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THE DIRICHLET PROBLEM FOR MONGE-AMPÈRE EQUATIONS IN NON-CONVEX DOMAINS AND SPACELIKE HYPERSURFACES OF CONSTANT GAUSS CURVATURE

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ABSTRACT. In this paper we extend the well known results on the existence and regularity of solutions of the Dirichlet problem for Monge-Ampère equations in a strictly convex domain to an arbitrary smooth bounded domain in $\mathbb{R}^n$ as well as in a general Riemannian manifold. We prove for the nondegenerate case that a sufficient (and necessary) condition for the classical solvability is the existence of a subsolution. For the totally degenerate case we show that the solution is in $C^{1,1}(\overline{\Omega})$ if the given boundary data extends to a locally strictly convex C^2 function on $\overline{\Omega}$. As an application we prove some existence results for spacelike hypersurfaces of constant Gauss-Kronecker curvature in Minkowski space spanning a prescribed boundary.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$. In this paper we consider the Dirichlet problem for Monge-Ampère equations

$$\det(u_{ij}) = \psi(x, u, Du) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\psi \geq 0$, $Du = (u_1, \cdots, u_2)$ denotes the gradient of u, $u_i = \partial u/\partial x_i$ and $u_{ij} = \partial^2 u/\partial x_i \partial x_j$.

When Ω is a strictly convex domain, this problem has received considerable study both in the non-degenerate case ($\psi > 0$) and in the degenerate case ($\psi = 0$ somewhere). A well known theorem (see Caffarelli, Nirenberg and Spruck [6], Ivochkina [17] and Krylov [19]) states that in the non-degenerate case $\psi > 0$, (1.1) has a strictly convex solution in $C^\infty(\overline{\Omega})$, provided Ω is strictly convex and there exists a strictly convex subsolution in $C^2(\overline{\Omega})$. (Please see, for example, [6], [12] and [22] for further references, including the earlier work of, among others, Pogorelov, Cheng and Yau, and P. L. Lions.) For the degenerate case ($\psi \geq 0$), counterexamples have been found showing that the Dirichlet problem, in general, does not have a solution in $C^2(\overline{\Omega})$; whether or not the weak solutions belong to $C^{1,1}(\overline{\Omega})$ has attracted a lot of attention. In the totally degenerate case $\psi \equiv 0$, the $C^{1,1}$ regularity was established by Caffarelli, Nirenberg and Spruck [7], who proved that if Ω is a strictly convex domain with $\partial\Omega \in C^{3,1}$ and $\varphi \in C^{3,1}(\partial\Omega)$, then the unique convex solution to the degenerate problem

$$\det(u_{ij}) = 0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega \tag{1.2}$$

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belongs to $C^{1,1}(\overline{\Omega})$. Earlier Trudinger and Urbas [26] obtained local $C^{1,1}$ regularity under the weaker hypotheses that $\partial\Omega \in C^{1,1}$ and $\varphi \in C^{1,1}(\partial\Omega)$. The recent work of Krylov [20], [21] provides a unified treatment of the non-degenerate and totally degenerate cases. The main purpose of the present paper is to extend the above mentioned results to non-convex domains.

The Monge-Ampère equations are closely related to problems involving Gauss-Kronecker curvature in differential geometry, such as the Minkowski and Weyl problems. From the viewpoint of geometric applications, it is of interest to study the Dirichlet problem for Monge-Ampère equations in non-convex domains. In his book [2], T. Aubin also raised the question of whether one can remove the hypothesis of convexity of the domain for a problem. Recently, an effort to extend the results of [6] to non-convex domains was made by J. Spruck et al. in [16] and [15]. It was proved in [15] that for $\psi > 0$ the Dirichlet problem (1.1) in an arbitrary smooth domain Ω admits a locally strictly convex solution in $C^\infty(\overline{\Omega})$ provided that $(\psi(x,z,p))^{1/n}$ is a convex function with respect to p and that there exists a locally strictly convex *strict* subsolution $\underline{u} \in C^2(\overline{\Omega})$ (i.e., assuming $\underline{u}$ satisfies the strict inequality in (1.4) below). This result applies to, for example, the Gauss curvature equation

$$\det(u_{ij}) = K(x,u)(1+|Du|^2)^{\frac{n+2}{2}} \tag{1.3}$$

for hypersurfaces in Euclidean space and has interesting geometric consequences (see, for example, [15], [23]). In this paper we will prove

Theorem 1.1. *Let $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\psi > 0$. Assume there exists a locally strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega})$ satisfying*

$$\det(\underline{u}_{ij}) \geq \psi(x, \underline{u}, D\underline{u}) \quad \text{in } \Omega, \quad \underline{u} = \varphi \quad \text{on } \partial\Omega. \tag{1.4}$$

Then there exists a locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ of (1.1) with $u \geq \underline{u}$. The solution is unique if $\psi_u \geq 0$.

Here a function $v \in C^2(\overline{\Omega})$ is said to be *locally strictly convex* if its Hessian matrix $\{v_{ij}\}$ is positive definite everywhere in $\overline{\Omega}$. Obviously, condition (1.4) in Theorem 1.1 cannot be removed even when Ω is strictly convex. In case $\psi \equiv \psi(x)$ or, more generally (due to P. L. Lions; see [6]), when ψ satisfies

$$0 < \psi(x,z,p) \leq C(1+|p|^2)^{n/2} \quad \text{for } x \in \overline{\Omega}, \ \ z \leq \max\varphi, \ \ p \in \mathbb{R}^n,$$

one can construct a strictly convex subsolution if Ω is strictly convex; this fails for non-convex Ω.

There has been some recent work on the Dirichlet problem for Monge-Ampère equations on general smooth Riemannian manifolds. In [14], Y. Y. Li and the author established an analogue of the result of Guan-Spruck [15] cited above. Some of the result was also obtained independently by A. Atallah and C. Zuily [1]. Building on [14] we will, in Section 5, extend Theorem 1.1 to general Riemannian manifolds.

The totally degenerate problem (1.2) is important to understanding hypersurfaces of vanishing Gauss-Kronecker curvature. It is of interest to study the regularity of its solution in non-convex domains. Applying Theorem 1.1 to the problem

$$\det(u_{ij}) = \varepsilon \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega \tag{1.5}$$

for sufficiently small $\varepsilon > 0$, by approximation we find that (1.2) has a unique locally convex weak solution (in the sense of Alexandrov, which is the same as weak solution in the viscosity sense; see [4]) in $C^{0,1}(\overline{\Omega})$, provided φ extends to a locally strictly convex function in $C^2(\overline{\Omega})$. We will prove that this solution actually belongs to $C^{1,1}(\overline{\Omega})$. More precisely, we have the following extension of the regularity theorem in [7].

Theorem 1.2. *Assume $\partial\Omega$ is in $C^{3,1}$ and $\varphi \in C^{3,1}(\partial\Omega)$. Suppose there exists a locally strictly convex function $\underline{u} \in C^2(\overline{\Omega})$ with $\underline{u} = \varphi$ on $\partial\Omega$. Then there is a unique locally convex weak solution of (1.2) in $C^{1,1}(\overline{\Omega})$.*

As we have observed, a major motivation for our studying the Dirichlet problem (1.1) in non-convex domains comes from its close connection with geometric problems. In this paper we consider one such problem, which concerns existence of spacelike hypersurfaces of constant Gauss-Kronecker curvature with specified boundary in Minkowski space $\mathbb{R}^{n,1}$. (Please see [15], [13] for related results for hypersurfaces in Euclidean space, and [23] in hyperbolic space.) We are interested in the following question: given a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of codimension-two closed smooth submanifolds of $\mathbb{R}^{n,1}$, decide whether there exists a spacelike hypersurface M of constant Gauss-Kronecker curvature with boundary $\partial M = \Gamma$. Locally M is given as the graph of a function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, satisfying the spacelike condition $|Du| < 1$ and the Monge-Ampère equation (1.1) with

$$\psi(x, u, Du) = \mathcal{K}(1 - |Du|^2)^{\frac{n+2}{2}}, \tag{1.6}$$

where $\mathcal{K}$ is the Gauss-Kronecker curvature of M. We note that the right hand side of (1.6), in contrast with that of (1.3), is not a convex function with respect to the gradient Du.

Theorem 1.3. *Suppose Γ bounds a compact C^2 spacelike locally strictly convex hypersurface $\tilde{M}$. Then for any constant $0 \le K \le \min_{q \in \tilde{M}} \mathcal{K}[\tilde{M}](q)$, where $\mathcal{K}[\tilde{M}]$ denotes the Gauss-Kronecker curvature of $\tilde{M}$, there exists a compact spacelike hypersurface M_K of constant Gauss-Kronecker curvature K with boundary $\partial M_K = \Gamma$. Moreover, M_K is C^∞ for $K > 0$ and M_0 is $C^{1,1}$.*

The corresponding problem for spacelike hypersurfaces with prescribed boundary value and mean curvature was treated by R. Bartnik and L. Simon [3]; see also [10] and references therein for related results.

This paper is organized as follows. In Section 2 we first derive *a priori* estimates for the C^2 norms of the desired solutions of (1.1) for the non-degenerate case. Then Theorem 1.1 is proved using the continuity method and degree theory based on these *a priori* estimates. Section 3 contains the proof of Theorem 1.2. Theorem 1.3 is proved in Section 4 as a consequence of more general theorems presented there. We note that since in general the hypersurface M_K is not globally a graph over a domain in $\mathbb{R}^n$, Theorem 1.3 does not follow directly from Theorems 1.1 and 1.2. To overcome this difficulty, we will reformulate the problem in a more general setting and appeal for its proof to Theorem 5.1, the extension of Theorem 1.1 to general Riemannian manifolds, which is proved in Section 5.

2. The non-degenerate Monge-Ampère equations

In this section we prove Theorem 1.1 using the method of continuity and degree theory. As usual, the proof is based on the establishment of global $C^{2,\alpha}$ *a priori* estimates for prospective solutions. A somewhat surprising fact to us is that in deriving these estimates one only needs the assumption that $\underline{u}$ is a locally strictly convex C^2 function; it does not necessarily satisfy (1.4). As we shall see, condition (1.4) is needed in the proof of Theorem 1.1 only to guarantee that, when $\psi_u \geq 0$, the unique solution u of (1.1) satisfies $u \geq \underline{u}$ in Ω by the maximum principle.

2.1. **A priori estimates.** In this subsection we only assume $\underline{u} \in C^2(\overline{\Omega})$ is a locally strictly convex function; thus there exists a constant $\varepsilon > 0$ (without loss of generality, we may assume $\varepsilon \leq 1$) such that

$$\{\underline{u}_{ij}\} \geq \varepsilon \{\delta_{ij}\} \quad \text{on } \overline{\Omega}. \tag{2.1}$$

Set

$$\mathcal{A} = \{w \in C^\infty(\overline{\Omega}) : \{w_{ij}\} > 0, w \geq \underline{u}, w|_{\partial\Omega} = \varphi\}.$$

As in [15] it is easy to see that

$$|w| + |Dw| \leq K_1, \quad \text{for any } w \in \mathcal{A}, \tag{2.2}$$

where the constant K_1 depends only on Ω, n, and $\|\underline{u}\|_{C^1(\overline{\Omega})}$. From (2.2) we have

$$0 < \psi_0 \equiv \inf_{x\in\overline{\Omega}, w\in\mathcal{A}} \psi(x, w(x), Dw(x)) \leq \sup_{x\in\overline{\Omega}, w\in\mathcal{A}} \psi(x, w(x), Dw(x)) \equiv \psi_1 < \infty. \tag{2.3}$$

Theorem 2.1. *Let $u \in \mathcal{A}$ be a solution of (1.1). Then*

$$|D^2u| \leq C \quad \text{on } \overline{\Omega}. \tag{2.4}$$

Here the constant C depends on Ω, n, ε, K_1, ψ_0, ψ_1, $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$, $\|\psi\|_{C^2(\overline{\Omega})}$ and $\|\underline{u}\|_{C^2(\overline{\Omega})}$.

Proof. It is shown in [6] how to derive (2.4) from C^2 estimates on the boundary. Thus we need only estimate D^2u on $\partial\Omega$. Consider any point 0 on $\partial\Omega$; we may assume it is the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at 0. Near the origin, $\partial\Omega$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}). \tag{2.5}$$

Since $u - \underline{u} = 0$ on $\partial\Omega$,

$$(u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_n(0)B_{\alpha\beta}, \quad \alpha, \beta < n. \tag{2.6}$$

It follows that

$$|u_{\alpha\beta}(0)| \leq C, \quad \alpha, \beta < n. \tag{2.7}$$

Next we estimate the mixed normal-tangential derivative $u_{\alpha n}(0)$. Rewrite equation (1.1) in the form

$$\log\det(u_{ij}) = \log\psi(x, u, Du) \equiv f(x, u, Du) \tag{2.8}$$

and let $\mathcal{L}$ denote the linear operator defined by

$$\mathcal{L}w = u^{ij}w_{ij} - f_{p_i}(x, u, Du)w_i \quad \text{for } w \in C^2(\overline{\Omega}), \tag{2.9}$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$ and $f_{p_i} = f_{p_i}(x, u, p)$. For fixed $\alpha < n$ consider the operator

$$T = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta\partial_n - x_n\partial_\beta).$$

As in [6] we have

$$|\mathcal{L}T(u-\underline{u})| \le C(1+\sum u^{ii}). \tag{2.10}$$

Since $|Du| \le K_1$, $|T(u-\underline{u})| \le C$ in $\overline{\Omega}$. Moreover, on $\partial\Omega$ near the origin

$$|T(u-\underline{u})| \le C|x|^2. \tag{2.11}$$

We will employ a barrier function of the form

$$v = (u-\underline{u}) + t(h-\underline{u}) - Nd^2, \tag{2.12}$$

where h is the harmonic function in Ω with $h|_{\partial\Omega} = \varphi$, d is the distance function from $\partial\Omega$, and t, N are positive constants to be determined. We may take $\delta > 0$ small enough so that d is smooth in $\Omega_\delta = \Omega \cap B_\delta(0)$. The key ingredient is the following:

Lemma 2.2. *For N sufficiently large and t, δ sufficiently small,*

$$\mathcal{L}v \le -\frac{\varepsilon}{4}(1+\sum u^{ii}) \quad \text{in } \Omega_\delta, \quad v \ge 0 \quad \text{on } \partial\Omega_\delta.$$

Proof. By (2.1) we have $u^{ij}(u_{ij} - \underline{u}_{ij}) \le n - \varepsilon \sum u^{ii}$. It follows that

$$\mathcal{L}(u-\underline{u}) \le C_0 - \varepsilon \sum u^{ii}. \tag{2.13}$$

Next, since $\Delta\underline{u} \ge n\varepsilon > 0$,

$$(h-\underline{u})(x) \ge c_0 d(x), \quad \text{for } x \in \Omega$$

for some uniform constant $c_0 > 0$. Moreover, we have

$$\mathcal{L}(h-\underline{u}) \le C_1(1+\sum u^{ii}),$$

for some constant $C_1 > 0$ under control. Thus

$$\mathcal{L}v \le C_0 + tC_1 + (tC_1 - \varepsilon)\sum u^{ii} - 2N(d\mathcal{L}d + u^{ij}d_i d_j) \quad \text{in } \Omega_\delta.$$

It is easy to see that

$$\mathcal{L}d \ge -C_2(1+\sum u^{ii}).$$

Furthermore, since $\{u^{ij}\}$ is positive definite and $d_n(0) = 1$, $d_\beta(0) = 0$ for all $\beta < n$, we have, for δ sufficiently small,

$$u^{ij}d_i d_j \ge u^{nn}d_n^2 + 2\sum_{\beta<n} u^{n\beta}d_n d_\beta \ge \frac{u^{nn}}{2} - C_3\delta\sum u^{ii} \quad \text{in } \Omega_\delta. \tag{2.14}$$

Let $\lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of $\{u_{ij}\}$. We have $\sum u^{ii} = \sum \lambda_i^{-1}$ and $u^{nn} \ge \lambda_n^{-1}$. By the inequality for arithmetic and geometric means,

$$\frac{\varepsilon}{4}\sum u^{ii} + Nu^{nn} \ge \frac{n\varepsilon}{4}(N\lambda_1^{-1}\cdots\lambda_n^{-1})^{\frac{1}{n}} \ge \frac{n\varepsilon}{4(\psi_1)^{1/n}}N^{\frac{1}{n}} \equiv c_1 N^{\frac{1}{n}}.$$

Now we fix $t > 0$ sufficiently small so that $tC_1 \le \frac{\varepsilon}{4}$ and fix N so that $c_1 N^{1/n} \ge C_0 + \varepsilon$. We obtain

$$\mathcal{L}v \le -\frac{\varepsilon}{4}(1+\sum u^{ii}) \quad \text{in } \Omega_\delta$$

if we require δ to satisfy $2(C_2+C_3)N\delta \le \frac{\varepsilon}{4}$ in Ω_δ.

It remains to examine the value of v on $\partial\Omega_\delta$. On $\partial\Omega \cap B_\delta(0)$ we have $v = 0$. On $\Omega \cap \partial B_\delta(0)$,

$$v \geq tc_0 d - Nd^2 \geq (tc_0 - N\delta)d \geq 0,$$

if we require, in addition, $N\delta \leq tc_0$. Now we can fix δ sufficiently small to complete the proof of Lemma 2.2. □

Using Lemma 2.2 we can choose $A \gg B \gg 1$ so that

$$\mathcal{L}(Av + B|x|^2 \pm T(u - \underline{u})) \leq 0 \quad \text{in} \ \ \Omega_\delta$$

and

$$Av + B|x|^2 \pm T(u - \underline{u}) \geq 0 \quad \text{on} \ \ \partial\Omega_\delta$$

by (2.10) and (2.11). It thus follows from the maximum principle that

$$|u_{\alpha n}(0)| \leq C. \tag{2.15}$$

Finally, the tangential strict convexity of u has been established in [15]; i.e.

$$\sum_{\alpha,\beta<n} w_{\alpha\beta}(0)\xi_\alpha\xi_\beta \geq c_0 > 0 \tag{2.16}$$

for any unit vector $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Thus it follows as in [6] that

$$|u_{nn}(0)| \leq C. \tag{2.17}$$

The proof of Theorem 2.1 is complete. □

From the Evans-Krylov theory (see [11], [24], [18] and [5]) we thus have an *a priori* bound for the $C^{2,\alpha}$ norm of u,

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \quad 0 < \alpha < 1, \tag{2.18}$$

with constant C depending only on Ω, n, ε, ψ_0, ψ_1, $\|\varphi\|_{C^{3,1}(\overline{\Omega})}$, $\|\psi\|_{C^3(\overline{\Omega})}$ and $\|\underline{u}\|_{C^2(\overline{\Omega})}$. By the standard Schauder theory, we thus obtain the *a priori* bound for each integer $k \geq 3$:

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \leq C. \tag{2.19}$$

With the aid of such estimates we can apply the continuity method and degree theory to prove Theorem 1.1 as in [6] with some modifications.

2.2. **Proof of Theorem 1.1.** We first assume that the subsolution $\underline{u}$ is in $C^\infty(\overline{\Omega})$ and prove the existence of a solution to (1.1) in $\mathcal{A}$ in two steps as follows.

(a) The special case: $\psi_u \geq 0$. For each fixed $t \in [0, 1]$ consider the Dirichlet problem

$$\det(u_{ij}) = t\psi(x, u, Du) + (1 - t)\det(\underline{u}_{ij}) \quad \text{in} \ \ \Omega, \quad u = \varphi \quad \text{on} \ \ \partial\Omega. \tag{2.20}$$

Note that since $\underline{u}$ is a subsolution of (2.20), it follows from the maximum principle that any locally strictly convex solution $u \in C^\infty(\overline{\Omega})$ of (2.20) satisfies $u \geq \underline{u}$ and hence (2.19), independent of t. Thus we can utilize the continuity method to show that for each $t \in [0, 1]$ there exists a locally strictly convex solution of (2.20) in $C^\infty(\overline{\Omega})$. The uniqueness follows from the maximum principle.

(b) Turning to the general case, we assume $\underline{u}$ is not a solution of (1.1) and let $u^0 \in \mathcal{A}$ be the unique solution of

$$\det(u_{ij}) = \psi_0 \quad \text{in} \ \ \Omega, \quad u = \varphi \quad \text{on} \ \ \partial\Omega, \tag{2.21}$$

where ψ_0 is as in (2.3). The existence of u^0 follows from part (a).

For $r > 0$ we consider the open convex set of functions in $C^5(\overline{\Omega})$

$$\mathcal{C}_r = \left\{ v \in C^5(\overline{\Omega}) : \|v\|_{C^5(\overline{\Omega})} < r,\ v > 0 \text{ in } \Omega,\ v|_{\partial\Omega} = 0 \text{ and } v_\nu > 0 \text{ on } \partial\Omega \right\}.$$

where ν is the unit interior normal to $\partial\Omega$. We want to prove that for each $t \in [0,1]$ there exists a solution in $\mathcal{A}$ of the form

$$u = \underline{u} + v, \quad v \in \mathcal{C}_r \text{ with } r \text{ sufficiently large,} \tag{2.22}$$

to the Dirichlet problem

$$\det(u_{ij}) = t\psi(x,u,Du) + (1-t)\psi_0 \equiv \psi^t(x,u,Du) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega. \tag{2.23}$$

As we observed, any solution $u \in \mathcal{A}$ of (2.23) satisfies the *a priori* bound

$$\|u\|_{C^5(\overline{\Omega})} \le C \quad \text{independent of } t. \tag{2.24}$$

Moreover, by the maximum principle and the Hopf lemma we have

$$u > \underline{u} \text{ in } \Omega \text{ and } (u - \underline{u})_\nu > 0 \text{ on } \partial\Omega. \tag{2.25}$$

Thus we can choose r sufficiently large so that (2.23) has no solution in $\mathcal{A}$ of the form (2.22) with $v \in \partial\mathcal{C}_r$, the boundary of $\mathcal{C}_r$.

Now for $0 \le t \le 1$ and fixed $v \in \overline{\mathcal{C}_r}$ consider the Dirichlet problem

$$\det(u_{ij}) = \psi^t(x,u,Du)e^{t\Lambda(u-\underline{u}-v)} \equiv \eta^t(x,u,Du) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega, \tag{2.26}$$

where

$$\Lambda = \frac{1}{\psi_0} \sup_{x\in\overline{\Omega}} \sup_{u\in\mathcal{A}} \psi_u(x,u,Du) < \infty, \quad \psi_0 \text{ as in (2.3).}$$

We observe that $\underline{u}$ is a subsolution of (2.26) and ${\eta^t}_u \ge 0$. Thus by part (a) there exists a unique solution $u^t \in \mathcal{A}$ for each $t \in [0,1]$. For $t = 0$, this solution is our u^0.

From elliptic theory, the map $T^t v = u^t - \underline{u}$ is compact in C^5. On the other hand, we have seen that there are no solutions of

$$v - T^t v = 0 \tag{2.27}$$

on the boundary of $\mathcal{C}_r$. Thus the degree

$$\deg(I - T^t, \mathcal{C}_r, 0) = \gamma \tag{2.28}$$

is well defined and independent of t. For $t = 0$, (2.27) has a unique solution $v^0 = u^0 - \underline{u}$. By the maximum principle, when $t = 0$ the linearized operator of (2.23), linearized at u^0, is invertible. Thus v^0 is a regular point of $I - T^0$. Consequently $\gamma = \pm 1$, and (2.27) has a solution $v^t \in \mathcal{C}_r$ for all $0 \le t \le 1$. The function $u^1 = \underline{u} + v^1$ is then a solution of (1.1). The elliptic regularity theory implies that $u^1 \in C^\infty(\overline{\Omega})$.

To finish the proof, we have to consider the case $\underline{u} \in C^2(\overline{\Omega})$. We may take a sequence of locally strictly convex functions $\underline{u}^m \in C^\infty(\overline{\Omega})$ converging to $\underline{u}$ in $C^2(\overline{\Omega})$, such that

$$\det(\underline{u}^m_{ij}) \ge \frac{m}{m+1}\psi(x, \underline{u}^m, D\underline{u}^m) \quad \text{in } \overline{\Omega}.$$

For each integer $m \ge 1$, set

$$\psi^m = \frac{m}{m+1}\psi, \quad \varphi^m = \underline{u}^m|_{\partial\Omega}$$

and consider the Dirichlet problem

$$\det(u_{ij}) = \psi^m \quad \text{in } \overline{\Omega}, \quad u = \varphi^m \quad \text{on } \partial\Omega. \tag{2.29}$$

Note that $\underline{u}^m$ is a subsolution of (2.29). Consequently, there exists a locally strictly convex solution $u^m \in C^\infty(\overline{\Omega})$ of (2.29) satisfying the *a priori* estimates

$$\|u^m\|_{C^k(\overline{\Omega})} \le C(m,k) \quad \text{for every } k \ge 1,$$

where the constant $C(m,k)$ only depends on, besides other known data, $\underline{u}^m$ and its derivatives up to second order. It follows that a subsequence of u^m converges to a solution of (1.1) in $C^\infty(\overline{\Omega})$. This completes the proof of Theorem 1.1.

3. The totally degenerate Monge-Ampère equation

The main purpose of this section is to prove Theorem 1.2. We will first obtain a weak solution in $C^{0,1}(\overline{\Omega})$ of the totally degenerate Monge-Ampère equation by approximation. The major part of the section is devoted to the proof of the $C^{1,1}$ regularity of this weak solution.

For each small constant $\lambda > 0$, we may first take smooth approximations of Ω and $\underline{u}$ and apply Theorem 1.1, then pass to the limit to obtain, under the hypothesis of Theorem 1.2, a locally strictly convex function $u^\lambda \in C^2(\overline{\Omega})$ satisfying

$$\det(u^\lambda_{ij}) = \lambda \quad \text{in } \Omega, \quad u^\lambda = \varphi \quad \text{on } \partial\Omega \tag{3.1}$$

and the C^1 estimate

$$\|u^\lambda\|_{C^1(\overline{\Omega})} \le C_0 \tag{3.2}$$

for some constant $C_0 > 0$ independent of λ. Therefore, there exists a sequence u^{λ_k} that converges to a locally convex function $u \in C^{0,1}(\overline{\Omega})$, and

$$\|u\|_{C^{0,1}(\overline{\Omega})} \le C_0. \tag{3.3}$$

By the maximum principle, u is the unique locally convex solution of (1.2). In the rest of this section we will modify the proof of [7] to show that u is in $C^{1,1}(\overline{\Omega})$.

We say a subset U of $\overline{\Omega}$ is relatively convex in $\overline{\Omega}$ if any segment contained in $\overline{\Omega}$ with endpoints in U lies completely in U. The relative convex hall, denoted by $\Gamma_{\overline{\Omega}}(U)$, of a set $U \subset \overline{\Omega}$ is the smallest relatively convex set in $\overline{\Omega}$ containing U.

Lemma 3.1. *Assume $x^0 \in \Omega$ is such that $u(x^0) = 0$ and $u \ge 0$ near x^0. Let S_{x^0} be the component of $\{x \in \overline{\Omega} : u(x) = 0\}$ containing x^0. Then $S_{x^0} = \Gamma_{\overline{\Omega}}(S_{x^0} \cap \partial\Omega)$.*

Proof. By the local convexity of u, we see that S_{x^0} is relatively convex in $\overline{\Omega}$. Thus we only have to show that $S_{x^0} \subset \Gamma_{\overline{\Omega}}(S_{x^0} \cap \partial\Omega)$. If not, there is a point, which we may assume to be the origin after translation and rotation of coordinates, in $\Omega \cap S_{x^0}$ such that 0 is the only point in $S_{x^0} \cap B_{\delta_0}(0)$ that lies in the half space $\{x_n \ge 0\}$, for some small $\delta_0 > 0$. Therefore, there is a constant $\delta_1 > 0$ small such that $u \ge a$ on $\partial B_{\delta_0}(0) \cap \{x_n \ge -\delta_1\}$ for some constant $a > 0$. But then the function

$$v = \delta_2(\delta_1 + 2x_n + \delta_3|x|^2) \quad \text{for } \delta_2, \delta_3 > 0 \text{ small}$$

satisfies

$$\det(v_{ij}) = 2\delta_2\delta_3 > 0 \quad \text{in } U, \quad v \le u \quad \text{on } \partial U,$$

where $U \equiv B_{\delta_0}(x^0) \cap \{x_n > -\delta_1\}$. Consequently, $u(0) \ge v(0) = \delta_1\delta_2 > 0$ by the maximum principle. This contradicts the fact that $u(0) = 0$. ∎

In this section a constant is said to be under control if it depends only on Ω, $\|\varphi\|_{C^{3,1}(\partial\Omega)}$ and $\underline{u}$ (up to its second derivatives).

Theorem 3.2. *There is a constant C under control such that for every $x^0 \in \Omega$ there exist $\delta(x^0) > 0$ and $V(x^0) \in \mathbb{R}^n$ so that for every $x \in \Omega$ with $|x - x^0| \le \delta(x^0)$, we have*

$$|u(x) - u(x^0) - (x - x^0) \cdot V(x^0)| \le C|x - x^0|^2. \tag{3.4}$$

We note that Theorem 3.2 implies that u is differentiable and $Du(x^0) = V(x^0)$. Thus it follows from (3.3) that

$$|Du| \le C_0 \quad \text{on} \quad \overline{\Omega}. \tag{3.5}$$

According to [7], Theorem 3.2 then implies that $u \in C^{1,1}(\overline{\Omega})$; namely, for some constant C under control,

$$|Du(x) - Du(y)| \le C|x - y| \quad \text{for all} \quad x, y \in \Omega. \tag{3.6}$$

Proof of Theorem 3.2. For any fixed point $x^0 \in \Omega$, since u is locally convex, there exists an affine function p such that $p(x^0) = u(x^0)$ and $p \le u$ near x^0. Let $V(x^0) = Dp(x^0)$; (3.4) is then equivalent to

$$|u(x) - p(x)| \le C|x - x^0|^2. \tag{3.7}$$

Without loss of generality, we may suppose $p \equiv 0$ and, therefore, $u(x^0) = 0$ and $u \ge 0$ near x^0. By Lemma 3.1, x^0 then lies in a simplex $S \subset \{x \in \overline{\Omega} | u(x) = 0\}$ of dimension $k \le n$ with vertices on $\partial\Omega$. According to [7], we only have to consider the case $k = 1$. So we assume S is a segment with end points $x^1, x^2 \in \partial\Omega$. By the local convexity of u we have $u \ge 0$ in a neighborhood of S.

Of the two end points of S, suppose x^2 is closer to x^0. We may assume $x^2 = 0$. After rotation of coordinates, we suppose the positive x_n-axis is interior normal to $\partial\Omega$ at 0 and $x^0 = (x_1^0, 0, \cdots, 0, x_n^0)$ with $x_1^0 \ge 0$, $x_n^0 = x_1^0 \tan\theta$, $0 \le \theta \le \frac{\pi}{2}$. Near the origin, $\partial\Omega$ is represented as a graph

$$x_n = \rho(x') = \frac{1}{2} \sum_{i,j<n} \rho_{ij}(0) x_i x_j + O(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}). \tag{3.8}$$

Lemma 3.3 below implies that if θ is sufficiently small, x^1 falls in that piece of $\partial\Omega$ given by (3.8). Set $\xi = (\cos\theta, 0, \cdots, 0, \sin\theta) \in \mathbb{R}^n$.

Lemma 3.3. *For some constants c_1, C_1 under control,*

$$0 < c_1\theta \le |x^1| \le C_1\theta. \tag{3.9}$$

Proof. We have

$$\underline{u}(x^1) = \underline{u}(0) + \underline{u}_\xi(0)|x^1| + \underline{u}_{\xi\xi}(tx^1)|x^1|^2, \quad \text{for some } 0 < t < 1.$$

But $\underline{u}(0) = \underline{u}(x^1) = 0$ and $\underline{u}_\xi(0) = \underline{u}_n(0)\sin\theta$, since $\underline{u}_i(0) = 0$ for $1 \le i \le n-1$; thus

$$\underline{u}_{\xi\xi}(tx^1)|x^1| = -\underline{u}_n(0)\sin\theta,$$

and (3.9) follows from the local strict convexity of $\underline{u}$ and the comparison principle. □

Lemma 3.3 also implies that $\theta > 0$; that is, S cannot be tangential to $\partial\Omega$ at 0.

Lemma 3.4. *There exist uniform positive constants ε_0 and θ_0 sufficiently small so that if $\theta \le \theta_0$ then*

$$\rho_{11}(0) \ge \varepsilon_0 > 0.$$

Proof. For any $\mu > 0$, there exists $u^\lambda \in C^{2,1}(\overline{\Omega})$ satisfying (3.1) for some $\lambda > 0$ such that

$$\|u^\lambda - u\|_{C^{0,1}(\overline{\Omega})} \le \mu.$$

Since $u(0) = u^\lambda(0) = 0$, it follows that

$$0 \le u(0,t) - u^\lambda(0,t) \le t\mu, \quad \text{for any } t > 0 \text{ with } (0,t) \in \Omega.$$

But $u(0,t) \ge 0$ for all t sufficiently small; thus

$$u_n^\lambda(0) \ge -\mu. \tag{3.10}$$

As in the proof of Lemma 3.3, we have

$$u_{\xi\xi}^\lambda(tx^1)|x^1| = -u_n^\lambda(0)\sin\theta, \quad \text{for some } 0 < t < 1,$$

since $u^\lambda(x^1) = u^\lambda(0) = 0$ and $u_\xi^\lambda(0) = u_n^\lambda(0)\sin\theta$. Since $u^\lambda \in C^2(\overline{\Omega})$, it follows from Lemma 3.3 that

$$|u_{\xi\xi}^\lambda(tx^1) - u_{\xi\xi}^\lambda(0)| < \mu, \quad \text{when } \theta \text{ is sufficiently small.}$$

By (3.10) and Lemma 3.3 we thus obtain

$$u_{\xi\xi}^\lambda(0) \le C_2\mu, \quad \text{for } \theta \text{ sufficiently small.} \tag{3.11}$$

Next,

$$u_{\xi\xi}^\lambda(0) = u_{11}^\lambda(0)\cos^2\theta + u_{nn}^\lambda(0)\sin^2\theta + 2u_{1n}^\lambda(0)\sin\theta\cos\theta.$$

We have $u_{nn}^\lambda(0) > 0$ and, from the proof of (2.15) in Section 2, $|u_{1n}^\lambda(0)| \le C$ for some constant C independent of λ. Thus, by (3.11),

$$u_{11}^\lambda(0) \le C_3\mu + C_4\theta, \quad \text{for } \theta \text{ sufficiently small.} \tag{3.12}$$

Finally, it follows from (see (2.6)) $\underline{u}_{11}(0) - u_{11}^\lambda(0) = (u_n^\lambda(0) - \underline{u}_n(0))\rho_{11}(0)$ and (3.2), (3.12) that

$$\rho_{11}(0) \ge c_0(\underline{u}_{11}(0) - C_3\mu - C_4\theta),$$

where $c_0 > 0$ is a uniform constant. By the strict convexity of $\underline{u}$ we can first fix μ small, then choose θ_0 sufficiently small to complete the proof of Lemma 3.4. □

Returning to the proof of Theorem 3.2, we first consider the case $\theta \le \theta_0$, where θ_0 is fixed such that Lemma 3.4 holds for some $\varepsilon_0 > 0$. To set up notation we fix a positive constant r_0 depending only on Ω such that $\Gamma \equiv \{(x', \rho(x')) : |x'| \le r_0\} \subset \partial\Omega$; by Lemma 3.3 we may assume θ_0 is sufficiently small so that $x^1 \in \Gamma$. As in [7], using Lemma 3.3 and the hypothesis that $\varphi \in C^{3,1}(\partial\Omega)$, one can prove that

$$|\varphi_{11}(0)| \le A\theta^2, \quad |\varphi_{1j}(0)| \le A\theta, \quad |\varphi_{ij}(0)| \le A, \quad 1 < i,j \le n-1, \tag{3.13}$$

where A is a constant under control.

It follows from Lemma 3.4 that for any point $x \in \Omega$ with $|x - x^0| \le \delta$ sufficiently small (depending on ε_0 and possibly on x^0), the ray from x^1 to x strikes Γ at a point $\bar{x} = (\bar{x}', \bar{x}_n)$ with $\bar{x}' = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{n-1})$ satisfying

$$\theta^2(\bar{x}_1)^2 + (\bar{x}_2)^2 + \cdots + (\bar{x}_{n-1})^2 \le C|x - x^0|^2 \tag{3.14}$$

for a constant C under control (here we also use the fact that $|x^1 - x^0| \geq |x^0|$). Since $u(x^1) = 0$ it follows from the local convexity of u that

$$u(x) \leq u(\bar{x}) = \varphi(\bar{x}') = \sum_{i,j=1}^{n-1} \varphi_{ij}\bar{x}_i\bar{x}_j + O(|\bar{x}'|^3). \tag{3.15}$$

Consequently by (3.13) and (3.14),

$$u(x) \leq C\left(A + \frac{\delta}{\theta^3}\right)|x - x^0|^2,$$

with C under control. Now we may fix $\delta = \delta(x^0) \leq \theta^3$ to obtain (3.7) for $\theta \leq \theta_0$.

The case $\theta > \theta_0$ is simple, and we refer the reader to [7] for the details.

The proof of Theorem 3.2 is complete, and thus so is that of Theorem 1.2. □

4. Spacelike hypersurfaces of prescribed Gauss curvature

Recall that Minkowski space $\mathbb{R}^{n,1}$ is the space $\mathbb{R}^n \times \mathbb{R}$ endowed with the Lorentz metric $ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2$, where $x = (x_1, \ldots, x_n)$ and x_{n+1} are the coordinates in $\mathbb{R}^n$ and $\mathbb{R}$. A spacelike hypersurface of $\mathbb{R}^{n,1}$ is a codimension-one submanifold whose induced metric is Riemannian. Locally a spacelike hypersurface M is given as the graph of a function $x_{n+1} = u(x)$ satisfying the spacelike condition $|Du| < 1$. (We will also denote the hypersurface by u when it is globally given as the graph of u). The first and second fundamental forms of M are given respectively by

$$g_{ij} = \delta_{ij} - u_iu_j, \quad A_{ij} = \frac{u_{ij}}{(1 - |Du|^2)^{\frac{1}{2}}}.$$

We say M is a *locally strictly convex* hypersurface if its second fundamental form is positive definite everywhere. The Gauss-Kronecker curvature of M has the expression

$$\mathcal{K}[M] = \frac{\det(u_{ij})}{(1 - |Du|^2)^{\frac{n+2}{2}}}.$$

Thus the equation

$$\det(u_{ij}(x)) = K(x, u(x))(1 - |Du(x)|^2)^{\frac{n+2}{2}} \tag{4.1}$$

locally describes spacelike hypersurfaces with prescribed Gauss-Kronecker curvature K. As an immediate consequence of Theorem 1.1, we first state an existence result for spacelike graphs with prescribed boundary value and Gauss-Kronecker curvature. (By a graph in $\mathbb{R}^{n,1}$, we always mean a submanifold, with or without boundary, that can be represented globally as the graph of a function $x_{n+1} = u(x)$ defined in a subset of $\mathbb{R}^n$.)

Theorem 4.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$. For given $\varphi \in C^\infty(\partial\Omega)$ and $K \in C^\infty(\overline{\Omega} \times \mathbb{R})$, $K > 0$, suppose there exists a spacelike locally strictly convex hypersurface $\underline{u} \in C^2(\overline{\Omega})$ with $\mathcal{K}[\underline{u}](x) \geq K(x, \underline{u}(x))$ for $x \in \overline{\Omega}$ and $\underline{u}|_{\partial\Omega} = \varphi$. Then there exists a spacelike locally strictly convex hypersurface $u \in C^\infty(\overline{\Omega})$ with prescribed boundary value $u|_{\partial\Omega} = \varphi$ and Gauss-Kronecker curvature $\mathcal{K}[u](x) = K(x, u(x))$ for $x \in \overline{\Omega}$, i.e., u satisfies (4.1) in Ω and the gradient bound*

$$|Du| < 1 \quad \text{on } \overline{\Omega}. \tag{4.2}$$

Proof. Note that since $\underline{u}$ is a locally strictly convex subsolution of (4.1), Theorem 4.1 will follow from Theorem 1.1 once the *a priori* gradient bound (4.2) is established. By the local convexity of u we see that $|Du|$ attains its maximum value on $\partial\Omega$. Next, we want to show that

$$\max_{\partial\Omega}|Du| \leq \max_{\partial\Omega}|D\underline{u}|, \tag{4.3}$$

which implies (4.2) since $\underline{u}$ is a spacelike hypersurface.

Let γ denote the interior normal vector to $\partial\Omega$ and let $\xi \in \mathbb{R}^n$ be a unit vector. Consider an arbitrary point $\bar{x} \in \partial\Omega$. If $\xi \cdot \gamma(\bar{x}) \leq 0$, then

$$u_\xi(\bar{x}) \leq \underline{u}_\xi(\bar{x}) \leq |D\underline{u}(\bar{x})|, \tag{4.4}$$

since $u \geq \underline{u}$ in Ω and $u = \underline{u}$ on $\partial\Omega$. Now suppose $\xi \cdot \gamma(\bar{x}) > 0$, and let $y \in \partial\Omega$ be the first point where the ray $\bar{x} + t\xi$, $t > 0$, touches $\partial\Omega$. Then we have

$$u_\xi(\bar{x}) \leq u_\xi(y) \leq \underline{u}_\xi(y) \leq |D\underline{u}(y)|. \tag{4.5}$$

The first inequality follows from the local convexity of u, the second from (4.4) since $\xi \cdot \gamma(y) \leq 0$. Finally, suppose $|Du(\bar{x})| \neq 0$ and take $\xi = Du(\bar{x})/|Du(\bar{x})|$. From (4.4) and (4.5) it follows that

$$|Du(\bar{x})| = u_\xi(\bar{x}) \leq \max_{\partial\Omega}|D\underline{u}|.$$

This proves (4.3). □

We note that if M is a compact spacelike hypersurface and ∂M is a graph over the boundary of a domain $\Omega \subset \mathbb{R}^n$, then M is necessarily a graph over Ω. Thus Theorem 1.3 follows from Theorem 4.1 and Theorem 1.2 when Γ is a graph. To prove Theorem 1.3 in the general situation, we formulate an extension of Theorem 4.1 as follows: Let U be a compact domain that immerses into $\mathbb{R}^n$ with smooth boundary ∂U, and let $\pi : U \to \mathbb{R}^n$ denote this immersion. Given a function $u : U \to \mathbb{R}$, one obtains a hypersurface of $\mathbb{R}^{n,1}$ defined by

$$X : U \to \mathbb{R}^{n,1}, \quad X(q) = (\pi(q), u(q)) \quad \text{for } q \in U. \tag{4.6}$$

Theorem 4.2. *Let $\varphi \in C^\infty(\partial U)$ and $K \in C^\infty(\mathbb{R}^{n+1})$, $K > 0$. Suppose there exists a spacelike locally strictly convex hypersurface $\tilde{M}$ of $\mathbb{R}^{n,1}$ represented by*

$$q \in U \mapsto (\pi(q), \underline{u}(q)) \in \mathbb{R}^{n,1} \tag{4.7}$$

with $\underline{u} \in C^2(\overline{U})$ and $\underline{u}|_{\partial U} = \varphi$, such that $\mathcal{K}[\tilde{M}](q) \geq K(\pi(q), \underline{u}(q))$ for $q \in U$. Then there exists a spacelike locally strictly convex hypersurface M given by (4.6) with $u \in C^\infty(\overline{U})$ satisfying

$$\mathcal{K}[M](q) = K(\pi(q), u(q)) \quad \textit{for } q \in U, \quad u|_{\partial U} = \varphi. \tag{4.8}$$

Proof. We observe that it suffices to prove that, with respect to the metric on U induced by the immersion $\pi : U \to \mathbb{R}^n$, the Monge-Ampère equation

$$\det(u_{ij}) = K(1 - |Du|^2)^{\frac{n+2}{2}} \quad \text{in } U, \quad u = \varphi \quad \text{on } \partial U \tag{4.9}$$

has a locally strictly convex solution in $C^\infty(\overline{U})$ that satisfies the spacelike condition

$$|Du| < 1 \quad \text{in } U. \tag{4.10}$$

The existence of such a solution will follow from Theorem 5.1 in Section 5 once (4.10) is derived. To complete the proof, one observes that (4.10) can be derived as in the proof of Theorem 4.1 with some slight modification. □

The case $K \equiv 0$ leads to the degenerate Monge-Ampère equation.

Theorem 4.3. *Let $\varphi \in C^{3,1}(\partial U)$ and suppose there exists a spacelike locally strictly convex hypersurface $\tilde{M}$ of $\mathbb{R}^{n,1}$ given by (4.7) with $\underline{u} \in C^2(\overline{U})$ and $\underline{u}|_{\partial U} = \varphi$. Then there exists a locally convex spacelike hypersurface M given by (4.6) with $u \in C^{1,1}(\overline{U})$ and $u|_{\partial U} = \varphi$, whose Gauss-Kronecker curvature vanishes everywhere.*

Proof. The existence of M of the form (4.6) with $u \in C^{0,1}(\overline{U})$ follows from Theorem 4.2 by approximation. In order to obtain the desired $C^{1,1}$ regularity, we observe that for an arbitrary point $q \in U$, since it is locally convex, M has a supporting hyperplane, T, at $Q \equiv (\pi(q), u(q)) \in M$. By Lemma 3.1, there is a simplex $S \subset M \cap T$, containing Q, of dimension $k \leq n$ with vertices on ∂M. In a neighborhood of S, M lies above T and is given as a graph $x_{n+1} = \tilde{u}(x)$ which solves $\det(\tilde{u}_{ij}) = 0$ weakly. Now we can repeat the proof of Theorem 3.2 to show that $\tilde{u}$ satisfies an inequality of the form (3.4), which implies that $u \in C^{1,1}(\overline{U})$ since, clearly, $\tilde{u} = u \circ \pi$ locally. □

Finally, in order to see that Theorem 1.3 is a consequence of Theorems 4.2 and 4.3, we take $U = \tilde{M}$ and let $\pi : U \to \mathbb{R}^n$ be the orthogonal projection from $\tilde{M} \subset \mathbb{R}^{n+1}$ to $\mathbb{R}^n$. By the spacelike condition, we see π is an immersion and $\tilde{M}$ can be represented in the form (4.7) with $\underline{u} \in C^2(\overline{U})$. Theorem 1.3 thus follows.

Entire spacelike hypersurfaces with constant or prescribed mean curvature have also been studied in Minkowski space and in more general Lorentzian manifolds as well; for references please see, for example, [10]. In [9], Cheng and Yau proved a Bernstein type theorem for entire maximal spacelike hypersurfaces. It seems of interest to study entire spacelike hypersurfaces of constant Gauss-Kronecker curvature.

5. Monge-Ampère equations on manifolds

In this section we extend Theorem 1.1 to Monge-Ampère equations on Riemannian manifolds. Let M^n be a smooth Riemannian manifold of dimension $n \geq 2$ and $\Omega \subset M^n$ a smooth domain with compact closure $\overline{\Omega}$. We consider the Dirichlet problem

$$g^{-1} \det(\nabla_{ij} u) = \psi(x, u, \nabla u) \quad \text{in} \quad \Omega, \quad u = \varphi \quad \text{on} \quad \partial\Omega, \tag{5.1}$$

where $g = \det(g_{ij}) > 0$, g_{ij} denotes the metric of M^n, ∇ is the Levi-Civita connection, and $\nabla_{ij} u$ denotes the Hessian of u (with respect to the metric g_{ij}). We assume $\varphi \in C^\infty(\partial\Omega)$ and $\psi > 0$ is C^∞ with respect to $(x, z, p) \in \overline{\Omega} \times \mathbb{R} \times T_x M$; here $T_x M$ denotes the tangent space at $x \in M$. The main result of this section is the following analogue of Theorem 1.1, which extends some of the results in [14].

Theorem 5.1. *There exists a locally strictly convex solution of (5.1) in $C^\infty(\overline{\Omega})$, provided that there exists a locally strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega})$ to (5.1). Furthermore, the solution is unique if $\psi_z \geq 0$.*

Proof. Let $\mathcal{A}$ be as in Section 2. We shall prove the existence of a solution of (5.1) in $\mathcal{A}$. It is clear that the proof of Theorem 1.1 in Section 2 still works in this general case once C^2 *a priori* estimates are established for such solutions. According to [14], we need only estimate the second derivatives at the boundary.

About a point $x_0 \in \partial\Omega$, let $e_1, \ldots, e_n$ be a local orthonormal frame on M^n obtained by parallel translation of a local orthonormal frame on $\partial\Omega$ and the interior unit normal vector field to $\partial\Omega$ along the geodesics perpendicular to $\partial\Omega$ on M^n. We assume e_n is the parallel translation of the unit normal field on $\partial\Omega$.

Let $u \in \mathcal{A}$ be a solution of (5.1). Since $u - \underline{u} = 0$ on $\partial\Omega$, it is straightforward to bound the pure tangential second derivatives

$$|\nabla_{\alpha\beta} u| \leq C \quad \text{on } \partial\Omega \text{ for } \alpha, \beta < n. \tag{5.2}$$

Next, we note that Lemma 2.2 readily extends to the present general case. For completeness we restate the lemma. As in (2.9), set

$$\mathcal{L} = u^{ij}\nabla_{ij} - f_{p_i}(x, u, \nabla u)\nabla_i,$$

where $\{u^{ij}\}$ is the inverse matrix of the Hessian $\{\nabla_{ij} u\}$, $f = \log \psi$. Let v be the function as defined in (2.12).

Lemma 5.2. *For N sufficiently large and t, δ sufficiently small,*

$$\mathcal{L}v \leq -\frac{\varepsilon}{4}(1 + \sum u^{ii}) \quad \text{in } \Omega_\delta, \quad v \geq 0 \text{ on } \partial\Omega_\delta,$$

where $\Omega_\delta = \Omega \cap B_\delta(x_0)$; here $B_\delta(x_0)$ denotes the geodesic ball of radius δ about x_0.

Proof. It is the same as that of Lemma 2.2 except that (2.14) takes a simpler form:

$$u^{ij}\nabla_i d\, \nabla_j d \geq u^{nn}(\nabla_n d)^2,$$

since $\nabla_\beta d = 0$ for all $\beta < n$. □

Using Lemma 5.2 one can estimate $\nabla_{n\alpha} u$ on $\partial\Omega$ for $\alpha \leq n$ as in [14]. For any fixed $\alpha \leq n$, differentiate equation (5.1) and use the formula for commuting the covariant derivatives

$$\nabla_{ijk} w - \nabla_{jik} w = R^l_{kji}\nabla_l w,$$

to find

$$|\mathcal{L}\nabla_\alpha(u - \underline{u})| \leq C(1 + \sum u^{ii}). \tag{5.3}$$

The mixed normal tangential derivatives $\nabla_{\alpha n} u(x_0)$, $\alpha < n$, can be estimated the same way as in Section 2. Namely, by (5.3), we may choose $A \gg B \gg 1$ such that

$$\mathcal{L}(Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta,$$

where $|x|$ denotes the (geodesic) distance between x and x_0, and

$$Av + B|x|^2 \pm \nabla_\alpha(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

since $\nabla_\alpha(u - \underline{u}) = 0$ on $\partial\Omega \cap B_\delta(x_0)$, and $|\nabla_\alpha(u - \underline{u})| \leq C$ in Ω. It follows from the maximum principle that $Av + B|x|^2 \geq |\nabla_\alpha(u - \underline{u})|$ in Ω_δ. Consequently,

$$|\nabla_{n\alpha} u(x_0)| \leq A\nabla_n v(x_0) + |\nabla_{n\alpha}\underline{u}(x_0)| \leq C, \quad \alpha < n. \tag{5.4}$$

For the double normal derivative $\nabla_{nn} u$, since u is locally convex, it suffices to derive an upper bound

$$\nabla_{nn} u \leq C \quad \text{on } \partial\Omega. \tag{5.5}$$

We use an idea of Trudinger [25]. For $x \in \partial\Omega$ let

$$\lambda(x) = \min_{|\xi|=1, \xi \in T_x(\partial\Omega)} \nabla_{\xi\xi} u(x),$$

and assume that $\lambda(x)$ is minimized at $x_0 \in \partial\Omega$ with $\xi = e_1(x_0)$, that is, $\nabla_{11}u(x_0) \leq \nabla_{\xi\xi}u(x)$ for all $x \in \partial\Omega$ and any unit vector $\xi \in T_x(\partial\Omega)$. As in [6], (5.5) will follow from

$$\nabla_{11}u(x_0) \geq c_0 > 0. \tag{5.6}$$

To show (5.6), we may assume $\nabla_{11}u(x_0) < \frac{1}{2}\nabla_{11}\underline{u}(x_0)$, since otherwise we are done as $\nabla_{11}\underline{u}(x_0) \geq c_1 > 0$ for some uniform $c_1 > 0$. We have

$$\nabla_{11}u = \nabla_{11}\underline{u} - B_{11}\nabla_n(u - \underline{u}) \quad \text{on } \partial\Omega, \tag{5.7}$$

where $B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$, $1 \leq \alpha, \beta \leq n-1$. It follows that

$$B_{11}(x_0)\nabla_n(u - \underline{u})(x_0) \geq \frac{1}{2}\nabla_{11}\underline{u}(x_0) \geq \frac{c_1}{2},$$

and for $x \in \partial\Omega$ near x_0, since $\nabla_{11}u|_{\partial\Omega}$ is minimized at x_0,

$$B_{11}(x)\nabla_n(u - \underline{u})(x) \leq \nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0).$$

Because B_{11} is smooth near $\partial\Omega$ and $0 < \nabla_n(u - \underline{u}) \leq C$, we must have $B_{11} \geq c_2 > 0$ on Ω_δ for some uniform $c_2 > 0$, if δ is chosen sufficiently small. Therefore,

$$\nabla_n(u - \underline{u})(x) \leq \Psi(x) \quad \text{for } x \in \Omega_\delta \cap \partial\Omega \;\text{ and }\; \nabla_n(u - \underline{u})(x_0) = \Psi(x_0) \tag{5.8}$$

where $\Psi(x) = B_{11}^{-1}(x)[\nabla_{11}\underline{u}(x) - \nabla_{11}\underline{u}(x_0) + B_{11}(x_0)\nabla_n(u - \underline{u})(x_0)]$.

We observe that since Ψ is smooth in Ω_δ, by (5.3), (5.8) and Lemma 5.2 we may choose $A \gg B \gg 1$ such that

$$Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

$$\mathcal{L}(Av + B|x|^2 + \Psi - \nabla_n(u - \underline{u})) \leq 0 \quad \text{in } \Omega_\delta.$$

By the maximum principle, $v + \Psi - \nabla_n(u - \underline{u}) \geq 0$ in Ω_δ, and therefore

$$\nabla_{nn}u(x_0) \leq C.$$

This shows that the eigenvalues of $\{\nabla_{ij}u(x_0)\}$ are all bounded (and all positive). On the other hand, equation (5.1) says the product of these eigenvalues is bounded below away from zero by a uniform positive constant (ψ_0 as in (2.3)). Thus each of them must be bounded below away from zero. In particular, we obtain the estimate (5.6), which in turn implies (5.5). □

We conclude this section by a remark on the following equation of Monge-Ampère type on $\mathbb{S}^n$:

$$g^{-1}\det(ug_{ij} + \nabla_{ij}u) = \psi(x, u, \nabla u). \tag{5.9}$$

This equation arises in various geometric problems related to Gauss curvature such as the Minkowski problem (see for example Cheng and Yau [8] and the references therein). The Dirichlet problem was studied, in connection with the boundary value problem of finding hypersurfaces in $\mathbb{R}^{n+1}$ of prescribed Gauss-Kronecker curvature, by J. Spruck and the author in [15] and [13] under the hypothesis that $\psi^{1/n}$ is a convex function with respect to the gradient ∇u. By employing a better barrier similar to that in Lemma 5.2, we may refine the argument in [13] to prove

Theorem 5.3. *Let $\Omega \subset \mathbb{S}^n$ be a smooth domain that does not contain any hemisphere. Let $\varphi \in C^\infty(\partial\Omega)$, and let $\psi > 0$ be a smooth function. Then (5.9) has a solution $u \in C^\infty(\overline{\Omega})$ satisfying $\{ug_{ij} + \nabla_{ij}u\} > 0$ in $\overline{\Omega}$ and $u = \varphi$ on $\partial\Omega$, provided that there exists a subsolution $\underline{u} \in C^2(\overline{\Omega})$ with $\{\underline{u}g_{ij} + \nabla_{ij}\underline{u}\} > 0$ in $\overline{\Omega}$ and $\underline{u} = \varphi$ on $\partial\Omega$.*

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