

Research Article

The Diophantine Equation $8^x + p^y = z^2$

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Let p be a fixed odd prime. Using certain results of exponential Diophantine equations, we prove that (i) if $p \equiv \pm 3 \pmod{8}$, then the equation $8^x + p^y = z^2$ has no positive integer solutions (x, y, z) ; (ii) if $p \equiv 7 \pmod{8}$, then the equation has only the solutions $(p, x, y, z) = (2^q - 1, (1/3)(q + 2), 2, 2^q + 1)$, where q is an odd prime with $q \equiv 1 \pmod{3}$; (iii) if $p \equiv 1 \pmod{8}$ and $p \neq 17$, then the equation has at most two positive integer solutions (x, y, z) .

1. Introduction

Let \mathbb{Z}, \mathbb{N} be the sets of all integers and positive integers, respectively. Let p be a fixed odd prime. Recently, the solutions (x, y, z) of the equation

$$8^x + p^y = z^2, \quad x, y, z \in \mathbb{N} \quad (1)$$

were determined in the following cases:

- (1) (Sroysang [1]) if $p = 19$, then (1) has no solutions;
- (2) (Sroysang [2]) if $p = 13$, then (1) has no solutions;
- (3) (Rabago [3]) if $p = 17$, then (1) has only the solutions $(x, y, z) = (1, 1, 5), (2, 1, 9)$, and $(3, 1, 23)$.

In this paper, using certain results of exponential Diophantine equations, we prove a general result as follows.

Theorem 1. *If $p \equiv \pm 3 \pmod{8}$, then (1) has no solutions (x, y, z) . If $p \equiv 7 \pmod{8}$, then (1) has only the solutions*

$$(p, x, y, z) = \left(2^q - 1, \frac{q+2}{3}, 2, 2^q + 1 \right), \quad (2)$$

where q is an odd prime with $q \equiv 1 \pmod{3}$.

If $p \equiv 1 \pmod{8}$ and $p \neq 17$, then (1) has at most two solutions (x, y, z) .

Obviously, the above theorem contains the results of [1, 2]. Finally, we propose the following conjecture.

Conjecture 2. *If $p \neq 17$, then (1) has at most one solution (x, y, z) .*

2. Preliminaries

Lemma 3. *If $2^n - 1$ is a prime, where n is a positive integer, then n must be a prime.*

Proof. See Theorem 1.10.1 of [4]. □

Lemma 4. *If p is an odd prime with $p \equiv 1 \pmod{4}$, then the equation*

$$u^2 - pv^2 = -1, \quad u, v \in \mathbb{N} \quad (3)$$

has solutions (u, v) .

Proof. See Section 8.1 of [5]. □

Lemma 5. *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \quad \gcd(X, Y) = 1, \quad Y > 1, \quad m > 1, \quad n > 2 \quad (4)$$

has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

Proof. See Theorem 8.4 of [6]. □

Lemma 6. Let D be a fixed odd positive integer. If the equation

$$u^2 - Dv^2 = -1, \quad u, v \in \mathbb{N} \tag{5}$$

has solutions (u, v) , then the equation

$$X^2 - D = 2^n, \quad X, n \in \mathbb{N}, n > 2 \tag{6}$$

has at most two solutions (X, n) , except the following cases:

- (i) $D = 2^{2r} - 3 \cdot 2^{r+1} + 1$, $(X, n) = (2^r - 3, 3)$, $(2^r - 1, r + 2)$, $(2^r + 1, r + 3)$, and $(3 \cdot 2^r - 1, 2r + 3)$, where r is a positive integer with $r \geq 3$;
- (ii) $D = ((1/3)(2^{2r+1} - 17))^2 - 32$, $(X, n) = ((1/3)(2^{2r+1} - 17), 5)$, $(1/3)(2^{2r+1} + 1, 2r + 3)$, and $((1/3)(17 \cdot 2^{2r+1} - 1), 4r + 7)$, where r is a positive integer with $r \geq 3$;
- (iii) $D = 2^{2r_1} + 2^{2r_2} - 2^{r_1+r_2+1} - 2^{r_1+1} - 2^{r_2+1} + 1$, $(X, n) = (2^{r_2} - 2^{r_1} - 1, r_1 + 2)$, $(2^{r_2} - 2^{r_1} + 1, r_2 + 2)$, and $(2^{r_2} + 2^{r_1} - 1, r_1 + r_2 + 2)$, where r_1, r_2 are positive integers with $r_2 > r_1 + 1 > 2$.

Proof. See [7]. □

Lemma 7. If D is an odd prime and D belongs to the exceptional case (i) of Lemma 6, then $D = 17$.

Proof. We now assume that D is an odd prime with $D = 2^{2r} - 3 \cdot 2^{r+1} + 1$. Then we have

$$(2^r - 1)^2 - 2^{r+2} = D, \tag{7}$$

$$(2^r + 1)^2 - 2^{r+3} = D. \tag{8}$$

If $2 \mid r$, since $r \geq 3$, then $r \geq 4$, and by (7), we have

$$(2^r - 1) + 2^{r/2+1} = D, \quad (2^r - 1) - 2^{r/2+1} = 1. \tag{9}$$

But, by the second equality of (9), we get $1 \equiv (2^r - 1) - 2^{r/2+1} \equiv -1 \pmod{8}$, a contradiction.

If $2 \nmid r$, then from (8) we get

$$(2^r + 1) + 2^{(r+3)/2} = D, \quad (2^r + 1) - 2^{(r+3)/2} = 1. \tag{10}$$

Further, by the second equality of (10), we have $2^r = 2^{(r+3)/2}$, $r = 3$, and $D = 17$. Thus, the lemma is proved. □

Lemma 8. If D is an odd prime and D belongs to the exceptional case (iii) of Lemma 6, then $D = 17$.

Proof. Using the same method as in the proof of Lemma 7, we can obtain this lemma without any difficulty. □

Lemma 9. If D belongs to the exceptional case (ii), then (6) has at most one solution (X, n) with $3 \mid n$.

Proof. Notice that, for any positive integer r , there exists at most one number of $5, 2r + 3$, and $4r + 7$ which is a multiple of 3. Thus, by Lemma 6, the lemma is proved. □

Lemma 10. The equation

$$X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{N}, \min \{X, Y, m, n\} > 1 \tag{11}$$

has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

Proof. See [8]. □

3. Proof of Theorem

We now assume that (x, y, z) is a solution of (1). Then we have $\gcd(2p, z) = 1$.

If $2 \mid y$, since $\gcd(z + p^{y/2}, z - p^{y/2}) = 2$, then from (1) we get

$$z + p^{y/2} = 2^{3x-1}, \quad z - p^{y/2} = 2, \tag{12}$$

where we obtain

$$z = 2^{3x-2} + 1, \tag{13}$$

$$p^{y/2} = 2^{3x-2} - 1. \tag{14}$$

Since $p > 1$, applying Lemma 10 to (14), we get

$$y = 2, \quad p = 2^{3x-2} - 1. \tag{15}$$

Further, by Lemma 3, we see from the second equality of (15) that

$$p = 2^q - 1, \quad q = 3x - 2 \tag{16}$$

is an odd prime with $q \equiv 1 \pmod{3}$.

Therefore, by (13), (15), and (16), we obtain the solutions given in (2).

Obviously, if p satisfies (2), then $p \equiv 7 \pmod{8}$. Otherwise, since $2 \nmid y$, we see from (1) that $p \equiv p^y \equiv z^2 - 8^x \equiv 1 \pmod{8}$. It implies that if $p \equiv \pm 3 \pmod{8}$, then (1) has no solutions (x, y, z) . If $p \equiv 7 \pmod{8}$, then (1) has only the solutions (2).

Here and below, we consider the remaining cases that $p \equiv 1 \pmod{8}$. By the above analysis, we have $2 \nmid y$. If $y > 1$, then $y \geq 3$ and (4) has the solution $(X, Y, m, n) = (z, p, 3x, y)$ with $3 \mid m$. But, by Lemma 5, it is impossible. Therefore, we have

$$y = 1. \tag{17}$$

Substituting (17) into (1), the equation

$$X^2 - p = 2^n, \quad X, n \in \mathbb{N}, n > 2 \tag{18}$$

has the solution $(X, n) = (z, 3x)$ with $3 \mid n$. Since $p \equiv 1 \pmod{8}$, by Lemma 4, (3) has solutions (u, v) . Therefore, by Lemmas 6–9, (1) has at most two solutions (x, y, z) . Thus, the theorem is proved.

Conflict of Interests

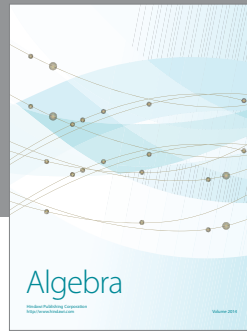
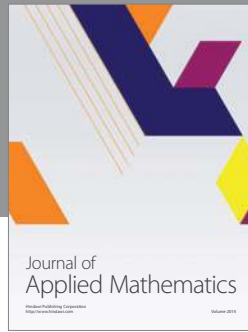
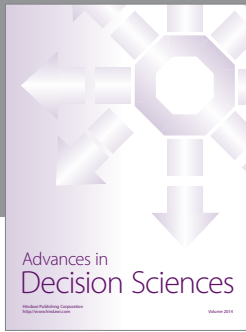
The authors declare that there is no conflict of interests regarding the publication of this paper.

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