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Abstract

In this paper we study the discrete time algebraic Riccati equation and its connection to the discrete time linear matrix inequality. We show that in general only a subset of the set of rank-minimizing solutions of the linear matrix inequality correspond to the solutions of the associated algebraic Riccati equation, and study under what conditions these sets are equal. In this process we also derive very weak assumptions under which a Riccati equation has a solution.

Keywords: discrete algebraic Riccati equation, linear matrix inequality, rank-minimizing solutions

1 Introduction

The algebraic Riccati equation and linear matrix inequality are important tools in linear systems and control theory. Since their introduction in control theory in the early sixties, they have appeared in an impressive range of problems in control theory including H_2 and H_∞ optimal control theory.
In this paper, we first establish some properties of the general discrete algebraic Riccati equation. Then we concentrate on the Riccati equation appearing in H_2 and linear quadratic control. Establishing a connection between the solutions of a linear matrix inequality and its associated algebraic Riccati equation has been a longstanding research problem. In continuous-time setting it is shown (see [2, 8, 23]) that the set of "boundary" solutions of the continuous linear matrix inequality coincides with the set of real symmetric solutions of an appropriately defined continuous algebraic Riccati equation. The "boundary" solutions of the continuous linear matrix inequality are those solutions which minimize the rank of the given matrix in the continuous linear matrix inequality and hence are known as rank-minimizing solutions. The rank-minimizing solutions of the continuous linear matrix inequality play a prominent role in H_2 optimal control theory, and their characterization in terms of an appropriately defined continuous algebraic Riccati equation is of significant interest.

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In this paper we would like to examine the connections between rank-minimizing solutions of the linear matrix inequality and solutions of the algebraic Riccati equation in a discrete time setting. Our first surprising observation is that, unlike the continuous-time case, in general the rank-minimizing solutions of the discrete linear matrix inequality cannot be obtained from solutions of an appropriately defined discrete algebraic Riccati equation. However, we show that a subset of rank-minimizing solutions of the discrete linear matrix inequality which we refer to as strong rank-minimizing solutions, coincides with the set of real symmetric solutions of the associated discrete algebraic Riccati equation. An algebraic Riccati equation can be associated to a matrix pencil and we will show that the rank-minimizing solutions of the linear matrix inequality have a one to one relationship with invariant subspaces of the matrix pencil. Moreover, the strongly rank-minimizing solutions are precisely those that are related to the invariant subspaces of this matrix pencil having only finite eigenvalues. We will also discuss results like the existence of solutions to an algebraic Riccati equation and in so doing we improve the existing results in the literature.
We will use a reduction technique to derive our results for the general case. This technique is presented in the appendix. Also some properties of matrix pencils, which we will need in our derivations, are presented in the appendix.

2 Discrete algebraic Riccati equation

In this section we will derive properties of the general discrete algebraic Riccati equation:

Definition 2.1 : *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{m\times n}$ with Q and R being symmetric be given. Then*

$$X = A^{\mathrm{T}}XA - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) + Q \tag{2.1a}$$

$$\mathit{Ker}\ (B^{\mathrm{T}}XB + R) \subseteq \mathit{Ker}\ (A^{\mathrm{T}}XB + S) \tag{2.1b}$$

where M^{+} denotes the Moore-Penrose generalized inverse of the matrix M is called the general discrete algebraic Riccati equation.

Definition 2.2 : *X is called a stabilizing solution of the algebraic Riccati equation if X satisfies (2.1) and is such that the rank of*

$$\begin{pmatrix} zI - A & -B \\ Q + A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB + S^{\mathrm{T}} \\ B^{\mathrm{T}}XA + S & B^{\mathrm{T}}XB + R \end{pmatrix} \tag{2.2}$$

is equal to its normal rank for all z outside or on the unit circle. We will call X a semi-stabilizing solution of the algebraic Riccati equation if X satisfies (2.1) and the rank of (2.2) is equal to its normal rank for all z outside the unit circle.

The following matrix plays an important role in the study of this equation:

$$L(X) := \begin{pmatrix} Q + A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB + S^{\mathrm{T}} \\ B^{\mathrm{T}}XA + S & B^{\mathrm{T}}XB + R \end{pmatrix} \tag{2.3}$$

Moreover, the following rational matrix will be important to us:

$$H(z) = \begin{pmatrix} B^{\mathrm{T}}(z^{-1}I - A^{\mathrm{T}})^{-1} & I \end{pmatrix} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} (zI - A)^{-1}B \\ I \end{pmatrix} \tag{2.4}$$

In the classical situation, which we will sometimes refer to as the positive semi-definite case, we have

$$\begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} = \begin{pmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix}, \tag{2.5}$$

and in that case we have $H = G^{\sim}G$ where G is the transfer matrix of (A, B, C, D) and $G^{\sim}(z) = G^{\mathrm{T}}(z^{-1})$.

Lemma 2.3 : *For any symmetric matrix X we have:*

$$H(z) = H(z, X) := \begin{pmatrix} B^{\mathrm{T}}(z^{-1}I - A^{\mathrm{T}})^{-1} & I \end{pmatrix} L(X) \begin{pmatrix} (zI - A)^{-1}B \\ I \end{pmatrix}$$

Proof : The identity can be verified by writing down a realization in descriptor form for H and $H(\cdot, X)$ and is a generalization of a fact already noted in continuous time by [18]. ■

Next we would like to see under what conditions a solution of the algebraic Riccati equation will be such that $R + B^{\mathrm{T}}XB$ is invertible. In the semi-definite case where we have (2.5) it was already well-known (see e.g. [19]) that a solution of the Riccati equation will be such that $R + B^{\mathrm{T}}XB$ is invertible if and only if (A, B, C, D) is left-invertible or, in other words, H has full rank. The amazing fact for this special case is that either all or none of the solutions will satisfy this extra invertibility assumption. We will see that this latter property holds in general. Just as a reminder, the inertia of a matrix are defined as the triple of the number of eigenvalues in the open left half plane, the number of eigenvalues on the imaginary axis and the number of eigenvalues in the open right half plane.

Theorem 2.4 : *Assume that a symmetric matrix X satisfying (2.1) exists. Then*

- *H has full normal rank if and only if $B^{\mathrm{T}}XB + R$ is invertible.*
- *The inertia of $B^{\mathrm{T}}XB + R$ are equal to the inertia of $H(z)$ for all but finitely many z on the unit circle.*
- *$B^{\mathrm{T}}XB + R \geq 0$ if and only if $H(z) \geq 0$ for any point z on the unit circle.*

Proof : Assume that a symmetric matrix X satisfies the algebraic Riccati equation. Then it can be checked straightforwardly that

$$V^{\mathrm{T}}(z^{-1})L(X)V(z) = \begin{pmatrix} H(z) & 0 \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

where V is given by

$$V(z) = \begin{pmatrix} (zI-A)^{-1}B & I \\ I & -(B^{\mathrm{T}}XB+R)^{\dagger}(B^{\mathrm{T}}XA+S) \end{pmatrix}$$

Note that V is square and invertible for almost all z. Hence, (2.6) implies that the rank of $L(X)$ equals the normal rank of H. This guarantees in particular that H has full normal rank if and only if $B^{\mathrm{T}}XB+R$ is invertible. Moreover, for all but finitely many points on the unit circle, the inertia of $L(X)$ is equal to the inertia of H together with an appropriate number of zero eigenvalues. Since the Schur complement of $B^{\mathrm{T}}XB+R$ in $L(X)$ is zero we find that the inertia of $B^{\mathrm{T}}XB+R$ equals the inertia of $H(z)$ for all but finitely many points on the unit circle. In particular, we have that $H(z) \geq 0$ on the unit circle if and only if $B^{\mathrm{T}}XB+R \geq 0$. ■

The last point in the above lemma is basically a special case of the second point but listed separately since it will play an important role in the rest of this paper. Note that the above lemma implies that a necessary condition for the existence of a solution to the discrete algebraic Riccati equation is that the inertia of $H(z)$ are independent of z except for possibly some singularities. A necessary condition for the existence of a solution with $B^{\mathrm{T}}XB+R \geq 0$ is that $H(z) \geq 0$ for all z on the unit disc. This condition was already presented for the discrete time in [16] and in [23] for the continuous time. Finally, note that H being of full normal rank guarantees that the generalized inverse in (2.1a) is a normal inverse and that (2.1b) is automatically satisfied. In other words, in that case we can simply focus on the equation:

$$X = A^{\mathrm{T}}XA - (A^{\mathrm{T}}XB+S^{\mathrm{T}})(B^{\mathrm{T}}XB+R)^{-1}(B^{\mathrm{T}}XA+S)+Q \tag{2.7}$$

We would like to understand the structure of the algebraic Riccati equation better and to this end we apply some algebraic manipulations to the algebraic Riccati equation. We assume that H has full rank and hence we can study equation (2.7) instead of (2.1). Obviously, we can find a matrix Z such that $R+B^{\mathrm{T}}ZB$ is invertible. Assume that we have a solution X of the algebraic Riccati equation (2.7). We find:

$$\begin{pmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I-LZ & L \\ -\tilde{A}^{\mathrm{T}}Z & \tilde{A}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} A_{cl} \tag{2.8}$$

where

$$\begin{aligned} L &= B(R+B^{\mathrm{T}}ZB)^{-1}B^{\mathrm{T}} \\ \tilde{A} &= A - B(R+B^{\mathrm{T}}ZB)^{-1}(B^{\mathrm{T}}ZA+S) \\ \tilde{Q} &= Q + A^{\mathrm{T}}ZA - (A^{\mathrm{T}}ZB+S^{\mathrm{T}})(R+B^{\mathrm{T}}ZB)^{-1}(B^{\mathrm{T}}ZA+S) \\ A_{cl} &= A - B(R+B^{\mathrm{T}}XB)^{-1}(B^{\mathrm{T}}XA+S) \end{aligned}$$

In fact, (2.8) states that

$$\mathcal{V} = \mathrm{Im} \begin{pmatrix} I \\ X \end{pmatrix} \tag{2.9}$$

is an invariant subspace of the regular matrix pencil:

$$\left[\begin{pmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{pmatrix}, \begin{pmatrix} I - LZ & L \\ -\tilde{A}^{\mathrm{T}} Z & \tilde{A}^{\mathrm{T}} \end{pmatrix} \right] \tag{2.10}$$

such that the eigenvalues of the pencil restricted to $\mathcal{V}$ are the eigenvalues of the matrix A_{cl} and hence are finite. Note that this is a symplectic pencil. In the above we have obtained the following lemma:

Lemma 2.5 : *Assume that H, given by (2.4), has full normal rank. A subspace $\mathcal{V}$ of the form (2.9) is an invariant subspace of the matrix pencil (2.10) such that the matrix pencil restricted to $\mathcal{V}$ has only finite eigenvalues if and only if X is a solution of the algebraic Riccati equation (2.1).*

The matrix Z plays an important role in the above formulation but can be chosen rather arbitrarily. We will transform our problem to get rid of the matrix Z.
Choose μ such that the matrix:

$$\begin{pmatrix} I + \mu A & \mu B & 0 \\ Q & S^{\mathrm{T}} & \mu I + A^{\mathrm{T}} \\ S & R & B^{\mathrm{T}} \end{pmatrix} \tag{2.11}$$

has full rank. The assumption that H has full normal rank guarantees that this is satisfied for all but finitely many μ. We define the following matrix pencil:

$$\left[\begin{pmatrix} A & 0 & B \\ -Q & I & -S^{\mathrm{T}} \\ S & 0 & R \end{pmatrix}, \begin{pmatrix} I & 0 & \mu B \\ 0 & A^{\mathrm{T}} & -\mu S^{\mathrm{T}} \\ 0 & -B^{\mathrm{T}} & \mu R \end{pmatrix} \right] \tag{2.12}$$

Note that H has full normal rank if and only if this pencil is regular. However, this pencil is no longer symplectic. For this pencil we will study invariant subspaces of the form

$$\mathcal{V} = \mathrm{Im} \begin{pmatrix} I \\ X \\ P \end{pmatrix} \tag{2.13}$$

We obtain the following result:

Theorem 2.6 : *Assume that the rational matrix H has full normal rank. Choose μ such that the matrix in (2.11) has full rank. Let the pencil (2.12) be given and define L by (2.3).*

(*i*) *If a symmetric matrix X is such that the rank of $L(X)$ is equal to m then there exists P such that $\mathcal{V}$ defined by (2.13) is an invariant subspace of (2.12). Conversely if (2.13) is an invariant subspace of (2.12) then X is such that the rank of $L(X)$ is equal to m.*

(*ii*) *A matrix X satisfies the Riccati equation (2.1) if and only if there exists P such that $\mathcal{V}$ defined by (2.13) is an invariant subspace of (2.12) and the eigenvalues of the pencil restricted to $\mathcal{V}$ are finite. Conversely if (2.13) is an invariant subspace of (2.12) and the eigenvalues of the pencil restricted to $\mathcal{V}$ are finite then X is a solution of the Riccati equation (2.1).*

The paper [11] connected solutions of the algebraic Riccati equation to invariant subspaces of (2.12) for $\mu = 0$. We generalize their results.

Proof : Assume $L(X)$ has rank m. Let L_1 and L_2 be such that:

$$L(X)\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = 0, \qquad \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \text{ injective.} \tag{2.14}$$

Note that since (2.11) has full rank we must have that $L_1 + \mu A L_1 + \mu B L_2$ is invertible. We choose:

$$V_1 = AL_1 + BL_2, \;\; V_1 = L_1 \text{ and } P = L_2(L_1 - \mu AL_1 - \mu BL_2)^{-1}.$$

Then it is easily checked that:

$$\begin{pmatrix} A & 0 & B \\ -Q & I & -S^{\mathrm{T}} \\ S & 0 & R \end{pmatrix}\begin{pmatrix} I \\ X \\ P \end{pmatrix} V_2 = \begin{pmatrix} I & 0 & \mu B \\ 0 & A^{\mathrm{T}} & -\mu S^{\mathrm{T}} \\ 0 & -B^{\mathrm{T}} & \mu R \end{pmatrix}\begin{pmatrix} I \\ X \\ P \end{pmatrix} V_1 \tag{2.15}$$

We know that $V_1 - \mu V_2 = (L_1 - \mu AL_1 - \mu BL_2)$ is invertible and hence (V_1, V_2) is a regular pencil. By definition, $\mathcal{V}$ defined by (2.13) is then an invariant subspace for (2.12).
To prove the converse in part (i) we assume that $\mathcal{V}$ is an invariant subspace of the pencil (2.12). But in that case we know there exists matrices V_1 and V_2 with $(V_1^{\mathrm{T}} \;\; V_2^{\mathrm{T}})$ surjective such that

$$\begin{pmatrix} A & 0 & B \\ -Q & I & -S^{\mathrm{T}} \\ S & 0 & R \end{pmatrix}\begin{pmatrix} I \\ X \\ P \end{pmatrix} V_2 = \begin{pmatrix} I & 0 & \mu B \\ 0 & A^{\mathrm{T}} & -\mu S^{\mathrm{T}} \\ 0 & -B^{\mathrm{T}} & \mu R \end{pmatrix}\begin{pmatrix} I \\ X \\ P \end{pmatrix} V_1$$

But after premultiplication with the matrix

$$W = \begin{pmatrix} I & 0 & 0 \\ -A^{\mathrm{T}}X & I & 0 \\ B^{\mathrm{T}}X & 0 & I \end{pmatrix}$$

we obtain that

$$L(X)\begin{pmatrix} V_2 \\ P(V_2 - \mu V_1) \end{pmatrix} = 0$$

Moreover, it is easy to check that

$$\begin{pmatrix} V_2 \\ P(V_1 - \mu V_2) \end{pmatrix}$$

is an injective matrix. Hence the rank of $L(X)$ is less than or equal to m. However, lemma 2.3 together with the assumption that H is of full rank guarantees that the rank of $L(X)$ is at least m.
For part (ii) we note that X satisfies the Riccati equation if and only if the rank of $L(X)$ is equal to the rank of $B^{\mathrm{T}}XB + R$ which is then invertible. Moreover, this is equivalent to the requirement that in (2.14) we can choose $L_1 = I$. On the other hand, $\mathcal{V}$ is an invariant subspace of the pencil (2.12) such that the eigenvalues of the pencil restricted to $\mathcal{V}$ are finite if and only if (2.15) is satisfied with $V_2 = I$. The same steps as in the proof of part (i) but with $V_2 = L_1 = I$ then yield a proof of part (ii). ■

We are also interested in (semi-)stabilizing solutions of the algebraic Riccati equation as defined in definition 2.2:

Theorem 2.7 : *Assume that H has full normal rank. A stabilizing solution, if it exists, is unique. Moreover, if a semi-stabilizing solution exists, it is actually a stabilizing solution if and only if*

$$\begin{pmatrix} Q & S^{\mathrm{T}} & z^{-1}I - A^{\mathrm{T}} \\ S & R & -B^{\mathrm{T}} \\ zI - A & -B & 0 \end{pmatrix} \tag{2.16}$$

has full rank for all z on the unit circle.

Proof : A stabilizing solution of the algebraic Riccati equation is clearly unique since solutions of the algebraic Riccati equation have a one to one relation with invariant subspaces of the symplectic pencil (2.10). Since the symplectic pencil has at most n stable eigenvalues, a stable n-dimensional subspace of the pencil is unique and hence also the associated solution to the algebraic Riccati equation is unique.
A semi-stabilizing solution of the algebraic Riccati equation is necessarily stabilizing if the matrix pencil (2.12) has no eigenvalues on the unit circle. It is easy to see that if (2.12) has an eigenvalue λ then (2.16) has a non-empty kernel for $z = \lambda$. Hence if (2.16) has full rank for all z on the unit circle then the matrix pencil (2.12) has no eigenvalues on the unit circle. ■

3 The linear matrix inequality and its associated algebraic Riccati equation

The algebraic Riccati equation studied in section 2 is very general and includes the Riccati equation studied in H_∞ control (see e.g. [3, 10, 21]) as well as the Riccati equation studied in linear quadratic control (see e.g. [1,12,16]). In the rest of the paper we will concentrate on the Riccati equation used in linear quadratic control and the linear matrix inequality associated to it. We therefore require that the solution of the Riccati equation satisfies the additional requirement that

$$B^{\mathrm{T}} X B + R \geq 0 \tag{3.1}$$

We know from section 2 that either all or none of the solutions of the algebraic Riccati equation satisfy this additional property. Basically, we need to assume that $H(z) \geq 0$ for all z on the unit circle, where H is defined by (2.4). Then we know all solutions of the discrete algebraic Riccati equation satisfy (3.1).

Definition 3.1 : *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{m\times n}$ with Q and R being symmetric be given. The matrix inequality for an unknown $n \times n$ matrix X of the form*

$$L(X) := \begin{pmatrix} Q + A^{\mathrm{T}} X A - X & A^{\mathrm{T}} X B + S^{\mathrm{T}} \\ B^{\mathrm{T}} X A + S & B^{\mathrm{T}} X B + R \end{pmatrix} \geq 0 \tag{3.2}$$

is called the discrete linear matrix inequality. Moreover a matrix X which satisfies (3.2) is referred to as a solution of the discrete linear matrix inequality.

We denote the set of real symmetric solutions of the discrete linear matrix inequality (3.2) as Γ, i.e.

$$\Gamma := \{ \; X \in \mathbb{R}^{n\times n} \mid X = X^{\mathrm{T}} \text{ and } L(X) \geq 0 \; \} \tag{3.3}$$

Next we define the notion of rank-minimizing solutions for the discrete linear matrix inequality:

Definition 3.2 : *A solution $X \in \Gamma$ is said to be rank-minimizing if*

$$\textit{rank}\ L(X) = \rho =: \min_{Y\in\Gamma}\ \textit{rank}\ L(Y)$$

Moreover, we denote the set of rank-minimizing solutions of the discrete linear matrix inequality as $\Gamma_{\min}$, i.e.

$$\Gamma_{\min} := \{ \; X \in \Gamma \mid \textit{rank}\ L(X) = \rho \; \} \tag{3.4}$$

Finally we need the concept of strongly rank-minimizing solutions.

Definition 3.3 :*A solution $X \in \Gamma$ is said to be a strongly rank-minimizing solution of the linear matrix inequality if:*

$$\text{rank } L(X) = \text{ rank } (B^{\mathrm{T}}XB + R) \tag{3.5}$$

Moreover, we denote the set of strongly rank-minimizing solutions of the linear matrix inequality as:

$$\mathcal{L}_{\min} := \{ \; X \in \Gamma \mid \text{rank } L(X) = \text{ rank } (B^{\mathrm{T}}XB + R) \; \}$$

The name suggests that strongly rank-minimizing solutions are also rank-minimizing. This property is indeed true as will be shown later. We can also define a stabilizing solution of the linear matrix inequality:

Definition 3.4 : *A solution $X \in \Gamma$ is said to be stabilizing if the rank of*

$$\begin{pmatrix} zI - A & -B \\ Q + A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB + S^{\mathrm{T}} \\ B^{\mathrm{T}}XA + S & B^{\mathrm{T}}XB + R \end{pmatrix} \tag{3.6}$$

is equal to its normal rank for all z outside or on the unit circle. $X \in \Gamma$ is called a semi-stabilizing solution if the rank of the matrix (3.6) is equal to its normal rank for all z outside the unit circle.

We define the discrete-time algebraic Riccati equation associated with the discrete linear matrix inequality (3.2) as follows:

Definition 3.5 : *The H_2 algebraic Riccati equation associated with the discrete linear matrix inequality (3.2) is defined as (2.1) with the additional requirement (3.1).*

The rest of this section will be devoted to the existence of solutions, rank-minimizing solutions, strongly rank-minimizing solutions and (semi-)stabilizing solutions of the linear matrix inequality. Moreover, we will derive relationships between the different kind of solutions to the linear matrix inequality as well as the relation with the H_2 algebraic Riccati equation.
The following lemma shows sufficient conditions for the existence of solutions to the linear matrix inequality, and it is a discrete time equivalent of continuous-time results in [5,23].

Lemma 3.6 : *Assume that $H(z) \geq 0$ for all z on the unit circle, and (A, B) is controllable. Then there exists a solution to the linear matrix inequality.*

Proof : We will study the following optimization problem:

$$\mathcal{J}^*(x_0) := \inf_u \ \{ \ \mathcal{J}(u, x_0) \mid x(k) \to 0 \text{ as } k \to \infty \ \}$$

subject to $x(k+1) = Ax(k) + Bu(k)$, $x(0) = 0$ where

$$\mathcal{J}(u, x_0) := \sum_{k=0}^{\infty} \begin{pmatrix} x(k)^{\mathrm{T}} & u(k)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}$$

We first have to prove that the infimum is finite. First we show that there exist inputs which make $\mathcal{J}(u, x_0)$ finite and which are such that $x(k) \to 0$ as $k \to 0$. Since (A, B) is controllable there exists a stabilizing state feedback $u = Fx$ and it is not hard to see that an input u generated in this way satisfies the requirements for any initial condition x_0. This implies the infimum is bounded from above.

We still have to show that the infimum is bounded from below. For zero-initial condition we can apply the Laplace transform and Parseval's theorem, to get for an arbitrary input u which makes $\mathcal{J}(u, x_0)$ finite while rendering $x(k) \to 0$ as $k \to 0$:

$$\mathcal{J}(u, x_0) = \int_0^{2\pi} \hat{u}\left(e^{i\theta}\right)^* H\left(e^{i\theta}\right) \hat{u}\left(e^{i\theta}\right) \, d\theta \geq 0$$

where $\hat{u}$ denotes the Laplace transform of u. Next let x_0 be an arbitrary initial condition. Then since (A, B) is controllable there exists an input $\bar{u}$ on an interval $[-T, -1]$ which steers the system from $x(-T) = 0$ to $x(0) = x_0$. Hence for an arbitrary input u on the interval $[0, \infty)$ which makes $\mathcal{J}(u, x_0)$ finite and is such that $x(k) \to 0$ as $k \to 0$ we get:

$$\sum_{k=-T}^{-1} \begin{pmatrix} x(k) \\ \bar{u}(k) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x(k) \\ \bar{u}(k) \end{pmatrix} + \sum_{k=0}^{\infty} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \geq 0$$

since on the extended interval $[-T, \infty)$ we have zero-initial condition and we can hence use the previous argument. In this way we get:

$$\mathcal{J}(u, x_0) \geq -\sum_{k=-T}^{-1} \begin{pmatrix} x(k)^{\mathrm{T}} & \bar{u}(k)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x(k) \\ \bar{u}(k) \end{pmatrix}$$

and we see that the infimum is indeed bounded from below.

Using an approach from [14], it is then straightforward to check that there exists a matrix X such that $\mathcal{J}^*(x_0) = x_0^{\mathrm{T}} X x_0$. We then get using a simple dynamic programming step:

$$x_0^{\mathrm{T}} X x_0 = \inf_{u(0)} \begin{pmatrix} x_0^{\mathrm{T}} & u(0)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x_0 \\ u(0) \end{pmatrix} + [Ax_0 + Bu(0)]^{\mathrm{T}} X [Ax_0 + Bu(0)]$$

for all initial conditions. This implies in particular that:

$$-x_0^{\mathrm{T}} X x_0 + \begin{pmatrix} x_0^{\mathrm{T}} & u(0)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \begin{pmatrix} x_0 \\ u(0) \end{pmatrix} + [Ax_0 + Bu(0)]^{\mathrm{T}} X [Ax_0 + Bu(0)] \geq 0$$

for all x_0 and $u(0)$ which is equivalent to $L(X) \geq 0$. In other words we have constructed a solution of the linear matrix inequality. ■

The controllability condition in lemma 3.6 cannot be weakened unless other assumptions are imposed. This can be seen by the following example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad R = 0, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The linear matrix inequality has no solution. We have $H(z) \geq 0$ for all z on the unit circle but (A, B) is not controllable. On the other hand (A, B) is stabilizable and hence we cannot even weaken our controllability assumption in the above lemma to stabilizability.

We will now have a closer look at rank-minimizing solutions of the linear matrix inequality. We obtain the following lemma:

Lemma 3.7 : *We have:*

$$\textit{rank } L(X) \geq \rho = \textit{normalrank } H \quad \forall X \in \Gamma. \tag{3.7}$$

Proof : This follows from lemma 2.3. Obviously, since $H(\cdot, X)$ has normal rank ρ, we must have that the rank of $L(X)$ is larger than ρ. ■

For the particular case where (A, B) is controllable and H has full rank, we can obtain a very explicit characterization of all rank-minimizing solutions of the linear matrix inequality. This is given in the following lemma which is a generalization of a result from [16].

Lemma 3.8 : *Let (A, B) be controllable. Moreover assume that H has full normal rank and $H(z) \geq 0$ for all z on the unit circle. Choose μ such that (2.11) has full rank. Let $\lambda_1, \ldots, \lambda_j$ be the eigenvalues unequal to μ outside the unit circle, and $\lambda_{j+1}, \ldots, \lambda_r$ be the eigenvalues unequal to μ on the unit circle of the matrix pencil (2.12), all of them without counting multiplicity. Then there exists a symmetric solution to the algebraic H_2 Riccati equation (2.7) with the additional requirement (3.1) such that the eigenvalues of*

$$A - B(R + B^{\mathrm{T}} X B)^{-1}(B^{\mathrm{T}} X A + S^{\mathrm{T}}) \tag{3.8}$$

are $\mu_1, \ldots \mu_j, \lambda_{j+1}, \ldots \lambda_r$ where for $i = 1, \ldots, j$ we have either $\mu_i = \lambda_i$ or $\mu_i = \lambda_i^{-1}$

Proof : By lemma 3.6 we know there exists at least one solution $\bar{X}$ of the linear matrix inequality. Let C and D satisfy (A.5). We can try to find a solution of the algebraic Riccati equation:

$$\tilde{X} = C^{\mathrm{T}} C + A^{\mathrm{T}} \tilde{X} A - (A^{\mathrm{T}} \tilde{X} B + C^{\mathrm{T}} D)(B^{\mathrm{T}} \tilde{X} B + D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} \tilde{X} A + D^{\mathrm{T}} C) \tag{3.9}$$

It is easy to check that X satisfies the Riccati equation (2.7) if and only if $\tilde{X} = X - \bar{X}$ satisfies the Riccati equation (3.9). Moreover, the eigenvalues of (3.8) are equal to the eigenvalues of

$$A - B(D^{\mathrm{T}} D + B^{\mathrm{T}} \tilde{X} B)^{-1}(B^{\mathrm{T}} \tilde{X} A + D^{\mathrm{T}} C)$$

Finally the eigenvalues of the matrix pencil (2.12) are equal to the eigenvalues of the following pencil:

$$\left[\begin{pmatrix} A & 0 & B \\ -C^{\mathrm{T}}C & I & -C^{\mathrm{T}}D \\ D^{\mathrm{T}}C & 0 & D^{\mathrm{T}}D \end{pmatrix}, \begin{pmatrix} I & 0 & \mu B \\ 0 & A^{\mathrm{T}} & -\mu C^{\mathrm{T}}D \\ 0 & -B^{\mathrm{T}} & \mu D^{\mathrm{T}}D \end{pmatrix}\right] \tag{3.10}$$

Next, we know there exists a feedback F and a suitable basis in the state space such that:

$$A+BF=\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, B=\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C+DF=\begin{pmatrix} 0 & C_2 \end{pmatrix},$$

where the eigenvalues of A_{11} are precisely the invariant zeros of (A,B,C,D) on the unit circle. We will try to find a solution of the algebraic Riccati equation of the form:

$$\tilde{X}=\begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix}$$

We find that X_{22} should satisfy the following algebraic Riccati equation:

$$X_{22}=A_{22}^{\mathrm{T}}X_{22}A_{22}-(A_{22}^{\mathrm{T}}X_{22}B_2+C_2^{\mathrm{T}}D)(B_2^{\mathrm{T}}X_{22}B_2+D^{\mathrm{T}}D)^{-1}(B_2^{\mathrm{T}}X_{22}A_{22}+D^{\mathrm{T}}C_2)+C_2^{\mathrm{T}}C_2 \tag{3.11}$$

such that the matrix

$$A_{22}-B_2(B_2^{\mathrm{T}}X_{22}B_2+D^{\mathrm{T}}D)^{-1}(B_2^{\mathrm{T}}X_{22}A_{22}+D^{\mathrm{T}}C_2) \tag{3.12}$$

has eigenvalues $\mu_1,\ldots,\mu_j$. Moreover, the matrix pencil associated to this Riccati equation is:

$$\left[\begin{pmatrix} A_{22} & 0 & B_2 \\ -C_2^{\mathrm{T}}C_2 & I & -C_2^{\mathrm{T}}D \\ D^{\mathrm{T}}C_2 & 0 & D^{\mathrm{T}}D \end{pmatrix}, \begin{pmatrix} I & 0 & \mu B_2 \\ 0 & A_{22}^{\mathrm{T}} & -\mu C_2^{\mathrm{T}}D \\ 0 & -B_2^{\mathrm{T}} & \mu D^{\mathrm{T}}D \end{pmatrix}\right] \tag{3.13}$$

and it is easy to check that this matrix pencil has no eigenvalues on the unit circle. Moreover, the eigenvalues of the matrix pencil (3.10) are precisely the eigenvalues of the reduced matrix pencil (3.13) together with the eigenvalues of A_{11}. This implies in particular that the eigenvalues of A_{11} are $\lambda_{j+1},\ldots,\lambda_r$. Let A_{22} be an $n_r\times n_r$ matrix and B_2 be a $n_r\times m_r$ matrix. Choose a n_r-dimensional invariant subspace

$$\mathcal{V}=\operatorname{im}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

of the pencil (3.13) associated with the eigenvalues $\mu_1,\ldots,\mu_j$. In other words, we have:

$$\begin{pmatrix} A_{22} & 0 & B_2 \\ -C_2^{\mathrm{T}}C_2 & I & -C_2^{\mathrm{T}}D \\ D^{\mathrm{T}}C_2 & 0 & D^{\mathrm{T}}D \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}V_2=\begin{pmatrix} I & 0 & \mu B_2 \\ 0 & A_{22}^{\mathrm{T}} & -\mu C_2^{\mathrm{T}}D \\ 0 & -B_2^{\mathrm{T}} & \mu D^{\mathrm{T}}D \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}V_1$$

Since $\mu_j \neq \mu$ for all j we have that $V_2 - \mu V_1$ is invertible. Then it is easy to check that $(X_1^T \; X_2^T)^T$ must be injective. Moreover

$$\mathcal{V}_r = \text{im} \begin{pmatrix} X_1 \\ X_2 - ZX_1 \end{pmatrix} = \text{im}\, X$$

is an invariant subspace of the symplectic pencil $[L_1, L_2]$ with

$$\begin{aligned} L_1 &= \begin{pmatrix} A_{22} - B_2\tilde{R}^{-1}\tilde{S} & 0 \\ -C_2^T C_2 + A_{22}^T Z A_{22} - \tilde{S}^T\tilde{R}^{-1}\tilde{S} - Z & I \end{pmatrix}, \\ L_2 &= \begin{pmatrix} I & B_2\tilde{R}^{-1}B_2^T \\ 0 & A_{22}^T - \tilde{S}^T\tilde{R}^{-1}B_2 \end{pmatrix} \end{aligned}$$

where $\tilde{S} = B_2^T Z A_{22} + D^T C_2$, $\tilde{R} = B_2^T Z B_2 + D^T D$ and Z is a symmetric matrix chosen such that $\tilde{R}$ is invertible. We have $L_1 X V_2 = L_2 X V_1$. Moreover, $[L_1, L_2]$ has no eigenvalues on the unit circle and hence $L_1 - L_2$ and $V_1 - V_2$ are both invertible. We find:

$$(T_1 - T_2)^{-1}(T_1 + T_2)X = X(V_1 + V_2)(V_1 - V_2)^{-1}$$

Using some algebraic manipulations we find that:

$$\bar{T} = (T_1 - T_2)^{-1}(T_1 + T_2) = \begin{pmatrix} \bar{A} & -\bar{R} \\ -\bar{Q} & -\bar{A}^T \end{pmatrix}$$

with $(\bar{A}, \bar{R})$ controllable, $\bar{R} \geq 0$ and $\bar{Q}$ symmetric. Moreover, we find:

$$\bar{T}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\bar{T}^T$$

We can then use the argument from [6, p. 87] to show that X_1 is invertible and that $X_2^T X_1$ is symmetric. In this way, we find that

$$\text{Im} \begin{pmatrix} I \\ X_2 X_1^{-1} \\ X_3 X_1^{-1} \end{pmatrix}$$

is an invariant subspace of (3.13). This guarantees that $X_{22} = X_2 X_1^{-1}$ satisfies (3.11). Moreover, since $X_2^T X_1$ is symmetric we find that X_{22} is symmetric. Finally, the eigenvalues of (3.12) are the required $\mu_1, \ldots, \mu_j$. Then

$$X = \bar{X} + \begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix}$$

satisfies the requirements of the lemma. ∎

The above can be used to obtain the following characterization of the set $\Gamma_{\min}$:

Lemma 3.9 : *Assume that H has full normal rank m and $H(z) \geq 0$ on the unit circle. Moreover, assume that the uncontrollable eigenvalues $\lambda_1, \lambda_2, \dots \lambda_J$ of (A, B) are such that $\lambda_i \lambda_j \neq 1$ for any i, j. Then the H_2 Riccati equation (2.7) with the additional requirement (3.1) has at least one symmetric solution. Moreover:*

$$\Gamma_{\min} = \{ X \in \Gamma \mid \text{rank } L(X) = m \} \tag{3.14}$$

Proof : We write A and B in Kalman canonical form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

Next decompose Q, S and a potential solution of the algebraic Riccati equation X compatibly:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}, \qquad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{pmatrix}.$$

We first note that the $1,1$-block of the equation (2.7) reduces to :

$$X_{11} = A_{11}^{\mathrm{T}} X_{11} A_{11} - (A_{11}^{\mathrm{T}} X_{11} B_1 + S_1)(B_1^{\mathrm{T}} X_{11} B_1 + R)^{-1}(B_1^{\mathrm{T}} X_{11} A_{11} + S_1) + Q_{11} \tag{3.15}$$

We see that X_{11} must be the solution of the discrete algebraic Riccati equation associated to the controllable subsystem. Our assumptions with respect to the uncontrollable eigenvalues of (A, B) guarantees that there are no uncontrollable eigenvalues on the unit circle and if λ is an uncontrollable eigenvalue then λ^{-1} is not an uncontrollable eigenvalue. From lemma 3.8 we know the existence of a solution X_{11} of the Riccati equation (3.15) such that the eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ of

$$A_x := A_{11} - B_1(B_1^{\mathrm{T}} X_{11} B_1 + D^{\mathrm{T}} D)^{-1} B_1^{\mathrm{T}} X_{11} A_{11} \tag{3.16}$$

are such that $\mu_i \lambda_j \neq 1$ for $i = 1, \dots, n$ and $j = 1, \dots, J$. Note that $\lambda_1, \dots, \lambda_J$ are precisely the eigenvalues of A_{22}.
Next we study the $2,1$ block of the Riccati equation (2.7). It can be written in the following form:

$$X_{12}^{\mathrm{T}} = A_{22}^{\mathrm{T}} X_{12}^{\mathrm{T}} A_x + [A_{12}^{\mathrm{T}} X_{11} A_x + Q_{21}]$$

Due to our condition on the relation between the eigenvalues of A_x and A_{22} we know this equation is uniquely solvable for X_{12} (see e.g. [13]). Finally, we have the $2,2$-block of the Riccati equation (2.7). It can be written in the following form:

$$X_{22} = A_{22}^{\mathrm{T}} X_{22} A_{22} + M$$

where M depends on X_{11} and X_{12} but is independent of X_{22}. We know A_{22} has no two eigenvalues which are the inverse of each other and hence this equation has a symmetric solution X_{22}. It is easily checked that the so-constructed X is a solution of the algebraic Riccati equation (2.7).
We know that X is a solution of the discrete linear matrix inequality. Moreover, it is easy to check that the rank of $L(X)$ is equal to m. Because of (3.7) we find that X is a rank-minimizing solution of the discrete linear matrix inequality. (3.14) is then an immediate consequence. ∎

Remark: Note that (3.14) does not hold in general. For instance if

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad R = 1, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then (3.14) is not true. All matrices X yield a matrix $L(X)$ with rank larger than or equal to 2. On the other hand $m = 1$.

We are of course also interested in an analogous result as of the above theorem for the case H has no longer full normal rank. We have:

Lemma 3.10 : *Assume that $H(z) \geq 0$ on the unit circle and that there exist at least one solution of the linear matrix inequality (3.2). Moreover, assume that the uncontrollable eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_J$ of (A, B) are such that $\lambda_i \lambda_j \neq 1$ for any i, j. Then the H_2 Riccati equation (2.1) with the additional requirement (3.1) has at least one symmetric solution. Moreover:*

$$\Gamma_{\min} = \{\ X \in \Gamma \mid \mathit{rank}\ L(X) = \rho\ \} \tag{3.17}$$

Proof : Since there exists a solution to the linear matrix inequality we can use the reduction scheme described in the appendix. Using theorem A.6 we know that we need to find a strongly rank-minimizing solution of $L^r(X_{22}) \geq 0$. The reduced system has an associated H^r, defined by (A.12), which has full normal rank. Moreover, the uncontrollable eigenvalues of (A_{22}, B_{22}) are a subset of the uncontrollable eigenvalues of (A, B) and, hence, also satisfy the above property that the product of two uncontrollable eigenvalues is always unequal to 1. By applying lemma 3.9, we then obtain the desired result. ∎

Remark: The above result also has a direct continuous time analogue. We look at the linear matrix inequality:

$$L_c(X) = \begin{pmatrix} A^{\mathrm{T}}X + XA + C^{\mathrm{T}}C & XB + D^{\mathrm{T}}C \\ B^{\mathrm{T}}X + C^{\mathrm{T}}D & D^{\mathrm{T}}D \end{pmatrix} \geq 0$$

There exists a rank-minimizing solution X which yields a rank of $L_c(X)$ equal to normal rank of (A, B, C, D) if the uncontrollable eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_J$ of (A, B) are such that $\lambda_i + \lambda_j \neq 0$ for any i, j. If D is injective, X satisfies the continuous algebraic Riccati equation:

$$A^{\mathrm{T}}X + XA + C^{\mathrm{T}}C - (XB + D^{\mathrm{T}}C)(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}X + C^{\mathrm{T}}D) = 0$$

This is a generalization of [18] which showed that (A, B) stabilizable is sufficient to guarantee the above.

We are interested in those solutions of the discrete linear matrix inequality which can be associated to solutions of the discrete algebraic Riccati equation.
Note that the assumptions of lemma 3.9 guarantees that the set $\mathcal{L}_{\min}$ is non-empty. We show that the set $\mathcal{L}_{\min}$ in fact coincides with the set of real symmetric solutions of the discrete algebraic Riccati equation associated with the discrete linear matrix inequality.
We have the following lemma:

Lemma 3.11 : *The set of solutions X of the linear matrix inequality $L(X) \geq 0$ coincides with the set of real symmetric solutions of the general algebraic inequality given by:*

- $Ker\,(R + B^{\mathrm{T}}XB) \subseteq Ker\,(A^{\mathrm{T}}XB + S^{\mathrm{T}})$,
- $B^{\mathrm{T}}XB + R \geq 0$,
- $A^{\mathrm{T}}XA - X - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) + Q \geq 0.$

Proof : For any x we have

$$\begin{pmatrix} 0 & x^{\mathrm{T}} \end{pmatrix} L(X) \begin{pmatrix} 0 \\ x \end{pmatrix} \geq 0$$

which implies that $B^{\mathrm{T}}XB + R = 0$. Moreover, since $L(X) \geq 0$ we have:

$$\begin{pmatrix} 0 & x^{\mathrm{T}} \end{pmatrix} L(X) \begin{pmatrix} 0 \\ x \end{pmatrix} = 0 \text{ which implies } L(X) \begin{pmatrix} 0 \\ x \end{pmatrix} = 0$$

and in this way we obtain the inclusion of the two kernels in our lemma. ■

Remark: The above lemma indicates that in searching for a connection between the linear matrix inequality and an algebraic Riccati equation we need to look for a H_2 Riccati equation with the additional condition (3.1) as an essential ingredient and not the general discrete time algebraic Riccati equation.

Corollary 3.12 :*The set of strongly rank-minimizing solutions of the linear matrix inequality coincides with the set of real symmetric solutions of the H_2 algebraic Riccati equation associated with the linear matrix inequality.*
In other words, any symmetric real matrix X satisfying the H_2 algebraic Riccati equation (2.1) and the additional condition (3.1) belongs to the set $\mathcal{L}_{\min}$. Conversely any $X \in \mathcal{L}_{\min}$ satisfies the discrete algebraic Riccati equation (2.1) and the additional condition (3.1).

Proof : According to lemma 3.11 condition (2.1b) and (3.1) are satisfied. Next we note that:

$$\begin{pmatrix} I & -(A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger} \\ 0 & I \end{pmatrix} L(X) \begin{pmatrix} I & 0 \\ -(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) & I \end{pmatrix}$$
$$= \begin{pmatrix} A^{\mathrm{T}}XA - X - (A^{\mathrm{T}}XB + S^{\mathrm{T}})(B^{\mathrm{T}}XB + R)^{\dagger}(B^{\mathrm{T}}XA + S) + Q & 0 \\ 0 & I \end{pmatrix}$$

where we used that $V(I - W^{\dagger}W) = 0$ if $\mathrm{Ker}\, W \subseteq \mathrm{Ker}\, V$. Hence the rank of $L(X)$ equals the sum of the rank of $B^{\mathrm{T}}XB + D^{\mathrm{T}}D$ and the rank of its Schur complement. Therefore the Schur complement, which is equal to the Riccati equation, must be 0 which implies (2.1a). ■

The above implies that strongly rank-minimizing solutions are indeed also rank-minimizing solutions:

Observation 3.13 : *We have $\mathcal{L}_{\min} \subseteq \Gamma_{\min}$.*

Proof : Let X be any strongly rank-minimizing solution. Then X satisfies the algebraic Riccati equation (2.1) and hence, according to lemma 2.4 the rank of $B^{\mathrm{T}}XB + R$ equals the normal rank of H. On the other hand by lemma (3.7) we have rank $B^{\mathrm{T}}XB + D^{\mathrm{T}}D =$ rank $L(X)$ is larger than or equal to the normal rank of H. ■

The real symmetric solutions of the algebraic Riccati equation are a subset of all the rank-minimizing solutions of the linear matrix inequality. Within the set of rank-minimizing solutions they are the ones that maximize the rank of $B^{\mathrm{T}}XB + R$.
Suppose we have the matrix pencil (2.12). We can ask ourselves whether we have a result equivalent to theorem 2.6 in case the rational matrix H has no longer full rank. H has full rank if and only if the pencil (2.12) is regular. Hence for the general case we have to work with singular pencils. Let ρ denote the normal rank of H. This time we study invariant subspaces of the form:

$$\mathcal{V} = \operatorname{Im} \begin{pmatrix} I & 0 \\ X & 0 \\ P & M \end{pmatrix} \tag{3.18}$$

where M is a $m \times (m-\rho)$ injective matrix. We can associate a matrix pencil (V_1, V_2) to the matrix pencil (2.12) restricted to the invariant subspace $\mathcal{V}$ in the sense that:

$$\begin{pmatrix} A & 0 & B \\ -Q & I & -S^{\mathrm{T}} \\ S & 0 & R \end{pmatrix} \begin{pmatrix} I & 0 \\ X & 0 \\ P & M \end{pmatrix} V_2 = \begin{pmatrix} I & 0 & \mu B \\ 0 & A^{\mathrm{T}} & -\mu S^{\mathrm{T}} \\ 0 & -B^{\mathrm{T}} & \mu R \end{pmatrix} \begin{pmatrix} I & 0 \\ X & 0 \\ P & M \end{pmatrix} V_1 \tag{3.19}$$

We obtain the following result:

Theorem 3.14 : *Choose μ such that the matrix in (2.11) has rank equal to $2n + \rho$. Assume that a solution of the linear matrix inequality exists. Let the pencil (2.12) be given and define L by (2.3).*

(*i*) *If a symmetric matrix X is such that the rank of $L(X)$ is equal to ρ then there exists a matrix $P \in \mathbb{R}^{m \times n}$ and an injective matrix $M \in \mathbb{R}^{m \times (m-\rho)}$ such that $\mathcal{V}$ defined by (3.18) is an invariant subspace of (2.12) such that we can associate to (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) without eigenvalue μ.*

Conversely, if (3.18) is an invariant subspace of (2.12) such that we can associate to (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) without eigenvalue μ. then X is such that the rank of $L(X)$ is equal to ρ.

(*ii*) *A matrix X satisfies the Riccati equation (2.1) if and only if there exists a matrix $P \in \mathbb{R}^{m \times n}$ and an injective matrix $M \in \mathbb{R}^{m \times (m-\rho)}$ such that $\mathcal{V}$ defined by (3.18) is an invariant subspace of (2.12) such that we can associate to (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) with only finite eigenvalues which are different from μ.*

Conversely if (3.18) is an invariant subspace of (2.12) such that we can associate to the pencil (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) with only finite eigenvalues different from μ then X is a solution of the Riccati equation (2.1).

Proof : Suppose that we have an invariant subspace of the pencil (2.12) of the form (3.18) such that (3.19) is satisfied for a regular pencil (V_1, V_2) with no eigenvalue in μ. After some algebraic manipulations we find:

$$L(X)\begin{pmatrix} \begin{bmatrix} I & 0 \end{bmatrix} V_2 \\ \begin{bmatrix} P & M \end{bmatrix} (V_2 - \mu V_1) \end{pmatrix} = 0$$

This guarantees that the rank of $L(X)$ is equal to ρ as soon as we have shown that the following matrix is injective:

$$\begin{pmatrix} \begin{bmatrix} I & 0 \end{bmatrix} V_2 \\ \begin{bmatrix} P & M \end{bmatrix} (V_2 - \mu V_1) \end{pmatrix} \tag{3.20}$$

Suppose x is in the kernel of this matrix. Then the equation (3.19) yields $\begin{bmatrix} I & 0 \end{bmatrix} V_1 x = 0$ and we get:

$$\begin{pmatrix} I & 0 \\ P & M \end{pmatrix} (V_2 - \mu V_1)x = 0$$

Since M is injective and $V_2 - \mu V_1$ is invertible we get $x = 0$ and hence the matrix (3.20) is injective and therefore the rank of $L(X)$ equals ρ.
We will use lemma 2.3. Let z be on the unit circle for which $H(z)$ has rank ρ. Define V by:

$$V = \begin{pmatrix} (zI - A)^{-1}B \\ I \end{pmatrix}$$

We have $V^* L(X) V = H(z) \geq 0$ and the rank of $V^* L(X) V$ equals the rank of $L(X)$. This implies $L(X) \geq 0$. Therefore X is a rank-minimizing solution of the linear matrix inequality.
On the other hand if V_2 is invertible then $\begin{bmatrix} I & 0 \end{bmatrix} V_2$ is surjective. This implies that we have

$$\begin{pmatrix} \begin{bmatrix} I & 0 \end{bmatrix} V_2 \\ \begin{bmatrix} P & M \end{bmatrix} (V_2 - \mu V_1) \end{pmatrix} = \begin{pmatrix} I & 0 \\ S_1 & S_2 \end{pmatrix} R$$

for some invertible matrix R. This guarantees:

$$L(X)\begin{pmatrix} I \\ S_1 \end{pmatrix} = 0$$

and it is easy to check that this, together with $L(X) \geq 0$, imply that X satisfies the algebraic Riccati equation.

Conversely assume that X is a rank-minimizing solution. We have to construct P, M, V_1 and V_2 satisfying (3.19). Since X is a solution of the linear matrix inequality, we can apply the reduction technique described in the appendix with $\bar{X} = X$. We get

$$A + BF = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}, C + DF = \begin{pmatrix} 0 & C_1 \end{pmatrix}, D = \begin{pmatrix} 0 & D_1 \end{pmatrix},$$

using a suitable feedback F and suitable bases. Moreover, C, D are defined by (A.5). We can find P_{22}, W_2 and W_1 satisfying

$$\begin{pmatrix} A_{22} & 0 & B_{22} \\ -C_1^T C_1 & I & -C_1^T D_1 \\ D_1^T C_1 & 0 & D_1^T D_1 \end{pmatrix} \begin{pmatrix} I \\ 0 \\ P_{22} \end{pmatrix} W_2 = \begin{pmatrix} I & 0 & \mu B_{22} \\ 0 & A_{22}^T & -\mu C_1^T D_1 \\ 0 & -B_{22}^T & \mu D_1^T D_1 \end{pmatrix} \begin{pmatrix} I \\ 0 \\ P_{22} \end{pmatrix} W_1$$

since $X_{22} = 0$ is a rank-minimizing solution of the reduced linear matrix inequality. If X is a strongly rank-minimizing solution we have W_2 invertible and otherwise W_2 is singular.
We can now easily construct suitable P, M, V_1 and V_2 in this basis:

$$P = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix} + F, M = \begin{pmatrix} I \\ 0 \end{pmatrix}, V_1 = \begin{pmatrix} A_{11} & 0 & B_{11} \\ 0 & W_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, V_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & I \end{pmatrix}$$

and we see that the invariant subspace indeed satisfies all the requirements of the theorem. ■

The main difference with theorem 2.6 is that the eigenvalues of the pencil (2.12) restricted to $\mathcal{V}$ are no longer unique. Hence we have to focus on the defining equation (3.19). One can determine invariant subspaces of a matrix pencil using numerical tools which are not very well developed for the singular case but it can still be done (see e.g. [20,22]).
We have to work with the invariant subspaces of the form (3.18) instead of invariant subspaces of the form (2.13). However, if we are only interested in strongly rank-minimizing solutions we can dispense with the matrix M:

Theorem 3.15 : *Choose μ such that the matrix in (2.11) has rank equal to $2n + \rho$. Assume that a solution of the linear matrix inequality exists. Let the pencil (2.12) be given and define L by (2.3).*
A matrix X satisfies the Riccati equation (2.1) if and only if there exists a matrix $P \in \mathbb{R}^{m \times n}$ such that $\mathcal{V}$ defined by (2.13) is an invariant subspace of (2.12) such that we can associate to (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) with only finite eigenvalues which are different from μ. Conversely if (2.13) is an invariant subspace of (2.12) such that we can associate to the pencil (2.12) restricted to $\mathcal{V}$ a pencil (V_1, V_2) with only finite eigenvalues different from μ then X is a solution of the Riccati equation (2.1).

Proof : Given a solution of the Riccati equation the existence of an invariant subspace of the form (2.13) with the desired properties is a trivial consequence of theorem 3.14.

For the converse we note that

$$L(X)\begin{pmatrix} V_2 \\ P(V_2-\mu V_1) \end{pmatrix} = 0$$

with V_2 invertible. This implies that

$$\begin{aligned} -X + A^{\mathrm{T}}XA + Q + (A^{\mathrm{T}}XB+S)G &= 0 \\ (B^{\mathrm{T}}XA+S^{\mathrm{T}}) + (B^{\mathrm{T}}XB+R)G &= 0 \end{aligned}$$

for $G = P(V_2-\mu V_1)V_2^{-1}$. Hence we immediately find:

$$X = A^{\mathrm{T}}XA + Q - (A^{\mathrm{T}}XB+S)(B^{\mathrm{T}}XB+R)^{\dagger}(B^{\mathrm{T}}XA+S^{\mathrm{T}}) \qquad \blacksquare$$

We know that each rank minimizing solution of the linear matrix inequality is associated to an invariant subspace of the matrix pencil (2.12). If the pencil is regular (i.e. H has full normal rank) then the eigenvalues of the matrix pencil restricted to that subspace are uniquely determined. In that case, it can be checked that λ is an eigenvalue of the matrix pencil restricted to $\mathcal{V}$ if and only if the matrix (2.2) has a zero for $z=\lambda$. Moreover ∞ is an eigenvalue of the matrix pencil restricted to $\mathcal{V}$ if and only if the the matrix (2.2) viewed as a polynomial matrix in z has an infinite zero. Note that (2.2) can be viewed as the Rosenbrock system matrix of some system and the invariant zeros of that system determine the eigenvalues of the matrix pencil restricted to $\mathcal{V}$. If X is a strongly rank-minimizing solution then $B^{\mathrm{T}}XB+R$ is invertible and the invariant zeros of (2.2) are precisely the eigenvalues of the matrix

$$A - B(B^{\mathrm{T}}XB+R)^{-1}(B^{\mathrm{T}}XA+S)$$

In the case where H has no longer full normal rank the eigenvalues of the regular pencil (V_1,V_2) satisfying (3.19) are no longer uniquely determined. The finite or infinite zeros of the matrix pencil (2.2) are eigenvalues of all regular pencils (V_1,V_2) satisfying (3.19). The rest of the eigenvalues can be chosen arbitrarily. If X is a strongly rank-minimizing solution then the eigenvalues of the pencil (V_1,V_2) are eigenvalues of the matrix

$$A - BT^{\dagger}(B^{\mathrm{T}}XA+S) - B(I-T^{\dagger}T)F \tag{3.21}$$

for some suitably chosen matrix F where $T = B^{\mathrm{T}}XB+R$.
In linear quadratic control or H_2 control we are particularly interested in the semi-stabilizing rank-minimizing solution of the discrete linear matrix inequality. This is defined (see lemma 3.4) as the rank-minimizing solution of the linear matrix inequality for which the matrix (2.2) has all zeros inside or on the unit circle. In particular, this means no infinite zeros and hence this must necessarily be a strongly rank-minimizing solution. Hence the semi-stabilizing solution can be alternatively defined as a rank-minimizing solution of the linear matrix inequality for which there exists a matrix F for which (3.21) has all eigenvalues in the closed unit disc.
We will derive some additional properties of the semi-stabilizing solution. For this, we need a preliminary lemma:

Lemma 3.16 : *Assume that* (A, B) *has no uncontrollable eigenvalues on the unit circle,* $Q \geq 0$*,* $R \geq 0$ *and* H *has full normal rank. Let the following algebraic Riccati equation be given:*

$$X = A^{\mathrm{T}}XA - A^{\mathrm{T}}XB(R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA + Q$$

If X *is a symmetric semi-stabilizing solution of this algebraic Riccati equation then* $X \geq 0$*.*

Proof : We can rewrite the algebraic Riccati equation in the following form:

$$X = \tilde{A}^{\mathrm{T}}X\tilde{A} + F^{\mathrm{T}}RF + Q \tag{3.22}$$

where $F = (R + B^{\mathrm{T}}XB)^{-1}B^{\mathrm{T}}XA$ and $\tilde{A} = A + BF$. Since X is a semi-stabilizing solution we know that $\tilde{A}$ has all eigenvalues inside or on the unit circle.
Assume that $(A - \lambda I)y \in \ker X \cap \ker Q$ with $|\lambda| = 1$. Then we get from (3.22) that

$$y^*Xy = y^*\tilde{A}^{\mathrm{T}}X\tilde{A}y + y^*F^{\mathrm{T}}RFy + y^*Qy = y^*Xy + y^*F^{\mathrm{T}}RFy + y^*Qy.$$

Since $Q, R \geq 0$, we get $Qy = 0$ and $RFy = 0$. Next we note that $B^{\mathrm{T}}X\tilde{A} = -RF$. We see from (3.22) that for $z = X\tilde{A}y$ we have $\tilde{A}^{\mathrm{T}}z = Xy = \lambda^{-1}XAy = \lambda^{-1}z$. Moreover $B^{\mathrm{T}}z = -RFy = 0$. Since (A, B) has no uncontrollable eigenvalues on the unit circle, the same holds for $(\tilde{A}, B)$. Finally, λ^{-1} is on the unit circle. Therefore, we find that $z = 0$. In other words, $Xy = 0$. Hence we see that $y \in \ker X \cap \ker Q$.
In conclusion the above guarantees recursively that all the eigenvectors and generalized eigenvectors of $\tilde{A}$ associated to eigenvalues on the unit circle are in the kernel of both X and Q. After all if $(A - \lambda I)y = 0 \in \ker X \cap \ker Q$ we get $y \in \ker X \cap \ker Q$. Generalized eigenvectors have the property that $(\tilde{A} - \lambda I)^k y = 0$ for some k. Hence we have to use the argument in the previous paragraph k times to get $y \in \ker X \cap \ker Q$.
Since $\tilde{A}$ has all eigenvalues in the closed unit disc we find that

$$Q\tilde{A}^k \to 0, X\tilde{A}^k \to 0 \text{ as } k \to \infty$$

Moreover the convergence is exponentially fast. But it is easily seen from (3.22) that for all K:

$$X = \sum_{k=0}^{K}(\tilde{A}^{\mathrm{T}})^k(Q + F^{\mathrm{T}}RF)\tilde{A}^k + (\tilde{A}^{\mathrm{T}})^{K+1}X\tilde{A}^{K+1}$$

When we take the limit as $K \to \infty$, we get:

$$X = \sum_{k=0}^{\infty}(\tilde{A}^{\mathrm{T}})^k(Q + F^{\mathrm{T}}RF)\tilde{A}^k$$

and we trivially see that $X \geq 0$. ■

The following theorem which is for a large part an extension of theorem 2.7 discusses the uniqueness of the semi-stabilizing solution:

Theorem 3.17 : *The semi-stabilizing, rank-minimizing solution of the linear matrix inequality, if it exists, is the largest solution of the linear matrix inequality, i.e. if $\tilde{X}$ is a semi-stabilizing, rank-minimizing solution of the linear matrix inequality and X is an arbitrary solution of the linear matrix inequality then we have $\tilde{X} \geq X$. In particular this implies that the semi-stabilizing, rank-minimizing solution of the linear matrix inequality is unique.*
If a semi-stabilizing solution exists, it is actually a stabilizing solution if and only if (2.16) does not lose rank for any z on the unit circle.

Proof : Assume that $\tilde{X}$ is a semi-stabilizing rank-minimizing solution of the algebraic Riccati equation and X is an arbitrary solution of the linear matrix inequality. We need to show that $\tilde{X} \geq X$. We will apply the reduction scheme presented in the appendix. We use as an initial solution of the linear matrix inequality $\bar{X} = X$. We factorize $L(\bar{X})$ as in (A.5). We obtain a new linear matrix inequality $\bar{L}$ given in (A.6) and we note that $\hat{X} = \tilde{X} - X$ is a semi-stabilizing solution of this linear matrix inequality. There exists a matrix F such that $\mathcal{R}^*(\Sigma)$ is then the largest $A + BF$-invariant subspace containing $B \ker D$ and contained in the kernel of $C + DF$. Using the basis as in (A.9) we get that

$$\tilde{X} - X = \begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix}$$

and X_{22} is such that

$$X_{22} = A_{22}^{\mathrm{T}} X_{22} A_{22} - A_{22}^{\mathrm{T}} X_{22} B_{22} (D_1^{\mathrm{T}} D_1 + B_{22}^{\mathrm{T}} X_{22} B_{22})^{-1} B_{22}^{\mathrm{T}} X_{22} A_{22}$$

and

$$A_{22} - B_{22}(D_1^{\mathrm{T}} D_1 + B_{22}^{\mathrm{T}} X_{22} B_{22})^{-1} B_{22}^{\mathrm{T}} X_{22} A_{22}$$

has all eigenvalues inside or on the unit circle. Lemma 3.16 then implies that $X_{22} \geq 0$ and hence $\tilde{X} \geq X$.
A proof that a semi-stabilizing solution is actually a stabilizing solution if (2.16) has no zeros on the unit circle can be given by first applying the reduction scheme and then applying theorem (2.7). It then only needs some algebraic manipulations to translate conditions for the matrix pencil associated to the reduced problem into the matrix pencil of the original system. ■

Note that the above theorem only states uniqueness of semi-stabilizing and rank-minimizing solutions of the algebraic Riccati equation. In general there can be more semi-stabilizing solutions but only one rank-minimizing and semi-stabilizing solution.
The next lemma establishes necessary and sufficient conditions under which the set of rank-minimizing solutions of the discrete linear matrix inequality, coincides with the set of real symmetric solutions of the discrete algebraic Riccati equation associated with the discrete linear matrix inequality.

Lemma 3.18 : *The set of rank-minimizing solutions of the linear matrix inequality* $\Gamma_{\min}$ *equals the set of strongly rank-minimizing solutions of the linear matrix inequality* $\mathcal{L}_{\min}$ *if the matrix pencil:*

$$\begin{pmatrix} Q & S & z^{-1}I - A^{\mathrm{T}} \\ S^{\mathrm{T}} & R & -B^{\mathrm{T}} \\ zI - A & -B & 0 \end{pmatrix} \tag{3.23}$$

has no infinite zeros.

Proof : According to theorem 3.14, any rank minimizing solution is connected to an invariant subspace $\mathcal{V}$ of the matrix pencil 2.12 and if there exists a pencil (V_1, V_2) with only finite eigenvalues for the pencil (2.12) restricted to $\mathcal{V}$ then it is a strongly rank-minimizing solution. The eigenvalues of the matrix pencil (V_1, V_2) are partially fixed and partially freely assignable (the latter case occurs if the pencil is singular). Hence $\mathcal{V}$ is associated to a strongly rank-minimizing solution if among these fixed eigenvalues there are no infinite eigenvalues. It is easy to check that these fixed eigenvalues must be zeros of the pencil (3.23). Hence a sufficient condition to guarantee that all rank-minimizing solutions are actually strongly rank-minimizing is that (3.23) has no infinite zeros. ■

A particular case is the question when $X = 0$ is the unique semi-stabilizing solution of the linear matrix inequality:

Lemma 3.19 : $X = 0$ *is a semi-stabilizing solution of the linear matrix inequality if and only if*

$$\begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix} \geq 0 \tag{3.24}$$

and all zeros of (A, B, C, D) *are inside the closed unit disc where*

$$\begin{pmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} Q & S^{\mathrm{T}} \\ S & R \end{pmatrix}$$

In particular (A, B, C, D) *should not have infinite zeros. Moreover,* $X = 0$ *is the unique rank-minimizing, semi-stabilizing solution of the linear matrix inequality if and only if additionally* (A, B, C, D) *is right-invertible.*

Proof : This can be checked straightforwardly. ■

This lemma basically solves the discrete-time perfect regulation problem. For the positive semi-definite case, where (3.24) is satisfied, we know that a solution $X = 0$ of the linear matrix inequality exists and we can use the reduction scheme described in the previous section

to reduce the problem of finding solutions of the general algebraic Riccati equation and linear matrix inequality to a reduced algebraic Riccati equation and a reduced linear matrix inequality both of which satisfy the assumption that the associated rational matrix H has full rank and we can solve these equations using classical techniques. The latter approach is numerically much better than determining invariant subspaces of singular pencils.
In the general case where (3.24) is not satisfied we can also apply the reduction scheme. But to use this as a numerical tool we have to find an initial solution $\tilde{X}$ of the linear matrix inequality. This could be done using convex optimization using e.g. numerically reliable interior point methods (see [15]). But this extra complication leads us to believe that in this case working with invariant subspaces of singular pencils has its advantages.

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4 Conclusion

In this paper we showed that in general not all rank-minimizing solutions of the discrete linear matrix inequality satisfy the associated algebraic Riccati equation. We identified that this problem was related to infinite eigenvalues of the symplectic pencil associated with the algebraic Riccati equations. Only invariant subspaces for which the pencil restricted to that subspace has finite eigenvalues can be associated to solutions of the Riccati equation. Otherwise, they will yield rank-minimizing solutions of the linear matrix inequality but they do not satisfy the Riccati equation.

A Reduction to the case that H has full rank

The algebraic Riccati equation studied in section 2 is very general and includes the Riccati equation studied in H_∞ control (see e.g. [3, 10, 21]) as well as the Riccati equation studied in linear quadratic control (see e.g. [1, 12, 16]). In this paper we mostly concentrate on the Riccati equation used in linear quadratic control and the linear matrix inequality associated to it. In this appendix we will present a technique to reduce problems where H, as defined in (2.4) does not have full rank to the case where H has full rank.
As it will become more clear in the next section, the key is to study solutions of the following inequality:

$$L(X) := \begin{pmatrix} Q + A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB + S^{\mathrm{T}} \\ B^{\mathrm{T}}XA + S & B^{\mathrm{T}}XB + R \end{pmatrix} \geq 0 \tag{A.1}$$

Moreover, for solutions X of the above inequality, we are interested in the zeros of the matrix pencil (2.2). We will need the following technical lemma:

Lemma A.1 : *Let (A, B) be controllable. Then $X = 0$ is the unique symmetric solution of the following linear matrix inequality:*

$$\begin{pmatrix} A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB \\ B^{\mathrm{T}}XA & B^{\mathrm{T}}XB \end{pmatrix} \geq 0 \tag{A.2}$$

Proof : Let X be an arbitrary solution of (A.2). For any matrix F we find:

$$\begin{pmatrix} (A+BF)^{\mathrm{T}}X(A+BF) - X & (A+BF)^{\mathrm{T}}XB \\ B^{\mathrm{T}}X(A+BF) & B^{\mathrm{T}}XB \end{pmatrix} =$$

$$= \begin{pmatrix} I & F^{\mathrm{T}} \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB \\ B^{\mathrm{T}}XA & B^{\mathrm{T}}XB \end{pmatrix} \begin{pmatrix} I & 0 \\ F & I \end{pmatrix} \geq 0 \tag{A.3}$$

Since (A, B) is controllable there exists a matrix F such that $A + BF$ is stable in which case (A.3) tells us that:

$$(A+BF)^{\mathrm{T}}X(A+BF) - X \geq 0 \tag{A.4}$$

Standard theory for the discrete time Lyapunov equation (see e.g [9]) then tells us that $X \geq 0$. Conversely, if we choose F such that $A + BF$ is antistable then we again obtain (A.4) but standard theory for the discrete time Lyapunov equation then tells us that $X \leq 0$. We have that X must be positive and negative semidefinite which clearly implies that X must be 0. ■

Next we need the controllability subspace (see e.g. [4]) of a linear system (A, B, C, D):

Definition A.2 : *A subspace $\mathcal{R}$ is called a controllability subspace for the system*

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

if for any initial condition $x(0) = x_0 \in \mathcal{R}$ there exists an input u which steers the state x to 0 in finite time while keeping the output y identically 0. Equivalently $\mathcal{R}$ is a controllability subspace if there exists a matrix F such that $\mathcal{R}$ is the smallest $A + BF$ invariant subspace containing Im $B \cap \mathcal{R}$ and contained in Ker $(C + DF)$.
$\mathcal{R}^*(\Sigma)$ is defined as the largest controllability subspace of the system Σ.

An important property of the largest controllability subspace is expressed in the following lemma (see e.g. [17]):

Lemma A.3 : *A system Σ with realization (A, B, C, D) is left invertible if and only if $\mathcal{R}^*(\Sigma) = \{0\}$ and $(B^{\mathrm{T}} \ \ D^{\mathrm{T}})^{\mathrm{T}}$ is injective.*

Without loss of generality we assume that there exists one solution $\bar{X}$, i.e. $\bar{X} \in \Gamma$. We factorize $L(\bar{X})$ as

$$L(\bar{X}) = \begin{pmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} \tag{A.5}$$

We define a new discrete linear matrix inequality:

$$\bar{L}(X) := \begin{pmatrix} C^{\mathrm{T}}C + A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}XA + D^{\mathrm{T}}C & B^{\mathrm{T}}XB + D^{\mathrm{T}}D \end{pmatrix} \geq 0 \tag{A.6}$$

and we have that

$$L(X) \geq 0 \text{ if and only if } \bar{L}(X - \bar{X}) \geq 0$$

We define $\mathcal{R}^*(\Sigma)$ according to definition A.2. Hence there exists a matrix F such that $\mathcal{R}^*(\Sigma)$ is $(A + BF)$-invariant and contained in Ker $(C + DF)$. We define the shifted linear matrix inequality as follows:

Definition A.4 : *The shifted discrete linear matrix inequality associated with the discrete linear matrix inequality (3.2) is defined as:*

$$L^s(X) \geq 0 \tag{A.7}$$

where

$$L^s(X) := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \bar{L}(X) \begin{pmatrix} I & 0 \\ F & I \end{pmatrix} = \begin{pmatrix} \tilde{C}^{\mathrm{T}}\tilde{C} + \tilde{A}^{\mathrm{T}}X\tilde{A} - X & \tilde{A}^{\mathrm{T}}XB + \tilde{C}^{\mathrm{T}}D \\ B^{\mathrm{T}}X\tilde{A} + D^{\mathrm{T}}\tilde{C} & B^{\mathrm{T}}XB + D^{\mathrm{T}}D \end{pmatrix} \tag{A.8}$$

and $\tilde{A} = A + BF$ and $\tilde{C} = C + DF$.

Observation A.5 : *Let $\bar{X}$ be a solution of the linear matrix inequality (A.1) and let F be such that $\mathcal{R}^*(\Sigma)$ is $(A+BF)$-invariant and contained in Ker $(C+DF)$ where Σ is the system with realization (A, B, C, D) with C, D defined by (A.5).*

- *(i) X is a solution of the discrete linear matrix inequality (A.1) if and only if $X - \bar{X}$ is a solution of the associated shifted discrete linear matrix inequality (A.7).*
- *(ii) Let X be a solution of the linear matrix inequality. Then the rank of $L(X)$ equals the rank of $L^s(X - \bar{X})$. In particular X is a rank-minimizing solution of the linear matrix inequality (A.1) if and only if $X - \bar{X}$ is a rank-minimizing solution of the associated shifted discrete linear matrix inequality (A.7).*

(*iii*) *Let X be a solution of the linear matrix inequality. The zeros of the matrix pencil (2.2) are equal to the zeros of the following matrix pencil:*

$$\begin{pmatrix} zI - A & -B \\ \tilde{C}^{\mathrm{T}}\tilde{C} + \tilde{A}^{\mathrm{T}}\tilde{X}\tilde{A} - \tilde{X} & \tilde{A}^{\mathrm{T}}\tilde{X}B + \tilde{C}^{\mathrm{T}}D \\ B^{\mathrm{T}}\tilde{X}\tilde{A} + D^{\mathrm{T}}\tilde{C} & B^{\mathrm{T}}\tilde{X}B + D^{\mathrm{T}}D \end{pmatrix}$$

where $\tilde{X} = X - \bar{X}$.

The above observation shows that without loss of generality we can focus on the shifted discrete linear matrix inequality which has more structure and is hopefully easier to handle. In particular if we choose a basis in the state space $\mathcal{X}_1 \oplus \mathcal{X}_2$ such that $\mathcal{X}_1 = \mathcal{R}^*(\Sigma)$ and a basis in the input space $\mathcal{U}_1 \oplus \mathcal{U}_2$ such that $\mathcal{U}_1 = B^{-1}\mathcal{R}^*(\Sigma)$. In that basis we get that A, B, C and D have a special form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}, C = \begin{pmatrix} 0 & C_1 \end{pmatrix}, D = \begin{pmatrix} 0 & D_1 \end{pmatrix}, \tag{A.9}$$

such that (A_{11}, B_{11}) is controllable, $C_1^{\mathrm{T}}D_1 = 0$ and $(A_{22}, B_{22}, C_1, D_1)$ left-invertible. We can then look at the linear matrix inequality restricted to $\mathcal{X}_1 \oplus \mathcal{U}_1$ to obtain:

$$\begin{pmatrix} A_{11}^{\mathrm{T}}X_{11}A_{11} - X_{11} & A_{11}^{\mathrm{T}}X_{11}B_{11} \\ B_{11}^{\mathrm{T}}X_{11}A_{11} & B_{11}^{\mathrm{T}}X_{11}B_{11} \end{pmatrix} \geq 0$$

which according to lemma A.1 implies that $X_{11} = 0$. Denote by V the projection onto $\mathcal{X}_1 \oplus \mathcal{U}_1$. Then we find that $V^{\mathrm{T}}L(X)V=0$ and $L(X) \geq 0$. Hence $L(X)V = 0$. When we write this out in terms of our decomposition and using that $X_{11} = 0$ we get:

$$\begin{pmatrix} A_{22}^{\mathrm{T}}X_{21}A_{11} - X_{21} & A_{22}^{\mathrm{T}}X_{21}B_{11} \\ B_{22}^{\mathrm{T}}X_{21}A_{11} & B_{22}^{\mathrm{T}}X_{21}B_{11} \end{pmatrix} = 0 \tag{A.10}$$

Let F be such that $A_{11} + B_{11}F$ has no eigenvalues in common with A_{22}, which is possible since (A_{11}, B_{11}) is controllable. It is easy to see that (A.10) implies that

$$A_{22}^{\mathrm{T}}X_{21}(A_{11} + B_{11}F) - X_{21} = 0$$

and the standard theory guarantees that this Sylvester equation has a unique solution $X_{21} = 0$. In other words, we only have to compute X_{22}. It is easy to see that the linear matrix inequality reduces to:

$$L^r(X_{22}) = \begin{pmatrix} A_{22}^{\mathrm{T}}X_{22}A_{22} - X_{22} + C_1^{\mathrm{T}}C_1 & A_{22}^{\mathrm{T}}X_{22}B_{22} + C_1^{\mathrm{T}}D_1 \\ B_{22}^{\mathrm{T}}X_{22}A_{22} + D_1^{\mathrm{T}}C_1 & B_{22}^{\mathrm{T}}X_{22}B_{22} + D_1^{\mathrm{T}}D_1 \end{pmatrix} \geq 0. \tag{A.11}$$

However, since $(A_{22}, B_{22}, C_1, D_1)$ is left-invertible, this is a linear matrix inequality such that the associated rational matrix H^r has full rank where

$$\begin{aligned} H^r(z) &= \begin{pmatrix} B_{22}^{\mathrm{T}}(z^{-1}I - A_{22}^{\mathrm{T}})^{-1} & I \end{pmatrix} \begin{pmatrix} C_1^{\mathrm{T}}C_1 & C_1^{\mathrm{T}}D_1 \\ D_1^{\mathrm{T}}C_1 & D_1^{\mathrm{T}}D_1 \end{pmatrix} \begin{pmatrix} (zI - A_{22})^{-1}B_{22} \\ I \end{pmatrix} \\ &= G^{\mathrm{T}}(-z)G(z) \end{aligned} \tag{A.12}$$

where G is the transfer matrix of $(A_{22}, B_{22}, C_1, D_1)$. This enables us to first derive results for the case that H has full rank and then use the above reduction step to derive results for the general case. The results of the above reduction scheme are put together in the following theorem:

Theorem A.6 : *Let $\bar{X}$ be a solution of the linear matrix inequality (A.1) and let F be such that $\mathcal{R}^*(\Sigma)$ is $(A + BF)$-invariant and contained in Ker $(C + DF)$ where Σ is the system with realization (A, B, C, D) with C, D defined by (A.5). Moreover, we assume that we have chosen the appropriate bases as described above.*

(i) X is a solution of the discrete linear matrix inequality (A.1) if and only if

$$X - \bar{X} = \begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix}$$

and X_{22} is a solution of reduced linear matrix inequality (A.11).

(ii) Let X be a solution of the linear matrix inequality. Then the rank of $L(X)$ equals the rank of $L^r(X_{22})$. In particular X is a rank-minimizing solution of the linear matrix inequality (A.1) if and only if X_{22} is a rank-minimizing solution of the associated reduced linear matrix inequality (A.11).

(iii) Let X be a solution of the linear matrix inequality. The zeros of the matrix pencil (2.2) are equal to the zeros of the following matrix pencil:

$$\begin{pmatrix} zI - A_{22} & -B_{22} \\ A_{22}^{\mathrm{T}} X_{22} A_{22} - X_{22} + C_1^{\mathrm{T}} C_1 & A_{22}^{\mathrm{T}} X_{22} B_{22} + C_1^{\mathrm{T}} D_1 \\ B_{22}^{\mathrm{T}} X_{22} A_{22} + D_1^{\mathrm{T}} C_1 & B_{22}^{\mathrm{T}} X_{22} B_{22} + D_1^{\mathrm{T}} D_1 \end{pmatrix}.$$

It is in general computationally not very attractive to use this method to determine solutions of the linear matrix inequality and Riccati equation since we first have to find an initial solution $\bar{X}$ of the linear matrix inequality. But it does yield a straightforward method to derive properties of the linear matrix inequality and the algebraic Riccati equations since all important features of solutions of the linear matrix inequality are preserved in the reduction scheme. If an initial solution of the linear matrix inequality is available (e.g. in the positive semi-definite case) then the above does yield a computationally attractive method to determine solutions of the Riccati equation.

B Matrix pencils and generalized eigenvalue problems

Matrix pencils and its properties presented in this appendix can be found in more detail in [7,20].
Consider a pair (H_1, H_2) of two square matrices. We will associate to this pair a matrix pencil $\mu H_1 - \lambda H_2$. We call the matrix pencil regular if the pencil is invertible for almost all λ and μ. We call λ an eigenvalue of the matrix pencil if $H_1 - \lambda H_2$ is not injective. We call ∞ an

eigenvalue if H_2 is not injective. The (algebraic) multiplicity of an eigenvalue λ is equal to the dimension of the kernel of $(H_1 - \lambda H_2)^m$ where m is the dimension of the matrix pencil. The multiplicity of an eigenvalue ∞ is equal to the dimension of the kernel of H_2^m. A regular m-dimensional matrix pencil has m eigenvalues. A singular pencil can have infinitely many eigenvalues. A regular $2n$-dimensional matrix pencil which satisfies the property that:

$$H_1 \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} H_1^{\mathrm{T}} = H_2 \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} H_2^{\mathrm{T}}$$

where I_n denotes the $n \times n$-identity matrix is called a symplectic pencil. It is easy to see that this implies that λ is an eigenvalue if and only if λ^{-1} is an eigenvalue.
A subspace $\mathcal{V}$ is called invariant or deflating with respect to the matrix pencil (H_1, H_2) if

$$\dim\ \{H_1\mathcal{V} + H_2\mathcal{V}\} \leq \dim\ \mathcal{V} \tag{B.1}$$

For regular pencils we have an equality in (B.1).
$\mathcal{V}$ is an invariant subspace if and only if there exists a regular matrix pencil (L_1, L_2) such that

$$H_1 V L_2 = H_2 V L_1$$

where V is an injective matrix such that $\operatorname{Im} V = \mathcal{V}$. If (H_1, H_2) is a regular pencil then the pencil (L_1, L_2) is unique up to pre- and post-multiplication by invertible matrices. For regular pencils the eigenvalues of the symplectic pencil (L_1, L_2) are eigenvalues of the symplectic pencil (H_1, H_2) and are called the eigenvalues of (H_1, H_2) restricted to $\mathcal{V}$. For singular pencils the eigenvalues of (L_1, L_2) are eigenvalues of (H_1, H_2) but are no longer uniquely determined by $\mathcal{V}$ and (H_1, H_2).

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