Tôhoku Math. J.
49 (1997), 565–575

THE DIRICHLET PROBLEM FOR HARMONIC MAPS BETWEEN DAMEK-RICCI SPACES

KEISUKE UENO

(Received May 15, 1996, revised October 9, 1996)

Abstract. A Damek-Ricci space has nonpositive curvature. Thus we can consider the Eberlein-O'Neill compactifications adding the sphere at infinity. In this paper, we prove the existence and uniqueness of a solution to the Dirichlet problem at infinity for harmonic maps between Damek-Ricci spaces.

1. Introduction. For a Hadamard manifold, that is, a simply connected, connected, complete Riemannian manifold of nonpositive curvature, one can consider the Eberlein-O'Neill compactification by adding the ideal boundary, which is the sphere at infinity defined by the asymptotic classes of geodesic rays. The Dirichlet problem at infinity for harmonic maps between these manifolds then consists of finding a harmonic map which extends a boundary map given on the sphere at infinity.

In their papers, Li and Tam [14] (for arbitrary dimensions) and Akutagawa [1] (for dimension two) have solved the Dirichlet problem at infinity for harmonic maps between real hyperbolic spaces. In particular, exploiting the heat equation method, Li and Tam [14] established a general theory for the existence and uniqueness of solutions to this problem. For example, they proved the following: Let the Hadamard manifolds under consideration satisfy appropriate geometric conditions on the curvatures, the volume of unit balls and the bottom spectrum of the Laplacian. If one can construct a suitable initial map for the parabolic harmonic map equation, then it converges to the desired harmonic map.

In 1994, Donnelly [9] discussed the problem for harmonic maps between rank one symmetric spaces of noncompact type. He investigated the unbounded models of these spaces, realized as upper half space models, and their global coordinates near the ideal boundary. Making use of this realization, he constructed "good" initial maps and solved the problem for boundary maps having sufficient regularity.

Noticing that all of these manifolds are symmetric and have strictly negative curvature, the following question arises naturally: For nonsymmetric or nonpositively curved manifolds, can one solve the Dirichlet problem at infinity for harmonic maps? As an answer to this question, we shall study in this paper the Dirichlet problem for harmonic maps between Damek-Ricci spaces.

Damek-Ricci spaces were first introduced by Damek [6] as a semidirect extension of the generalized Heisenberg groups discovered by Kaplan [12]. These spaces may be

1991 *Mathematics Subject Classification.* Primary 58E20.

regarded as a generalization of rank one symmetric spaces of noncompact type, since they have geometry similar to them. However, Damek-Ricci spaces are not symmetric in general and have nonpositive curvature. In fact, nonsymmetric Damek-Ricci spaces admit vanishing sectional curvatures for certain two-plane.

The purpose of this paper is to show that Donnelly's existence and uniqueness results can be extended to the case of Damek-Ricci spaces. Thus the geometric conditions such as symmetry and strict negativity of curvature are not essential. Thus it becomes worthwhile to study the problem for more general Hadamard manifolds.

The author is grateful to Professor Seiki Nishikawa for his valuable discussions and encouragement during the preparation of this paper and to Professor Lieven Vanhecke for his comments on the latest results concerning the geometry of the Damek-Ricci spaces. He also thanks the referee for valuable comments and suggestions.

2. Damek-Ricci spaces.

2.1. Generalized Heisenberg algebra. We start this section with a brief review of generalized Heisenberg algebras due to Kaplan [12].

Let $(\mathfrak{n}, \langle\,,\rangle_{\mathfrak{n}})$ be a 2-step nilpotent Lie algebra with an inner product, $\mathfrak{z}$ its center and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ with respect to $\langle\,,\rangle_{\mathfrak{n}}$. Since $\mathfrak{n}$ is 2-step nilpotent, $\operatorname{ad} v|_{\mathfrak{v}}$ is a map from $\mathfrak{v}$ to $\mathfrak{z}$ for any $v\in\mathfrak{v}$. We set $\mathfrak{k}_v := \{u\in\mathfrak{v} \mid \operatorname{ad} v(u)=[v,u]=0\}$ and consider the orthogonal decomposition $\mathfrak{v}=\mathfrak{k}_v\oplus\mathfrak{v}_v$ with respect to $\langle\,,\rangle_{\mathfrak{n}}$. We say $(\mathfrak{n}, \langle\,,\rangle_{\mathfrak{n}})$ to be a generalized Heisenberg algebra if $\operatorname{ad} v|_{\mathfrak{v}_v}:\mathfrak{v}_v\to\mathfrak{z}$ is a surjective isometry for any unit vector $v\in\mathfrak{v}$.

Generalized Heisenberg algebras can be constructed systematically in the following fashion:

Let $(\mathfrak{u}, \langle\,,\rangle_{\mathfrak{u}})$, $(\mathfrak{v}, \langle\,,\rangle_{\mathfrak{v}})$ be real vector spaces equipped with inner product. Let $\mu:\mathfrak{u}\times\mathfrak{v}\to\mathfrak{v}$ be a composition of quadratic forms, that is, μ is a bilinear map satisfying $|\mu(u,v)|_{\mathfrak{v}}=|u|_{\mathfrak{u}}|v|_{\mathfrak{v}}$ for any $u\in\mathfrak{u}$, $v\in\mathfrak{v}$. Note that $\mu:\mathfrak{u}\times\mathfrak{v}\to\mathfrak{v}$ satisfies $\mu(u_0,v)=v$ for some $u_0\in\mathfrak{u}$. Indeed, for any given $u_0\in\mathfrak{u}$. set $T:\mathfrak{v}\to\mathfrak{v}$ by $T(v):=\mu(u_0,v)$. Then $\mu'(u,v):=\mu(u,T^{-1}(v))$ satisfies $\mu'(u_0,v)=v$. Hence we may suppose $\mu(u_0,v)=v$. Define $\phi:\mathfrak{v}\times\mathfrak{v}\to\mathfrak{u}$ by

$$\langle u, \phi(v,v')\rangle_{\mathfrak{u}}=\langle\mu(u,v),v'\rangle_{\mathfrak{v}}\,.$$

Let $\mathfrak{z}$ be the orthogonal complement of $\boldsymbol{R}u_0=\{ru_0 \mid r\in\boldsymbol{R}\}$ in $\mathfrak{u}$ (where $\boldsymbol{R}$ is the field of real numbers), $\pi:\mathfrak{u}\to\mathfrak{z}$ the orthogonal projection and $\mathfrak{n}$ the direct sum $\mathfrak{n}:=\mathfrak{v}\oplus\mathfrak{z}$ of $\mathfrak{v}$ and $\mathfrak{z}$ with the natural inner product. Define a Lie bracket on $\mathfrak{n}$ by

$$[v+z, v'+z'] := \pi\circ\phi(v,v')\,. \tag{2.1}$$

THEOREM 2.1 (cf. Kaplan [12]). *$\mathfrak{n}$, equipped with the above inner product and a Lie bracket, is a generalized Heisenberg algebra. Conversely, any generalized Heisenberg algebra arises in this manner.*

Regarding the existence of a composition of quadratic forms, we remark the following. Let ρ be a function defined on positive integers by

$$\rho(n) := 8p + 2^q \quad \text{if } n = (\text{odd number}) \times 2^{4p+q}, \ 0 \le q \le 3 .$$

Then it is known by Hurwitz, Radon and Eckmann that a composition of quadratic forms $\mu : \mathfrak{u} \times \mathfrak{v} \to \mathfrak{v}$ exists if and only if $0 < \dim \mathfrak{u} \le \dim \mathfrak{v}$. Therefore we may conclude (see [12]):

Given integers m, $n > 0$, there exists an $(n+m)$-dimensional generalized Heisenberg algebra with m-dimensional center if and only if $0 < m < \rho(n)$.

As a consequence, we can construct a rich variety of generalized Heisenberg algebras and Damek-Ricci spaces as well.

2.2. Damek-Ricci space. Let $(\mathfrak{n}, \langle , \rangle_{\mathfrak{n}})$ be a generalized Heisenberg algebra, $\mathfrak{a}$ a one-dimensional real vector space and h a unit vector in $\mathfrak{a}$. Since $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, any element of $\mathfrak{n}$ is written uniquely as $v + z$ with $v \in \mathfrak{v}$, $z \in \mathfrak{z}$. Now we define on the linear space $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n}$ a canonical inner product and a Lie bracket by

$$\langle th + v + z, t'h + v' + z' \rangle := tt' + \langle v + z, v' + z' \rangle_{\mathfrak{n}} ,$$

$$[th + v + z, t'h + v' + z'] := tv' - t'v + 2tz' - 2t'z + 2[v + z, v' + z']_{\mathfrak{n}} ,$$

where $[,]_{\mathfrak{n}}$ is the Lie bracket on $\mathfrak{n}$ defined by (2.1). In this way, $\mathfrak{s}$ becomes a Lie algebra with an inner product. The simply connected Lie group S associated with $\mathfrak{s}$, equipped with the left-invariant metric g_S, is called a Damek-Ricci space.

EXAMPLE. Let $\mathfrak{v} = \boldsymbol{R}^{2k}$, $\mathfrak{u} = \boldsymbol{R}^2$, and $\mu(z, v) = (-y, x)$, $\mu(w, v) = (x, y)$, where $\{z, w\}$ is an orthogonal basis of $\boldsymbol{R}^2$ and $v = (x, y) \in \boldsymbol{R}^k \oplus \boldsymbol{R}^k$. Then $\mathfrak{n}$ is a (classical) Heisenberg algebra and (S, g_S) is isometric to the complex hyperbolic space whose sectional curvature K satisfies $-4 \le K \le -1$. The quaternionic hyperbolic space and the Cayley hyperbolic plane are also Damek-Ricci spaces. These three spaces are called *classical* (cf. [6]). Thus Damek-Ricci spaces are generalizations of rank one symmetric spaces of noncompact type.

Let ∇ be the Levi-Civita connection on (S, g_S). Using the formula

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle$$

for X, Y, $Z \in \mathfrak{s}$, one obtains

$$\nabla_h X = 0 , \quad \nabla_X h = -[h, X] ,$$

$$\nabla_v v' = \langle v, v' \rangle h + \frac{1}{2} [v, v'] ,$$

$$\nabla_v z = \nabla_z v = -\mu(z, v) ,$$

$$\nabla_z z' = 2\langle z, z' \rangle h ,$$

where $X \in \mathfrak{s}$, $v, v' \in \mathfrak{v}$, $z, z' \in \mathfrak{z}$ and μ is the composition of quadratic forms in the definition of $\mathfrak{n}$. Then Damek proved the following:

THEOREM 2.2 (Damek [5], [6]). (1) *A Damek-Ricci space is a symmetric space if and only if it is classical.*

(2) *The sectional curvature of Damek-Ricci space is nonpositive. Moreover, there exists a Damek-Ricci space whose sectional curvature vanishes for some two-dimensional plane.*

Damek constructed a simple example for which the second assertion holds. Other examples have been given in [3]. Further, it has been proved in [2] that a Damek-Ricci space is symmetric if and only if it has strictly negative curvature. (The proof needed a correction which has now been given in [10], [13].) So, for nonsymmetric Damek-Ricci space the sectional curvature vanishes for some two-dimensional plane.

We refer to [3] for more information about the geometry, in particular about the curvature, of the Damek-Ricci spaces.

Finally, we estimate the bottom spectrum $\lambda_1(S)$ of the Laplacian for S with the left-invariant metric.

LEMMA 2.3. *Let (r, ω) be the polar coordinate on (S, g_S) around an origin. Then the volume form dv_S of S is given by*

$$dv_S = 2^{-m}(\sinh r)^n(\sinh 2r)^m dr d\sigma(\omega)\,,$$

where $n = \dim \mathfrak{v}$, $m = \dim \mathfrak{z}$ and $d\sigma(\omega)$ is the surface element of the unit sphere S^{n+m}.

From this lemma and the fact $\lambda_1(S) \geq (\inf \Delta_S r)^2/4$, Δ_S being the positive Laplacian, we obtain:

COROLLARY 2.4.

$$\lambda_1(S) \geq \frac{1}{4}(n+2m)^2\,.$$

NOTE. Damek and Ricci [8] proved that the equality holds in the above inequality.

3. Harmonic maps on Damek-Ricci spaces. In this section, following Donnelly [9], we shall prove the existence and uniqueness of solutions of the Dirichlet problem at infinity for harmonic maps.

3.1. Harmonic map equation. Let $u: M \to M'$ be a C^2-map between complete Riemannian manifolds (M, g) and (M', h). For any relatively compact domain $D \subset M$, we set

$$E_D(u) := \frac{1}{2}\int_D |d_x u|^2 dv_g\,,$$

where $|d_x u|$ denotes the Hilbert-Schmidt norm of the differential map $d_x u: T_x M \to T_{u(x)} M'$ and dv_g is the volume measure on (M, g). We say u to be a harmonic map if it is a critical point of $E_D(u)$ for any D. In other words, u is a harmonic map if and only if it satisfies the Euler-Lagrange equation

$$\tau(u) := \sum_{i=1}^{n} (\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)) = 0 ,$$

where $\{e_i\}_{i=1}^{n}$ is an orthonormal frame field of M, ∇ is the Levi-Civita connection of the tangent bundle TM and $\tilde{\nabla}$ is the induced connection of the pull-back bundle $u^{-1}TM'$. $\tau(u)$ is called the tension field of u.

For later reference, we compute explicitly the tension field of a harmonic map between Damek-Ricci spaces. Let (S, g_S) be a Damek-Ricci space with left invariant metric g_S and set $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$. Take the unit vector $h \in \mathfrak{a}$ such that $\mathrm{ad}\, h(v) = v$, $\mathrm{ad}\, h(z) = 2z$ for any $v \in \mathfrak{v}$, $z \in \mathfrak{z}$ and an orthonormal basis $\{h, v_1, \dots, v_n, z_1, \dots, z_m\}$ of $\mathfrak{s}$, $\{v_i\}$ (resp. $\{z_j\}$) being an orthonormal basis of $\mathfrak{v}$ (resp. $\mathfrak{z}$). We define a map $\varphi : \boldsymbol{R} \times N \to S$ by $\varphi(t, n) := n(\exp(th))$, where N is the simply connected Lie group associated with $\mathfrak{n}$ and exp is the exponential map on $\mathfrak{a}$. Then the induced metric via φ is given by $dt^2 + e^{-2t} g_{\mathfrak{v}} + e^{-4t} g_{\mathfrak{z}}$, where $g_{\mathfrak{v}} + g_{\mathfrak{z}}$ is a left invariant metric on N. Setting $y = \exp t$, one can easily verify that (S, g_S) is isometric to $M := \boldsymbol{R}_+ \times N$ with the Riemannian metric

$$g_M := y^{-2} dy^2 + y^{-2} g_{\mathfrak{v}} + y^{-4} g_{\mathfrak{z}} ,$$

where $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} \mid r > 0\}$. Straightforward calculation then yields that the Levi-Civita connection ∇ on (S, g_S) is given by

$$\nabla_\eta \eta = -y^{-1} \eta , \quad \nabla_\eta v_i = -y^{-1} v_i , \quad \nabla_\eta z_k = -2y^{-1} z_k ,$$

$$\nabla_{v_i} v_j = y^{-1} \delta_{ij} \eta + \frac{1}{2} \sum_{k=1}^{m} a_i{}^k{}_j z_k , \tag{3.1}$$

$$\nabla_{v_i} z_k = \frac{1}{2} y^{-2} \sum_{j=1}^{n} a_j{}^k{}_i v_j , \quad \nabla_{z_k} z_l = 2y^{-3} \delta_{kl} \eta ,$$

where $a_i{}^k{}_j$ are the structure constants, $[v_i, v_j] = \sum_{k=1}^{m} a_i{}^k{}_j z_k$, $\eta = \partial/\partial y$ and, v_i and z_k are regarded as left invariant vector fields on S.

Now we are in a position to compute the tension field. Let S, S' be Damek-Ricci spaces and $u \in C^2(S, S')$. We represent (S, g_S) (resp. $(S', g_{S'})$) as $(\boldsymbol{R}_+ \times N, y^{-2} dy^2 + y^{-2} g_{\mathfrak{v}} + y^{-4} g_{\mathfrak{z}})$ (resp. $(\boldsymbol{R}_+ \times N', \bar{y}^{-2} d\bar{y}^2 + \bar{y}^{-2} g_{\mathfrak{v}'} + \bar{y}^{-4} g_{\mathfrak{z}'})$), where $y, \bar{y} \in \boldsymbol{R}_+$ and $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ (resp. $\mathfrak{n}' = \mathfrak{v}' + \mathfrak{z}'$) with $\dim \mathfrak{v} = n$ (resp. $\dim \mathfrak{v}' = n'$), $\dim \mathfrak{z} = m$ (resp. $\dim \mathfrak{z}' = m'$). Take an orthonormal basis $\{v_1, \dots, v_n, z_1, \dots, z_m\}$ (resp. $\{\bar{v}_1, \dots, \bar{v}_{n'}, \bar{z}_1, \dots, \bar{z}_{m'}\}$) of $\mathfrak{n}$ (resp. $\mathfrak{n}'$) as above and denote their left invariant extensions on S by the same letters. Let $e_0 = \partial/\partial y$, $e_i = v_i$ $(1 \le i \le n)$, $e_i = z_{i-n} (n+1 \le i \le n+m)$; $f_0 = \partial/\partial \bar{y}$, $f_\alpha = \bar{v}_\alpha$ $(1 \le \alpha \le n')$, $f_\alpha = \bar{z}_{\alpha-n'}$ $(n'+1 \le \alpha \le n'+m')$ and $u_j^\alpha = f_\alpha^*(du(e_i))$, $u_{ij}^\alpha = e_j \cdot u_i^\alpha$, $\tau^\alpha(u) = f_\alpha^*(\tau(u))$, where f_α^* is the dual frame of f_α. Then we get

$$
\begin{aligned}
\tau^0(u) &= \sum_{i=0}^{n+m} g^{ii} u^0_{ii} + (1-n-2m)u^0_0 y - (u^0)^{-1} \sum_{i=0}^{n+m} g^{ii}(u^0_i)^2 \\
&\quad + (u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} \sum_{\beta=1}^{n'} (u^\beta_i)^2 + 2(u^0)^{-3} \sum_{i=0}^{n+m} g^{ii} \sum_{\beta=n'+1}^{n'+m'} (u^\beta_i)^2 , \\
\tau^\alpha(u) &= \sum_{i=0}^{n+m} g^{ii} u^\alpha_{ii} + (1-n-2m)u^\alpha_0 y - 2(u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} u^0_i u^\alpha_i \\
&\quad + (u^0)^{-2} \sum_{i=0}^{n+m} g^{ii} \sum_{\beta=1}^{n'} \sum_{\gamma=n'+1}^{n'+m'} u^\gamma_i u^\beta_i a_\alpha{}^{\gamma-n'}{}_\beta , \qquad (1 \le \alpha \le n') , \\
\tau^\alpha(u) &= \sum_{i=0}^{n+m} g^{ii} u^\alpha_{ii} + (1-n-2m)u^\alpha_0 y - 4(u^0)^{-1} \sum_{i=0}^{n+m} g^{ii} u^0_i u^\alpha_i , \\
&\qquad (n'+1 \le \alpha \le n'+m') .
\end{aligned}
\tag{3.2}
$$

Here (g_{ij}) denotes the matrix component of the metric g_M, (g^{ij}) its inverse matrix and $u^0 = \bar{y}(u)$. Note that $u^\alpha_{ij} \neq u^\alpha_{ji}$ because $[v_i, v_j] \neq 0$.

3.2. Uniqueness theorem. Let S and S' be Damek-Ricci spaces and assume that they are represented as in the previous section. We denote by $\bar{S}$ (resp. $\bar{S}'$) the Eberlein-O'Neill compactification of S (resp. S'). Then $\{y=0\} \times N$ (resp. $\{\bar{y}=0\} \times N'$) represents the ideal boundary $\partial\bar{S}$ (resp. $\partial\bar{S}'$) except a point in $\partial\bar{S}$ (resp. $\partial\bar{S}'$) (cf. [4]).

Let $u \in C^2(S, S')$ be a proper harmonic map. First, we investigate a necessary condition for the existence of a C^2-extension $u: \bar{S} \to \bar{S}'$ of u, and then prove the following uniqueness theorem.

THEOREM 3.1 (Uniqueness theorem). *Let u and w be proper harmonic maps between Damek-Ricci spaces. Suppose $u, w \in C^2(\bar{S}, \bar{S}')$ and $f := u_{|\partial\bar{S}} = w_{|\partial\bar{S}}$. If*

$$
\sum_{j=n+1}^{n+m} \sum_{\gamma=n'+1}^{n'+m'} (f^\gamma_j)^2 > 0 \qquad \text{on} \quad \partial\bar{S} ,
$$

then $u = w$ on $\bar{S}$.

To prove this theorem the following lemma plays an important role.

LEMMA 3.2 (cf. [9], [15]). *Suppose that $\omega \in C^1 A^1 S \cap C^0 A^1 \bar{S}$ is a 1-form defined on a neighborhood of $p \in \bar{S}$. Let $\omega = \sum_{i=0}^{n+m} \omega_i e^*_i$, where $\{e^*_i\}$ is the dual coframe of $\{e_i\}$. Then there exists a sequence of points $\{q_k\} \subset S$ such that $q_k \to p$ and $(y^{-1} \sum_{j=0}^{n+m} g^{jj} \omega_{jj})(q_k) \to 0$.*

Using this lemma together with $\tau(u) = 0$, we obtain the following necessary condition.

LEMMA 3.3. *Let $u \in C^2(\bar{S}, \bar{S}')$ be a proper harmonic map. Then at the boundary we* *have*

(1) $\sum_{j=0}^{n} \sum_{\beta=n'+1}^{n'+m'} (u_j^\beta)^2 = 0,$

(2) $(n+2m)(u_0^0)^4 - \sum_{j=0}^{n} \sum_{\beta=1}^{n'} (u_j^\beta)^2 (u_0^0)^2 - 2 \sum_{j=0}^{n} \sum_{\beta=n'+1}^{n'+m'} (u_{j0}^\beta)^2 - 2 \sum_{j=n+1}^{n+m} \sum_{\beta=n'+1}^{n'+m'} (u_j^\beta)^2 = 0,$

(3) $(1+n+2m) u_0^\alpha (u_0^0)^2 - \sum_{j=0}^{n} \sum_{\beta=1}^{n'} \sum_{\gamma=n'+1}^{n'+m'} a_\alpha{}^{\gamma-n'}{}_\beta u_j^\beta u_{j0}^\gamma = 0, \qquad (1 \le \alpha \le n'),$

(4) $(2+n+2m) u_0^0 u_{00}^\alpha = 0, \qquad (n'+1 \le \alpha \le n'+m').$

PROOF. Since $u \in C^2(\bar{S}, \bar{S}')$ is proper, we have $u^0 = O(y)$ and $\lim_{y\to 0} u^0 y^{-1} = u_0^0$. Multiplying by $(u^0)^3 y^{-2}$ both sides of the first equation in (3.2), let $y \to 0$. Then, by virtue of Lemma 3.2, the first term on the right hand side tends to 0. Hence we obtain (1).

The rest of the statement follows similarly if we multiply the first, second and third equations in (3.2) by $(u^0)^3 y^{-4}$, $(u^0)^2 y^{-3}$ and $u^0 y^{-3}$, respectively. □

Since $u_{0j}^\gamma = 0$ $(\gamma \ge n'+1)$ at the boundary and $ddu = 0$, we have

$$u_{j0}^\gamma = -\sum_{\mu,\nu=1}^{n'} a_\mu{}^{\gamma-n'}{}_\nu u_0^\mu u_j^\nu \qquad (0 \le j \le n,\ n'+1 \le \gamma \le n'+m'). \tag{3.3}$$

Substituting this in the equation (3) of Lemma 3.3 and adding the result in α, we get

$$(1+n+2m)(u_0^0)^2 \sum_{\alpha=1}^{n'} (u_0^\alpha)^2 + \sum_{j=0}^{n} \sum_{\gamma=n'+1}^{n'+m'} \left(\sum_{\alpha,\beta=1}^{n'} a_\alpha{}^{\gamma-n'}{}_\beta u_0^\alpha u_j^\beta \right)^2 = 0. \tag{3.4}$$

Now we can deduce the following:

PROPOSITON 3.4. *Let $u \in C^2(\bar{S}, \bar{S}')$ be a proper harmonic map. If $f := u_{|\partial \bar{S}}$ satisfies*

$$\sum_{j=n+1}^{n+m} \sum_{\gamma=n'+1}^{n'+m'} (f_j^\gamma)^2 > 0,$$

then at the boundary u must satisfy $u_0^0 > 0$, $u_0^\alpha = 0$ $(1 \le \alpha \le n'+m')$ and $u_{k0}^\beta = u_{00}^\beta = 0$ $(1 \le k \le n,\ n'+1 \le \beta \le n'+m')$.

PROOF. The assumption on f implies that the fourth term in the equation (2) of Lemma 3.3 never vanishes. Thus $u_0^0 > 0$, which implies

$$u_0^\alpha = 0, \quad u_{00}^\alpha = 0, \quad \sum_{\alpha,\beta=1}^{n'} a_\alpha{}^{\gamma-n'}{}_\beta u_0^\alpha u_j^\beta = 0$$

from the equation (4) of Lemma 3.3 and (3.4). Finally, (3.3) implies $u_{k0}^\beta = 0$. □

When S and S' are real two-dimensional hyperbolic spaces, this proposition means that a proper harmonic map must be conformal at the ideal boundary.

By Proposition 3.4 we get:

COROLLARY 3.5. *We have*

$$u_0^\alpha = O(y)\ (1 \le \alpha \le n'), \quad u_0^\alpha = o(y)\ (n'+1 \le \alpha \le n'+m').$$

We shall now give a sketch of the proof of Theorem 3.1.

If we write

$$u(y, n) = (\bar{y}(u), \bar{n}(u)), \quad w(y, n) = (\bar{y}(w), \bar{n}(w)), \quad f(n) = \bar{n}(f),$$

then

$$\begin{aligned} d(u, w) \le\ & d((\bar{y}(u), \bar{n}(u)), (\bar{y}(u), \bar{n}(f))) + d((\bar{y}(u), \bar{n}(f)), (\bar{y}(w), \bar{n}(f))) \\ & + d((\bar{y}(w), \bar{n}(f)), (\bar{y}(w), \bar{n}(w))). \end{aligned} \tag{3.5}$$

From the explicit expression for the metrics and $\bar{n}(f) = \bar{n}(u(0, n))$, we see that the first term on the right hand side of (3.5) is

$$\int_0^y \left| \frac{\partial \bar{n}}{\partial t}(u(t, n)) \right| dt \le \int_0^y \left[t^{-1} \sum_{\alpha=1}^{n'} |u_0^\alpha| + t^{-2} \sum_{\alpha=n'+1}^{n'+m'} |u_0^\alpha| \right] dt = o(1).$$

The last estimate in the above follows from Corollary 3.5. Similarly, the third term is $o(1)$. On the other hand, the second term on the right hand side of (3.5) is

$$\left| \log \frac{\bar{y}(u)}{\bar{y}(w)} \right| = \left| \log \frac{u^0(y, n)}{w^0(y, n)} \right|.$$

Since $u_0^0(0, n)$ and $w_0^0(0, n)$ are uniquely determined by f and positive, we have

$$\left| \log \frac{\bar{y}(u)}{\bar{y}(w)} \right| = \left| \log \frac{u_0^0(0, n)y + o(y)}{w_0^0(0, n)y + o(y)} \right| = o(1).$$

Therefore $d(u, w) = 0$ at the boundary. Hence the maximal principle (cf. [17]) implies Theorem 3.1.

3.3. Existence theorem. To show the existence of a solution of the Dirichlet problem at infinity, we use the heat flow method due to Li and Tam [14]. As we saw in §2, Damek-Ricci spaces are homogeneous spaces of nonpositive curvature and have positive bottom spectrum of the Laplacian. Therefore, we can apply the general existence theory [14, Theorem 5.2] if there exists a suitable initial map. Indeed, h in Proposition 3.7 can be taken as our initial map.

The decay order of the tension field of the initial map can be estimated by the following:

LEMMA 3.6. *Let* $h \in C^{2,\varepsilon}(\partial\bar{S}, \partial\bar{S}')$ $(0 < \varepsilon < 1)$. *If* h *satisfies* (1)–(4) (*with* u *replaced by* h) *in Lemma* 3.3 *at the boundary, then its tension field* $\tau(h)$ *has the following decay order as* $y \to 0$:

$$\tau^{\alpha}(h)=O(y^{1+\varepsilon})\ (0\le\alpha\le n')\,,\quad \tau^{\alpha}(h)=O(y^{2+\varepsilon})\ (n'+1\le\alpha\le n'+m')\,.$$

This lemma follows easily from the Taylor expansion of $\tau^{\alpha}(h)$ in (3.2) with respect to y.

PROPOSITION 3.7. *Suppose that $f\in C^{3,\varepsilon}(\partial\bar{S},\partial\bar{S}')$ $(0<\varepsilon<1)$ satisfies $f_j^{\gamma}=0$ $(1\le j\le n$, $n'+1\le\gamma\le n'+m')$, and $\sum_{j=n+1}^{n+m}\sum_{\gamma=n'+1}^{n'+m'}(f_j^{\gamma})^2>0$. Then there exists an $h\in C^{2,\varepsilon}(\bar{S},\bar{S}')$ such that $h=f$ at the boundary and $\|\tau(h)\|=O(y^{\varepsilon})$, where $\|\tau(h)\|$ is the norm of the tension field in the Riemannian metric.*

PROOF. Let $\phi>0$ be the unique solution of

$$(n+2m)\phi^4-\sum_{j=1}^{n}\sum_{\beta=1}^{n'}(f_j^{\beta})^2\phi^2-2\sum_{j=n+1}^{n+m}\sum_{\beta=n'+1}^{n'+m'}(f_j^{\beta})^2=0\,.$$

Since $f\in C^{3,\varepsilon}$, we have $\phi\in C^{2,\varepsilon}(\partial\bar{S})$. Set $h(y,n):=(y\phi(n),f(n))$. Then $h\in C^{2,\varepsilon}(\bar{S},\bar{S}')$. It is easy to verify that h satisfies $h_0^0=\phi$, $h_0^{\alpha}=0$ $(1\le\alpha\le n'+m')$, $h_{00}^{\beta}=0$ $(n'+1\le\beta\le n'+m')$, $h_{j0}^{\gamma}=0$ $(1\le j\le n,\ n'+1\le\gamma\le n'+m')$ and $h=f$ at the boundary. Lemma 3.6 and the explicit expression for the metric then imply $\|\tau(h)\|=O(y^{\varepsilon})$. □

Next we construct a comparison function.

LEMMA 3.8. *For sufficiently large r_0 and some constant s, define*

$$\psi(r):=\begin{cases} e^{-sr_0} & r\le r_0\\ e^{-sr} & r\ge r_0\,.\end{cases}$$

If $0<s\le n+2m$, then ψ is a superharmonic function on S. Here r is the distance function as in lemma 2.3.

PROOF. It follows from Lemma 2.3 that the Laplacian on (S,g_S) has the form

$$\Delta_S=-\frac{\partial^2}{\partial r^2}-\left(n\frac{\cosh r}{\sinh r}+2m\frac{\cosh 2r}{\sinh 2r}\right)\frac{\partial}{\partial r}+(\text{spherical part})\,.$$

Straightforward calculation then yields the lemma. □

Now we prove the main theorem of this section.

THEOREM 3.9 (Existence theorem). *Suppose that $f\in C^{3,\varepsilon}(\partial\bar{S},\partial\bar{S}')$ $(0<\varepsilon<1)$ satisfies $f_j^{\gamma}=0$ $(1\le j\le n,\ n'+1\le\gamma\le n'+m')$, and $\sum_{j=n+1}^{n+m}\sum_{\gamma=n'+1}^{n'+m'}(f_j^{\gamma})^2>0$. Then there exists a harmonic map $u:\bar{S}\to\bar{S}'$, which is continuous up to the ideal boundary and has f as boundary value.*

PROOF. For a map $v:S\times[0,\infty)\to S'$, consider the heat equation for harmonic maps

$$\begin{cases} \dfrac{\partial v}{\partial t}(x, t) = \tau(v)(x, t), & (x, t) \in S \times [0, \infty), \\ v(x, 0) = h(x), & x \in S, \end{cases}$$

where h is the map constructed in Proposition 3.7. Since the initial map h satisfies the conditions in [14, Theorem 5.2], a solution v exists globally in time t and converges to a harmonic map v_∞ as $t \to \infty$. We shall show that $u = v_\infty$ has f as the boundary value. Let ψ be a supersolution in Lemma 3.8, then (cf. [11])

$$\left(\frac{\partial}{\partial t} + \Delta_S\right)(\|v_t\|^2 - c\psi) \leq 0$$

and

$$\|v_t(\cdot, 0)\|^2 - c\psi = \|\tau(h)\|^2 - c\psi,$$

where $v_t = \partial v/\partial t$, Taking $s = 2\varepsilon$ and c sufficiently large, we may assume

$$\|v_t(\cdot, 0)\|^2 - c\psi = \|\tau(h)\|^2 - c\psi < 0.$$

Therefore the parabolic maximum principle implies

$$\|v_t(x, t)\| \leq ce^{-\varepsilon r}, \qquad \text{for all } (x, t) \in S \times [0, \infty).$$

By virtue of [14, Lemma 5.1], we have

$$\|v_t(x, t)\| \leq c_1 e^{-c_2 t}$$

for some constants c_1, $c_2 > 0$. Then for any $T > 0$

$$\begin{aligned} d_{S'}(u(x), h(x)) &\leq \int_0^\infty \|v_t(x, t)\| dt \leq \int_0^T \|v_t(x, t)\| dt + \int_T^\infty \|v_t(x, t)\| dt \\ &\leq ce^{-\varepsilon r(x)} T + c_1 e^{-c_2 T}. \end{aligned}$$

Choosing $T = \varepsilon r$, we get

$$ce^{-\varepsilon r(x)} T, \quad c_1 e^{-c_2 T} \to 0 \qquad \text{as} \quad r \to \infty.$$

Therefore, $d_{S'}(u(x), h(x)) \to 0$ as $x \to \partial\bar{S}$ and $u = h = f$ at the boundary. □

NOTE. Our result remains true, with necessary modifications, when the base or the target manifold is replaced by a real hyperbolic space.

REFERENCES

[1] K. AKUTAGAWA, Harmonic diffeomorphisms of the hyperbolic plane, Trans. Amer. Math. Soc. 342 (1994), 325–342.

[2] J. BOGGINO, Generalized Heisenberg groups and solvmanifolds naturally associated, Rend. Sem. Mat. Univ. Politec. Torino 43 (1995), 529–547.

[3] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Math. 1598, Springer Verlag, Berlin, Heidelberg, New York, 1995.

[4] M. Cowling, A. H. Dooley, A. Korányi and F. Ricci, *H*-type groups and Iwasawa decompositions, Adv. Math. 87 (1991), 1–41.

[5] E. Damek, Curvature of a semi-direct extension of a Heisenberg type nilpotent group, Colloq. Math. 53 (1987), 249–253.

[6] E. Damek, The geometry of a semi-direct extension of a Heisenberg type nilpotent group, Colloq. Math. 53 (1987), 255–268.

[7] E. Damek and F. Ricci, A class of nonsymmetric harmonic Riemannian spaces, Bull. Amer. Math. Soc. (N.S.) 27 (1992), 139–142.

[8] E. Damek and F. Ricci, Harmonic analysis on solvable extensions of *H*-type groups, J. Geom. Anal. 2 (1992), 213–248.

[9] H. Donnelly, Dirichlet problem at infinity for harmonic maps: rank one symmetric spaces, Trans. Amer. Math. Soc. 344 (1994), 713–735.

[10] I. Dotti, On the curvature of certain extensions of *H*-type groups, Proc. Amer. Math. Soc., to appear.

[11] P. Hartman, On homotopic harmonic mappings, Canad. J. Math. 19 (1967), 673–687.

[12] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, Trans. Amer. Math. Soc. 258 (1980), 147–153.

[13] M. Lanzendorf, Einstein metrics with nonpositive sectional curvature on extensions of Lie algebras of Heisenberg type, preprint. 1995.

[14] P. Li and L.-F. Tam, The heat equation and harmonic maps of complete manifolds, Invent. Math. 105 (1991), 1–46.

[15] P. Li and L.-F. Tam, Uniqueness and regularity of proper harmonic maps, Ann. of Math. 137 (1993), 167–201.

[16] P. Li and L.-F. Tam, Uniqueness and regularity of proper harmonic maps II, Indiana Univ. Math. J. 42 (1993), 591–635.

[17] R. Schoen and S. T. Yau, Compact group actions and the topology of the manifolds with nonpositive curvature, Topology 18 (1979), 361–380.

Institute of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990
Japan

E-mail address: ueno@kszaoh3.kj.yamagata-u.ac.jp