The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian

by

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Dedicated to Lars Gårding on his 65th birthday

This is a sequel to [1] and [2]. We will study the Dirichlet problem in a bounded domain Ω in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{aligned} F(D^2u) &= \psi \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The function F is of a very special nature. It is represented by a smooth symmetric function $f(\lambda_1, \dots, \lambda_n)$ of the eigenvalues $\lambda=(\lambda_1, \dots, \lambda_n)$ of the Hessian matrix $D^2u=\{u_{ij}\}$, which we denote by $\lambda(u_{ij})$. The equation is assumed to be elliptic for the functions under consideration, i.e.

$$\frac{\partial f}{\partial \lambda_i} > 0, \quad \forall i \tag{2}$$

and to satisfy:

$$f \text{ is a concave function.} \tag{3}$$

As we will see in section 3, this means F is a concave function of the arguments $\{u_{ij}\}$.

The function f will be required to satisfy various conditions. First of all it is assumed to be defined in an open convex cone $\Gamma \subsetneqq \mathbf{R}^n$, with vertex at the origin, containing the positive cone: $\{\lambda \in \mathbf{R}^n \mid \text{each component } \lambda_i > 0\}$, and to satisfy (2), (3) in

Γ. Γ is assumed to be invariant under interchange of any two λ_i; i.e. it is symmetric in the λ_i. It follows easily that

$$\Gamma \subset \left\{ \sum \lambda_i > 0 \right\}. \tag{4}$$

We distinguish two types of cones.

Definition. Γ is said to be of type 1 if the positive λ_i axes belong to $\partial\Gamma$; otherwise it is called of type 2.

We assume $\psi \in C^\infty(\bar{\Omega})$, $\varphi \in C^\infty(\partial\Omega)$ and, for convenience,

$$\psi > 0 \quad \text{in } \bar{\Omega}. \tag{5}$$

Set

$$0 < \psi_0 = \min_{\bar{\Omega}} \psi \leqslant \max_{\bar{\Omega}} \psi = \psi_1.$$

We assume that for some $\bar{\psi}_0 < \psi_0$,

$$\overline{\lim_{\lambda \to \lambda_0}} f(\lambda) \leqslant \bar{\psi}_0 \quad \text{for every } \lambda_0 \in \partial\Gamma. \tag{6}$$

In addition we assume that for every $C>0$ and every compact set K in Γ there is a number $R=R(C,K)$ such that

$$f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geqslant C \quad \text{for all } \lambda \in K, \tag{7}$$

$$f(R\lambda) \geqslant C \quad \text{for all } \lambda \in K. \tag{8}$$

From (8) and concavity it follows that

$$\sum \lambda_i f_{\lambda_i} > 0 \quad \text{in } \Gamma. \tag{8$'$}$$

Using (3) and (6) we may conclude that there is a positive number δ such that

$$\sum \lambda_i \geqslant \delta \quad \text{in the set } T = \{\lambda \in \Gamma \mid f(\lambda) \geqslant \psi_0\}. \tag{9}$$

Namely, the set T is closed, convex and symmetric in the λ_i. The unique closest point in T to the origin is therefore of the form $(b, \ldots, b)$ with $b>0$. It follows that (9) holds with $\delta = nb$.

Consider now the domain Ω. In case $f=\log \Pi \lambda_i$ we assumed in [1] that Ω is strictly convex. On the other hand if $f=\Sigma \lambda_i$, any Ω with smooth boundary should be allowed. What kind of Ω should be admitted in the general case?

We will suppose that there exists a number R sufficiently large such that at every point $x\in\partial\Omega$, if $\varkappa_1,\ldots,\varkappa_{n-1}$ represent the principal curvatures of $\partial\Omega$ (relative to the interior normal), then

$$(\varkappa_1,\ldots,\varkappa_{n-1},R)\in\Gamma. \tag{10}$$

As we will see, this is a natural condition for type 1. Note the following

LEMMA A. *Assume that Γ is of type* 1 *and that* (10) *holds. Then $\partial\Omega$ is necessarily connected.*

Proof. Suppose $\partial\Omega$ has more than one component. Let Ω_1 be the component which is the boundary of the unbounded component of Ω^c, the complement of $\bar\Omega$. By shrinking a large sphere enclosing $\bar\Omega$ so that it first touches a component of $\partial\Omega$ other than Ω_1 we find a point on $\partial\Omega$ where $\varkappa_1,\ldots,\varkappa_{n-1}<0$. According to (10), at that point $(\varkappa_1,\ldots,\varkappa_{n-1},R)\in\Gamma$. Consequently $(0,\ldots,0,R)$ belongs to Γ—contradicting the fact that the positive λ_n axis does not.

Definition. A function $u\in C^2(\bar\Omega)$ with $u=\varphi$ on $\partial\Omega$ is called *admissible* if at every $x\in\bar\Omega$, $\lambda(u_{ij})(x)\in\Gamma$.

We now state our main results. The case $\varphi\equiv$constant is much easier to treat than the general case so we consider that first.

THEOREM 1. *Assume conditions* (2), (3), (6–8) *and $\varphi\equiv$constant. There exists a unique admissible solution $u\in C^\infty(\bar\Omega)$ of* (1) *if and only if* (10) *holds at every point of* $\partial\Omega$.

The sufficiency statement of Theorem 1 is a special case of

THEOREM 2. *For general φ, there exists a unique admissible solution $u\in C^\infty(\bar\Omega)$ of* (1) *if* (2), (3), (6–8) *and* (10) *are satisfied.*

THEOREM 2′. *In case $\psi\equiv$constant, Theorem* 2, *holds even if condition* (7) *is dropped.*

Remark. Cones of type 2 are rather simple to treat because the orthogonal projection Γ' of Γ onto $\mathbf{R}^{n-1}$ (along the λ_n axis) is the entire space $\mathbf{R}^{n-1}$. Thus condition (10) holds automatically and furthermore, condition (8) implies condition (7).

Examples. (1) Here is a simple example with $n=2$ and Γ of type 2:

$$f(\lambda_1,\lambda_2)=((\lambda_1+\lambda_2)^2-\tau(\lambda_1-\lambda_2)^2)^{1/2}, \quad 0<\tau<1.$$

This is simply obtained from the (Monge-Ampère) case $(\lambda_1\lambda_2)^{1/2}$ by a linear transformation. With $k=4\tau/(1-\tau)$, it follows from Theorem 2 that for any bounded Ω in $\mathbf{R}^2$, $\partial\Omega$ smooth, the problem

$$(\Delta u)^2+k(u_{xx}u_{yy}-u_{xy}^2)=\psi(x,y)>0 \text{ in } \bar{\Omega}, \quad u=\varphi \text{ on } \partial\Omega,$$

with ψ, $\varphi\in C^\infty$, has a solution $u\in C^\infty(\bar{\Omega})$.

(2) An interesting example of a function f satisfying the conditions of Theorem 2 is

$$f(\lambda)=[\sigma^{(k)}(\lambda)]^{1/k}$$

where

$$\sigma^{(k)}(\lambda)=\sum_{i_1<\ldots<i_k}\lambda_{i_1}\ldots\lambda_{i_k}$$

is the kth elementary symmetric function. The fact that this is an example follows from the paper [3] of L. Gårding concerning hyperbolic polynomials—of which $\sigma^{(k)}$ is an example. This will be explained in the next section.

For $\sigma^{(k)}$, $1<k\leq n$, we will prove the following result:

THEOREM 3. *The Dirichlet problem*

$$\sigma^{(k)}(\lambda(u_{ij}))=\psi>0 \textit{ in } \bar{\Omega},\ k>1, \quad u=\varphi \textit{ on } \partial\Omega \tag{1)'$$

admits a (unique) admissible solution $u\in C^\infty(\bar{\Omega})$ provided

$$\partial\Omega \textit{ is connected, and at every point } x\in\partial\Omega,\ \sigma^{(k-1)}(\varkappa_1,\ldots,\varkappa_{n-1})>0. \tag{10)'$$

In case $\varphi\equiv$constant, condition (10)′ is also necessary for existence of a solution in $C^2(\bar{\Omega})$.

Proof. The necessity follows from Proposition 1.3, Remark 1.1 just following it, and Lemma A. In Remark 1.2 we show that (10) follows from (10)′. Using Proposition 1.1, the sufficiency then follows from Theorem 2.

The Monge-Ampère equation is the case $k=n$:

$$\sigma^{(n)}(\lambda)\equiv\prod\lambda_i\equiv\det(u_{ij})=\psi.$$

In Theorems 1–3 the uniqueness follows from ellipticity and the maximum principle. We will also make use of the following, somewhat unusual, form of the maximum principle. Here u is an admissible solution of (1) in Ω and $v\in C^2(\Omega)\cap C(\bar\Omega)$.

LEMMA B (Maximum principle). *Assume that at every point x in Ω, $\lambda(v_{ij}(x))$ lies outside the set $\bar\Gamma(x)=\{\lambda\in\Gamma\,|\,f(\lambda)\geqslant\psi(x)\}$. If $u\leqslant v$ on $\partial\Omega$, then*

$$u\leqslant v \quad \textit{in } \Omega.$$

Proof. If not, $v-u$ achieves a negative minimum at some point $x\in\Omega$. Since, as matrices, $\{v_{ij}\}\geqslant\{u_{ij}\}$ at x, the eigenvalues of $\{v_{ij}\}$ are not smaller than the corresponding ones for $\{u_{ij}\}$. However $\lambda(u_{ij})\in\bar\Gamma(x)$, and it follows that $\lambda(v_{ij})\in\bar\Gamma(x)$, a contradiction.

In [5], N. M. Ivočkina studied the Dirichlet problem (1)′: She established a priori estimates for the C^2 norm of *convex* solutions having boundary values $\varphi\equiv$constant in strictly convex Ω. As she remarks, it is not reasonable to expect the solutions to be convex.

In Section 1 we describe hyperbolic polynomials P. By the corollary to Proposition 1.1, Theorems 1 and 2 apply to a wide class of these. In addition we prove necessity in Theorems 1 and 3.

In Section 2 we show that if (10) holds then there exist smooth admissible functions in $\bar\Omega$. In particular there is one, $\underline{u}$, which is a subsolution:

$$F(\underline{u}_{ij})\geqslant\psi \quad \text{in } \bar\Omega;$$

in fact $F(\underline{u}_{ij})$ may be made arbitrarily large.

Section 3 contains a proof that concavity of f implies concavity of $F(u_{ij})$; this proof, due to the referee, is simpler than our original one. Sufficiency in Theorems 1 and 2 is proved in Sections 4–7 via the continuity method and a priori estimates for the C^2 norm of u. In Section 4 the estimates $|u|_{C^1}\leqslant C$ are proved. The estimate for the second derivatives, in particular at the boundary, are established in Section 5 for Γ of type 2, and completed in Section 6 for type 1, and in Section 7 for Theorem 2′.

In Section 9 we present an example of an equation (1) for which (2), (3), (5) and (6) hold but not (7), (8). Here $\psi\in C^k(\bar\Omega)$, where $k>0$ is any given integer. There is a unique convex solution in class $C^2(\Omega)\cap C(\bar\Omega)$ but it is not in $C^2(\bar\Omega)$.

(3) Theorem 2′ has some interesting applications. Let G be an open convex region in λ-space, $\mathbf{R}^n$, with smooth (C^∞) boundary Σ satisfying:

$$G \text{ is symmetric in the } \lambda_i\text{, and the interior normal at every point of } \Sigma \text{ lies in the positive cone.} \tag{11}$$

$$\text{The origin is not in } \bar{G} \text{ and it lies on the opposite side, from } G\text{, of every tangent hyperplane } P \text{ to } \Sigma. \tag{12}$$

It follows that:

$$\text{The cone } \Gamma \text{ with vertex at the origin generated by points of } \Sigma \text{ contains the positive cone.} \tag{13}$$

We see that G is necessarily unbounded.

In a bounded domain Ω in $\mathbf{R}^n$, with $\partial\Omega$ smooth, consider the Dirichlet problem: find a function $u \in C^\infty(\bar{\Omega})$ satisfying

$$\lambda(u_{ij}) \in \Sigma \text{ for every } x \in \Omega, \quad u = \varphi \text{ on } \partial\Omega,\ \varphi \in C^\infty. \tag*{(1)''}$$

Remark. The condition (11) corresponds to ellipticity, the convexity of G to the concavity condition (3).

THEOREM 4. *The Dirichlet problem* (1)″ *admits a solution* $u \in C^\infty(\bar{\Omega})$ *provided* $\partial\Omega$ *satisfies condition* (10).

(Recall that (10) is automatic if Γ is of type 2.)

Proof. In Γ define f as the function which equals 1 on Σ and is positive homogeneous of degree one. To solve (1)″, we solve

$$f(\lambda(u_{ij})) = 1 \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega. \tag{14}$$

We see that f satisfies (6), with $\bar{\psi}_0=0$, and (8). By (11) we see that f satisfies condition (2). Condition (3) is easily verified using the convexity of Σ. Theorem 2′ then yields a solution of (14).

Reese Harvey and H. Blaine Lawson Jr. (in [4], Sections III.2.A and B) have taken up the differential equations

$$\operatorname{Im}\det(\delta_{ij}+iu_{ij}) \equiv \sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma^{(2k+1)}(\lambda(u_{ij})) = 0. \tag{15}$$

They showed that it is elliptic at every solution u and (see their Theorem 2.7) that if u is a solution, then the graph of ∇u is an absolutely volume-minimizing submanifold of $\mathbf{R}^{2n}$. They asked the question: does the Dirichlet problem

$$u \text{ satisfies (15) in } \Omega, \quad u=\varphi \text{ on } \partial\Omega, \tag{16}$$

have solutions?

As we'll see in Section 8, the solution set of

$$g(\lambda) \equiv \operatorname{Im} \prod_{j=1}^{n} (1+i\lambda_j) = 0 \tag{17}$$

has exactly n components—smooth hypersurfaces—in case n is odd, and $(n-1)$ in case n is even. One might conjecture that there should be the same respective number of solutions for suitable domains Ω. The case $n=3$ looks particularly inviting:

$$\det u_{ij} - \sum u_{ii} = 0 \text{ in } \Omega, \quad u=\varphi \text{ on } \partial\Omega. \tag{16$'$}$$

As an application of Theorem 4 we prove

THEOREM 5. *The Dirichlet problem* (16) *has at least two solutions belonging to* $C^\infty(\bar{\Omega})$ *in the following cases:*

(i) *n is odd and $\partial\Omega$ is strictly convex,*

(ii) *n is even and $\partial\Omega$ satisfies* (10) *with* $k=n-1$, i.e.,

$$\partial\Omega \textit{ is connected and } \sigma^{(n-2)}(\varkappa_1, \ldots, \varkappa_{n-1}) > 0 \textit{ on } \partial\Omega. \tag{18}$$

The proof uses

LEMMA C. *One of the components Σ of the solution set of* (17) *is the boundary of a convex region G satisfying* (11), (12), (13). *In case n is odd the corresponding cone Γ is the positive cone. In case n is even the cone Γ is the cone $\Gamma(\sigma^{(n-1)}, a)$ of Section* 1, i.e. *the component in $\mathbf{R}^n$, containing $a=(1, \ldots, 1)$, in which $\sigma^{(n-1)}$ is positive.*

Proof of Theorem 5 (i). In this case Theorem 4 yields a solution u of (1)$''$ and hence of (16). It also yields a solution v of

$$\lambda(v_{ij}) \in \Sigma \text{ in } \Omega, \quad v=-\varphi \text{ on } \partial\Omega.$$

The function $-v$ is then another solution of (16).

(ii) In case n is even we obtain two solutions by using Theorem 4 in the same way. That condition (18) implies (10) is shown in Remark 1.2 of the next section.

Note that for (16)$'$, with $\varphi=0$, neither of our two solutions is the solution $u=0$.

Lemma C is proved in Section 8.

We wish to express our thanks to the referee for helpful comments.

Added in proof. We have learned that the concavity result of our Section 3 is contained in Theorem 5.1 (i) of the paper by J. Ball: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal., 63 (1977), 337–403. We also wish to call attention to a recent paper by N. V. Krylov: On degenerate nonlinear elliptic equations, II. Mat. Sbornik, 121 (163) (1983), 211–232; english translation: Math. USSR Sbornik, 49 (1984) 207–228, which treats various nonlinear second order equations.

1. Hyperbolic polynomials

In [3] Gårding proved a beautiful inequality for homogeneous hyperbolic polynomials. In this section we will summarize some of the results of [3]—which we will use.

Hyperbolic polynomials. A homogeneous polynomial $P(\lambda)$ of degree k defined in $\mathbf{R}^n$ is called hyperbolic with respect to a direction $a\in\mathbf{R}^n$, abbreviated, hyp a, provided for every $\lambda\in\mathbf{R}^n$ the polynomial in t,

$$P(ta+\lambda),$$

has exactly k real roots.

Necessarily $P(a)\neq0$. We will always suppose $P(a)>0$; it follows that the coefficients of P are real. It is easily seen that if P is hyp a so is

$$Q=\sum a_j\frac{\partial}{\partial\lambda_j}P. \tag{1.1}$$

Since $\sigma^{(n)}(\lambda)=\Pi\lambda_i$ is hyp a for $a=(1,\dots,1)$ it follows readily that so is $\sigma^{(k)}(\lambda)$, $k<n$.

Assuming $P(a)>0$, let $\Gamma=\Gamma(P,a)$ denote the component in $\mathbf{R}^n$, containing a, in which $P>0$. It is proved in [3] that Γ is a convex cone$\neq\mathbf{R}^n$, with vertex at the origin, and that P is hyp b for every $b\in\Gamma$. Furthermore, for Q given by (1.1), we have $\Gamma(P,a)\subset\Gamma(Q,a)$.

In particular,

$$\Gamma(\sigma^{(k)})\subset\Gamma(\sigma^{(k-1)}). \tag{1.2}$$

On the positive λ_i axes we have $\sigma^{(k)}(\lambda)=0$ for $k>1$ and therefore for $\sigma^{(k)}$, the cone Γ, which clearly contains the positive cone, is of type 1.

The main result proved in [3] is an inequality involving the completely polarized form M of P: For k vectors $\lambda^1, \dots, \lambda^k$ in $\mathbf{R}^n$ (λ^j has components $\lambda_1^j, \dots, \lambda_n^j$),

$$M(\lambda^1, \dots, \lambda^k) = \frac{1}{k!} \prod_j \left(\sum \lambda_i^j \frac{\partial}{\partial \lambda_i} \right) P(\lambda).$$

The inequality states that for $\lambda^1, \dots, \lambda^k \in \Gamma$,

$$M(\lambda^1, \dots, \lambda^k) \geq P(\lambda^1)^{1/k} \dots P(\lambda^k)^{1/k}.$$

A particular case (from which the general case is derived) is the inequality: For λ and μ in Γ.

$$\frac{1}{k} \sum \mu_j \frac{\partial}{\partial \lambda_j} P(\lambda) \geq P(\mu)^{1/k} P(\lambda)^{1-1/k}. \tag{1.3}$$

This is the form we will use. It is equivalent to the assertion that $P^{1/k}(\lambda)$ is a concave function on Γ.

This is easily proved. The statement that $P^{1/k}(\lambda)$ is concave in Γ can be expressed analytically as: For μ, $\lambda \in \Gamma$,

$$P^{1/k}(\mu) \leq P^{1/k}(\lambda) + \sum (\mu_j - \lambda_j) \frac{\partial}{\partial \lambda_j} (P^{1/k}(\lambda))$$

i.e.

$$P^{1/k}(\mu) \leq P^{1/k}(\lambda) + \frac{1}{k} P^{1/k-1}(\lambda) \sum \mu_j P_{\lambda_j}(\lambda) - P^{1/k}(\lambda)$$

since $P^{1/k}$ is positive homogeneous of degree one, i.e.

$$\frac{1}{k} \sum \mu_j P_{\lambda_j}(\lambda) \geq P^{1/k}(\mu) P^{1-1/k}(\lambda),$$

which is (1.3).

PROPOSITION 1.1. *Suppose that P is hyp a for $a=(1, \dots, 1)$, that Γ contains the positive cone, and that P is symmetric in the λ_i. Then $f=P^{1/k}$ satisfies in Γ conditions* (2), (3) *and* (6)–(9).

COROLLARY. *Theorems* 1 *and* 2 *apply for such f, in particular for $P=\sigma^{(k)}$.*

Proof. (2) is easily proved: Since $P^{1/k}$ is concave and positive in Γ we must have $P_{\lambda_i} \geq 0$ in Γ. Suppose $P_{\lambda_i}=0$ for some i at some point λ in Γ. By concavity again we find

$P_{\lambda_i} \equiv 0$ if we increase λ_i. Suppose $i=1$. Then by analyticity, $P(\mu, \lambda_2, \dots, \lambda_n)$ is independent of μ. Therefore the whole line L: $(\mu, \lambda_2, \dots, \lambda_n)$, $-\infty<\mu<\infty$, belongs to Γ. Using the convexity of Γ it is easily seen that any doubly infinite line parallel to L and close to it also belongs to Γ. On any such line $P^{1/k}>0$ and is concave. This is only possible if P is constant on the line. Hence we find $P_{\lambda_1} \equiv 0$ in a full neighbourhood of L. By analyticity it follows that P is independent of λ_1. Since P is symmetric, this is impossible; (2) is proved.

We have already proved (3); (6) follows from the fact that $f=0$ on $\partial\Gamma$. Consider, next, (7); if it did not hold there would be a constant C and a sequence $\lambda^j \in K$, and $R^j \to +\infty$, such that

$$f(\lambda_1^j, \dots, \lambda_{n-1}^j, \lambda_n^j + R) \leq C \quad \text{for } 0 \leq R \leq R^j,$$

(here we use (2)). It follows that for some $\lambda \in K$,

$$f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \leq C \quad \text{for } 0 \leq R < \infty.$$

But $f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R)$ is a polynomial in R which, by (2), is strictly increasing for $R>0$—a contradiction. Finally, (3) and (6) yield (9), and (8) follows by homogeneity. Proposition 1.1 is proved.

Our next result shows that condition (10) is a necessary condition in Theorem 1 for $f=P^{1/k}$ with P a hyperbolic polynomial.

PROPOSITION 1.3. *Let P be as in Proposition* 1.1. *Assume that there is a smooth function u in $\bar{\Omega}$ vanishing on $\partial\Omega$ and such that $P(\lambda(u_{ij}))>0$ in $\bar{\Omega}$. Then necessarily $\partial\Omega$ satisfies condition* (10).

Remark 1.1. The necessity of condition (10)′ in Theorem 3 follows from Proposition 1.3 and Lemma A in the introduction. For if $(\varkappa_1, \dots, \varkappa_{n-1}, R)$ belongs to $\Gamma(\sigma^{(k)}, a)$ for $a=(1, \dots, 1)$ then by Proposition 1.1,

$$\sigma^{(k-1)}(\varkappa_1, \dots, \varkappa_{n-1}) = \frac{\partial}{\partial \lambda_n} \sigma^{(k)}(\varkappa_1, \dots, \varkappa_{n-1}, R) > 0.$$

Proof of Proposition 1.3. (i) Clearly $u \not\equiv 0$. Suppose $u>0$ somewhere. If k is even we may replace u by $-u$ and so $u<0$ somewhere. In case k is odd, $v=-u$ satisfies

$$P(\lambda(v_{il})) < 0 \quad \text{in } \bar{\Omega}. \tag{1.4}$$

At the point where v takes a negative minimum each eigenvalue λ_j of $\{v_{il}\}$ is nonnegative and hence

$$\lambda(v_{il}) \in \bar{\Gamma},$$

contradicting (1.4). Thus in this case u cannot be positive anywhere. Consequently in any case we may suppose $u<0$ somewhere. At the point where u takes its minimum we have

$$\lambda_j(u_{rs}) \geqslant 0 \quad \text{for every } j.$$

Hence $\lambda \in \bar{\Gamma}$ and, since $P(\lambda)>0$, we see that $\lambda \in \Gamma$. Since Ω is connected we conclude that at every point $x \in \bar{\Omega}$,

$$\lambda(u_{ij}) \in \Gamma. \tag{1.5}$$

(ii) From property (9), which holds by Proposition 1.1, we have

$$\Delta u > 0 \quad \text{in } \bar{\Omega}.$$

By the Hopf lemma we may assert that the interior normal derivative

$$u_\nu < -a < 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

for some positive constant a.

In proving (10) for any boundary point of Ω we may suppose the point is the origin, that the postive x_n axis is interior normal there and that the boundary near there is represented by

$$x_n = \varrho(x') = \frac{1}{2}\sum_{1}^{n-1} \varkappa_\alpha x_\alpha^2 + O(|x'|^3), \tag{1.7}$$

where $x'=(x_1, \ldots, x_{n-1})$ and $\varkappa_\alpha$ are the principal curvatures of $\partial\Omega$ at 0.

We have $u=0$ on $\partial\Omega$, i.e.

$$u(x', \varrho(x')) = 0;$$

it follows, on differentiation, that at the origin for α, $\beta<n$,

$$u_{\alpha\beta} = -u_n \varrho_{\alpha\beta} = -u_n \varkappa_\alpha \delta_{\alpha\beta}. \tag{1.8}$$

For ε small we see from (1.5) that at 0, $\lambda(v_{ij}) \in \Gamma$ where

$$v = u - \frac{\varepsilon}{2}\sum_{\alpha<n} x_\alpha^2.$$

For t large, consider the function

$$w=\frac{1}{t}(e^{tv}-1).$$

At the origin,

$$w_{ij}=v_{ij}+tu_i u_j.$$

Since $t\{u_i u_j\}$ has nonnegative eigenvalues it follows from (1.5) that at the origin

$$\lambda(w_{ij})\in\Gamma.$$

We shall make use of the following:

LEMMA 1.2. *Consider the $n\times n$ symmetric matrix*

$$M=\left(\begin{array}{cccc|c} d_1 & & & \bigcirc & a_1 \\ & d_2 & & & \\ & & \ddots & & \vdots \\ & \bigcirc & & d_{n-1} & a_{n-1} \\ \hline a_1 & & \cdots & a_{n-1} & a \end{array}\right)$$

with $d_1,\dots,d_{n-1}$ fixed, $|a|$ tending to infinity and

$$|a_i|\leq C,\quad i=1,\dots,n-1.$$

Then the eigenvalues $\lambda_1,\dots,\lambda_n$ behave like

$$\lambda_\alpha=d_\alpha+o(1),\quad \alpha\leq n-1 \tag{1.9}$$

$$\lambda_n=a\left(1+O\left(\frac{1}{a}\right)\right), \tag{1.10}$$

where the $o(1)$ and $O(1/a)$ are uniform—depending only on $d_1,\dots,d_{n-1}$ and C.

Proof. The eigenvalues λ of M satisfy

$$\det\left(\begin{array}{ccc|c} d_1-\lambda & & \bigcirc & \\ & \ddots & & \dfrac{a_\alpha}{a} \\ \bigcirc & & d_{n-1}-\lambda & \\ \hline & a_\alpha & & 1-\dfrac{\lambda}{a} \end{array}\right)=0.$$

Hence for $|a|=\infty$ the numbers $d_1, \ldots, d_{n-1}$ are roots. By continuity of the roots it follows that there are roots given by (1.9). To find the last eigenvalue, set $\lambda=a\mu$. Then μ satisfies

$$\det\left(\begin{array}{ccc|c} \frac{d_1}{a}-\mu & & \bigcirc & \\ & \ddots & & \frac{a_\alpha}{a} \\ \bigcirc & & \frac{d_{n-1}}{a}-\mu & \\ \hline & \frac{a_\alpha}{a} & & 1-\mu \end{array}\right)=0.$$

For $|a|=\infty$, we see that $\mu=1$ is a simple root. By the implicit function therorem it follows that for $|a|$ large there is a root

$$\mu=1+O\left(\frac{1}{a}\right)$$

i.e.

$$\lambda=a\left(1+O\left(\frac{1}{a}\right)\right).$$

The lemma is proved.

Returning to (w_{ij}) we see that at the origin, for $-u_n=b>a$, by (1.6), we have

$$(v_{ij})=\begin{pmatrix} b\varkappa_1-\varepsilon & \bigcirc & & u_{1n} \\ & \ddots & & \\ \bigcirc & & b\varkappa_{n-1}-\varepsilon & u_{n-1,n} \\ u_{n,1} & & u_{n,n-1} & u_{nn}+tb^2 \end{pmatrix}.$$

Since $\lambda(v_{ij})\in\Gamma$ it follows from the lemma that for t sufficiently large

$$(b\varkappa_1-\varepsilon+o(1), \ldots, b\varkappa_{n-1}-\varepsilon+o(1), u_{nn}+tb^2+O(1))\in\Gamma$$

and hence for t sufficiently large (so that $o(1)<\varepsilon$),

$$(b\varkappa_1, \ldots, b\varkappa_{n-1}, \delta t)\in\Gamma$$

for some $\delta>0$. Since $b>a$ and Γ is a cone, it follows that

$$(\varkappa_1, \ldots, \varkappa_{n-1}, R)\in\Gamma$$

for R large, but under control; (10) is proved.

Remark 1.2. For $P=\sigma^{(k)}$, suppose (10)′ holds, then $\partial\Omega$ also satisfies (10).

This is easily verified: Since

$$\sigma^{(k)}(\varkappa_1,\dots,\varkappa_{n-1},R)=R\sigma^{(k-1)}(\varkappa_1,\dots,\varkappa_{n-1})+\sigma^{(k)}(\varkappa_1,\dots,\varkappa_{n-1})$$

we see that for R large,

$$\sigma^{(k)}(\varkappa_1,\dots,\varkappa_{n-1},R)>0 \quad \text{at every } x\in\partial\Omega.$$

On the other hand $\partial\Omega$ contains a point where all the $\varkappa_\alpha$ are positive so that at that point $(\varkappa_1,\dots,\varkappa_{n-1},R)\in\Gamma$. Since $\partial\Omega$ is connected, (10) follows at every point on $\partial\Omega$.

We conclude this section with the

Proof of necessity in Theorem 1. The proof is just like that of Proposition 1.3. From condition (9) we have $\Delta u>\delta$ and so $u<0$ in Ω and by the Hopf lemma, (1.6) holds. One may now follow the rest of the proof of Proposition 1.3. Q.E.D.

2. The existence of admissible functions

Assuming that $\partial\Omega$ satisfies (10) we will construct an admissible function $\underset{\sim}{u}$ which is also a subsolution of (1).

Suppose as in the preceding section that $0\in\partial\Omega$, and that the positive x_n axis is interior normal there, and $\partial\Omega$ is represented locally by (1.7). Near $\partial\Omega$ let $d(x)$ denote the distance to $\partial\Omega$. At the origin we have

$$d_{ij}=\left(\begin{array}{ccc|c} -\varkappa_1 & & & \\ & \ddots & & 0 \\ & & -\varkappa_n & \\ \hline & 0 & & 0 \end{array}\right)$$

For large t consider

$$v=\frac{1}{t}(e^{-td}-1);$$

at the origin,

$$v_{ij}=-d_{ij}+t\,d_i d_j=\left(\begin{array}{ccc|c} \varkappa_1 & & & \\ & \ddots & & 0 \\ & & \varkappa_{n-1} & \\ \hline & 0 & & t \end{array}\right).$$

By (10) we see that for t sufficiently large, $\lambda(v_{ij})\in\Gamma$ at, and hence near, the origin. Since the origin could be any boundary point we infer that for t large

$$\lambda(v_{ij})\in\Gamma$$

in a neighbourhood of the boundary—in particular in a region $0\leqslant d\leqslant\delta$—corresponding to $-\varepsilon\leqslant v\leqslant 0$ for some $\varepsilon>0$.

Let $g(s)$ be a C^∞ convex function defined on $s\leqslant 0$ satisfying

$$g\equiv -1 \quad \text{for } s\leqslant -\varepsilon$$
$$g(0)=0$$
$$\dot g(s)>0 \quad \text{on } -\varepsilon<s<0.$$

Then

$$w=g(v)$$

is well defined in Ω and satisfies

$$w_{ij}=\dot g v_{ij}+\ddot g v_i v_j.$$

Since $\ddot g\geqslant 0$ we see that the eigenvalues of w_{ij} are not less than the corresponding eigenvalues for $\dot g v_{ij}$ and hence

$$\lambda(w_{ij})\in\Gamma \quad \text{in the region } \left\{-\frac{\varepsilon}{2}\leqslant v\leqslant 0\right\}$$

and

$$\lambda(w_{ij})\in\bar\Gamma \quad \text{everywhere.}$$

Let $\zeta\geqslant 0$ be in $C^\infty(\Omega)$ with compact support in Ω such that $\zeta\equiv 1$ in the complement of the region $\{-\varepsilon/2\leqslant v\leqslant 0\}$. For $c>0$ small, consider the function

$$u=\frac{c}{2}\zeta|x|^2+w.$$

Then

$$u_{ij}=c\zeta\delta_{ij}+w_{ij}+c(\zeta_i x_j+\zeta_j x_i+\tfrac{1}{2}|x|^2\zeta_{ij}).$$

In the complement in Ω of the boundary strip $\{-\varepsilon/2\leqslant v\leqslant 0\}$, we have $\zeta\equiv 1$ and therefore

$$u_{ij}=c\delta_{ij}+w_{ij}.$$

Since $c\lambda(\mathrm{Id}) \in \Gamma$ and $\lambda(w_{ij}) \in \bar{\Gamma}$ it follows that

$$\lambda(u_{ij}) \in \Gamma.$$

In the boundary strip $\{-\varepsilon/2 \leqslant v \leqslant 0\}$ we have

$$\lambda(w_{ij}) \in \Gamma.$$

Hence for c sufficiently small, at every point in the strip,

$$\lambda(u_{ij}) \in \Gamma.$$

In case $\varphi \equiv 0$ the function u is admissible. Consider now the case of general φ. We may suppose that φ has been extended smoothly inside Ω. Set

$$\underline{u} = Au + \varphi$$

with A large, and observe that as x varies in $\bar{\Omega}$ the set of points $\lambda(u_{ij})$ fill out a compact set in Γ. Furthermore for δ_0 sufficiently small, the set of points $\lambda(u_{ij}+\delta\varphi_{ij})$, $0 \leqslant \delta \leqslant \delta_0$, fill out a compact set in Γ. By property (8) we see that for R sufficiently large, and $0 \leqslant \delta \leqslant \delta_0$,

$$\lambda(Ru_{ij}+R\delta\varphi_{ij}) \in \Gamma$$

and

$$f(\lambda(Ru_{ij}+R\delta\varphi_{ij})) \geqslant \psi_1 + 1$$

Taking $R=A$ so large that $A\delta_0 \geqslant 1$, we see that

$$\lambda(Au_{ij}+\varphi_{ij}) \in \Gamma \quad \text{i.e.} \quad \lambda(\underline{u}_{ij}) \in \Gamma.$$

Furthermore we see that

$$f(\lambda(\underline{u}_{ij})) \geqslant \psi_1 + 1$$

i.e. we have constructed a subsolution of (1).

3. The concavity condition

In this section we will verify that conditions (2) and (3) imply that the function

$$F(D^2u) = f(\lambda(u_{ij}))$$

is a concave function of the elements of the symmetric matrix $\{u_{ij}\}$ in the set where $\lambda(u_{ij})\in\Gamma$. We leave it to the reader to verify that condition (2) implies that F is elliptic at all admissible functions u.

Let $\lambda_1(U)\leq\lambda_2(U)\leq\ldots\leq\lambda_n(U)$ be the eigenvalues of the $n\times n$ symmetric matrix U with corresponding eigenvectors $u_1,\ldots,u_n$. By the min-max characterization of λ_1, $\lambda_1(U)$ is clearly a concave function U. More generally, from U we construct the self-adjoint operator

$$U^{[k]}=\sum_{i=1}^{k}1\otimes\ldots\otimes \underset{i}{U}\otimes\ldots\otimes 1$$

acting on the exterior powers Λ^k by

$$U^{[k]}\omega_1\wedge\ldots\wedge\omega_k=\sum_{i=1}^{k}\omega_1\wedge\ldots\wedge U\omega_i\wedge\ldots\wedge\omega_k$$

with eigenvalues $\lambda_{i_1}+\ldots+\lambda_{i_k}$ and eigenvectors $u_{i_1}\wedge\ldots\wedge u_{i_k}$, $i_1<i_2<\ldots<i_k$. Then $\lambda_1+\ldots+\lambda_k$ is a concave function of U.

Now $f(\lambda)$ is the infimum of linear functions of the form $\Sigma\mu_j\lambda_j+\mu_0$, with $\mu_j\geq 0, j\geq 0$. By the symmetry of f we may take the μ_j decreasing for $j>0$ (see Lemma 6.2). Then

$$\sum\mu_j\lambda_j+\mu_0=\sum_{1}^{n-1}(\mu_j-\mu_{j+1})(\lambda_1+\ldots+\lambda_j)+\mu_n(\lambda_1+\ldots+\lambda_n)+\mu_0$$

is a concave function of U so $f(\lambda)$ is a concave function of U.

4. Preliminary a priori estimates

We have proved necessity in Theorems 1 and 3 and shown how sufficiency in Theorem 3 follows from Theorem 1. Now we begin the proofs of the sufficiency in Theorems 1, 2 and 2'.

As in [1] and [2], the proofs go via the continuity method and a priori estimates. Set

$$u^0=\frac{k}{2}|x|^2$$

with $k>0$ chosen so that $f(k,\ldots,k)=\psi_1$. By conditions (6) and (8) there is such a unique k. We use the continuity method to find for every t in $0\leq t\leq 1$ the admissible solution u^t of

$$\begin{aligned} f(\lambda(u^t_{ij})) &= (1-t)\,\psi_1 + t\psi =: \psi^t \\ u^t &= t\varphi + (1-t)\,u^0 =: \varphi^t \quad \text{on } \partial\Omega. \end{aligned} \tag{1^t}$$

For $t=0$ the solution is u^0; for $t=1$ it is our desired solution of (1).

In [2] and N. V. Krylov [6] (¹) it was shown how, from a priori estimates

$$|u^t|_{C^2} \le C \tag{4.1}$$

and uniform ellipticity of the linearized operator $L=\Sigma F_{u_{ij}}(\{u_{kl}\})\,\partial_{ij}$, one may derive estimates

$$|u^t|_{C^{2+\mu}(\bar\Omega)} \le \bar C.$$

So it suffices to derive a priori estimates (4.1) and to show that the set of values $\lambda(u^t_{ij})$ for the solutions u^t lie in a compact set in Γ.

The a priori estimates

$$|u^t|_{C^1} \le C \tag{4.2}$$

in Theorems 1 and 2′ are easily established.

In section 2 we have constructed smooth subsolutions. Thus for each t we have a subsolution $\underline{u}^t$ satisfying (2.1). Clearly $|\underline{u}^t|_{C^1} \le C$ for $0\le t\le 1$. Using the maximum principle, and (9), we find

$$\underline{u}^t \le u^t \le v^t,$$

where v^t is the harmonic function in Ω which equals φ^t on $\partial\Omega$. Note that v^t need not be admissible. Clearly $|v^t|_{C^1} \le C$ for $0\le t\le 1$.

Consequently, for the interior normal derivative at any point on $\partial\Omega$,

$$\underline{u}^t_\nu \le u^t_\nu \le v^t_\nu.$$

Thus in each theorem we have $|u|\le C$ and

$$|\nabla u| \le C \quad \text{on } \partial\Omega. \tag{4.3}$$

(¹) These results make use of the purely interior estimates of the form (4.1) due to L. C. Evans and simplified by N. S. Trudinger; see the references [12], [13] and [16] in our paper [2].

Ignoring t we will derive further estimates for the admissible solution of (1). Differentiating the equation with respect to x_j we find

$$Lu_j = \psi_j,$$

where L is the linearized elliptic operator

$$L = F_{u_{ik}} \partial_{ik}.$$

So

$$|Lu_j| \leqslant C. \tag{4.4}$$

Using the concavity we find for our subsolution $\underline{u}$ satisfying (2.1),

$$F(\underline{u}_{ij}) \leqslant F(u_{ij}) + L(\underline{u} - u)$$

so that, by (2.1),

$$L(\underline{u} - u) \geqslant 1. \tag{4.5}$$

Consequently

$$L(C(\underline{u} - u) \pm u_j) \geqslant 0$$

and this implies that

$$C(\underline{u} - u) \pm u_j$$

takes its maximum on the boundary. From (4.3) we then conclude (4.2).

In the next sections we will establish the a priori estimates

$$|u_{ij}| \leqslant C_1 \quad \text{on } \partial\Omega. \tag{4.6}$$

Using (4.5) we conclude this section by establishing

$$|u_{ij}| \leqslant C_2 \quad \text{in } \bar{\Omega}. \tag{4.7}$$

From (6), and the fact that $F(u_{ij}^t) = \psi^t \leqslant \psi$, it then follows that the set of values of $\lambda(u_{ij}^t)$ lie in a compact subset of Γ and hence that L is uniformly elliptic.

To prove (4.7), let

$$\partial_\xi = \sum \xi_i \partial_{x_i}, \quad \sum \xi_i^2 = 1$$

be any fixed directional differential operator. Applying ∂_ξ twice to the equation (1) we find:

$$L\partial_\xi u = \partial_\xi \psi$$
$$L\partial_\xi^2 u \geq \partial_\xi^2 \psi \geq -C,$$

the second being a consequence of the concavity of F (as in [1] and [2]). From (4.5) it follows that

$$C(\underline{u}-u)+\partial_\xi^2 u$$

takes its maximum on $\partial\Omega$. By (4.6) we infer that

$$\partial_\xi^2 u \leq C \quad \text{in } \bar{\Omega}$$

(with a different constant C).

This is true for every such directional derivative. From $\Delta u > 0$ we may infer that for any such operator we also have

$$-\partial_\xi^2 u \leq (n-1)\, C$$

so that

$$|\partial_\xi^2 u| \leq (n-1)\, C.$$

In particular

$$|u_{ii}| \leq (n-1)\, C.$$

Taking

$$\partial_\xi = \frac{1}{\sqrt{2}}(\partial_i \pm \partial_j) \quad \text{for } i \neq j$$

we conclude that

$$|u_{ij}| \leq (n-1)\, C$$

and (4.7) is proved.

5. Estimates for some second derivatives on the boundary. Proofs of Theorem 1 and of Theorem 2(a)

In this section we will estimate some second derivatives at any boundary point. As in the proof of Proposition 1.3 we may suppose the boundary point is the origin, that the positive x_n axis is interior normal there and that the boundary is locally represented by (1.7). As before we find at the origin—assuming φ has been extended smoothly to $\bar{\Omega}$ with $\varphi_n(0)=0$—

$$u_{\alpha\beta}=\varphi_{\alpha\beta}-u_n\varrho_{\alpha\beta}, \quad \alpha,\beta<n. \tag{5.1}$$

Thus we have

$$|u_{\alpha\beta}(0)|\leq C, \quad \alpha,\beta<n. \tag{5.2}$$

We will establish next the estimate

$$|u_{\alpha n}(0)|\leq C \quad \text{for } \alpha<n \tag{5.3}$$

in Theorems 1, 2 or 2′. The proof is an extension of the argument in [1]. Observe first that since F depends only on the eigenvalues of u_{ij} it is invariant under rotation of coordinates. It follows that for the operator

$$x_i\partial_j-x_j\partial_i, \quad i\neq j,$$

which is the infinitesimal generator of a rotation, we have

$$L(x_i\partial_j-x_j\partial_i)\,u=(x_i\partial_j-x_j\partial_i)\,\psi. \tag{5.4}$$

By subtracting a linear function we may suppose $u=\varphi=0$ and $u_\beta=\varphi_\beta=0$ at the origin for $\beta<n$. For $\alpha<n$ let

$$T=\partial_\alpha+\varkappa_\alpha(0)\,(x_\alpha\partial_n-x_n\partial_\alpha),$$

so that on $\partial\Omega$, near the origin,

$$T(u-\varphi)=(\partial_\alpha+\varrho_\alpha\partial_n)\,(u-\varphi)+O(|x'|^2)=O(|x'|^2). \tag{5.5}$$

From (4.4) and (5.4) it follows that

$$|L(Tu)|\leq C_0$$

and

$$|L(T(u-\varphi))|\leq C_0+C_0\sum_i F_{u_{ii}}. \tag{5.4$'$}$$

We are now going to make use of the condition (10), according to which we may choose $\delta>0$ sufficiently small and R sufficiently large so that

$$(\varkappa_1-2\delta,\varkappa_2-2\delta,\ldots,\varkappa_{n-1}-2\delta,R)\in\Gamma. \tag{5.6}$$

The numbers δ and R may be fixed so that this holds for every point on $\partial\Omega$. Near the origin set

$$v=\sigma+\frac{\delta}{2}|x|^2;$$

where

$$\sigma=\sum_{\alpha<n}\frac{\varkappa_\alpha-2\delta}{2}x_\alpha^2+\frac{R}{2}x_n^2-x_n.$$

Set

$$w=Av-u=A\sigma-u+\frac{A\delta}{2}|x|^2, \tag{5.7}$$

A will be chosen large. In fact, by concavity,

$$L(A\sigma-u)\geq f(\lambda(A\sigma_{ij}))-F(u_{ij}).$$

By (5.6) and condition (8) we may choose A so large that

$$f(\lambda(A\sigma_{ij}))\geq C_0 \tag{5.8}$$

where C_0 is the constant in (5.4)$'$. In addition we have

$$L\frac{A\delta}{2}|x|^2=A\delta\sum F_{u_{ii}}$$

and thus for A sufficiently large (depending on δ, which is fixed) we have (by (5.4)$'$)

$$Lw\geq|LT(u-\varphi)|.$$

Hence in the closure of

$$B_\varepsilon=\Omega\cap\{|x|<\varepsilon\},$$

the function $w\pm T(u-\varphi)$ attains its maximum on ∂B_ε.

We will choose ε small and then A sufficiently large so that

$$w\pm T(u-\varphi)\leq 0 \quad \text{on } \partial B_\varepsilon. \tag{5.9}$$

It then follows that

$$|u_{\alpha n}(0)-\varphi_{\alpha n}(0)|\leq|w_n(0)|,$$

so that (5.3) holds.

To prove (5.9) consider first $\partial B_\varepsilon \cap \partial\Omega$; there we have

$$v \leqslant -\frac{\delta}{4}|x'|^2$$

if ε is sufficiently small (depending on δ and R). Hence by (5.5),

$$w \pm T(u-\varphi) \leqslant -\frac{A\delta}{4}|x'|^2 + C|x'|^2 \leqslant 0$$

for A sufficiently large.

On $\partial B_\varepsilon \cap \Omega$ we distinguish two cases.

Case 1. $\frac{1}{2}\delta|x'|^2 > (R+\delta)x_n^2 = (R+\delta)(\varepsilon^2 - |x'|^2)$. Then

$$\begin{aligned} w \pm Tu &\leqslant Av + C \\ &\leqslant A\left(\sum \frac{\varkappa_\alpha - \delta}{2} x_\alpha^2 - \varrho + \frac{R+\delta}{2} x_n^2\right) + C \\ &\leqslant A\left(-\frac{\delta}{2}|x'|^2 + O(|x'|)^3) + \frac{R+\delta}{2} x_n^2\right) + C \\ &\leqslant A\left(-\frac{\delta}{4}|x'|^2 + O(|x'|^3)\right) + C, \end{aligned}$$

since we are in case 1. Thus

$$w \pm Tu \leqslant -\frac{A\delta}{8}|x'|^2 + C \leqslant -c_1 A\varepsilon^2 + C$$

for sufficiently small ε, with a positive number c_1 (depending on δ and R). When $c_1 A\varepsilon^2 \geqslant C$ we obtain $w \pm Tu \leqslant 0$.

Case 2. $\frac{1}{2}\delta|x'|^2 \leqslant (R+\delta)x_n^2$. On this portion of $\partial B_\varepsilon \cap \Omega$ we have

$$x_n \geqslant c_2\varepsilon \quad \text{with } c_2 > 0.$$

So

$$\begin{aligned} w \pm Tu &\leqslant Av + C \\ &\leqslant A\left(C|x'|^2 + \frac{R+\delta}{2} x_n^2 - x_n\right) + C \\ &\leqslant A(C_1 x_n^2 - x_n) + C \end{aligned}$$

in our case 2, where C_1 depends on δ and R.

If $\varepsilon c_1 < 1/2$ it follows that

$$w \pm Tu \leqslant C - \frac{A}{2} x_n.$$

Now fix $\varepsilon > 0$ to satisfy all the requirements we have made. Then in this case we find

$$\begin{aligned} w \pm Tu &\leqslant C - \frac{A}{2} c_2 \varepsilon \\ &\leqslant 0 \quad \text{for } A \text{ sufficiently large.} \end{aligned}$$

If we finally fix A (depending on ε) to satisfy all the requirements we have imposed, we obtain (5.9) and hence (5.3).

To complete the proofs of Theorems 1, 2 and 2′ we have to establish the bound

$$|u_{nn}(0)| \leqslant C. \tag{5.10}$$

At the origin, $\Sigma_{\alpha<n} u_{\alpha\alpha} + u_{nn} > 0$ by (9) and so $u_{nn}(0) \geqslant -C$ and we have only to prove

$$m = u_{nn}(0) \leqslant C. \tag{5.11}$$

We first prove this for Theorem 1. We may suppose $u = \varphi = 0$ on $\partial\Omega$. Then we have (1.8) at the origin, i.e.

$$u_{\alpha\beta} = -u_n \varkappa_\alpha \delta_{\alpha\beta} \tag{5.12}$$

Since, by (9), $\Delta u > \delta$ in Ω we see by the Hopf lemma that

$$-u_n(0) \geqslant a > 0 \quad \text{for some positive constant } a.$$

Suppose that $u_{nn}(0) = m$ is very large. We may then apply Lemma 1.2 and infer that the eigenvalues of (u_{ij}) are

$$\begin{aligned} \lambda_\alpha &= -u_n(0) \varkappa_\alpha(0) + o(1), \quad \text{for } \alpha < n \\ \lambda_n &= m\left(1 + O\left(\frac{1}{m}\right)\right). \end{aligned}$$

It follows from (10) and (7) that

$$f(\lambda_1, \ldots, \lambda_n) > \max \psi$$

which is a contradiction. Hence (5.11) holds with some constant C, under control.

Consider next Theorem 2 for Γ of type 2. It follows from (5.2) that at the origin,

$$|\lambda_\alpha| \leqslant C \quad \text{for } \alpha < n.$$

Since Γ is of type 2, it follows that for some constant M, depending on C and on Γ, $(\lambda_1, \ldots, \lambda_{n-1}, M)$ lies in a compact subset of Γ. Using condition (7) we infer that at the origin, λ_n is bounded from above. Hence so is $u_{nn}(0)$ and (5.11) is proved in this case.

Theorem 1 is completely proved and Theorem 2 is proved for Γ of type 2.

6. Completion of the proof of Theorem 2

It remains to prove (5.11):

$$m = u_{nn}(0) \leqslant C.$$

Our proof is rather tricky and long—there should be a shorter one.

Let Γ' denote the projection to $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$ of Γ. Since Γ is of type 1, Γ' is an open convex cone in $\mathbf{R}^{n-1}$ which is not all of $\mathbf{R}^{n-1}$. At the origin we have (5.1) and $u=\varphi=0$, $u_\alpha=\varphi_\alpha=0$. (From now on derivatives are computed at the origin.) Using condition (10) we see that for large positive and negative t, the eigenvalues λ' of the $(n-1)$ by $(n-1)$ matrix $\{\varphi_{\alpha\beta}+t\varrho_{\alpha\beta}\}$ belong to Γ' and the complement of $\bar{\Gamma}'$ respectively. Let t_0 be the first value of t as we decrease t from $+\infty$ such that

$$\lambda'(\sigma_{\alpha\beta}) \in \partial\Gamma',$$

where

$$\sigma = \tfrac{1}{2}(\varphi_{\alpha\beta}(0) + t_0\,\varrho_{\alpha\beta}(0))\,x_\alpha x_\beta. \tag{6.1}$$

Then $|t_0| \leqslant C$, for C under control. Without loss of generality we may suppose $\{\sigma_{\alpha\beta}\}$ to be diagonal with the diagonal elements $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots \leqslant \tilde{\lambda}_{n-1}$.

Suppose m is very large. By Lemma 1.2 the first $n-1$ eigenvalues of $u_{ij}(0)$, $\lambda'(u_{ij})$, are given by

$$\lambda'(u_{\alpha\beta}) + o(1).$$

From (5.1):

$$u_{\alpha\beta} = \varphi_{\alpha\beta} - u_n\,\varrho_{\alpha\beta},$$

it follows that $-u_n$ cannot be much lower than t_0. Our aim is to establish the estimate

$$-u_n(0) \geqslant t_0 + \eta \tag{6.2}$$

for some fixed $\eta>0$. From the definition of t_0 it then follows that $\lambda'(u_{\alpha\beta})$ is in Γ' and its distance to $\partial\Gamma'$ is greater than some positive constant η'. If m is very large, $\lambda'(u_{ij})$ is close to $\lambda'(u_{\alpha\beta})$ and so its distance to $\partial\Gamma'$ is greater than $\eta'/2$. So for some constant M under control, $(\lambda'(u_{ij}), M)$ belongs to a compact set in Γ. But condition (7) then yields a bound on λ_n, and hence on $m=u_{nn}(0)$.

On $\partial\Omega$ near 0 we have (recall that derivatives of φ and ϱ are evaluated at the origin, and summation over Greek letters goes up to $n-1$):

$$u=\varphi=\tfrac{1}{2}\varphi_{\alpha\beta}x_\alpha x_\beta+t_0(\tfrac{1}{2}\varrho_{\alpha\beta}x_\alpha x_\beta-\varrho(x'))+P(x')+R(x') \tag{6.3}$$

where P is a homogeneous cubic in x' and

$$|R|\leq C|x'|^4. \tag{6.4}$$

At $\lambda'(\sigma_{\alpha\beta})=\tilde{\lambda}$, the cone Γ' has a plane of support, i.e.

$$\Gamma' \text{ lies in } \left\{\sum\mu_\alpha(\lambda_\alpha-\tilde{\lambda}_\alpha)>0\right\}, \quad \sum\mu_\alpha=1. \tag{6.5}$$

We shall make use of

LEMMA 6.1. *If $\tilde{\lambda}_1\leq\ldots\leq\tilde{\lambda}_{n-1}$, there is such a plane of support with*

$$\mu_1\geq\mu_2\geq\ldots\geq\mu_{n-1}\geq 0. \tag{6.6}$$

Proof. Since Γ' contains the positive cone in $\mathbf{R}^{n-1}$ it is clear that for any plane of support, all $\mu_\alpha\geq 0$. Suppose first that the components of $\tilde{\lambda}$ are all unequal, i.e. $\tilde{\lambda}_1<\tilde{\lambda}_2<\ldots$. By symmetry $(\tilde{\lambda}_2,\tilde{\lambda}_1,\tilde{\lambda}_3,\ldots,\tilde{\lambda}_n)$ belongs to $\overline{\Gamma'}$ and hence

$$\mu_1(\tilde{\lambda}_2-\tilde{\lambda}_1)+\mu_2(\tilde{\lambda}_1-\tilde{\lambda}_2)\geq 0$$

and so $\mu_1\geq\mu_2$. Similarly we find (6.6). To prove the lemma in general it suffices to show that any point $\tilde{\lambda}\in\partial\Gamma'$ can be approached by points on $\partial\Gamma'$ with no two components equal. If this were not the case, then near $\tilde{\lambda}$, the boundary $\partial\Gamma'$ would have to lie in a hyperplane of the form $\lambda_j=\lambda_k$, say $\lambda_1=\lambda_2$. But then near $\tilde{\lambda}$, $\partial\Gamma'$ must coincide with that hyperplane, and the hyperplane is then necessarily a plane of support of Γ'. Thus in Γ' we would have, say, $\lambda_1-\lambda_2>0$ — contradicting the symmetry of Γ' in the λ's. The lemma is proved.

Since $t\tilde{\lambda}\in\partial\Gamma'$ for $t>0$ we have $\Sigma\mu_\alpha\tilde{\lambda}_\alpha(t-1)\geq 0$ for all $t>0$ and hence $\Sigma\mu_\alpha\tilde{\lambda}_\alpha=0$.

From now on we may assume:

$$\Gamma' \text{ lies in } \left\{\sum \mu_\alpha \lambda_\alpha > 0\right\}, \text{ with } \mu_1 \geq \ldots \geq \mu_{n-1} \geq 0, \text{ and } \sum \mu_\alpha = 1. \quad \sum \mu_\alpha \tilde{\lambda}_\alpha = 0. \tag{6.7}$$

We will also make use of

LEMMA 6.2. *Let $A=\{a_{ij}\}$ be a square $n\times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Let $\mu_1 \geq \ldots \geq \mu_n \geq 0$ be given numbers. Consider an orthonormal basis of vectors $b^1, \ldots, b^n$ and set*

$$a^i = \sqrt{\mu_i}\, b^i, \quad 1 \leq i \leq n.$$

Then

$$\sum_i \langle A a^i, a^i\rangle \geq \sum \mu_i \lambda_i.$$

In particular, we have

$$\sum \mu_i a_{ii} \geq \sum \mu_i \lambda_i.$$

The lemma is a special case of a result of M. Marcus [7]. For convenience we include a short proof.

Proof. We may suppose the matrix A is diagonal. Then if $b^i=(b_1^i, \ldots, b_n^i)$, we have

$$J = \sum (Av^i, v^i) = \sum \lambda_j \mu_i (b_j^i)^2 = \sum c_{ij} \lambda_j \mu_i,$$

where the matrix $c_{ij} = (b_j^i)^2$ is a doubly stochastic matrix. So the minimum over the convex set of all doubly stochastic matrices of $\Sigma\, c_{ij}\mu_i\lambda_j$ is achieved at an extreme point, i.e., at a doubly stochastic matrix where each element is either 0 or 1. Thus we find

$$J \geq \sum \lambda_j \sigma_j$$

where $(\sigma_1, \ldots, \sigma_n)$ is a permutation of $(\mu_1, \ldots, \mu_n)$. By induction on n it is easily verified that min $\Sigma\, \lambda_j \sigma_j$ for such permutations is achieved by $\Sigma\, \lambda_j \mu_j$. Q.E.D.

Returning to our domain Ω near 0, let S be a surface represented locally by

$$x_n = \varrho(x') - \frac{\tau}{2}|x'|^2 \quad \text{with } 0<\tau \text{ small,} \tag{6.8}$$

and let $d(x)$ denote the distance of $x \in \Omega$ from S. At the origin the eigenvalues of d_{ij} are (not in order)

$$\lambda(d_{ij}) = (\tau - \varkappa_1, \dots, \tau - \varkappa_{n-1}, 0) \tag{6.8$'$}$$

By our critical hypothesis (10), $(\varkappa_1, \dots, \varkappa_{n-1})$ is in Γ', and hence for fixed positive τ sufficiently small we have

$$\sum \mu_a \varkappa_a - \tau \geqslant a > 0, \tag{6.9}$$

for some fixed positive constant a (independent of the particular point on $\partial\Omega$ which we have chosen as origin). This holds for any ordering of the $\varkappa_\alpha$. We take τ to be so fixed from now on.

For $\alpha=1,\dots,n-1$, let $b^\alpha(x)$ be smooth orthonormal vector fields in $\bar\Omega$ near 0 tangent to the level surfaces d=constant (i.e., orthogonal to ∇d) and such that $b^\alpha(0,x_n)$ is the unit vector in the x_α direction. Set

$$a^\alpha(x) = \sqrt{\mu_a}\, b^\alpha(x), \quad \alpha = 1, \dots, n-1, \quad \text{and} \quad \Lambda = \sum_{i,j} \sum_a a_i^\alpha a_j^\alpha \partial_i \partial_j. \tag{6.10}$$

Recall that $\sigma_{\alpha\beta}$ is diagonal, with eigenvalues $\tilde\lambda$, and that also $\Sigma \mu_\alpha \tilde\lambda_\alpha = 0$. For h small let

$$D_h = \{x \in \Omega \mid |x'| < h,\ x_n < h^2\}.$$

In D_h, where for $h \leqslant h_0$ small,

$$d \leqslant C_1 h^2, \tag{6.11}$$

we will employ the barrier function

$$v = w + \eta(C_0|x'|^2 - x_n), \tag{6.12}$$

where

$$w = \sigma(x') - t_0 x_n + P(x') + l(x')\left(\frac{\tau}{2}|x'|^2 - d\right) + \frac{M}{2} d^2. \tag{6.13}$$

Here C_0 is fixed so that $C_0|x'|^2-x_n\geq 0$ on $\partial\Omega\cap D_{h_0}$; $l(x')$ is a suitable linear function of x', M and η will be chosen, respectively, as large and small positive constants, so that we will have

$$u\leq v \quad \text{on } \partial D_h, \tag{6.14}$$

and at every point in D_h:

$$\lambda(v_{ij})\notin \bar{\Gamma}=\{\lambda\in\Gamma\,|\,f(\lambda)\geq\psi_0\}. \tag{6.15}$$

According to the maximum principle, Lemma B, it then follows that

$$u\leq v \quad \text{in } D_h,$$

which yields (6.2).

With the a^α as in (6.10) we will first determine $l(x')$ and then choose M large so as to guarantee that for h small:

$$u\leq w \quad \text{on } \partial\Omega\cap\bar{D}_h, \tag{6.14$'$}$$

$$u-w\leq -1 \quad \text{on } \partial D_h\cap\Omega, \tag{6.14$''$}$$

and

$$\Lambda w<0 \quad \text{in } D_h. \tag{6.16}$$

Inequality (6.16) has the important consequence: it implies that at every point in D_h,

$$\lambda(w_{ij})\notin\Gamma \tag{6.17}$$

independent of the size of M. For if $\nu_1\leq\ldots\leq\nu_n$ are the eigenvalues of $w_{ij}(x)$ then we know by Lemma 6.2 with $\mu_n=0$ that

$$\sum\mu_\alpha\nu_\alpha<0$$

and hence $(\nu_1,\ldots,\nu_{n-1})$ is not in Γ'.

Once $l(x')$, h and M are fixed so that (6.14)$'$, (6.14)$''$ and (6.16) hold, then, by fixing η sufficiently small, we see that (6.14) and (6.15) hold and we are through.

To establish (6.16) observe first that by Lemma 6.2 and (6.8), (6.9),

$$-\Lambda d=-\sum d_{ij}a_i^\alpha a_j^\alpha\geq a \quad \text{at the origin.} \tag{6.18}$$

From (6.7) we see that $\Lambda\sigma=0$, at $(0, x_n)$.

Hence at any point in D_h we have

$$\Lambda\sigma = m_1(x')+O(|x'|^2)$$

where m_1 is a linear function. Furthermore we find in D_h,

$$\Lambda P = m_2(x')+O(|x'|^2)$$

where $m_2(x')$ is linear. Thus for $m(x')=m_1+m_2$, in D_h,

$$|\Lambda(\sigma+P)-m(x')| \leq C|x'|^2. \tag{6.19}$$

The linear function $l(x')$ in (6.13) is now determined via the following lemma (recall (6.8)):

LEMMA 6.3. *Let $m(x')$ be a linear function with coefficients bounded by K. There exists a unique linear function $l(x')$ such that*

$$\left|\Lambda\left[l(x')\left(d-\frac{\tau}{2}|x'|^2\right)\right]-m(x')\right| \leq CK|x|^2.$$

Here C is a constant independent of K, and the coefficients of l are bounded by CK.

Proof. Set $l=\Sigma l_\alpha x_\alpha$. Since the vectors a^α are perpendicular to ∇d we see that $\Lambda(ld)=l\Lambda d$. If we expand

$$I=\Lambda\left[l\left(d-\frac{\tau}{2}|x'|^2\right)\right]$$

near the origin we find

$$I=l(x')\left[(\Lambda d)(0)-\tau\sum\mu_\alpha\right]-2\tau\sum\mu_\alpha l_\alpha x_\alpha+O(|x|^2).$$

(Here the constant in $O(|x|^2)$ depends on the bound on the coefficients of l.) By (6.18), we have $(\Lambda d)(0)\leq -a$, and it follows that the lemma holds with

$$l(x')=\sum[(\Lambda d)(0)-\tau-2\tau\mu_\alpha]^{-1}m_\alpha x_\alpha. \qquad \text{Q.E.D.}$$

With the function l determined, we now require M to be large enough to ensure (6.16). Namely, by (6.19), and the lemma, we have (recall a^α is perpendicular to ∇d)

$$\Lambda w \leq C|x|^2+Md\Lambda d.$$

For h sufficiently small we have $\Lambda d \leqslant -a/2$ in D_h and hence in D_h

$$\Lambda w \leqslant C|x|^2 - \frac{Ma}{2} d.$$

Since

$$d \geqslant c_0(|x'|^2 + |x_n|) \quad \text{in } D_h \text{ for some } c_0 > 0, \tag{6.20}$$

we obtain (6.16) i.e. $\Lambda w < 0$ in D_h, and (6.17) holds, provided

$$M \geqslant M_0 \quad \text{sufficiently large.} \tag{6.21}$$

Next we establish (6.14)'. On $\partial\Omega \cap \bar{D}_h$ we have for h small,

$$\left| \frac{\tau}{2} |x'|^2 - d \right| \leqslant C|x'|^3.$$

Using (6.3), (6.4) and (6.13) we find on $\partial\Omega \cap \bar{D}_h$, with, as usual, a different constant C (under control)

$$u - w \leqslant C|x'|^4 - \frac{M}{2} d^2 \leqslant 0$$

provided M is sufficiently large, i.e. (6.14)' holds.

Turn to (6.14)''. On $\partial D_h \cap \Omega$ we have, from (6.20),

$$d \geqslant c_0 h^2.$$

and we now finally fix M (depending on h which has been fixed) satisfying (6.21) and the other requirements, so that (6.14)'' holds.

The proof is complete.

7. Completion of the proof of Theorem 2′

Here ψ=constant. The proof of the final estimate (5.11):

$$m = u_{nn}(0) \leqslant C,$$

is a modification of that of Section 6, and we will use the same notation. At the origin we have (5.1) and $u = \varphi = u_\alpha = \varphi_\alpha = 0$.

Let Γ_0' denote the projection to λ' of the convex set

$$\Gamma_0 = \{\lambda \in \Gamma \mid f(\lambda) > \psi\}.$$

Γ_0' is a convex subset of the cone Γ'. By condition (8) we see that for any compact set K in Γ' there is a constant t such that $tK \subset \Gamma_0'$. Also $t\Gamma_0' \subset \Gamma_0'$ for all $t>1$.

Remark 7.1. If the projection K' to λ'-space of a compact set K in λ-space is disjoint from Γ_0' then, for some positive $\varepsilon=\varepsilon(K)$,

$$K \text{ is disjoint from } \{\lambda \in \Gamma \mid f(\lambda) \geq \psi-\varepsilon\}. \tag{7.1}$$

Proof. If not there would be a sequence of points λ^j in K with $\lambda^j \in \Gamma$, and $f(\lambda^j) \to \psi$. Choosing a subsequence converging to $\lambda=(\lambda', \lambda_n) \in K$ we would conclude that $\lambda \in \Gamma$ (recall (6)) and $f(\lambda)=\psi$. But then we would have $f(\lambda', \lambda_n+1)>\psi$ and hence $\lambda' \in \Gamma_0'$, contradicting the fact that λ' has to be disjoint from Γ_0'.

Using condition (8) and (10) we see that for large positive and negative t, the eigenvalues λ' of the $(n-1)$ by $(n-1)$ matrix $\{\varphi_{\alpha\beta}+t\varrho_{\alpha\beta}\}$ belong to Γ_0' and the complement of $\bar{\Gamma}_0'$ respectively. Let t_0 be the first value of t as we decrease t from $+\infty$ such that

$$\lambda'(\sigma_{\alpha\beta}) \in \partial\Gamma_0',$$

where

$$\sigma=\tfrac{1}{2}(\varphi_{\alpha\beta}(0)+t\varrho_{\alpha\beta}(0))\, x_\alpha x_\beta.$$

This differs from the definition of t_0 in Section 6 since Γ' has been replaced by Γ_0'; $|t_0| \leq C$, for C under control. From now on we take $t=t_0$ in σ and without loss of generality we may suppose $\{\sigma_{\alpha\beta}\}$ to be diagonal with the diagonal elements $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_{n-1}$.

Suppose m is very large. By Lemma 1.2 the first $n-1$ eigenvalues of $u_{ij}(0)$, $\lambda'(u_{ij})$, are given by

$$\lambda'(u_{\alpha\beta})+o(1).$$

From (5.1):

$$u_{\alpha\beta}=\varphi_{\alpha\beta}-u_n \varrho_{\alpha\beta},$$

it follows that $-u_n$ cannot be much lower than t_0. Our aim, as before, is to establish the estimate

$$-u_n(0) \geq t_0+\eta \tag{7.2}$$

for some fixed $\eta>0$. From the definition of t_0 it then follows that $\lambda'(u_{\alpha\beta})$ is in Γ_0' and its distance to $\partial\Gamma_0'$ is greater than some positive constant η'. If m is very large, $\lambda'(u_{ij})$ is close to $\lambda'(u_{\alpha\beta})$ and so its distance to $\partial\Gamma_0'$ is greater than $\eta'/2$. But then $f(\lambda(u_{ij}(0)))>\psi+\varepsilon$ for some fixed $\varepsilon>0$, a contradiction. Hence $m=u_{nn}(0)$ must be bounded.

On $\partial\Omega$ near 0 we have (6.3) and (6.4). At $\lambda'(\sigma_{\alpha\beta})=\bar{\lambda}$, the set Γ_0' has a plane of support, i.e.

$$\Gamma_0' \text{ lies in } \left\{\sum \mu_\alpha(\lambda_\alpha-\bar{\lambda}_\alpha)>0\right\}, \quad \sum \mu_\alpha=1. \tag{7.3}$$

Since for any λ' in the positive cone in $\mathbf{R}^{n-1}$, $t\lambda'\in\Gamma_0'$ for t sufficiently large it is clear that each $\mu_\alpha\geq 0$. Lemma 6.1 continues to hold. Furthermore, since $t\bar{\lambda}\in\overline{\Gamma_0'}$ for $t\geq 1$, we have $\Sigma\mu_\alpha\bar{\lambda}_\alpha(t-1)\geq 0$ for all $t>1$, and hence $k=\Sigma\mu_\alpha\bar{\lambda}_\alpha\geq 0$. From now on we may assume

$$\begin{gathered}\Gamma_0' \text{ lies in } \sum \mu_\alpha\lambda_\alpha-k>0 \text{ with } \mu_1\geq\ldots\geq\mu_{n-1}\geq 0 \quad\text{and}\\ \sum\mu_\alpha=1; \quad k=\sum\mu_\alpha\bar{\lambda}_\alpha\geq 0.\end{gathered} \tag{7.4}$$

We will use d and (6.8), (6.8)$'$ as in Section 6. As we have remarked, since $\varkappa=(\varkappa_1,\ldots,\varkappa_{n-1})$ is in Γ', for some $t\geq 1$, under control, we have $t\varkappa\in\Gamma_0'$. By (7.4), $t\Sigma\mu_\alpha\varkappa_\alpha\geq k+b\geq b$ for some fixed $b>0$. Hence for fixed positive τ sufficiently small,

$$\sum\mu_\alpha\varkappa_\alpha-\tau\geq a>0 \tag{7.5}$$

for some fixed positive constant a (independent of the particular point on $\partial\Omega$ which we have chosen as origin).

In

$$D_h=\{x\in\Omega\,|\,|x'|<h,\ x_n<h^2\};$$

as before, with h so small that (6.11) holds, we will use b^α and a^α as in (6.10), and the operator

$$\Lambda=\sum_{i,j}\sum_\alpha a_i^\alpha a_j^\alpha\partial_i\partial_j. \tag{7.6}$$

Now $\Lambda\sigma(0,x_n)=k$ and therefore

$$\Lambda\sigma-k=m_1(x')+O(|x'|^2).$$

In place of (6.19) we have

$$|\Lambda(\sigma+P)-k-m(x')|\leqslant C|x'|^2. \tag{7.7}$$

With the new value of t_0 we use again the barrier function

$$v=w+\eta(C_0|x'|^2-x_n) \tag{7.8}$$

of (6.12), where, as before,

$$w=\sigma(x')-t_0\,x_n+P(x')+l(x')\left(\frac{\tau}{2}|x'|^2-d\right)+\frac{M}{2}d^2. \tag{7.9}$$

Here the linear function l is the same one as in Section 6 — determined by Lemma 6.1.

We will choose h small, M large and then η small so that (6.14) holds, i.e. $u\leqslant v$ on ∂D_h, and also the analogue of (6.15): at every point in D_h,

$$\lambda(v_{ij})\notin\{\lambda\in\Gamma\,|\,f(\lambda)\geqslant\psi\}. \tag{7.10}$$

As before it follows from Lemma B that $u\leqslant v$ in D_h; (7.2) then follows.

To prove (6.14) and (7.10) we will establish (6.14)′, (6.14)″ and the analogue of (6.16):

$$\Lambda w-k<0 \quad \text{in } D_h. \tag{7.11}$$

To establish these we follow the arguments of the preceding section. Having determined l we have

$$Lw-k\leqslant C|x|^2+MdLd$$

and as before, for h small and M large we obtain (7.11). (6.14)′ and (6.14)″ are then obtained as before by taking M sufficiently large. M is now fixed.

From (7.11) follows the crucial fact that at every point in $\bar{D}_h$

$$\lambda(w_{ij})\notin\Gamma_0.$$

For if $\nu_1\leqslant\ldots\leqslant\nu_n$ are the eigenvalues of $w_{ij}(x)$ then, by Lemma 6.2,

$$\sum\mu_\alpha\nu_\alpha-k\leqslant 0 \quad \text{in } \bar{D}_h$$

and hence $(\nu_1,\ldots,\nu_{n-1})$ is not in Γ_0'. Consequently, having fixed M, it follows from Remark 7.1 that for some fixed $\varepsilon>0$, and every point in D_h,

$$\lambda(w_{ij})\notin\{\lambda\in\Gamma\,|\,f(\lambda)\geqslant\psi-\varepsilon\}. \tag{7.12}$$

Thus for fixed positive small η we obtain (6.14) and (7.10) from (6.14)′, (6.14)″ and (7.11).

The proof of (7.1) and hence of Theorem 2′ is complete.

8. Proof of Lemma C

Observe that for the function g in (17), $g(\lambda)=0$ is equivalent to

$$\tilde{g}(\lambda) = \sum \arg(1+i\lambda_j) = l\pi, \quad l \in \mathbf{Z}, \ |l| < \frac{n}{2}, \tag{8.1}$$

and

$$\tilde{g}_j = \frac{1}{1+\lambda_j^2} > 0. \tag{8.2}$$

Consequently the set $g^{-1}(0)$ is a complete analytic hypersurface; it may have more than one component. No component can be bounded for if it were it would necessarily be a compact hypersurface. Then λ_1 would have a maximum on it, and at that point, ∇g would be parallel to the λ_1-axis—contradicting the fact that $g_2 \neq 0$ there.

Case 1. $n=2k+1$ is odd.

(i) For any λ in Γ_+, the positive cone, $\arg(1+it\lambda_j)$ varies monotonically from 0 to $\pi/2$ as t goes from 0 to $+\infty$. So there are exactly k positive values of t, $0<t_1<t_2<\ldots<t_k$ such that $g(t\lambda)=0$, namely

$$\sum_j \arg(1+it_\alpha\lambda_j) = \alpha\pi, \quad \alpha = 1, \ldots, k. \tag{8.3}$$

The functions t_α are continuous functions of λ and decreasing in each λ_j. The corresponding points $t_\alpha\lambda$ lie on k different components of $g^{-1}(0)$. The points $-t_\alpha\lambda$ lie on k more components. In addition there is the component containing the origin, on which $\arg\pi(1+i\lambda_j)=0$. If we take $\lambda=(1,\ldots,1)$ we thus obtain n components of $g^{-1}(0)$; it is clear that each one is symmetric in the λ_i.

To see that there are no other components, observe that by (8.2), any component S of $g^{-1}(0)$ may be represented as a graph over an open subset $\tilde{S}$ of the plane $\Sigma\lambda_i=0$. We claim that $\tilde{S}$ is the whole plane, for if not, there would be a sequence of points μ^j in S, with $\mu^j\to\mu$, and a sequence of real numbers t^j with $|t^j|\to\infty$, say $t^j\to+\infty$, such that

$$\lambda^j = \mu^j + t^j(1, \ldots, 1) \in S.$$

But then $\tilde{g}(\lambda^j)\to n\pi/2$—contradicting (8.1). Therefore, S contains a point on the diagonal and must then be one of the n components described above.

(ii) For $\Gamma_+^1=\{\lambda\in\Gamma_+ \mid |\lambda|=1\}$, consider the smooth hypersurface

$$\Sigma=\{t_k(\lambda)\,\lambda,\ \text{for } \lambda\in\Gamma_+^1\}. \tag{8.4}$$

We claim that Σ is an entire component of $g^{-1}(0)$. This follows from the assertion:

$$\text{if } \lambda\in\Gamma_+^1,\ \lambda\to\partial\Gamma_+ \quad \text{then } t_k(\lambda)\to+\infty. \tag{8.5}$$

(8.5) is easily proved: Suppose a sequence λ^j in Γ_+^1 tends to $\mu\in\partial\Gamma_+$ and $t^j=t_k^j(\lambda^j)\to s<\infty$. We have

$$\arg(1+it^j\lambda_1^j)+\ldots+\arg(1+it^j\lambda_n^j)=\frac{n-1}{2}\pi=k\pi.$$

Since the smallest components λ_1^j tend to zero we find

$$\arg(1+is\mu_2)+\ldots+\arg(1+is\mu_n)=\frac{n-1}{2}\pi.$$

But this is impossible since $\arg(1+is\mu_j)<\pi/2$ for each j.

Since Σ is a complete hypersurface lying in Γ_+ it is the boundary of the unbounded set

$$G=\{t\lambda \mid \lambda\in\Gamma_+^1,\ t>t_k(\lambda)\}; \tag{8.6}$$

clearly $0\notin\bar{G}$.

(iii) Σ is strictly convex at every point.

To verify this it suffices to show that at any point on Σ, for any nonzero vector ξ perpendicular to ∇g, the quadratic

$$\sum \tilde{g}_{jm}\xi_j\xi_m$$

is definite. Differentiating (8.2) we find

$$\tilde{g}_{jm}\xi_j\xi_m=-2\sum\xi_j^2\frac{\lambda_j}{(1+\lambda_j^2)^2}. \tag{8.7}$$

The right hand side is clearly negative definite in Γ_+ and (iii) is proved.

(iv) We may now conclude that G, given by (8.6) is convex and satisfies the conditions (11), (12) of Theorem 4 with the cone $\Gamma=\Gamma_+$. It is clear that Γ is Γ_+; condition (11) follows from (8.2), and condition (12) also follows easily.

Case 2. $2<n=2k+2$ is even.

(i) As before, for $\lambda\in\Gamma_+$, there are k values of $t>0$, $t_1,\dots,t_k$ such that $g(t_\alpha\lambda)=0$, i.e.,

$$\arg\prod(1+it_\alpha\lambda_j)=\alpha\pi,\quad \alpha=1,\dots,k.$$

We also have $g(-t_\alpha\lambda)=0$, and of course $g(0)=0$. Again it follows that $g^{-1}(0)$ has exactly $2k+1=n-1$ components each of which is a smooth complete hypersurface which is symmetric in the λ_i.

(ii) Consider now the convex cone $\Gamma=$the connected component containing $(1,\dots,1)$ in which $\sigma^{(n-1)}$ is positive. This is described in Section 1; it contains Γ_+.

Claim. For every $\lambda\in\Gamma$ there is a positive $t=t_k$ so that

$$\arg\prod(1+it\lambda_j)=k\pi=\frac{n-2}{2}\pi. \tag{8.8}$$

Proof. We already know this for $\lambda\in\Gamma_+$ so we need only consider $\lambda\in\Gamma\setminus\Gamma_+$. For such $\lambda=(\lambda_1,\dots,\lambda_n)$ with $\lambda_1\leq\lambda_2\dots$, necessarily $\lambda_1\leq 0<\lambda_2$. For if $\lambda_1\leq\lambda_2\leq 0$ then, since, in Γ, $\sigma^{(n-1)}$ is increasing in each λ_i (see Section 1), it would follow that

$$\sigma^{(n-1)}(0,0,\lambda_3,\dots,\lambda_n)>0,$$

but in fact this is zero.

Suppose $\lambda_1=0$, then

$$\arg\prod_1^n(1+it\lambda_j)=\arg\prod_2^n(1+it\lambda_j)\to\frac{n-1}{2}\pi$$

so clearly there is a $t>0$ for which (8.8) holds. Suppose $\lambda_1<0$; in this case

$$\lim_{t\to+\infty}\arg\prod_1^n(1+it\lambda_j)=\frac{n-2}{2}\pi, \tag{8.9}$$

so we have to examine this case a bit more closely. For large t we have

$$\begin{aligned}\prod(1+it\lambda_j)&=i^nt^n\prod\lambda_j+i^{n-1}t^{n-1}\sigma^{(n-1)}(\lambda)+O(t^{n-2})\\&=i^{n-2}pt^n\left(1+\frac{i}{pt}\sigma^{(n-1)}(\lambda)+O(t^{-2})\right)\end{aligned} \tag{8.10}$$

where $p=-\Pi\lambda_j>0$. Since $\sigma^{(n-1)}(\lambda)>0$ we see that $\arg\Pi(1+it_j)>(n-2)/2$ for t large, and so for some $t>0$ we have (8.8).

Next we prove the analogue of (8.5). Set $\Gamma^1=\Gamma\cap S^{n-1}$.

$$\text{If } \lambda\in\Gamma^1,\ \lambda\to\partial\Gamma, \quad \text{then } t_k(\lambda)\to+\infty. \tag{8.5$'$}$$

As before this is easily proved. If it is not true there is a sequence $\lambda^j\in\Gamma^1$ tending to $\mu\in\partial\Gamma$ and $t^j=t^j_k(\lambda^j)\to s<\infty$, such that $\arg\Pi(1+is\mu_r)=k\pi$. The polynomial in t, $g(t\mu)$ then has k positive roots. Their negatives are also roots, as is $t=0$. So $g(t\mu)$ has $2k+1=n-1$ roots. However $\mu\in\partial\Gamma$, and so $\sigma^{(n-1)}(\mu)=0$. From (8.10) we see that

$$g(t\mu)=\operatorname{Im}\prod(1+it\mu_r)$$

is a polynomial of degree $n-2$; it cannot have $(n-1)$ roots—(8.5)$'$ is proved.

We conclude that $\Sigma=\{t_k(\lambda)\lambda\,|\,\lambda\in\Gamma^1\}$ is a complete hypersurface lying in Γ.

(iii) To show that

$$G=\{t\lambda\,|\,\lambda\in\Gamma^1,\ t>t_k(\lambda)\}$$

satisfies the conditions of Theorem 4, and hence to conclude the proof of Lemma C, we prove finally

LEMMA 8.1. *Σ is strictly convex at every point.*

Proof. We must prove that on Σ,

$$\sum \tilde g_{jm}\xi_j\xi_m \text{ is definite} \quad \text{if } \sum\xi_j\tilde g_j=0,\ \xi\neq0.$$

According to (8.7) we must show that

$$Q:=\sum\frac{\xi_j^2\lambda_j}{(1+\lambda_j^2)^2}>0 \quad \text{if } \sum\frac{\xi_j}{1+\lambda_j^2}=0,\ \xi\neq0. \tag{8.11}$$

If all $\lambda_j>0$ there is nothing to prove. So we need only consider the case $\lambda_1\leq0<\lambda_2\dots$. Then

$$\frac{\xi_1}{1+\lambda_1^2}=-\sum_{\alpha>1}\frac{\xi_\alpha}{1+\lambda_\alpha^2},$$

and so (α, β always sum from 2 to n)

$$\frac{\xi_1^2}{(1+\lambda_1^2)^2} \leqslant \sum \frac{\xi_\alpha^2 \lambda_\alpha}{1+\lambda_\alpha^2} \cdot \sum \frac{1}{\lambda_\beta}.$$

Hence

$$Q \geqslant \left(\lambda_1 \sum \frac{1}{\lambda_\beta} + 1\right) \sum \frac{\xi_\alpha^2 \lambda_\alpha}{1+\lambda_\alpha^2}.$$

Now

$$\sigma^{(n-1)}(\lambda) = \prod_{\alpha>1} \lambda_\alpha \cdot \left(1+\lambda_1 \sum \frac{1}{\lambda_\beta}\right).$$

It follows that

$$Q \geqslant \frac{\sigma^{n-1}(\lambda)}{\prod_{\alpha>1} \lambda_\alpha} \sum \frac{\xi_\alpha^2 \lambda_\alpha}{1+\lambda_\alpha^2} > 0.$$

(8.11) and also Lemma C are proved.

Remark. No other component S of $g^{-1}(0)$ satisfies conditions (11)–(13) on Σ. For if it did then it would lie in the half space $\Sigma \lambda_i > ns$ with $(s, \dots, s)$ on S. But for large t, the point $(t, \dots, t)$ lies in the convex region G bounded by Σ, and it would then follow that $g(t, \dots, t, \lambda_n)$ has at least two roots, corresponding to points on S and on Σ — contradicting the fact that g is linear affine as a function of λ_n.

8. An example

In the disc $\Omega = r = |x| < 1$ in the plane consider the convex function

$$u = r^2 - 1 + \tfrac{1}{3}(1-r^2)^{3/2}. \tag{9.1}$$

For a function of r, the eigenvalues of the Hessian matrix are $\ddot{u}$ and $\dot{u}/r$. Thus

$$\lambda_1 = \frac{\dot{u}}{r} = 2-(1-r^2)^{1/2} \geqslant 1,$$

$$\lambda_2 = 2-(1-r^2)^{1/2} + r^2(1-r^2)^{-1/2} \geqslant \lambda_1$$

i.e.

$$\lambda_2 = (1-r^2)^{-1/2}(1+2(1-r^2)^{1/2} - 2(1-r^2))$$

Given any positive integer k, let $g(s)$ be a C^∞ concave function defined for $s>0$ and satisfying

$$\dot{g}(s)>0$$
$$g(s)=2k\log s, \quad \text{for } s\geq 1,$$
$$g(s)\to-\infty \quad \text{as } s\to 0.$$

The symmetric function

$$f(\lambda_1,\lambda_2)=2-e^{-g(\lambda_1)-g(\lambda_2)} \tag{9.2}$$

is concave and satisfies $f_{\lambda_i}>0$, $i=1,2$.

Thus the nonlinear partial differential operator for functions u of two variables,

$$F(u_{ij})=f(\lambda_1(u_{ij}),\lambda_2(u_{ij})), \tag{9.3}$$

is elliptic at every strictly convex function u. For u given by (9.1) set

$$F(u_{ij})=\psi(r). \tag{9.4}$$

Since $1\leq\lambda_1\leq\lambda_2$ for this function, we find

$$\psi(r)=2-(\lambda_1\lambda_2)^{-2k}\geq 1.$$

Now $\psi\in C^\infty$ for $r<1$ but not in $r\leq 1$. In fact for r close to 1 we have

$$\lambda_1\lambda_2=(1-r^2)^{-1/2}(2+3(1-r^2)^{1/2}-6(1-r^2)+2(1-r^2)^{3/2})$$

and therefore ψ is in $C^k(\bar{\Omega})$ but not in $C^{k+1}(\bar{\Omega})$.

Thus the function u given by (9.1) is a solution of

$$F(u_{ij})=\psi\in C^k(\bar{\Omega})$$
$$u=0 \quad \text{on } \partial\Omega,$$

and for $\Gamma=\Gamma_+$, the positive cone, we find that conditions (2), (3), (5) and (6) hold. Conditions (7) and (8) do not. Furthermore u, the unique convex solution in class $C^2(\Omega)\cap C(\bar{\Omega})$, belongs to $C^1(\bar{\Omega})$ but not to $C^2(\bar{\Omega})$.

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