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THE DIRICHLET PROBLEM FOR THE EQUATION OF PRESCRIBED GAUSS CURVATURE

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We treat necessary and sufficient conditions for the classical solvability of the Dirichlet problem for the equation of prescribed Gauss curvature in uniformly convex domains in Euclidean n space. Our methods simultaneously embrace more general equations of Monge-Ampère type and we establish conditions which ensure that solutions have globally bounded second derivatives.

1. Introduction

Let Ω be a domain in Euclidean n space $\mathbb{R}^n$ and u a function in $C^2(\Omega)$ with graph $S \subset \mathbb{R}^{n+1}$. The Gauss curvature of S at a point $(x, z) = (x, u(x)) \in S$ is given by the formula

$$K(x, z) = \frac{\det D^2u(x)}{\left(1+|Du(x)|^2\right)^{(n+2)/2}} . \tag{1.1}$$

Here Du, D^2u denote respectively the gradient and Hessian of u. In this paper we are concerned with the problem of recovering the function u from the prescription of K, and given boundary values on $\partial\Omega$, which is equivalent to the Dirichlet problem for the equation (1.1). We shall prove the following sharp theorem.

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THEOREM 1.1. *Let* Ω *be a uniformly convex* $C^{1,1}$ *domain in* $\mathbb{R}^n$ *and* K *a positive function in* $C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$. *Then the classical Dirichlet problem,*

$$\det D^2u = K(x)\left(1+|Du|^2\right)^{(n+2)/2}, \quad u = \phi \text{ on } \partial\Omega, \tag{1.2}$$

is uniquely solvable for arbitrary $\phi \in C^{1,1}(\overline{\Omega})$, *with convex solution* $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$, *if and only if the following two conditions hold,*

$$\int_\Omega K < \omega_n, \tag{1.3}$$

$$K = 0 \text{ on } \partial\Omega. \tag{1.4}$$

More generally we consider the Dirichlet problem for equations of Monge-Ampère type,

$$\det D^2u = f(x, u, Du) \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega, \tag{1.5}$$

where f is a positive function in $C^{1,1}\left(\Omega \times \mathbb{R} \times \mathbb{R}^n\right)$ subject to the following structure conditions

$$f_z \geq 0 \text{ in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{1.6}$$

$$f(x, -N, p) \leq \frac{g(x)}{h(p)} \text{ for all } x \in \Omega, \; p \in \mathbb{R}^n, \tag{1.7}$$

$$f\left(x, \phi(x), p\right) \leq \mu d^\beta \left(1+|p|^2\right)^{\alpha/2} \text{ for all } x \in N, \; p \in \mathbb{R}^n, \tag{1.8}$$

where N, μ, α, β are non-negative constants such that $\beta \geq \alpha - n - 1$, g and h are positive functions in $L^1(\Omega)$, $L^1_{loc}(\mathbb{R}^n)$ respectively such that

$$\int_\Omega g < \int_{\mathbb{R}^n} h, \tag{1.9}$$

$d = \text{dist}(x, \partial\Omega)$, and N is some neighbourhood of $\partial\Omega$. We then have the following existence result.

THEOREM 1.2. *Let* Ω *be a uniformly convex* $C^{1,1}$ *domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\overline{\Omega})$ *and let* f *be a positive function in* $C^{1,1}\left(\Omega \times \mathbb{R} \times \mathbb{R}^n\right)$

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satisfying (1.6), (1.7) *and* (1.8). *Then the Dirichlet problem* (1.5) *is uniquely solvable with convex solution* $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$.

The work of Lions [11], [12] reduces the classical solvability of the Dirichlet problem (1.5) to the existence of a generalized subsolution in the sense of Aleksandrov [2]. Results pertaining to the solvability of the generalized Dirichlet problem for equations of the above type are treated by Bakel'man [4] and established for equations in two variables in [5], [6], [7]. We provide in the next section a direct derivation of Theorem 2 from one of Lions' basic results thereby avoiding any consideration of generalized solutions. At the same time we are able to infer that our solution is uniformly Lipschitz. We also indicate an alternative derivation of Theorem 1.2, from the recent work of Caffarelli, Nirenberg and Spruck [8] on globally smooth solutions.

The necessity of conditions (1.3) and (1.9) is readily shown; (see [10]). For, suppose that the function f satisfies an inequality of the form

(1.10) $$f(x, z, p) \geq \frac{g(x)}{h(p)} \quad \text{for all} \quad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n ,$$

where g and h are positive functions in $L^1(\Omega)$, $L^1_{\text{loc}}(\mathbb{R}^n)$ respectively and let u be a convex $C^2(\Omega)$ solution of the equation (1.5). Since $Du : \Omega \to \mathbb{R}^n$ is one-to-one, we obtain

(1.11) $$\begin{aligned} \int_\Omega g &\leq \int_\Omega h(Du) \det D^2u \\ &= \int_{Du(\Omega)} h(p)dp \\ &\leq \int_{\mathbb{R}^n} h \end{aligned}$$

with strict inequality holding if $Du(\Omega)$ is bounded, that is if $u \in C^{0,1}(\bar{\Omega})$. The necessity of inequality (1.3) follows since

$$\int_{\mathbb{R}^n} \frac{dp}{(1+|p|^2)^{(n+2)/2}} = \omega_n .$$

Using an argument similar to that of Serrin [13] we shall show in Section 3 that conditions (1.4) and (1.8) are necessary in the following sense.

THEOREM 1.3. *Let* Ω *be a uniformly convex* $C^{1,1}$ *domain in* $\mathbb{R}^n$ *and* f *a positive function on* $\Omega \times \mathbb{R} \times \mathbb{R}^n$ *satisfying*

(1.12) $f(x, z, p) \geq \nu d^\beta (1+|p|^2)^{\alpha/2}$ *for all* $x \in N_y$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$,

where N_y *is a neighbourhood of some point* $y \in \partial\Omega$, ν *is a positive constant and* α *and* β *are non-negative constants satisfying* $\beta < \alpha - n - 1$. *Then there exists a function* $\phi \in C^\infty(\bar{\Omega})$ *for which the Dirichlet problem* (1.5) *is not solvable for convex* $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$.

In the last section of this paper we treat further global regularity of the solution of (1.2) and (1.5), giving conditions on K and f which imply the boundedness of second derivatives.

Finally we remark that all notation in this paper, unless otherwise indicated, is as in [*10*].

2. Existence

In this section we prove Theorem 1.2 and consequently the sufficiency of conditions (1.3) and (1.4) in Theorem 1.1. We first need two lemmas ensuring *a priori* bounds for solutions of the Dirichlet problem (1.5). The first, taken from Bakel'man [*4*], [*6*] (see also [*10*], Theorem 17.4), provides an estimate for the solution while the second is a gradient estimate.

LEMMA 2.1. *Let* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *be a convex solution of the Dirichlet problem* (1.5) *where the function* f *satisfies conditions* (1.6) *and* (1.7). *Then we have the estimate*

$$\inf_{\partial\Omega} \phi - N - C \operatorname{diam} \Omega \leq u \leq \sup_{\partial\Omega} \phi \tag{2.1}$$

where C *depends on* n, g *and* h .

LEMMA 2.2. *Let* Ω *be a uniformly convex domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\bar{\Omega})$ *and let* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *be a convex solution of the Dirichlet problem* (1.5) *where the function* f *satisfies conditions* (1.6) *and* (1.8). *Then* $u \in C^{0,1}(\bar{\Omega})$ *and we have the estimate*

$$\sup_{\Omega} |Du| \le C \,, \tag{2.2}$$

where C *depends on* $n, \mu, \alpha, \beta, N, \Omega$, $|u|_{0;\Omega}$ *and* $|\phi|_{1,1;\Omega}$.

Proof. Without loss of generality we may replace ϕ by a convex function. Let us fix at a point $y \in \partial\Omega$ an enclosing ball $B = B_R(\bar{y}) \supset \Omega$ with $\partial B \cap \partial\Omega = \{y\}$. We now take as a barrier, the convex function

$$w = \phi - \psi(d)$$

where $d(x) = \operatorname{dist}(x, \partial B)$ and ψ is given by

$$\psi(d) = \frac{1}{\nu} \log(1+kd)$$

where ν and k are positive constants to be determined. Using a principal coordinate system for ∂B at y , we may then estimate in $\Omega \cap B$,

$$\begin{aligned} \det D^2 w &\ge \det\left(-D^2\psi\right) \\ &= -\psi''\left(\frac{\psi'}{|x-\bar{y}|}\right)^{n-1} \\ &\ge -\psi''(\psi'/R)^{n-1} \,, \end{aligned}$$

while from the structure conditions (1.6), (1.8) we have

$$\begin{aligned} f(x, w, Dw) &\le f(x, \phi, Dw) \\ &\le \mu d^{\beta}\left(1+|Dw|^2\right)^{\alpha/2} \\ &\le \mu d^{\beta}\left(1+|D\phi|^2+|\psi'|^2\right)^{\alpha/2} \\ &\le 2^{\alpha}\mu d^{\beta}|\psi'|^{n+1+\beta} \end{aligned}$$

provided $x \in N$ and

$$(\psi')^2 \ge \mu_0 = \sup_{\Omega} \left(1+|D\phi|^2\right) .$$

We now choose

$$\nu = 1 + 2^{\alpha} R^{n-1} \mu \,,$$

so that $\psi' d \le 1$, and then k and $a > 0$ such that $N_a = \Omega \cap \{d < a\} \subset N$ and

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$$ka = e^{\nu M} - 1 \ , \quad k \geq \mu_0 \nu e^{\nu M} \ ,$$

where $M = \sup\limits_{\Omega} |u|$. It then follows that

$$\det D^2 w \geq f(x, w, Dw) \quad \text{in } N_a \ , \quad w \leq u \quad \text{on } \partial N_a \ ,$$

and hence by the comparison principle, $w \leq u$ in N_a. Thus we obtain a lower bound

$$\frac{u(x)-u(y)}{|x-y|} \geq -C$$

for any $y \in \partial\Omega$, $x \in \Omega$, where C depends on the same quantities as in (2.2). A two sided bound,

$$\frac{|u(x)-u(y)|}{|x-y|} \leq C \ , \tag{2.3}$$

follows immediately since u is subharmonic. Finally, by the convexity o u, the estimate (2.3) extends to all x, $y \in \Omega$, $x \neq y$. //

Theorem 2.1 can now be deduced from the following result of Lions [12].

THEOREM 2.3. *Let* Ω *be a uniformly convex* $C^{1,1}$ *domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\bar{\Omega})$, *and let* f *be a positive function in* $C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *with* $f_z \geq 0$ *and*

$$f(x, z, p) \leq \mu(1+|p|^2)^{n/2} \ , \tag{2.4}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, *for some positive constant* μ. *Then the Dirichlet problem* (1.5) *is uniquely solvable, with convex solution* $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$.

Proof of Theorem 1.2. To use Theorem 2.3, we truncate the function f with respect to z and p. Accordingly let f_m be a sequence of positive functions in $C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^m)$ satisfying $f_{mz} \geq 0$, $f_m \leq f$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with $f_m = f$ for $|z| + |p| \leq m$. By Theorem 2.3, there exists a sequence $\{u_m\} \subset C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ of solutions of the Dirichlet problems,

(2.5) $\det D^2u_m = f_m(x, u_m, Du_m)$ in Ω , $u = \phi$ on $\partial\Omega$,

and since the functions f_m satisfy by hypothesis the conditions (1.7), (1.8) uniformly in m , we obtain a uniform bound

$$|u|_{0;\Omega} + |Du|_{0;\Omega} \le C ,$$

by virtue of Lemmas 2.1 and 2.2. But this means that u_m will solve the given Dirichlet problem for sufficiently large m . //

The above proof is similar to that given in [*10*] for the case $\phi = 0$ where we need only assume Ω is uniformly convex and bounded.

We conclude this section by pointing out that Theorem 2.3 may also be derived by approximation from the following existence theorem of Caffarelli, Nirenberg and Spruck for globally smooth solutions, rather than the penalization method of Lions [*12*].

THEOREM 2.4. *Let* Ω *be a uniformly convex* $C^{3,1}$ *domain in* $\mathbb{R}^n$ *,* $\phi \in C^{3,1}(\bar{\Omega})$ *and let* f *be a positive function in* $C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ *satisfying* $f_z \ge 0$ *and* (2.4). *Then the Dirichlet problem* (1.5) *is uniquely solvable with convex solution in* $C^2(\bar{\Omega})$.

The passage from Theorem 2.4 to 2.3 may be accomplished by first obtaining a generalized solution by approximation and then deducing its regularity as in [*9*] or [*12*] or alternatively by direct use of the interior second derivative estimates in [*15*].

3. Nonexistence

In this section we prove Theorem 1.3. We shall use the following comparison lemma.

LEMMA 3.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* Γ *a relatively open* C^1 *portion of* $\partial\Omega$. *Let* $u \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \Gamma)$ *and* $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *be uniformly convex functions satisfying*

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(3.1) $$\frac{\det D^2u}{(1+|Du|^2)^{\alpha/2}} \geq \frac{\det D^2v}{(1+|Du|^2)^{\alpha/2}} \quad in \quad \Omega\ ,$$

$u \leq v$ on $\partial\Omega - \Gamma$ and $\partial v/\partial\nu = \infty$ on Γ, where ν is the outer unit normal to Γ. Then $u \leq v$ in Ω.

Proof. By the comparison principle we have

$$\sup_{\Omega}\ (u-v) \leq \sup_{\Gamma}\ (u-v)^+\ .$$

Since $(\partial/\partial\nu)(u-v) = -\infty$ on Γ, the function $u - v$ cannot achieve a maximum value on Γ. Hence $u \leq v$ in Ω. //

Proof of Theorem 1.3. Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a convex solution of the equation

$$\det D^2u = f(x,\ u,\ Du) \quad \text{in} \quad \Omega\ ,$$

where f satisfies (1.12). Let $B = B_R(\bar{y})$ be an interior sphere at y. We may assume that $B_R(y) \cap \Omega \subset N_y$ and $R \leq 1$.

Since u is convex, we have

(3.2) $$\sup_{\Omega-B_{R/2}(y)} u \leq \sup_{\partial\Omega-B_{R/2}(y)} u\ .$$

Let $\rho \in (R/2,\ R)$ and

$$w = \psi(r) = \sup_{\partial\Omega-B_{R/2}(y)} u + AR^\gamma - Ar^\gamma\ ,$$

where $r = \text{dist}(x,\ \partial B_\rho(\bar{y}))$ and $\gamma \in (0,\ 1)$ and A are to be chosen. Then for $x \in B_\rho(\bar{y}) \cap B_{R/2}(y)$ we have

$$\begin{aligned}\frac{\det D^2w}{(1+|Dw|^2)^{\alpha/2}} &= \frac{\psi''(-\psi'/|x-\bar{y}|)^{n-1}}{(1+|\psi'|^2)^{\alpha/2}} \\ &\leq 2^{n-1}(1-\gamma)(A\gamma)^{n-\alpha}r^{(\alpha-n)(1-\gamma)-1}R^{1-n} \\ &\leq \nu d^\beta \\ &\leq \frac{\det D^2u}{(1+|Du|^2)^{\alpha/2}}\ ,\end{aligned}$$

provided $(\alpha-n)(1-\gamma) - 1 > \beta$ and $2^{n-1}(1-\gamma)(A\gamma)^{n-\alpha} \le \nu R^{n-1}$. Thus from Lemma 3.1 we obtain $u \le w$ in $B_\rho(\bar{y}) \cap B_{R/2}(y)$. Letting $\rho \to R$ we obtain

$$u(y) \le \sup_{\partial\Omega - B_{R/2}(y)} u + AR^\gamma , \tag{3.3}$$

from which Theorem 1.3 follows. //

4. Further regularity

When the curvature K vanishes sufficiently rapidly at the boundary we can infer that the solutions of the Dirichlet problem (1.2) lie in the space $C^{1,1}(\bar{\Omega})$. More generally we shall assume in (1.5) that the function $g = f^{1/n}$ is convex with respect to the p variables and that for any $L > 0$, we have

$$\sup_{|z|+|p|\le L} \left(|g(x, z, p)|+|Dg(x, z, p)|\right) \le \mu_1 , \tag{4.1}$$

$$\sup_{|z|+|p|\le L} |D^2 g(x, z, p)| \le \mu_2(x) ,$$

for all $x \in \Omega$, where μ_1 is constant and $\mu_2 \in L^n(\Omega)$. We now have

THEOREM 4.1. *Let Ω be a uniformly convex $C^{2,1}$ domain in $\mathbb{R}^n$, $\phi = 0$ on $\partial\Omega$ and let f be a positive function in $C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfying $f_z \ge 0$ and (4.1). Then if $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ is convex and solves the Dirichlet problem (1.5) we have $u \in C^{1,1}(\bar{\Omega})$, together with the estimate*

$$\sup_\Omega |D^2 u| \le C \tag{4.2}$$

where C depends on $n, \Omega, |u|_{1;\Omega}, \mu_1, \mu_2$.

Proof. Let us first derive the estimate (4.2) for solutions of (1.5) which lie in $C^2(\bar{\Omega})$. Local regularity considerations (see [*10*], Lemma 17.16), ensure that they also lie in the Sobolev space $W^{4,n}_{\text{loc}}(\Omega)$. We now write the equation (1.5) in the form

$$F(D^2u) = (\det D^2u)^{1/n} = g(x, u, Du) \tag{4.3}$$

and differentiate twice with respect to x_k , $k = 1, \ldots, n$, to obtain

$$F_{ij}D_{ijkk}u + F_{ij,lm}D_{ijk}uD_{lmk}u = g_{x_kx_k} + 2g_{x_kz}D_ku + 2g_{x_kp_i}D_{ik}u$$
$$+ g_zD_{kk}u + g_{zz}(D_ku)^2 + 2g_{zp_i}D_iuD_{ik}u + g_{p_i}D_{ikk}u + g_{p_ip_j}D_{ik}uD_{jk}u \tag{4.4}$$

where

$$F_{ij}(r) = \frac{\partial}{\partial r_{ij}} F(r) = \frac{1}{n} Fr^{-1} ,$$

$$F_{ij,lm}(r) = \frac{\partial^2 F}{\partial r_{ij}\partial r_{lm}} .$$

Using the concavity of F , the convexity of g and u and the conditions (4.1), we therefore obtain the differential inequality

$$-F_{ij}D_{ij}\Delta u + g_{p_i}D_i\Delta u \le C(\mu_1+\mu_2)(1+\Delta u) \tag{4.5}$$

where C depends on n and $|u|_{1;\Omega}$. Since $\det|F_{ij}| = n^{-n}$, we can now deduce from the Aleksandrov maximum principle (see [3] or [10], Theorem 9.1),

$$\sup_{\Omega} \Delta u \le \sup_{\partial\Omega} \Delta u + C\|(\mu_1+\mu_2)(1+\Delta u)\|_{L^n(\Omega)}$$

where C depends on n, $|u|_{1;\Omega}$, diam Ω and $\|g_p\|_{L^n(\Omega)}$. We now make use of the interpolation type inequality

$$\|fg\|_{L^n(\Omega)} \le \varepsilon \sup_{\Omega} |f| + C_\varepsilon \int_{\Omega} |f| \tag{4.6}$$

which holds for any $f \in L^\infty(\Omega)$, $g \in L^n(\Omega)$ and $\varepsilon > 0$; the constant C depending on both ε and g . Accordingly we obtain

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$$\sup_{\Omega} \Delta u \le 2 \sup_{\partial\Omega} \Delta u + C \int_{\Omega} (1+\Delta u) \tag{4.7}$$

$$\le 2 \sup_{\partial\Omega} \Delta u + C\left(1 + \int_{\partial\Omega} |Du|\right)$$

$$\le 2 \sup_{\partial\Omega} \Delta u + C$$

where C depends on n, Ω, $|u|_{1;\Omega}$, μ_1 and μ_2 .

The estimation of Δu on the boundary $\partial\Omega$ can be accomplished with only minor modification to the method presented in [*10*], Theorem 17.20 (see also [*1*], [*8*]). We insert the proof here for completeness. Let $y \in \partial\Omega$ and let N be a neighbourhood of y such that each $x \in N$ has a unique nearest point on $\partial\Omega$. Let $\nu(x)$ be the outwards directed normal to $\partial\Omega$ at this point. Since $\partial\Omega \in C^{2,1}$, N can be chosen so that $\nu \in C^{1,1}(N)$. For $k = 1, \ldots, n$, let δ be the tangential gradient operator in $\partial\Omega$ given by

$$\delta_k = D_k - \nu_k \nu_l D_l \ .$$

Differentiating (4.3) with respect to x_k we get

$$F_{ij} D_{ijk} u = g_{x_k} + g_z D_k u + g_{p_i} D_{ik} u \ . \tag{4.8}$$

Using (4.8) and the equality

$$F_{ij} D_{jl} u = \frac{1}{n} g \delta_{il} \ ,$$

we thus obtain

$$F_{ij} D_{ij} \delta_k u = \delta_k g + g_z \delta_k u + g_{p_i} D_i (\delta_k u) \tag{4.9}$$

$$+ g_{p_i} D_i (\nu_k \nu_l) D_l u - F_{ij} D_{ij} (\nu_k \nu_l) D_l u - \frac{2}{n} g D_l (\nu_k \nu_l) \ .$$

Since

$$\det(F_{ij}) = n^{-n} \ ,$$

we have

$$T = \operatorname{trace}(F_{ij}) \ge 1 \ .$$

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Therefore from [10], Corollary 14.5, we conclude

$$|D_n \delta_k u(y)| \le C \,, \tag{4.10}$$

where C depends on n, μ_1, $N \cap \partial\Omega$, $|Du|_{0;\Omega}$ and $|\phi|_{2,1;\Omega}$. A similar estimate then holds for the mixed partial derivatives $D_{nk}u(y)$, $k = 1, \ldots, n-1$, with respect to a principal coordinate system at y. The partial derivatives $D_{jk}u(y)$, $j, k = 1, \ldots, n-1$ are bounded by $|D^2\phi|_{0;\Omega}$, so it remains only to estimate $D_{nn}u(y)$. In a principal coordinate system at y we have

$$D^2u - D^2\phi = \begin{bmatrix} D_n(u-\phi)\kappa_1 & 0 & \ldots & 0 & D_{n1}(u-\phi) \\ 0 & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ 0 & \ldots & \ldots & D_n(u-\phi)\kappa_{n-1} & \vdots \\ D_{n1}(u-\phi) & \ldots & \ldots & \ldots & D_{nn}(u-\phi) \end{bmatrix} \tag{4.11}$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at y. We may assume that $-\phi$ is convex, so that we have

$$|D_n(u-\phi)(y)| \ge \big(\sup_\Omega |u-\phi|\big)/(\text{diam}\,\Omega) \,. \tag{4.12}$$

Using (4.11) to solve (4.3) for $D_{nn}u(y)$, and taking $\phi \equiv 0$, we thus obtain an upper bound for $D_{nn}u(y)$ and hence also for $\Delta u(y)$, which on combination with (4.7) yields (4.2).

Finally to get the regularity assertion of Theorem 4.1, we truncate the function f as in the proof of Theorem 1.2, and solve by means of Theorem 2.4, the Dirichlet problems,

$$\det D^2u_{ml} = f_m\big(x, u_{ml}, Du_{ml}\big) \text{ in } \Omega_l \,, \quad u = \phi \text{ on } \partial\Omega_l \,, \tag{4.13}$$

where ϕ is redefined so that it lies in the space $C^4(\Omega) \cap C^{2,1}(\bar\Omega)$ and $\{\Omega_l\}$ is an increasing sequence of uniformly convex C^4 domains, with union Ω. By Lemmas 2.1, 2.2 and the estimate (4.2), we obtain that the norms $|u_{ml}|_{2;\Omega}$ are bounded independently of l and hence using also the interior second derivative Hölder estimates (see [14] or [10], Theorem 17.14), we obtain a subsequence of $\{u_{ml}\}_{l=1}^{\infty}$ converging in $C^2(\Omega) \cap C^1(\bar{\Omega})$ to a solution $u_m \in C^2(\Omega) \cap C^{1,1}(\bar{\Omega})$ of the Dirichlet problem (2.5). But by uniqueness u_m must coincide with our given solution u for sufficiently large m.

For the special case of the equation of prescribed Gauss curvature we now have

COROLLARY 4.2. *If in addition to the hypotheses of Theorem* 1.1, *the function* $K^{1/n} \in W^{2,n}(\Omega)$, Ω *is* $C^{2,1}$ *and* $\phi = 0$ *on* $\partial\Omega$, *then the second derivatives of the solution* u *of* (1.2) *are bounded in* Ω.

Note that when f is positive in $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ (so that $\beta = 0$) and $\partial\Omega$ and ϕ are sufficiently smooth we can infer further global regularity of solutions of (1.5) from the work of Caffarelli, Nirenberg and Spruck.

Finally to conclude this paper we remark that the growth conditions (1.8) and (1.12) in Theorems 1.2 and 1.3 may be extended slightly by adapting the barrier constructions of Serrin [13]. Furthermore by using barriers of the form

$$w = \phi + kd^{\gamma}\,, \quad k > 0\,, \quad 0 < \gamma < 1\,, \tag{4.14}$$

we may deduce the existence of uniformly Hölder continuous solutions of (1.5) when (1.6), (1.7), (1.8) are satisfied with possibly $\beta < \alpha-n-1$, provided the curvature of $\partial\Omega$ is sufficiently large.

The equation of prescribed Gauss curvature without boundary conditions is treated in the forthcoming paper [16].

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