check that $\dim(S_n) = n$, $|S_n| = 2n$, and $\Delta(S_n) = n - 1$. The problem of determining bounds on the dimension for various classes of ordered sets has been a major topic in the theory of ordered sets over the past 20 years. The reader is refered to the survey article [6] of Kelly and Trotter for further background information.

In this article we consider upper bounds on the dimension of an ordered set P in terms of its cardinality and maximum degree, and then study how tight these bounds actually are. There is one elementary bound of this type. Hiraguchi [5] proved that $\dim(P) \leq |P|/2$, when $|P| \geq 4$. The standard example shows that this is an optimal bound. Recently, Füredi and Kahn [3] used probabilistic techniques to prove the following inequalities.

Theorem 1.1. *There exists a positive absolute constant c_1 so that if $P = (X, <)$ is an ordered set, then $\dim(P) \leq c_1 \Delta(P) \log|P|$.* ∎

For an integer $k \geq 2$, let $m(k) = \max\{\dim(P): \Delta(P) = k\}$.

Theorem 1.2. *There exists a positive absolute constant c_2 so that if $P = (X, <)$ is an ordered set, then $\dim(P) \leq c_2 \Delta(P) \log^2(\Delta(P))$, i.e., $m(k) \leq c_2 k \log^2 k$, for all $k \geq 2$.* ∎

To establish inequality (1.1), Füredi and Kahn first showed that the dimension of P is the least t such that there exists a collection Σ' of linear orders with the property that, for all $x, y \in X$, either $x < y$ in P or $y < U[x]$ in L for some $L \in \Sigma'$, where $U[x] = \{z \in X: x \leq z$ in $P\}$. We shall call such a collection a *quasi-realizer.* They proved Theorem 1.1 by showing that almost all collections of linear orders of the given size are quasi-realizers. They proved Theorem 1.2 with a clever application of the Lovász Local Lemma [2].

Previously, the best known example of an ordered set with small maximum degree and cardinality, but large dimension, was the standard example. In order to find better examples, we set out to analyze the probability space $\Omega(n, p)$ of bipartite ordered sets with n minimal elements A, and n maximal elements A', where the events that any minimal element is less than any maximal element are independently distributed and each event has probability $p = p(n)$. We establish the following inequalities.

Theorem 1.3. *For every $\epsilon > 0$, there exist $\delta > 0$ so that if*

$$(\log^{1+\epsilon} n)/n < p \leq 1/\log n\,,$$

then $\dim(P) > \delta pn \log pn$ *for almost all $P \in \Omega(n, p)$.* ∎

Theorem 1.4. *For every $\epsilon > 0$, there exist $\delta, c > 0$ so that if*

$$1/\log n \le p < 1 - n^{-1+\epsilon},$$

then $\dim(P) > \max\{\delta n, n - cn/(p \log n)\}$ *for almost all* $P \in \Omega(n, p)$. ∎

The inequalities in (1.3) and (1.4) follow easily from the following comprehensive inequality.

Theorem 1.5. *For every $\epsilon > 0$, there exist $\delta > 0$ so that if*

$$(\log^{1+\epsilon} n)/n < p < 1 - n^{-1+\epsilon},$$

then $\dim(P) > (\delta pn \log pn)/(1 + \delta p \log pn)$ *for almost all* $P \in \Omega(n, p)$. ∎

A simple calculation shows that if $p > (\log^{2+\epsilon} n)/n$, then $|\Delta(P) - pn| = o(pn)$ for almost all $P \in \Omega(n, p)$; furthermore, when $p > n^{-1+\epsilon}$, $\log np$ and $\log n$ are of the same order, so we have the following result.

Corollary 1.6. *For every $\epsilon > 0$, there exists a $\delta > 0$ so that if $n^{-1+\epsilon} < p \le 1/\log n$, then* $\dim(P) > \delta\Delta(P) \log|P|$ *for almost all* $P \in \Omega(n, p)$. ∎

This corollary shows that the Furedi/Kahn inequality (1.1) is best possible up to the value of the constant c_1. However, our techniques do not allow us to determine the correct exponent on the $\log(\Delta(P))$ term in the Füredi/Kahn inequality (1.2). In fact when p is small, we are unable to answer whether the lower bound on $\dim(P)$ provided by Theorem 1.3 is accurate to within a multiplicative constant. As for Theorem 1.4, we derive in Section 2 an upper bound on the expected value of $\dim(P)$ which shows that the inequality $\dim(P) > n - cn/(p \log n)$, for almost all $P \in \Omega(n, p)$, is best possible up to the value of c, when p is constant. There is a gap in our bounds when $1/\log n \le p = o(1)$. We consider the problem of narrowing the gaps in the known bounds for the expected value of $\dim(P)$ as an important topic for further research.

Another probability model for ordered sets assigns equal probability to every labelled ordered set on n points. Using results of Kleitman and Rothschild [8] on the structure of random ordered sets in this model, we prove the following inequalities.

Theorem 1.7. *There exist absolute constants $c_1, c_2 > 0$ so that*

$$\frac{n}{4}\left(1 - \frac{c_1}{\log n}\right) < \dim(P) < \frac{n}{4}\left(1 - \frac{c_2}{\log n}\right)$$

for almost all labelled ordered sets on n points. ∎

The remainder of the paper is organized as follows. In Section 2 the probability space $\Omega(n, p)$ is introduced, and some easy upper bounds are proved. These bounds serve to illustrate some of the issues addressed in the proof of the main theorem and to suggest the form of the lower bounds which follow. The main results on the dimension of random bipartite ordered sets are proved in Section 3. In Section 4 the probability space $\mathscr{L}(n)$ of all labelled ordered sets on n points is introduced, and the dimension of a random ordered set in this model is studied. Some concluding remarks are made in Section 5. In the remainder of this section, we review our notation.

Notation. We consider an *ordered set* P as a pair $(X, <)$, where the *universe* X is a set (always finite in this paper) and $<$ is an irreflexive and transitive binary relation on X. Such relations are called *strict partial orders*. When $P = (X, <)$ is an ordered set, we let $P^* = (X, <^*)$ denote the *dual* of P, i.e., $a < b$ in P^* exactly when $b < a$ in P. If a and b are distinct elements of X and either $a < b$ in P or $b < a$ in P, we say a and b are *comparable*; otherwise a and b are *incomparable* and we write $a \| b$ in P. An ordered set $(X, <)$ is *linearly* ordered if there are no incomparable pairs. A linearly ordered set is also called a *chain*. A subset A of X is called an *antichain* if $a_1 \| a_2$ in P for every $a_1, a_2 \in X$. We denote by $\mathrm{MAX}(P)$ the antichain of *maximal* elements of P, i.e., $x \in \mathrm{MAX}(P)$ if $x \in X$ and there is no $y \in X$ for which $x < y$ in P. Dually, $\mathrm{MIN}(P)$ is the antichain of *minimal* elements of P.

Let $P = (X, <)$ be an ordered set and let $S \subset X$. S is called an *up-set* if $a \in S$ and $a < b$ always imply $b \in S$. *Down*-sets are defined dually. We denote the *restriction* of P to S by $P \restriction S$. If $a \in X$ and $a \not\in S$, we will write $a < S$, we will write $a < S$ in P when $a < b$ in P for every $b \in S$. Similarly, if S_1 and S_2 are disjoint subsets of X, we write $S_1 < S_2$ in P if $a < S_2$ in P for every $a \in S_1$. The notations $a \| S$ and $S_1 \| S_2$ are defined analogously. When $L_1 = (X_1, <_1)$ and $L_2 = (X_2, <_2)$ are linearly ordered sets, and $X_1 \cap X_2 = \emptyset$, we use the notation $L_1 \oplus L_2$ for the linear order $L = (X_1 \cup X_2, <)$ with $L \restriction X_i = L_i$ for $i = 1, 2$ and $X_1 < X_2$ in L. When $P = (X, <)$ is an ordered set and $a \in X$, we use the notation $U[a]$ to denote the up-set $\{b \in X : a \leq b\}$. The down-set $\{b \in X : b \leq a\}$ is denoted $D[a]$, while $U(a) = U[a] - \{a\}$ and $D(a) = D[a] - \{a\}$. All logarithms in the paper are natural logarithms (base e).

As mentioned previously, the *cardinality* of an ordered set $P = (X, <)$ is the cardinality $|X|$ of the universe. The *degree* of a point $a \in X$, denoted $\deg_P(a)$, is $|\{b \in X : a < b \text{ or } b < a\}|$. The *maximum degree* of P, denoted $\Delta(P)$, is $\max\{\deg_P(a) : a \in X\}$.

2. RANDOM BIPARTITE ORDERED SETS

We consider the following probability model for random bipartite ordered sets. For each positive integer n and each real number p (in general, p will be a function of n) with $0 \leq p \leq 1$, the sample space $\Omega = \Omega(n, p)$ consists of all labelled bipartite ordered sets $P = (A \cup A', <)$ such that $A = \{a_1, \ldots, a_n\}$, $A' = \{a'_1, \ldots, a'_n\}$, $A \subseteq \mathrm{MIN}(P)$, and $A' \subset \mathrm{MAX}(P)$. If P has q relations, then $\Pr(P) = p^q(1-q)^{n^2-q}$. We let $(a_i < a'_j)$ denote the event consisting of all ordered sets $P \in \Omega$ satisfying $a_i < a'_j$ in P. Similarly, $(a_i \| a'_j)$ denotes the event consisting of all $P \in \Omega$ with $a_i \| a_j$ in P. Thus $\Pr(a_i < a'_j) = p$, $\Pr(a_i \| a'_j) = 1 - p$ and the set of events $\{(a_i < a'_j) : 1 \leq i, j \leq n\}$ is independent. Recall that if $p > (\log^{2+\epsilon} n)/n$, then $|\Delta(P) - pn| = o(pn)$, for almost all $P \in \Omega$.

Consider $P \in \Omega$. It is clear from Hiraguchi's bound that $\dim(P) \le n$. In the next two inequalities, we bound $\dim(P)$ away from n. When $p \le 1/\log n$, we use the upper bound on $\dim(P)$ provided by the following trivial corollary to the Füredi/Kahn inequalities (1.1) and (1.2).

Corollary 2.1. *For each $\epsilon > 0$, there exist a positive absolute constant c so that if $(\log^{2+\epsilon} n)/n < p \le 1/(c \log n)$, then $\dim(P) \le \min\{cpn \log^2 pn,\ cpn \log n\}$ for almost all P.* ∎

When $p \ge 1/\log n$, we need a better bound. This will follow from the following preliminary result. The additive constant in this inequality can easily be removed; however, since it makes the proof clearer and does not affect the asymptotics, we leave it in.

Lemma 2.2. *Let $p \ge 1/\log n$ and let $t = t(n)$ be a function of n with $0 < t < n$. If*

$$\lim_{n\to\infty} (1 - (1-p)p^{(2t-n)/(n-t)})^{n(n-t-1)} = 1\,,$$

then $\dim(P) \le \lceil t + 2 \rceil$ for almost all $P \in \Omega(n, p)$.

Proof. Let $t' = \lceil t \rceil$. We actually show that there exists a collection $\Sigma = \{L_1, L_2, \ldots, L_{t'+2}\}$ so that Σ is a quasi-realizer of P for almost all P. To accomplish this, let $F' = \{a_i' : 1 \le i \le t'\}$ and $M' = \{a_j' : t' < j \le n\}$. Also let $\{S_j : t' < j \le n\}$ be a partition of $\{1, \ldots, t'\}$ such that $||S_j| - |S_k|| \le 1$, for all j and k. Observe that $|S_j| \ge \lfloor t'/(n - t') \rfloor \ge \lfloor t/(n-t) \rfloor \ge t/(n-t) - 1 = (2t-n)/(n-t)$, for each j with $t' < j \le n$. Then choose a collection $\Sigma = \{L_1, \ldots, L_{t'+2}\}$ of linear orders such that

(i) a_i' is the least element in L_i, for $1 < i \le t'$;
(ii) a_j' is the second least element in L_i, when $a_j' \in M'$ and $i \in S_j$, for $1 \le i \le t'$ and $t' < j \le n$; and
(iii) $L_{t'+1} = K \oplus K'$ and $L_{t'+2} = K^* \oplus (K')^*$, where K and K' are any linear orders on A and A', respectively.

It remains to check that for almost all $P \in \Omega$, Σ is a quasi-realizer of P. It is clear that $y < U[x]$ in either L_{t+1} or L_{t+2}, for all $(x, y) \in (A \times A) \cup (A' \times A) \cup (A' \times A')$. Let $Q_i(a, a')$ be the event that $a' < U[a]$ in L_i, and let $R(a, a')$ be the event $(a < a') \cup \bigcup_{i=1}^{t'} Q_i(a, a')$. We must show that $\Pr(\bigcap_{(a,a') \in A \times A'} R(a, a'))$ approaches 1.

Note that $a_i' < U[a]$ in L_i, for all $(a, a_i') \in A \times F'$. Thus it suffices to show that $\Pr(\bigcap_{(a,a') \in A \times M'} R(a, a'))$ approaches 1. For $(a, a_j') \in A \times M'$ and $i \in S_j$, $a_j' < U[a]$ in L_i iff $a \| a_i'$, since a_i' is the only element of $A \cup A'$ below a_j' in L_i. Thus the event $T(a, a_j') = (a < a_j') \cup \bigcup_{i \in S_j} (a \| a_i')$ is contained in $R(a, a')$. The set of events $\{(a < a_j')\} \cup \{(a \| ae_i') : i \in S_j\}$ is independent. Since the S_j are pairwise disjoint and $i \in S_j$ implies $a_i' \not\in M'$, the set of events $\{T(a, a_j') : (a, a_j') \in A \times M'\}$ is independent. Also, $|A \times M'| \ge n(n - t - 1)$. Thus

$$\Pr(\neg R(a, a_j')) \le \Pr(\neg T(a, a_j')) = (1-p)p^{|S_j|} \le (1-p)p^{(2t-n)/(n-t)};$$

$$\Pr(T(a, a_j')) \ge 1 - (1-p)p^{(2t-n)/(n-t)}; \text{ and}$$

$$\Pr\Big(\bigcap_{(a,a')\in A\times M'} R(a,a')\Big) \ge \Pr\Big(\bigcap_{(a,a')\in A\times M'} T(a,a')\Big)$$
$$\ge (1-(1-p)p^{(2t-n)/(n-t)})^{n(n-t-1)}.$$

The right-hand side of this inequality approaches 1, which completes the proof. ∎

The following result now follows from Lemma 2.2 by an elementary calculation. The lower bound on p in this result only serves to limit the inequality on $\dim(P)$ to the range where the Füredi/Kahn bound is no longer useful.

Theorem 2.3. *Let $\epsilon > 0$ and let $p = p(n)$ satisfy $1/\log n \le p < 1$. Then*

$$\dim(P) < n\Big(1 - \frac{1}{2+\epsilon}\frac{\log(\frac{1}{p})}{\log n}\Big)$$

for almost all $P \in \Omega(n, p)$.

Proof. Let

$$t = n\Big(1 - \frac{1}{2+\epsilon}\frac{\log(\frac{1}{p})}{\log n}\Big).$$

Then $0 < t < n$. Also,

$$(2t-n)/(n-t) = ((2+\epsilon)\log(n)/(\log\tfrac{1}{p})) - 2.$$

Therefore,

$$(1-(1-p)p^{(2t-n)/(n-t)})^{n(n-t-1)} \ge \Big(1 - \frac{(1-p)}{p^2}\,p^{(2t-n)/(n-t)+2}\Big)^{n^2} \ge e^{-\frac{2n^{-\epsilon}}{p^2}},$$

which goes to 1. ∎

Corollary 2.4. *When p is a constant with $0 < p < 1$, there exists a positive constant $c = c(p)$ so that*

$$\dim(P) < n - \frac{cn}{\log n}$$

for almost all $P \in \Omega(n, p)$. ∎

We have stated the special case in Corollary 2.4 to emphasize that our upper and lower bounds on the expected value of $\dim(P)$ have the same form when p is a constant.

3. LOWER BOUNDS ON DIMENSION

In this section we prove the following technical result, which is then used to obtain the inequalities (1.3), (1.4), and (1.5) discussed in the introduction.

Theorem 3.10. *Let $t = t(n)$ be a nonnegative function of n, and let $p = p(n)$ be a function of n with $0 < p < 1$. Suppose*

(1) $\lim_{n\to\infty} pn = \infty$,

(2) $\lim_{n\to\infty} \frac{t}{p(n-t)} = \infty$, and

(3) $\lim_{n\to\infty} e^{\frac{4t \log n \log pn}{p}} [1 - (1-p)e^{-\frac{24t}{p(n-t)}}]^{\frac{p^2(n-t)^3}{512t \log^2 pn}} = 0$.

Then $\dim(P) > t$ for almost all $P \in \Omega(n, p)$. ∎

The proof of Theorem 3.1 will follow from a series of lemmas. As we proceed we will explain our approach and point out some of the difficulties which must be overcome. Naively, we would like to show that the number of t-collections of linear orders times the probability that $P \in \Omega$ is realized by a particular t-collection of linear orders is extremely small. However, this is not true. The number of such t-collections that realize some $P \in \Omega$ is greater than $|\Omega|$. If, for example, $p = 1/2$, each of these ordered sets is equally likely, and this approach would yield only a lower bound of the form $\epsilon n/\log n$. To get around this problem, we introduce the notions of short pairs, short realizers and short dimension. We first show that for almost all $P \in \Omega$, the short dimension of P is less than or equal to the dimension of P. We then complete the proof by showing that the number of t-collections of short pairs times the probability that a t-collection realizes $P \in \Omega$ is extremely small.

Definition 3.2. A bipartite ordered set $P \in \Omega(n, p)$ is *s-mixed* if for all s-subsets $B \subset A$ and $B' \subset A'$, there exist $b \in B$ and $b' \in B'$ such that $b < b'$.

Lemma 3.3. *If $s = \lfloor (2 \log pn)/p \rfloor$ and $\lim_{n\to\infty} pn = \infty$, then almost all $P \in \Omega(n, p)$* are s-mixed.

Proof. If P is not s-mixed, there exist s-subsets $B \subset A$ and $B' \subset A'$ such that $b \| b'$ for all $b \in B$ and $b' \in B'$. Thus

$$\Pr[P \text{ is not } s\text{-mixed}] \le \binom{n}{s}^2 (1-p)^{s^2} \le n^{2s} e^{2s} s^{-2s} e^{-ps^2},$$

which goes to 0 when

$$s = \lfloor (2 \log pn)/p \rfloor \text{ and } \lim_{n\to\infty} pn = \infty. \qquad ∎$$

Definition 3.4. In the remainder of this section, we let $s = \lfloor (2 \log pn)/p \rfloor$, and we assume $\lim_{n\to\infty} pn = \infty$.

Definition 3.5. A *short linear order* $\sigma = (B, <)$ of a set S consists of an s-element subset $B \subset S$ and a linear order $<$ on B, i.e., a short linear order of S is a permutation of s distinct elements selected from S. In what follows, we call (σ, σ') *a short pair* when σ is a short linear order of A and σ' is a short linear order of A'.

Now let $P \in \Omega(n, p)$, let (σ, σ') be a short pair with $\sigma = (B, <)$ and $\sigma' = (B', <')$, and let $(a, a') \in A \times A'$. We say (σ, σ') *realizes* (a, a') if any of the following conditions hold:

(1) $a < a'$ in P.
(2i) $a \in B$ and $b \| a'$, for all $b \in B$ with $b > a$ in σ.
(2ii) $a' \in B'$ and $b' \| a$, for all $b' \in B'$ with $b' < a'$ in σ'.

A collection Σ of short pairs is called a *short realizer* of P if for every $(a, a') \in A \times A'$, there is some $(\sigma, \sigma') \in \Sigma$ so that (σ, σ') realizes (a, a'). The *short dimension* of P, denoted $s\dim(P)$, is the least s such that P has a short realizer Σ containing t short pairs. The reason for introducing short linear orders and short realizers is to eliminate the overcount noted previously when we consider linear orders on the entire ground set.

Lemma 3.6. $s\dim(P) \leq \dim(P)$ *for almost all* $P \in \Omega(n, p)$.

Proof. Let $P \in \Omega(n, p)$ and let $t = \dim(P)$. Choose a realizer $\{L_1, L_2, \ldots, L_t\}$ of P. For each $i = 1, 2, \ldots, t$, let σ_i be the restriction of L_i to the subset B_i containing the s largest elements of $L_i \restriction A$, and let σ_i' be the restriction of L_i to the subset B_i' containing the s smallest elements of $L_i \restriction A'$. Then let $\Sigma = \{(\sigma_i, \sigma_i') : 1 \leq i \leq t\}$. We claim Σ is a short realizer for P, provided P is s-mixed.

Suppose that $(a, a') \in A \times A'$, but that (a, a') is not realized by any short pair $(\sigma_i, \sigma_i') \in \Sigma$. Then $a \| a'$ in P, so there is some j with $a' < a$ in L_j. This requires $b \| a'$ for all $b \in A$ with $b > a$ in L_j, and $b' \| a$ for all $b' \in A'$ with $b' < a'$ in L_j. We conclude that $a \not\in B_j$ and $a' \not\in B_j'$. Thus $b' < a' < a < b$ in L_j and $b' \| b$ for every $b' \in B_j'$ and $b \in B_j$. This of course implies P is not s-mixed. Since almost all $P \in \Omega(n, p)$ are s-mixed, the lemma follows. ∎

In order to guarantee that we are not giving up too much in working with the short dimension of ordered sets in $\Omega(n, p)$, we give the following elementary bound.

Lemma 3.7. $\dim(P) \leq 1 + 2s\dim(P)$ *for all* $P \in \Omega(n, p)$.

Proof. Suppose that $\Sigma = \{(\sigma_1, \sigma_1'), (\sigma_2, \sigma_2'), \ldots, (\sigma_t, \sigma_t')\}$ is a family of short pairs and Σ is a short realizer of some $P \in \Omega(n, p)$. For each $i = 1, 2, \ldots, t$, let $\sigma_i = (B_i, <_i)$ and $\sigma_i' = (B_i', <_i')$. Form a realizer $R = \{M_1, \ldots, M_t, M_1', \ldots, M_t', N\}$ of P as follows. Let M_i restricted to A be formed by putting all the elements of B_i ordered by σ_i over all the elements of $A - B_i$ ordered arbitrarily. Extend this order to X by inserting the elements of A' as low as possible with respect to P. Notice that if $a \| a'$ and the short pair (σ_i, σ_i') realizes (a, a') by virtue of (2i), then $a' < a$ in M_i. Similarly, let M_i' restricted to A' be formed by putting all the elements of B_i' ordered by σ_i' under all the elements of $A' - B_i'$ ordered arbitrarily. Extend this order to X by inserting the elements of A as high as possible with respect to P. Notice that if $a \| a'$ and the short pair (σ_i, σ_i') realizes (a, a') by virtue of $(2ii)$, then $a' < a$ in M_i'. Finally, form N by putting all the elements of A, ordered by M_1^* under all the elements of A', also ordered by M_1^*. Clearly R is a realizer of P of the desired size. ∎

Next we observe that the number of families of short pairs is relatively small.

Lemma 3.8. *For each n, the number of t-collections $\Sigma = \{(\sigma_1, \sigma_1'), (\sigma_2, \sigma_2'), \ldots, (\sigma_t, \sigma_t')\}$ of short pairs is less than $e^{2st \log n}$.*

Proof. The number of t-collections of short pairs is at most $\left(\binom{n}{s}s!\right)^{2t}$ which is less than $e^{2st \log n}$.

Now we consider fixed values of n and p and a fixed t-collection $\Sigma = \{(\sigma_1, \sigma_1'), (\sigma_2, \sigma_2'), \ldots, (\sigma_t, \sigma_t')\}$ of short pairs. For each $i = 1, 2, \ldots, t$ let $\sigma_i = (B_i, <_i)$ and $\sigma_i' = (B_i', <_i')$ Then for each $a \in B_i$, let $\sigma_i(a)$ denote $|\{b \in B_i : a < b \text{ in } \sigma_i\}|$, and for each $a' \in B_i'$, let $\sigma_i'(a')$ denote $|\{b' \in B_i' : b' < a' \text{ in } \sigma_i'\}|$. Our goal is to produce an upper bound on the probability that Σ is a short realizer of a random $P \in \Omega(n, p)$. The analysis will be simplified by the following elementary property.

Fact. Among all t-collections $\Sigma = \{(\sigma_1, \sigma_1'), (\sigma_2, \sigma_2'), \ldots, (\sigma_t, \sigma_t')\}$ of short pairs, the probability that Σ is a short realizer of a random $P \in \Omega(n, p)$ is maximum for some Σ for which:

(1) $|\{a \in A$: there is some i with $1 \le i \le t$ for which $a \in B_i$ and $\sigma_i(a) = 0\}| = t$, and

(2) $|\{a' \in A'$: there is some i with $1 \le i \le t$ for which $a' \in B_i'$ and $\sigma_i'(a') = 0\}| = t$.

This fact follows from the observation that once an element $a \in A$ has occurred in first place in some σ_i, there is no advantage to having a occur in any B_j with $j \neq i$. Such an element can be safely deleted from B_j. The elements preceding a are advanced one position in σ_j and a new first element is chosen from among those elements of A not previously belonging to B_j. This exchange can only increase the set of $P \in \Omega(n, p)$ for which Σ is a short realizer.

Accordingly, in what follows, we will assume that Σ satisfies conditions (1) and (2) of this fact. We then let R be the event that Σ is a short realizer for a random $P \in \Omega(n, p)$. We will show that $\Pr(R)$ is very small, in fact much less than $e^{-2st \log n}$. To accomplish this, we express R in terms of simpler events. We have

$$R = \bigcap_{(a, a') \in A \times A'} R(a, a'),$$

where $R(a, a')$ is the event that some short pair in Σ realizes (a, a'). Then

$$R(a, a') = (a < a') \cup Q(a, a') \cup Q'(a, a'),$$

where

$$Q(a, a') = \bigcup_{i=1}^{t} Q_i(a, a'), \quad Q_i(a, a') = \bigcap \{(b \| a') : a < b \text{ in } \sigma_i\}$$

$$\text{if } a \in B_i, \text{ and } Q_i(a, a') = \emptyset \text{ if } a \not\in B_i; \text{ and}$$

$$Q'(a, a') = \bigcup_{i=1}^{t} Q_i'(a, a'), \quad Q_i'(a, a') = \bigcap \{(b' \| a) : b' < a' \text{ in } B_i'\}$$

$$\text{if } a' \in B_i', \text{ and } Q_i'(a, a') = \emptyset \text{ if } a' \not\in B_i'.$$

We seek a small lower bound for the probability of R. For each $(a, a') \in A \times A'$, it is reasonably straightforward to calculate a good lower bound for $\Pr(\rceil R(a, a'))$. Since $\rceil R(a, a') = (a \| a') \cap \rceil Q(a, a') \cap \rceil Q'(a, a')$, and these events are independent (they are defined in terms of disjoint sets of possible comparabilities in P),

$$\Pr(\rceil R(a, a')) = (1 - p) \Pr(\rceil Q(a, a')) \Pr(\rceil Q'(a, a')).$$

Next we consider the probability of $\rceil Q(a, a') = \bigcap_{i=1}^{t} \rceil Q_i(a, a')$. The events $\rceil Q_i(a, a')$ are not independent, but as we shall see, are positively correlated. Thus

$$\begin{aligned} \Pr(\rceil Q(a, a')) &\geq \prod_{i=1}^{t} \Pr \rceil Q_i(a, a') \\ &\geq \prod_{i=1}^{t} (1 - (1 - p)^{\sigma_i(a)}). \end{aligned}$$

Similarly, $\Pr(\rceil Q'(a, a')) \geq \Pi_{i=1}^{t}(1 - (1 - p)^{\sigma_i'(a')})$. Two problems remain. We would like to estimate $\Pr(R)$ by $\Pi_{(a, a') \in A \times A'} \Pr(R(a, a'))$. However, the set $\{\rceil R(a, a') : (a, a') \in A \times A'\}$ is neither independent nor positively correlated. Moreover, our lower bound for $\Pr(\rceil R(a, a'))$ depends on Σ. These problems will be circumvented by defining a subset $\mathcal{S} \subset A \times A'$, and for each $(a, a') \in \mathcal{S}$ an event $T(a, a')$ so that:

(1) $R(a, a') \subset T(a, a')$;
(2) $\Pr(T(a, a'))$ is a reasonable upper bound for $\Pr(R(a, a'))$ and is approximately equal to the average value of $\Pr(T(a, a'))$ for $(a, a') \in A \times A'$;
(3) The events in $\mathcal{S} = \{T(a, a') : (a, a') \in \mathcal{S}\}$ are independent; and
(4) $\mathcal{S}$ is a relatively large subset of $A \times A'$.

These conditions imply that $\Pr(R) \leq \Pi_{(a, a') \in \mathcal{S}} \Pr(T(a, a'))$. The price we pay is that $|\mathcal{S}| < |A \times A'|$. When p is very small, we apparently give up too much.

Consider subsets $X \subset A$ and $X' \subset A'$. For each $(a, a') \in X \times X'$, we define the event $R(a, a'/X, X')$ as follows.

$$R(a, a'/X, X') = (a < a') \cup Q(a, a'/X) \cup Q'(a, a'/X'),$$

where

$$Q(a, a'/X) = \bigcup_{i=1}^{t} Q_i(a, a'/X)\,;$$

$$Q_i(a, a'/X) = \bigcap \{(b\|a') : b \in X \cap B_i \text{ and } a < b \text{ in } \sigma_i\}\,;$$

$$Q'(a, a'/X') = \bigcup_{i=1}^{t} Q'_i(a, a'/X')\,;$$

and

$$Q'_i(a, a'/X') = \bigcap \{(b'\|a) : b' \in X' \cap B'_i \text{ and } b' < a' \text{ in } \sigma'_i\}\,.$$

The events $T(a, a')$ will have the form $R(a, a'/X(a), X'(a'))$. Note that for every $X \subset A$, $X' \subset A'$ and $(a, a') \in X \times X'$, we have $R(a, a') \subset R(a, a'/X, X')$, $Q(a, a') \subset Q(a, a'/X)$, and $Q'(a, a') \subset Q'(a, a'/X')$, since $Q_i(a, a') \subset Q_i(a, a'/X)$ and $Q'_i(a, a') \subset Q'_i(a, a'/X')$.

For $a \in X$, we define the *height* of a over X in σ_i by

$$h_i(a/X) = \begin{cases} |\{b \in X \cap B_i : a < b \text{ in } \sigma_i\}| & \text{if } a \in B_i \\ s & \text{if } a \not\in B_i\,. \end{cases}$$

Similarly, for $a' \in X'$, the height of a' over X' in σ'_i is

$$h_i(a'/X') = \begin{cases} |\{b' \in X' \cap B'_i : b' < a' \text{ in } \sigma'_i\}| & \text{if } a' \in B'_i \\ s & \text{if } a' \not\in B'_i\,. \end{cases}$$

For $a \in X$, and for k satisfying $1 \le k \le s$, let the *k-multiplicity* of a over X be $u_k(a/X) = |\{i : h_i(a/X) = k\}|$. The k-multiplicity of $a' \in X'$ over X' is defined similarly. Finally, we define a weight function w, which will allow us to choose a pair $(a, a') \in X \times X'$ so that the probability of the event $R(a, a'/X, X')$ can be accurately approximated.

Let the *weight* of a over X be $w(a/X) = w_1(a/X) + w_2(a/X)$, where

$$w_1(a/X) = \sum_{1 \le k < \frac{1}{p}} u_k(a/X) \log \frac{2}{kp}$$

and

$$w_2(a/X) = \sum_{\frac{1}{p} \le k \le s} 2u_k(a/X) e^{-pk}\,.$$

The *weight* $w(a'/X')$ of a' over X' is defined similarly. It is important to note that the value $k = 0$ is excluded in the definition of weight. This exclusion results from the special role played by the last elements in the σ_i's and the first elements of the σ'_i's.

Next we shall bound $\Pr(T(a, a'/X, X'))$ in terms of $w(a/X)$ and $w(a'/X')$. We will need the following special case of the Ahlswede and Daykin [1] inequality, which was first proved by Kleitman [7].

Proposition 3.9. *If $P=(X,<)$ is the family of subsets of a finite set ordered by inclusion, and U_1 and U_2 are up-sets of X, then $|U_1|/|X| \le |U_1 \cap U_2|/|U_2|$.* ■

In applying Proposition 3.9, we view Ω as a subset lattice as follows: For $P, P' \in \Omega$, $P \subset P'$ iff $a < a'$ in P implies $a < a'$ in P', for all pairs $(a, a') \in A \times A'$. Thus if E_1 and E_2 are events which happen to be up-sets of Ω, $\Pr(E_1) \le \Pr(E_1 | E_2)$.

Lemma 3.10. *Let $X \subset A$, $X' \subset A'$ and $(a, a') \in X \times X'$ with $h_i(a/X) \ne 0 \ne h_i(a'/X')$ for $i = 1, 2, \ldots, t$. Then $\Pr(R(a, a'/X, X')) \le 1-(1-p)e^{-w(a/X)-w(a'/X')}$.*

Proof. Recall that $R(a, a'/X, X')$ can be expressed as

$$R(a, a'/X, X') = (a < a') \cup Q(a, a'/X) \cup Q'(a, a'/X')\,. \tag{1}$$

The three events $(a < a')$, $Q(a, a'/X)$, and $Q'(a, a'/X')$ are independent because they are defined in terms of disjoint sets of comparability relations of P.

$$\Pr(Q_i(a, a'/X)) = (1-p)^{h_i(a/X)}. \tag{2}$$

Now let $E_i = \bigcap_{j=1}^{i-1} \rceil Q_i(a, a'/X)$ for each $i = 1, 2, \ldots, t$. Notice that the events $\rceil Q_i(a, a'/X)$ and E_i are both up-sets in Ω. Thus using Proposition 3.9 and (2) we obtain:

$$\begin{aligned}
\Pr(\rceil Q(a, a'/X)) &= \Pr\Big[\bigcap_{i=1}^{t} \rceil Q_i(a, a'/X)\Big] \qquad (3)\\
&= \Pr(\rceil Q_1(a, a'/X))\Pr(\rceil Q_2(a, a'/X)|E_2)\ldots\Pr(\rceil Q_t(a, a'/X)|E_t)\\
&\ge \prod_{i=1}^{t} \Pr(\rceil Q_i(a, a'/X))\\
&\ge \prod_{i=1}^{t} (1-(1-p)^{h_i(a/X)}) = \prod_{k=1}^{s} (1-(1-p)^k)^{u_k(a/X)}\\
&\ge \prod_{1 \le k < \frac{1}{p}} \Big(\frac{pk}{2}\Big)^{u_k(a/X)} \prod_{\frac{1}{p} \le k \le s} (1-e^{-pk})^{u_k(a/X)}\\
&\ge e^{-w_1(a/X)-w_2(a/X)} = e^{-w(a/X)}\,.
\end{aligned}$$

Similarly,

$$\Pr(\rceil Q'(a, a'/X')) \ge e^{-w(a'/X')}\,. \tag{4}$$

Thus by (1) and independence $\Pr(R(a, a'/X, X')) \le 1-(1-p)e^{-w(a/X)-w(a'/X')}$. ■

Lemma 3.11. *Let X and X' be subsets of A and A', respectively. The mean value of $w(a/X)$, for $a \in X$, is at most $\frac{3t}{p|X|}$, and the mean value of $w(a'/X')$ for $a' \in X'$, is at most $\frac{3t}{p|X'|}$.*

Proof. The total weight of all $a \in X$ is

$$\begin{aligned}
\sum_{a\in X} w(a/X) &= \sum_{a\in X}\sum_{i=1}^{t}\begin{cases}\log \dfrac{2}{ph_i(a/X)} & \text{if } h_i(a/X) < \dfrac{1}{p}\\ 2e^{-ph_i(a/X)} & \text{else}\end{cases}\\
&\le \sum_{i=1}^{t}\Big(\sum_{1\le k\le \frac{1}{p}} \log\frac{2}{pk} + \sum_{\frac{1}{p}\le k\le s} 2e^{-pk}\Big)\\
&\le t\Big(\frac{\log 2e}{p} + \frac{2}{e}\,\frac{1}{1-e^{-p}}\Big)\\
&\le \frac{3t}{p}\,.
\end{aligned}$$

Thus the mean value of $w(a/X)$, for $a \in X$, is at most $\frac{3t}{p|X|}$. A similar argument holds for $w(a'/X')$, $a' \in X'$. ∎

For each $(a, a') \in X \times X'$, we define the *witness* set of (a, a') over (X, X') to be $W(a, a'/X, X') = (\{a\} \times W'(a'/X')) \cup (W(a/X) \times \{a'\})$, where

$$W(a/X) = \{b \in X : a \le b \text{ in some } \sigma_i\}$$

and

$$W'(a'/X') = \{b' \in X' : b' \le a' \text{ in some } \sigma_i'\}\,.$$

We call $W(a/X)$ and $W'(a'/X')$ the *witness* sets of a over X and a' over X', respectively.

Key Observation. In the argument to follow, we observe that a set of events $\{R(a, a'/X(a), X'(a')) : (a, a') \in \mathcal{S}\}$ is independent if the witness sets $\{W(a, a'/X(a), X'(a)) : (a, a') \in \mathcal{S}\}$ are pairwise disjoint.

Lemma 3.12. *There exist a set $\mathcal{S} \subset A \times A'$ and events $T(a, a')$ for $(a, a') \in \mathcal{S}$ such that*

(1) $R(a, a') \subset T(a, a')$;
(2) $\Pr(T(a, a')) \le 1 - (1 - p)e^{-\frac{24t}{p(n-t)}}$
(3) $\{T(a, a') : (a, a') \in \mathcal{S}\}$ is independent; and
(4) $|\mathcal{S}| = \dfrac{(n-t)^3}{128 s^2 t}$.

Proof. We will construct $\mathcal{S}$ so that for each $(a, a') \in \mathcal{S}$, there exist $X(a) \subset A$ and $X'(a') \subset A'$ such that

(*i*) $w(a/X(a)), w(a'/X'(a')) \leq \dfrac{12t}{p(n-t)}$ and $h_i(a/X(a)) \neq 0 \neq h_i(a'/X'(a'))$, for $1 \leq i \leq t$, and

(*ii*) $W(a, a'/X(a), X'(a')) \cap W(b, b'/X(b), X'(b')) = \emptyset$, for any $(b, b') \in \mathcal{S} - \{(a, a')\}$.

Then setting $T(a, a') = R(a, a'/X(a), X'(a'))$, we will have (1) by our earlier remark, (2) by (*i*) and Lemma 3.10, and (3) by (*ii*) and our key observation. We shall write $w(a)$, $W(a)$, $W(a, a')$, etc. for $w(a/X(a))$, $W(a/X(a))$, $W(a, a'/X(a), X'(a'))$, etc.

The set $\mathcal{S}$ will have the form

$$\mathcal{S} = \bigcup_{i=1}^{r} (M_i \times M_i') ,$$

where for $1 \leq i < j \leq r$,

(*iii*) $M_i \cap M_j = \emptyset = M_i' \cap M_j'$;

(*iv*) $|M_i| = |M_j'| = m = \dfrac{(n-t)^2}{16s^2t}$; and

(*v*) $r = \dfrac{2s^2t}{n-t}$.

Then (4) will follow from (*iii*)–(*v*).

We shall ensure (*ii*) by maintaining the following conditions:

(*iia*) If $a, b \in M_i$, for $1 \leq i \leq r$ and $a \neq b$, then $W(a) \cap W(b) = \emptyset$, and similarly for $a', b' \in M_i'$.

(*iib*) If $a \in M_i$ and $b \in M_j$, for $1 \leq i < j \leq r$, then $a \not\in W(b)$, and similarly for $a' \in M_i'$ and $b' \in M_j'$.

To see that (*iia*) and (*iib*) imply (*ii*), let (a, a') and (b, b') be distinct elements of $\mathcal{S}$. Say $a \neq b$. If (a, a') and (b, b') are both in $M_i \times M_i'$ for some $1 \leq i \leq r$, then by (*iia*) $W(a, a') \cap W(b, b') = \emptyset$. Otherwise $(a, a') \in M_i \times M_i'$ and $(b, b') \in M_j \times M_j'$, for some $i \neq j$, and thus $a' \neq b'$. Suppose $(c, c') \in W(a, a') \cap W(b, b')$. Then without loss of generality, $a = c$ and $b' = c'$. Thus $a \in W(b)$ and $b' \in W(a')$. But then (*iib*) implies $j < i$ and $i < j$, which is a contradiction.

The advantage of (*iia*) and (*iib*) over (*ii*) is that they allow us to construct the M_i and M_i' independently. We now construct the M_i, for $i = 1, \ldots, r$. The M_i', $i = 1, \ldots, r$ are constructed analogously. The M_i will have the form $M_i = \{x_{i,1}, \ldots, x_{i,m}\}$, where the elements $x_{i,j}$ are constructed by recursion on the (i, j) ordered lexicographically.

We first need a preliminary step. Let $X_0 = \{a \in A : \sigma_i(a) \neq 0$, for all $i = 1, \ldots, t\}$. Then $|X_0| \geq n - t$. For $a \in X_0$, we define the multiplicity of a, denoted $\mu(a)$ by $\mu(a) = |\{i : a \in B_i\}|$. Since there are at most st pairs (a, i) such that $a \in X_0 \cap B_i$, $\Sigma_{a \in X_0} \mu(a) \leq st$. Thus the set $X = \left\{a \in X_0 : \mu(a) \leq \frac{2st}{n-t}\right\}$ has cardinality at least $\frac{n-t}{2}$. Note that if $a \in X(a) \subset X$, then $|W(a)| \leq \frac{2s^2t}{n-t}$.

Now suppose that for some $i \leq r$ and $j \leq m$, we have constructed $x_{u,v}$ and $X(x_{u,v})$, for all $(u, v) < (i, j)$, such that (i), (iia), and (iib) hold. Set $Y = X - ((\bigcup_{1 \leq u < i} M_u) \cup (\bigcup \{B_k : x_{iv} \in B_k,$ for some $v < j$ and $k \leq t\}))$. Then $|Y| \geq (n-t)/2 - rn - 2ms^2t/(n-t) \geq (n-t)/4$. By Lemma 3.11, there exists $y \in Y$ such that $w(y/Y) = 3t/(p|Y|) \leq 12t/(p(n-t))$. Let $x_{ij} = y$ and $X(x_{ij}) = Y \cup \{a : \sigma_k(a) = 0$, for some k with $x_{ij} \in B_k \cap Y\}$. Then x_{ij} and $X(x_{ij})$ satisfy (i), (iia) and (iib). This completes the proof. ∎

We collect our results in the following lemma.

Lemma 3.13. $\Pr(R) \leq [1 - (1-p)e^{-\frac{24t}{p(n-t)}}]^{\frac{(n-t)^3}{128s^2t}}$.

Proof. Let $\mathcal{S}$ and $T(a, a')$, for $(a, a') \in \mathcal{S}$ be as in Lemma 3.12. Then

$$R \subset \bigcap_{(a,a') \in \mathcal{S}} R(a, a') \subset \bigcap_{(a,a') \in \mathcal{S}} T(a, a'), \quad \text{and} \quad \Pr(T(a, a')) \leq 1 - (1-p)e^{\frac{-24t}{p(n-t)}}.$$

Since $\{T(a, a') : (a, a') \in \mathcal{S}\}$ is independent and $|\mathcal{S}| = \frac{(n-t)^3}{128s^2t}$, the result follows. ∎

Proof of Theorem 3.1. By Lemma 3.3, almost all $P \in \Omega(n, p)$ are s-mixed. Thus by Lemma 3.6, it suffices to show that $s\dim P \geq t$. By Lemma 3.8, the number of t-collections of short linear extensions is at most $l = e^{2st \log n}$. Using Lemma 3.13 and hypothesis (2) of Theorem 3.1, the probability that any one of these t-collections realizes P is at most $q = [1 - (1-p)e^{-\frac{24t}{p(n-t)}}]^{\frac{(n-t)^3}{128s^2t}}$. Thus it suffices to show that $lq \to 0$ as $n \to \infty$. But, after substituting $s = \lfloor (2 \log pn)/p \rfloor$, this is exactly what Theorem 3.1 asserts. ∎

We are now ready to derive the lower bounds on $\dim(P)$ discussed in the introduction.

Proof of Theorem 1.5. Let $\epsilon > 0$ and set $t = \frac{\delta pn \log pn}{1 + \delta p \log pn}$. We show $lq \to 0$ provided δ is sufficiently small in comparison to ϵ and $(\log^{1+\epsilon} n)/n < p < 1 - n^{-1+\epsilon}$. Observe that:

$$n - t = \frac{n}{1 + \delta p \log pn} ;$$

$$\frac{t}{p(n-t)} = \delta \log pn ;$$

$$\frac{4t \log n \log pn}{p} = \frac{4\,\delta n(\log^2 pn) \log n}{1 + \delta p \log pn} ; \text{ and}$$

$$\frac{p^2(n-t)^3}{512t \log^2 pn} = \frac{pn^2}{512\delta \log^3 pn(1 + \delta pn \log pn)^2} .$$

Thus:

$$q[1 - (1-p)e^{-\frac{24t}{p(n-t)}}]^{\frac{p^2(n-t)^3}{512t\log^2 pn}} \le e^{-\frac{pn^2(1-p)(pn)^{-24\delta}}{512\delta \log^3 pn(1+\delta p \log pn)^2}} .$$

Now

$$l = e^{\frac{4t \log n \log pn}{p}} = e^{\frac{4\delta n \log^2 pn \log n}{1+\delta p \log pn}}$$

In order to force $lq \to 0$, it suffices to require

$$\frac{pn^2(1-p)(pn)^{-24\delta}}{512\delta \log^3 pn(1 + \delta p \log pn)^2} \ge \frac{8\delta n \log^2 pn \log n}{1 + \delta p \log pn}$$

which is equivalent to:

$$(pn)^{1-24\delta}(1-p) \ge 4096\delta^2(\log^5 pn)(1 + \delta p \log pn) \log n . \tag{1}$$

Case 1. $p \le 1/\log n$.

In this case, we note that $1 + \delta p \log pn \le 1 + \delta$. Since $pn \to \infty$, inequality (1) can be replaced by:

$$(pn)^{1-24\delta} \ge \log^6 pn \log n . \tag{1$'$}$$

Now (1$'$) is satisfied if $p > (\log^{1+100\delta} n)/n$ and n is sufficiently large.

Case 2. $p > 1/\log n$.

In this case, $1 + \delta p \log pn \le 2\, p \log pn$ so (1) can be replaced by:

$$n^{1-24\delta}(1-p) \ge \log^7 n . \tag{1$''$}$$

Again this inequality is satisfied if $p < 1 - n^{-1+100\delta}$ and n is large.

We summarize the two cases with the observation that (1$'$) and (1$''$) are satisfied when $\delta = \epsilon/100$. ∎

4. THE DIMENSION OF A RANDOM LABELLED ORDERED SET

In this section we consider the probability space $\mathscr{L} = \mathscr{L}(n)$ of all labelled ordered sets on n elements with $\Pr(P) = 1/|\mathscr{L}(n)|$ for every $P \in \mathscr{L}(n)$, i.e., each labelled ordered set is equally likely. The following result is simply a restatement of the Kleitman/Rothschild [8] estimate for the number of labelled ordered sets on n elements. In fact, we use weak estimates for the error terms since these estimates suffice for our purposes.

Theorem 4.1. *Almost all ordered sets $P \in \mathscr{L}(n)$ satisfy the following properties*:

(1) The set A of minimal elements and the set A'' of maximal elements each contain at least $n/4 - n^{2/3}$ and at most $n/4 + n^{2/3}$ points;
(2) $A < A''$ in P; and
(3) $A' = P - A - A''$ is an antichain in P. ∎

In view of Theorem 4.1, we let $\mathscr{T} = \mathscr{T}(n)$ denote the subspace of $\mathscr{L}(n)$ containing all ordered sets P satisfying (1), (2), and (3) as given in the conclusion of the theorem. Theorem 1.7 will then follow from (4.1) and the following more technical result.

Theorem 4.2. *There exist absolute positive constants c_1 and c_2 so that*:

$$n/4 - c_1 n/\log n < \dim(P) < n/4 - c_2 n/\log n$$

for almost all $P \in \mathscr{T}(n)$. ∎

Before proceeding with the proof of Theorem 4.2, we need to develop some additional preliminary material on matchings in bipartite graphs.

Let S_1 and S_2 be disjoint nonempty n-element sets, and let $p = p(n)$ satisfy $0 \le p \le 1$. We then let $\mathscr{B}(S_1, S_2, p)$ denote the probability space of all bipartite graphs on the vertex set $S_1 \cup S_2$ where the edge set is a subset of $S_1 \times S_2$. If $G \in \mathscr{B}(S_1, S_2, p)$ and G has m edges, we set $\Pr(G) = p^m(1-p)^{n^2-m}$. Now let $G \in \mathscr{B}(S_1, S_2, p)$; recall that a *perfect matching* in G is a bijection $f: S_1 \to S_2$ so that $(s_1, f(s_1))$ is an edge of G for every $s_1 \in S_1$. In this paper, we need a randomized version of the matching problem. For the sake of completeness, we give the following elementary result.

Lemma 4.3. *If $p > (\log^{1+\epsilon} n)/n$, then almost all bipartite graphs in $\mathscr{B}(S_1, S_2, p)$ have a perfect matching.*

Proof. If $G \in \mathscr{B}(S_1, S_2, p)$ and G does not have a matching, then by Hall's theorem [5], there is a subset $S_1' \subset S_1$ for which the set $S_2' = \{s_2 \in S_2$: there is some $s_1' \in S_1'$ with (s_1', s_2) an edge in $G\}$ satisfies $|S_2'| < |S_1'|$. When $p > (\log^{1+\epsilon} n)/n$, this almost never happens. ∎

We are now ready to proceed with the argument for (4.2).

Proof of Theorem 4.2. We first establish the lower bound. Let $P \in \mathcal{T}(n)$ and let A_1 and A_1' be subsets of A and A' with $|A_1| = |A_1'| = n/4 - n^{2/3} = m_1$. Then the ordered set Q determined by restricting P to $A_1 \cup A_1'$ behaves just like our original bipartite model $\Omega(m_1, p)$ with $p = 1/2$. Now $\dim(P) \geq \dim(Q)$ since Q is contained in P, and for almost all Q in $\Omega(m_1, p)$, we have $\dim(Q) \geq m_1 - c_1 m_1/\log m_1$, where c_1 is the constant provided by Theorem 1.4 when $p = 1/2$. The desired inequality follows since $n^{2/3} = o(n/\log n)$.

We now establish the upper bound. Let $t = n/4 - \epsilon n/\log n$ where $\epsilon = 10^{-4}$. It is enough to show $\dim(P) \leq t + 1$ for almost all $P \in \mathcal{T}(n)$. To accomplish this, we set $m = n/4 + n^{2/3}$ and let $\mathcal{U}(4m)$ be the subspace of $\mathcal{T}(4m)$ consisting of all ordered sets $P \in \mathcal{T}(4m)$ with $|A| = |A''| = m$ and $(A') = 2m$. It suffices to show that $\dim(P) \leq t + 1$ for almost all $P \in \mathcal{U}(4m)$. To prove this statement, we let $\mathcal{S} = (A \times A') \cup (A' \times A'')$ and we consider the event R_1 consisting of all $P \in \mathcal{U}(4m)$ for which there exists a family $\Sigma = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that for every $(u, v) \in \mathcal{S}$ with $u \| v$ in P, there is some $i \leq t$ with $v < u$ in L_i.

Every ordered set $P \in R_1$ satisfies $\dim(P) \leq t + 1$, since we can construct a realizer Σ' of P by adding one new linear extension L_{t+1} to Σ. The linear order L_{t+1} is defined by $L_{t+1} \restriction A = L_1^* \restriction A$, $L_{t+1} \restriction A' = L_1^* \restriction A'$, $L_{t+1} \restriction A'' = L_1^* \restriction A''$ and $A < A' < A''$ in L_{t+1}. We now proceed to show that $\Pr(R_1) \to 1$. As before, we find it convenient to express R_1 in terms of simpler events. Without loss of generality, we assume t is an even integer.

Let F and F'' be t-element subsets of A and A'', respectively. Then let F_1' and F_2' be disjoint t-element subsets of A'. To preserve independence in future arguments, we partition each of these t-elements sets into two subsets each containing $t/2$ elements. To distinguish between the two subsets, we call the points in one of them *red* points, while points in the other subset are called *blue* points. The four subsets of red points are denoted F_r, $F_{1,r}'$, $F_{2,r}'$, and F_r'', and the subsets of blue points are denoted F_b, $F_{1,b}'$, $F_{2,b}'$, and F_b''.

Now define events S_r and S_b as follows. The event S_r consists of all ordered sets $P \in \mathcal{U}(4m)$ for which the red points can be labelled as $F_r\{b_1, b_2, \ldots, b_{t/2}\}$, $F_{1,r}' = \{c_1', c_2', \ldots, c_{t/2}'\}$, $F_{2,r}' = \{d_1', d_2', \ldots, d_{t/2}'\}$, and $F_r'' = \{e_1'', e_2'', \ldots, e_{t/2}''\}$ so that for all $i = 1, 2, \ldots, t/2$, the following condition is satisfied:

$$b_i \| c_i',\ d_i' \| e_i'',\ b_i < d_i' \text{ and } c_i' < e_i'' \text{ in } P. \qquad (*)$$

Similarly, S_b consists of those $P \in \mathcal{U}(4m)$ for which the blue points can be labelled so that condition $(*)$ holds for $i = t/2 + 1, t/2 + 2, \ldots, 2t$. Clearly, S_r and S_b are independent. We show that $\Pr(S_r) \to 1$ and $\Pr(S_b) \to 1$. This will be accomplished by finding perfect matchings and using these matchings to determine the labellings.

First, we consider the red points. Let $B_1 \in \mathcal{B}(F_r, F_{1,r}', 1/2)$ be the bipartite graph where $(a, a') \in F_r \times F_{1,r}'$ is an edge in B_1 if and only if $a \| a'$ in P. Similarly, let $B_2 \in \mathcal{B}(F_r'', F_{2,r}', 1/2)$ be the bipartite graph where $(a'', a') \in F_r'' \times F_{2,r}'$ is an edge in B_2 if and only if $a'' \| a'$ in P. For $i = 1, 2$, let S_i be the event that B_i has a perfect matching. Then S_1 and S_2 are independent, $\Pr(S_1) \to 1$ and $\Pr(S_2) \to 1$.

Let S_1 and S_2 be given, and let $f_1 : F_r \to F_{1,r}'$ and $f_2 : F_r'' \to F_{2,r}'$ be perfect matchings in the bipartite graphs B_1 and B_2, respectively. Then let $B_3 \in \mathcal{B}(F_r, F_r''$, 1/4) be the bipartite graph where $(a, a'') \in F_r \times F''_r$ is an edge in B_3 if and only if $f_1(a) < a''$ in P and $a < f_2(a'')$ in P. Let S_3 be the event that, given S_1, S_2 and the perfect matchings f_1 and f_2, there exists a perfect matching in B_3. Then $\Pr(S_3) \to 1$.

Now let $f_3 : F''_r \to F'_{2,r}$ be a perfect matching in B_3. We use f_1, f_2 and f_3 to label the red points as follows. Begin with an arbitrary labelling of F_r as $\{b_1, b_2, \ldots, b_{t/2}\}$. Then label the other red points by $c'_i = f_1(b_i)$, $e''_i = f_3(b_i)$ and $d'_i = f_2(f_3(b_i))$ for $i = 1, 2, \ldots, t/2$. Clearly, this labelling satisfies condition $(*)$, which proves that $\Pr(S_r) \to 1$. Similarly, $\Pr(S_b) \to 1$.

Now set $R_2 = R_1 | S_r S_b$. We need to show $\Pr(R_2) \to 1$. To accomplish this, we assume the red and blue points have been labelled so that condition $(*)$ is satisfied for $i = 1, 2, \ldots, t$. We then construct a family $\Sigma = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P. The construction will have both explicit and random components.

Set $M = A - F$, $M' = A' - (F'_1 \cup F'_2)$ and $M'' = A'' - F''$. For each $i = 1, 2, \ldots, t$, we perform the following random trial. Choose at random an element $x \in M$, an element $w'' \in M''$ and distinct elements $y', z' \in M'$. Call the trial a *success* if the following requirements are satisfied.

Success. $\quad z' \| w''$, $w'' \| d'_i$, $z' \| e''_i$, $x \| y'$, $x \| c'_i$ and $y' \| b_i$ in P.

Otherwise, the trial is a *failure*. Evidently, the probability of success is 1/64.

If the trial is a failure, we take L_i as any linear order on $A \cup A' \cup A''$ satisfying:

When trail i is a failure

$$D(c'_i) < c'_i < A - (\{b_i\} \cup D(c'_i)) < F_2 - U(b_i) < b_i < F_2 \cap U(b_i) < M' <$$
$$F_1 \cap D(e''_i) < e''_i < F_1 - D(e''_i) < A'' - (\{e''_i\} \cup U(d'_i)) < d'_i < U(d'_i) \text{ in } L_i .$$

It is clear that any such L_i is a linear extension of P. However, observe that L_i also satisfies:

(1) If $a \in A$ and $a \| c'_i$, then $c'_i < a$ in L_i;
(2) If $a'' \in A''$ and $d'_i \| a''_i$, then $a'' < d'_i$ in L_i;
(3) If $c \in F_1$ and $c' \| e''_i$, then $e''_i < c'$ in L_i; and
(4) If $d \in F_2$ and $b_i \| d'$, then $d' < b_i$ in L_i.

If the trial results in success, then for each $a' \in M' - \{y', z'\}$, we flip a fair coin. We let H' consist of those a' for which the coin toss results in heads, and we let T' consist of those a' for which the coin toss results in tails. Then let L_i be any linear order on $A \cup A' \cup A''$ satisfying:

When trial i is a success

$$D(c'_i) < c'_i < D(y') - D(c'_i) < y' < A - (\{b_i, x\} \cup D(y') \cup D(c'_i))$$
$$< (F'_2 \cup T') - (U(x) \cup U(b_i)) < x < ((F'_2 \cup T') - U(b_i)) \cap U(x) < b_i$$

$$< ((F_2' - \{d_i'\}) \cup T') \cap U(b_i) < ((F_1' - \{c_i'\}) \cup H') \cap D(e_i'') < e_i''$$
$$< ((F_1' \cup H') - D(e_i'')) \cap D(w'') < w'' < ((F_1' \cup H') - (D(w'') \cup D(e_i''))$$
$$< A'' - (\{e_i'', w''\} \cup U(z') \cup U(d_i')) < z' < U(z') - U(d_i') < d_i' < U(d_i') \quad \text{in} \quad L_i .$$

First, observe that any such L_i is a linear extension of P, and L_i satisfies (1), (2), (3), and (4). However, in this case L_i satisfies the following additional properties:

(5) If $a \in A$ and $a \| y'$ in P, then $y' < a$ in L_i unless $a < c_i'$ in P;
(6) If $a'' \in A''$ and $a'' \| z'$ in P, then $a'' < z'$ in L_i unless $a'' > d_i'$ in P;
(7) If $a' \in F_1' \cup H'$, $a'' \in \{w'', e_i''\}$ and $a' \| a''$, then $a'' < a'$ in L_i unless $a'' = w''$ and $a' < e_i''$ in P; and
(8) If $a' \in F_2' \cup T'$, $a \in \{x, b_i\}$ and $a \| a'$, then $a' < a$ in L_i unless $a = x$ and $b_i < a'$ in P.

Now the number of pairs in $\mathcal{S} = (A \times A') \cup (A' \times A'')$ is less than n^2. To complete the proof, we define for each $(u, v) \in \mathcal{S}$ the event $Q(u, v)$ by $Q(u, v) = (u < v) \cup Q'(u, v)$ where $Q'(u, v)$ holds when $u \| v$ in P and there is some L_i in Σ with $v < u$ in L_i. We need only show that $\Pr(\rceil Q(u, v)) \leq n^{-3}$ for all $(u, v) \in \mathcal{S}$. The argument will be simplified by the observation that if $(u, v) \in \mathcal{S}$ and u and v have different colors, then $\Pr(u < v) = 1/2$. Furthermore, since the construction of Σ is obviously symmetric, we need only consider the case where $(u, v) \in A \times A'$. We now subdivide the argument into cases depending on the location of v.

Case 1. $v \in F_1'$.

In this case, $v = c_i'$ for some $i \leq t$. If $u < v$ in P, then $Q(u, v)$ holds. If $u \| v$ in P, we see that $v < u$ in L_i by property (1). Thus, $\Pr(\rceil Q(u, v)) = 0$.

Case 2a. $v \in F_2'$ and $u \in F$.

Choose $i, j \leq t$ so that $v = d_i'$ and $u = b_j$. If $u < v$ in P, then $Q(u, v)$ holds. If $u \| v$, we must have $i \neq j$. Thus $v < u$ in L_j by property (4), and $\Pr(\rceil Q(u, v)) = 0$.

Case 2b $v \in F_2'$ and $u \in M$.

For each $j \leq t$ for which b_j and v have different colors, let $Q_j(u, v)$ be the event which holds if Trial j is a success, the point $u = x$ is selected from M and $b_j \| v$ in P. Clearly, $\mathcal{F} = \{Q_j(u, v)\colon \text{color}(b_j) \neq \text{color}(v)\}$ is a family of independent events and $\Pr(Q_j(u, v)) = \dfrac{1}{128(m-t)}$ for each event in $\mathcal{F}$. Furthermore, if any $Q_j(u, v) \in \mathcal{F}$ holds, then either $u < v$ in P, or $v < u$ in L_j by property (8).

Now $m - t \leq 2\epsilon n/\log n$ and $|\mathcal{F}| = t/2 \geq n/10$. Thus

$$\Pr(\rceil Q(u, v)) \leq \left(1 - \frac{\log n}{256\epsilon n}\right)^{\frac{n}{10}} < n^{-\frac{1}{3000\epsilon}}$$

$$< n^{-3} \text{ as required .}$$

(Recall that $\epsilon = 10^{-4}$.)

Case 3a. $v \in M'$ and $u \in F$.

Choose $i \leq t$ for which $u = b_i$. This time, for each $j \leq t$ with $\text{color}(b_i) \neq \text{color}(c_j')$, let $Q_j(u, v)$ be the event which holds if Trial j is a success, the point $y' = v$ is selected from M' and $b_i \| c_j'$. Also let $\mathcal{G}$ denote the family of all such events. If any $Q_j(u, v) \in \mathcal{G}$ holds, then either $u < v$ in P, or $v < u$ in L_j by property (5). By the same analysis as in the preceding case, we see that $\Pr(\rceil Q(u, v)) < n^{-3}$.

Case 3b. $v \in M'$ and $u \in M$.

In this case, for each $j \leq t$, let $Q_j(u, v)$ be the event which holds if Trial j is a success, $x = u$ is selected from M, $v \in T'$ and $b_j \| v$. Let $\mathcal{H}$ be the family of all such events. If an event $Q_j(u, v) \in \mathcal{H}$ holds, then either $u < v$ in P, or $v < u$ in L_j by property (5).

Now each event $Q_j(u, v) \in \mathcal{H}$ satisfies $\Pr(Q_j(u, v)) = \dfrac{1}{256} \dfrac{2m - 2t - 2}{2m - 2t} \dfrac{1}{m - t}$ and there are t events in $\mathcal{H}$. As in the preceding two cases, we conclude that $\Pr(\rceil Q(u, v)) < n^{-3}$. With this observation, the proof of the theorem is complete. ∎

5. CONCLUDING REMARKS AND QUESTIONS

Our original motivation in studying the probability space $\Omega(n, p)$ came from the possibility that for properly chosen p, a typical ordered set from $\Omega(n, p)$ would show that the Füredi/Kahn bounds given in Theorems 1.1 and 1.2 are best possible. In the case of Theorem 1.1, this goal has been accomplished. For Theorem 1.2 our results are not tight. We show that $\dim(P) = \Omega(\Delta \log(\Delta))$, for almost all $P \in \Omega(n, p)$, while Theorem 1.2 gives $\dim(P) = O(\Delta \log^2(\Delta))$. These considerations give rise to the following specific problems.

(5.1) For p satisfying $\log^2 pn = o(\log n)$, find improved bounds for the expected value $E(\dim(P))$ of the dimension of P. As of now, we know $c_2 pn \log pn < E(\dim(P)) < c_1 pn \log^2 pn$.

(5.2) Can random methods be used to find a (possibly quite rare) ordered set P with $\dim(P)$ close to $\Delta(P) \log^2 \Delta(P)$?

(5.3) Alternately, can the Füredi/Kahn inequality $\dim(P) < c\Delta(P) \log^2 \Delta(P)$ be improved by lowering the exponent on the $\log \Delta(P)$ term?

Because of our model's sensitivity to the choice of p, we believe it will provide a rich source of examples for dimension theoretic problems. We mention one such application. Trotter [9] asked what is the maximum value $f(n, k)$ of the dimension of an ordered set P such that $|P| = n$ and P does not contain a standard example S_k. No nontrivial lower bound for $f(n, k)$ was known. Our techniques show that $f(n, k) > n^{1-\frac{1}{k}}$ when n is large relative to k, and k itself is large. To see how this inequality is derived, observe that if $p = n^{-\frac{1}{k}}$, then almost all $P \in \Omega(n, p)$ satisfy $\dim(P) > n^{1-\frac{1}{k}}$ and do not contain a standard example S_k.

(5.4) Is the inequality $f(n, k) > n^{1-\frac{1}{k}}$ best possible?

We also consider it important to study the evolution of random ordered sets.

(5.5) As p increases, when does P first satisfy $\dim(P) \geq k$, for $k = 3, 4, 5, \ldots$?

(5.6) Is the expected value of $\dim(P)$ unimodal as p increases?

(5.7) When does $\lim_{n\to\infty} \dfrac{E(\dim(P))}{pn \log pn}$ exist, and what is its value?

There are several technical questions suggested by our approach.

(5.8) How does $\dim(P)$ behave when p is very close to 1?

(5.9) Evaluate $\lim_{n\to\infty} \dfrac{\dim(P)}{n}$ when $p = 1/\log n$.

(5.10) How tight is the inequality $s\dim(P) \leq \dim(P)$? We suspect that this inequality is very tight for almost all $P \in \Omega(n, p)$.

(5.11) How does $\dim(P)$ behave if we consider a bipartite model with n minimal elements and m maximal elements with $m > n$?

(5.12) How do the results change when the comparabilities in P are the union of k random edge disjoint matchings? This approach may prove particularly useful when k is very small.

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involving a_i. Azuma's inequality then gives $\Pr(|M_n - M_o| \geq t) \leq 2e^{-t^2/n}$. Now M_n is the actual dimension of the ordered set and M_o is the expected dimension.

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