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The Discrete Coagulation-Fragmentation Equations: Existence, Uniqueness, and Density Conservation

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The discrete coagulation-fragmentation equation describes the kinetics of cluster growth in which clusters can coagulate via binary interactions to form larger clusters or fragment to form smaller ones. These models have many applications in pure and applied science ranging from cluster formation in galaxies to the kinetics of phase transformations in binary alloys. Our results relate to existence, uniqueness, density conservation and continuous dependence and they generalise the corresponding results in [ref. 2] for the Becker-Doring equations for which the processes are restricted to clusters gaining or shedding one particle. Examples are given which illustrate the role of the assumptions on the kinetic coefficients and show the rich set of analytic phenomena supported by the general discrete coagulation-fragmentation equations.

KEY WORDS: Existence theorems; admissibility; coagulation; fragmentation; clustering.

1. INTRODUCTION

In this paper we discuss the mathematical theory of a model for the dynamics of cluster growth. Such models arise in a wide variety of situations; examples include astrophysics, atmospheric physics, biology, colloidal chemistry, polymer science, and the kinetics of phase transitions in binary alloys. The basis for the models is that the system under consideration can be viewed as consisting of a large number of clusters that can coagulate to form larger clusters or fragment to form smaller ones. In the model analyzed in this paper the clusters are assumed to be *discrete*, that is, they consist of a finite number of smaller *particles*. The particles may be atoms, molecules, cells, etc., depending on the application.

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If $c_j(t) \geqslant 0$, $j = 1, 2,...$, denotes the expected number of clusters consisting of j particles (j-clusters) per unit volume, then the discrete coagulation-fragmentation equations are

$$\dot{c}_j = \frac{1}{2} \sum_{k=1}^{j-1} (a_{j-k,k} c_{j-k} c_k - b_{j-k,k} c_j) - \sum_{k=1}^{\infty} (a_{j,k} c_j c_k - b_{j,k} c_{j+k}) \quad (1.1)$$

for $j = 1, 2,...$. The coagulation rates $a_{j,k}$ and fragmentation rates $b_{j,k}$ are nonnegative constants with $a_{j,k} = a_{k,j}$ and $b_{j,k} = b_{k,j}$. In Eq. (1.1) the first two terms represent the rate of change of the j-cluster due to the coalescence of smaller clusters and the breakup of the j-cluster into smaller clusters. The final two terms represent the change due to coalescence of the j-cluster with other clusters and the breakup of larger clusters into j-clusters. For a derivation of this equation and its analogue in which the cluster size is a continuous variable see ref. 7. The model neglects (among other things) the geometrical location of clusters and spatial fluctuations in cluster density. For further information on these effects see refs. 5 and 6.

Since particles are neither created nor destroyed in the interactions described by (1.1), we expect the density $\rho = \sum_{j=1}^{\infty} jc_j(t)$ to be a conserved quantity. Mathematically, this is equivalent to

$$\lim_{n \to \infty} \int_0^t \sum_{j=1}^{n} j \sum_{k=n-j+1}^{\infty} W_{j,k}(c(s)) \, ds = 0$$

where $W_{j,k}(c) \stackrel{\text{def}}{=} a_{j,k} c_j c_k - b_{j,k} c_{j+k}$. In certain circumstances, however, the density is not conserved. To illustrate this and other phenomena, we consider some special cases.

$\langle a \rangle$ *Pure coagulation.* Here $b_{j,k} = 0$ for all j and k. We further specialize to the following two idealized forms of coagulation kernel:

$$a_{j,k} = j^\alpha + k^\alpha \quad (1.2)$$

$$a_{j,k} = (jk)^\alpha \quad (1.3)$$

where $\alpha > 0$. The additive form (1.2) arises if we assume that binary interactions of clusters occur randomly with a rate proportional to the total effective surface area of the coagulating clusters. For compact clusters in d dimensions $\alpha = 1 - d^{-1}$, but other values of α are also of interest.[8] The multiplicative form (1.3) might apply to situations in which bond linking was the dominant mechanism. Note that, for the kernel (1.2), the rates for large–large and large–small interactions have the same order of magnitude (i.e., $a_{j,k} \cong a_{j,j}$ for large j and small k), whereas for (1.3), large–large interactions dominate.

If $\alpha > 1/2$, then for the kernel (1.3) density conservation can break down in finite time.[8,9] This is interpreted as the appearance of an infinite cluster or gel. For the kernel (1.2) we prove in Theorem 3.6 that if a solution exists, then density is conserved.

To gain some insight into the dependence of the rate of growth of clusters, we use a technique due to Leyvraz and Tschudi[10] to relate solutions of (1.1) having different initial data. We first consider the kernel (1.2), so that (1.1) takes the form

$$\dot{c}_j = \frac{1}{2} \sum_{k=1}^{j-1} [(j-k)^\alpha + k^\alpha] c_{j-k} c_k - \sum_{k=1}^{\infty} (j^\alpha + k^\alpha) c_j c_k \tag{1.4}$$

Let c_j^1 be a solution of (1.4) with initial data $c_j^1(0) = \delta_{j,1}$. For positive integers n, define $c^n(t) = (c_j^n(t))$, $j = 1, 2, \ldots$, by

$$\begin{aligned} c_{nj}^n(t) &= n^{-1} c_j^1(n^{\alpha-1} t) \\ c_r^n(t) &= 0, \qquad r \text{ not a multiple of } n \end{aligned} \tag{1.5}$$

It is then easy to check that $c^n(t)$ is a solution of (1.4) with initial data given by $c_j^n(0) = n^{-1}\delta_{j,n}$. From (1.5) we see that the time scale for this class of solutions depends on the sign of $\alpha - 1$. In fact, if $\alpha \leqslant 1$, we get global existence for the general initial value problem (with initial data having finite density), while if $\alpha > 1$, we have nonexistence of global solutions.

For the kernel (1.3), let c^1 be the solution of (1.1) with initial data $c_j^1(0) = \delta_{j,1}$. It is shown in ref. 10 that the appropriate scaling is

$$c_{nj}^n(t) = n^{-1} c_j^1(n^{2\alpha-1} t), \qquad c_r^n(t) = 0 \quad \text{otherwise} \tag{1.6}$$

From (1.6), we see that $\alpha = 1/2$ is the critical parameter value. Global solutions for the initial value problem exist for $\alpha \leqslant 1$ (see ref. 9 for a proof), but density conservation breaks down after a finite time if $\alpha > 1/2$. It is interesting to note that if $\alpha > 1$, we can still have global existence for this case.[10]

(b) *Pure fragmentation.* Here $a_{j,k} = 0$ for all j, k, so that (1.1) becomes linear. For any initial data with finite density, (1.1) has a density-conserving solution. However, for a large class of fragmentation coefficients (for example, $b_{j,k} = (j+k)^\beta$, $\beta > -1$), there are solutions with density $e^{t\lambda}$ for any $\lambda > 0$; in particular, solutions need not be unique. These spurious solutions are not of physical interest, and this leads to the problem of finding a criterion for selecting the correct solution for the general equation (1.1).

(c) *The Becker–Döring equations.* Here $a_{j,k} = b_{j,k} = 0$ if both j and k are greater than 1. The mathematical theory of these equations has been studied in ref. 2. In this case the density is always a conserved quantity. The asymptotic behavior of solutions is interesting both mathematically and for applications. Under certain hypotheses on the rate coefficients and the density of the initial data, we have that

$$\rho = \sum_{j=1}^{\infty} jc_j(t) > \sum_{j=1}^{\infty} j \lim_{t \to \infty} c_j(t) = \rho_s$$

The excess density $\rho - \rho_s$ corresponds to the formation of larger and larger clusters as $t \to \infty$, and may be interpreted as a transition from microscopic to macroscopic clusters. See also ref. 12 for an analysis of metastable solutions and refs. 3 and 14 for some technical refinements.

The aim of this paper is to obtain some of the fundamental results needed to extend the work on the Becker–Döring equation to more realistic models in which all interactions are allowed. The class of kinetic coefficients that we have in mind are $a_{j,k} = O(j^\alpha + k^\alpha)$ with $\alpha \leqslant 1$ and $b_{j,k} = a_{j,k} Q_j Q_k (Q_{j+k})^{-1}$, where $Q_j \simeq z_s^{-j} \exp(-\lambda j^p)$ and z_s, λ, and p are positive constants with $p < 1$. (See Section 6 for a discussion.) In particular, we will not study situations in which coagulation can lead to density breakdown. In ref. 4 we use the theory developed here to study the asymptotic behavior of solutions.

Before outlining our results, we review what is known about the mathematical theory of solutions of (1.1). Spouge[15] has proved existence under the assumption $a_{j,k} = o(jk)$ and a technical condition on $b_{j,k}$ which implies that it is bounded (see also ref. 9). White[17] has proved existence under the assumptions $b_{j,k} = 0$, $a_{j,k} \leqslant j^\alpha + k^\alpha$, $\sum_{j=1}^{\infty} j^m c_j(0) < \infty$, where $0 \leqslant \alpha \leqslant 1$ and $m > \alpha$ is an integer. Aizenmann and Bak[1] construct a complete mathematical theory for the continuous analogue of (1.1) for the case in which the kinetic rates are 1. In particular, they single out the physical solution by using semigroup theory and choosing an appropriate domain for the linear operator associated with the fragmentation. Finally, Stewart[16] has extended some of the results in this paper to the continuous analogue of (1.1).

Our results relate to existence, uniqueness, density conservation, and continuous dependence. The existence result (Theorem 2.4) generalizes the corresponding result in ref. 2 for the Becker–Döring equations and gives global existence when the initial data has finite density and $a_{j,k} = O(j+k)$. This is proved by taking the limit $N \to \infty$ of the system corresponding to (1.1) in which the maximum cluster size is N. Theorem 2.5 shows that solutions of (1.1) constructed in this way conserve density, thus excluding the nonphysical solutions mentioned earlier. It is also useful to have conditions under which all solutions conserve density. Such a result is given in

Theorem 3.6. The condition on the fragmentation rates needed for this result has an interesting physical interpretation (see Section 6).

We give two uniqueness results; the first (Theorem 4.1) concerns all solutions and the second (Theorem 4.2) applies only to solutions which conserve density. We note that while Theorem 4.1 implies uniqueness for our motivating example (see Section 6), it would be desirable to extend our uniqueness results to a larger class of kinetic rates.

In Section 5 we study the differentiability of solutions and continuous dependence of solutions on initial data. Finally, in Section 6 we give a number of examples which illustrate the role of the assumptions on the kinetic coefficients.

2. EXISTENCE OF DENSITY-CONSERVING SOLUTIONS

We first introduce some notation. Let

$$X=\{y=(y_r)\colon \|y\|<\infty\}, \qquad \|y\|=\sum_{r=1}^{\infty} r\,|y_r|$$

$(X, \|\cdot\|)$ is a Banach space. We write $y\geqslant 0$ if $y_r\geqslant 0$ for each r and set $X^+=\{y\in X\colon y\geqslant 0\}$.

As well as strong (norm) convergence in X, we will make use of weak* convergence: a sequence y^m converges in the weak* sense to y in X (symbolically $y^m \overset{*}{\rightharpoonup} y$) if (i) $\sup_m \|y^m\|<\infty$ and (ii) $y_r^m\to y_r$ as $m\to\infty$ for each r.

Definition. Let $0<T\leqslant\infty$. A solution $c=(c_j)$ of (1.1) on $[0, T)$ is a function $c\colon [0, T)\to X^+$ such that:

(i) Each $c_j\colon [0, T)\to\mathbb{R}$ is continuous and $\sup_{t\in[0,T)}\|c(t)\|<\infty$.

(ii) For $j=1, 2,\dots$,

$$\int_0^t \sum_{k=1}^{\infty} a_{j,k}c_k(s)\,ds<\infty, \qquad \int_0^t \sum_{k=1}^{\infty} b_{j,k}c_{j+k}(s)\,ds<\infty$$

for all $t\in[0, T)$.

(iii) For $j=1, 2,\dots$,

$$c_j(t)=c_j(0)+\int_0^t\left[\frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k}(c(s))-\sum_{k=1}^{\infty} W_{j,k}(c(s))\right]ds$$

for all $t\in[0, T)$.

It follows easily from the above definition that if c is a solution on $[0, T)$, then each c_r is absolutely continuous, so that c satisfies (1.1) for a.e. $t \in [0, T)$.

As in earlier work on similar equations,[11,13,15] we prove existence of solutions by taking a limit of solutions of the finite-dimensional system

$$\dot{c}_j = \frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k}(c) - \sum_{k=1}^{n-j} W_{j,k}(c), \qquad 1 \leqslant j \leqslant n \tag{2.1}$$
$$c_j(0) \geqslant 0, \qquad 1 \leqslant j \leqslant n$$

The following identity will be useful for finding bounds on solutions of (2.1).

Lemma 2.1. Let c be a solution of (2.1) and let (g_j) be a sequence of real numbers. Then for $1 \leqslant m \leqslant n$,

$$\sum_{j=m}^{n} g_j \dot{c}_j = \frac{1}{2}\sum_{T_1} (g_{j+k} - g_j - g_k) W_{j,k} + \frac{1}{2}\sum_{T_2} g_{j+k} W_{j,k} + \sum_{T_3} (g_{j+k} - g_k) W_{j,k} \tag{2.2}$$

where

$$T_1 = \{(j, k): j, k \geqslant m, j+k \leqslant n\}$$
$$T_2 = \{(j, k): m \leqslant j+k \leqslant n, \; j, k < m\}$$
$$T_3 = \{(j. k): 1 \leqslant j \leqslant m-1, k \geqslant m, j+k \leqslant n\}$$

with the sums equal to zero if the associated region is empty. (See Fig. 1.)

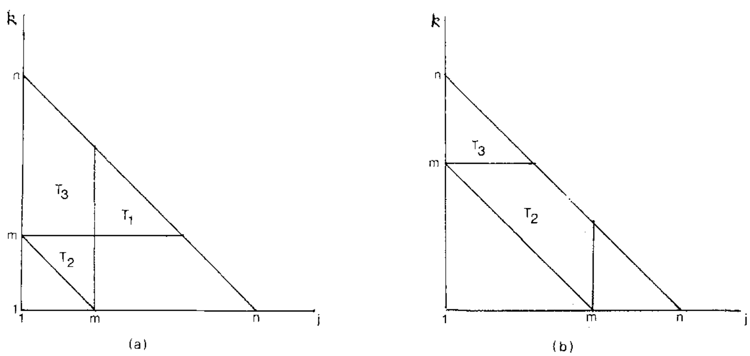

Fig. 1. Location of the region T_k for the cases (a) $2m < n$, (b) $2m > n$.

Proof. Let $T_4 = \{(j, k)\colon 1 \leqslant k \leqslant m-1, j \geqslant m, j+k \leqslant n\}$. Then the T_i, $i = 1, ..., 4$, are disjoint, and $\bigcup_{i=1}^{4} T_i = \{(j, k)\colon m \leqslant j+k \leqslant n\}$. Using the symmetry of the coagulation and fragmentation coefficients and (2.1), we have that

$$\sum_{j=m}^{n} g_j \dot{c}_j = \frac{1}{2} \sum_{m \leqslant j+k \leqslant n} g_{j+k} W_{j,k} - \frac{1}{2} \sum_{j=m}^{n-1} g_j \sum_{k=m}^{n-j} W_{j,k} - \frac{1}{2} \sum_{k=m}^{n-1} g_k \sum_{j=1}^{n-k} W_{j,k}$$

the last two terms being equal. The result follows from grouping the above terms into common regions in j–k space. ∎

Lemma 2.2. The system (2.1) has a unique solution for $t \geqslant 0$ with $c_j(t) \geqslant 0$, $1 \leqslant j \leqslant n$, and $\sum_{j=1}^{n} jc_j(t) = \sum_{j=1}^{n} jc_j(0)$.

Proof. The nonnegativity of each $c_j(t)$ may be proved in exactly the same way as the corresponding result in ref. 2 (see also ref. 15 for an alternative proof). The fact that $\sum_{j=1}^{n} jc_j(t)$ is a constant of the motion follows by setting $g_j = j$ in Lemma 2.1, and the global existence follows from the bounds $0 \leqslant c_j(t) \leqslant j^{-1} \sum_{j=1}^{n} jc_j(0)$. ∎

Lemma 2.3. Assume that $a_{j,k} \leqslant K_0 jk$ for all $j, k \geqslant 1$, where K_0 is a constant. Let c^n be a solution of (2.1) and let $\rho^n(0) = \sum_{j=1}^{n} jc_j(0)$. Then

$$\frac{d}{dt}\left\{e^{-t}\left[\sum_{j=m}^{n} jc_j^n(t) + 2mK_0 \rho^n(0)^2\right]\right\} \leqslant 0$$

for all $m \leqslant n$, $t \geqslant 0$.

Proof. By Lemma 2.1,

$$\frac{d}{dt} \sum_{j=m}^{n} jc_j^n(t) = \frac{1}{2} \sum_{T_2} (j+k) W_{j,k}(c^n) + \sum_{T_3} jW_{j,k}(c^n)$$

Hence

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{-t}\left[\sum_{j=m}^{n} jc_j^n(t) + 2mK_0 \rho^n(0)^2\right]\right\} \\
&\quad = e^{-t}\left[\frac{1}{2} \sum_{T_2} (j+k) W_{j,k} + \sum_{T_3} jW_{j,k} - \sum_{j=m}^{n} jc_j^n(t) - 2mK_0 \rho^n(0)^2\right] \\
&\quad \leqslant e^{-t}\left[\frac{1}{2} \sum_{T_2} (j+k) a_{j,k} c_j^n c_k^n + \sum_{T_3} ja_{j,k} c_j^n c_k^n - 2mK_0 \rho^n(0)^2\right] \\
&\quad \leqslant K_0 e^{-t}\left[\frac{1}{2} \sum_{T_2} (j+k) jkc_j^n c_k^n + \sum_{T_3} j^2 kc_j^n c_k^n - 2mK_0 \rho^n(0)^2\right]
\end{aligned}$$

Now

$$\sum_{T_2}(j+k)\,jkc_j^n c_k^n \leqslant 2m \sum_{T_2} jkc_j^n c_k^n \leqslant 2m\rho^n(0)^2$$

and

$$\sum_{T_3} j^2kc_j^n c_k^n \leqslant m \sum_{T_3} jkc_j^n c_k^n \leqslant m\rho^n(0)^2$$

The result follows. ▮

The following existence theorem generalizes to the full set of discrete coagulation-fragmentation equations the corresponding result in ref. 2, Corollary 2.3, for the Becker–Döring equations. The proof follows a similar pattern, the main difficulty being to find appropriate generalizations of the *a priori* estimates. However, one difference is the use of Helly's theorem together with the preceding Lemma 2.3 to extract a convergent subsequence of approximating solutions; this simplifies the corresponding argument in ref. 2.

Theorem 2.4. Assume that $a_{j,k} \leqslant K(j+k)$, for all $j, k \geqslant 1$, where K is a constant. Let $c_0 \in X^+$. Then there exists a solution c of (1.1) on $[0, \infty)$ with $c(0) = c_0$.

Proof. Let $c^n(0) = (c_{01}, c_{02}, ..., c_{0n})$. By Lemma 2.2, the system (2.1) has a unique solution c^n on $[0, \infty)$ with $c_j^n(t) \geqslant 0$ for $1 \leqslant j \leqslant n$ and

$$\sum_{j=1}^{n} jc_j^n(t) = \sum_{j=1}^{n} jc_j(0) \qquad \text{for all} \quad t \geqslant 0 \tag{2.3}$$

We regard $c^n(t)$ as an element of X^+ by defining $c_j^n(t) = 0$ if $j > n$. Thus, $\|c^n(t)\| \leqslant \|c_0\|$ and $0 \leqslant c_j^n(t) \leqslant j^{-1}\|c_0\|$ for all j and n. Let

$$\vartheta_m^n(t) = e^{-t}\left[\sum_{j=m}^{n} jc_j^n(t) + 2mK_0\rho^n(0)^2\right]$$

where $K_0 = 2K$. By (2.3) and Lemma 2.3, for each fixed m, the functions $\vartheta_m^n(\cdot)$, $n \geqslant m$, are of uniformly bounded variation on $[0, \infty)$. Hence, by Helly's theorem, there exists a subsequence, again denoted by ϑ_m^n, such that $\vartheta_m^n(t) \to \vartheta_m(t)$ as $n \to \infty$ for each $t \geqslant 0$, for some function ϑ_m of bounded variation. Since

$$c_j^n(t) = j^{-1}e^t[\vartheta_j^n(t) - \vartheta_{j+1}^n(t)] + 2j^{-1}K_0\rho^n(0)^2$$

it follows that there exist a subsequence, which we continue to denote by c^n, and functions $c_j\colon [0, \infty) \to \mathbb{R}$, each of bounded variation on every compact subset of $[0, \infty)$, such that $c_j^n(t) \to c_j(t)$ as $n \to \infty$ for each $t \geqslant 0$. Clearly, $c_j(t) \geqslant 0$ and

$$\sum_{j=1}^{\infty} jc_j(t) \leqslant \|c_0\| \qquad \text{for all} \quad t \geqslant 0 \tag{2.4}$$

In order to pass to the limit in the integrated form of the Jth equation in (2.1), we will prove that for each $T \geqslant 0$, $\varepsilon > 0$, and positive integer J, there exist $M > J$ and $N_0 > 3M$ such that

$$\int_0^T [p_M^n(t) + x_M^n(t)] \, dt < \varepsilon \qquad \text{for all} \quad n \geqslant N_0 \tag{2.5}$$

where

$$x_m^n \stackrel{\text{def}}{=} \sum_{j=m}^{n} jc_j^n, \qquad p_m^n \stackrel{\text{def}}{=} \sum_{j=1}^{m-1} \sum_{k=2m}^{n-1} jb_{j,k} c_{j+k}^n$$

Applying Lemma 2.1 with $g_j = j$, we obtain

$$\dot{x}_m^n(t) = \sum_{T_3} jW_{j,k}(c^n) + \frac{1}{2} \sum_{T_2} (j+k) \, W_{j,k}(c^n) \tag{2.6}$$

Let $2m < n$. To obtain an estimate on the terms in the sum over T_2 in (2.6), we apply Lemma 2.1 with

$$g_j = \begin{cases} j & \text{for} \quad m \leqslant j \leqslant 2m \\ 2m & 2m+1 \leqslant j \leqslant n \end{cases}$$

Then, with an obvious notation,

$$\sum_{j=m}^{2m} j\dot{c}_j^n + 2m \sum_{j=2m+1}^{n} \dot{c}_j^n$$
$$= \frac{1}{2} \sum_{T_1} \mu_{j,k} W_{j,k}(c^n) + \frac{1}{2} \sum_{T_2} (j+k) \, W_{j,k}(c^n) + \sum_{T_3} \lambda_{j,k} W_{j,k}(c^n) \tag{2.7}$$

For $(j, k) \in T_1$ we have that $\mu_{j,k} = \mu_{k,j}$ and

$$\mu_{j,k} = \begin{cases} 2m - (j+k) & \text{for} \quad j, k \leqslant 2m \\ -j & \text{for} \quad j \leqslant 2m, \quad k > 2m \\ -2m & \text{for} \quad j > 2m, \quad k > 2m \end{cases}$$

We note that

$$0 \leqslant -\mu_{j,k} \leqslant 2m \qquad \text{for all} \quad (j,k) \in T_1 \tag{2.8}$$

For $(j, k) \in T_3$ we have that

$$\lambda_{j,k} = \begin{cases} 0 & \text{for} \quad k \geqslant 2m \\ j & \text{for} \quad j+k \leqslant 2m \\ 2m-k & \text{for} \quad j+k \geqslant 2m+1, \quad k < 2m \end{cases}$$

Furthermore,

$$0 \leqslant \lambda_{j,k} \qquad \text{and} \qquad 0 \leqslant j - \lambda_{j,k} \qquad \text{for all} \quad (j,k) \in T_3 \tag{2.9}$$

From (2.6)–(2.7), we obtain for $2m < n$,

$$\begin{aligned} x_m^n(t) &= x_m^n(0) + q_m^n(t) - q_m^n(0) \\ &\quad + \int_0^t \left[\sum_{T_3} (j - \lambda_{j,k}) - \sum_{T_1} \mu_{j,k} \right] W_{j,k}(c^n(s))\, ds \end{aligned} \tag{2.10}$$

where

$$q_m^n \overset{\text{def}}{=} \sum_{j=m}^{2m} j c_j^n + 2m \sum_{j=2m+1}^{n} c_j^n$$

Since by (2.3), (2.4),

$$\left| \sum_{j=L}^{\infty} [c_j^n(t) - c_j(t)] \right| \leqslant L^{-1} \sum_{j=L}^{\infty} j[c_j^n(t) + c_j(t)] \leqslant 2L^{-1} \|c_0\|$$

we deduce that for each m and all $t \geqslant 0$,

$$\lim_{n \to \infty} q_m^n(t) = \sum_{j=m}^{2m} j c_j(t) + 2m \sum_{j=2m+1}^{\infty} c_j(t) \overset{\text{def}}{=} q_m(t) \tag{2.11}$$

Let $T > 0$. Then $0 \leqslant q_m(t) \leqslant \sum_{j=m}^{\infty} j c_j(t)$, so that

$$\lim_{m \to \infty} q_m(t) = 0, \qquad |q_m(t)| \leqslant \text{const} \qquad \text{for all} \quad t \in [0, T]$$

Thus, given $\varepsilon > 0$, there exists $M > J$ such that

$$\int_0^T q_M(t)\, dt < \varepsilon \qquad \text{and} \qquad x_M^n(0) = \sum_{j=M}^{n} j c_{0j} < \varepsilon / T \qquad \text{for all} \quad n \geqslant M \tag{2.12}$$

By (2.11), (2.12), there exists $N_0 > 3M$ such that for $n \geqslant N_0$,

$$\int_0^t q_M^n(s)\,ds < 2\varepsilon \qquad \text{for all} \quad t \in [0, T] \tag{2.13}$$

Returning to (2.10) and using (2.8), (2.9), we have that for $t \in [0, T]$, $n \geqslant N_0$,

$$\begin{aligned} &x_M^n(t) + \int_0^t p^n(s)\,ds \\ &\qquad \leqslant \varepsilon/T + q_M^n(t) + \int_0^t \left(\sum_{T_3} j + 2M \sum_{T_1}\right) a_{j,k} c_j^n(s)\, c_k^n(s)\,ds \end{aligned} \tag{2.14}$$

From (2.13) we obtain for $t \in [0, T]$, $n \geqslant N_0$,

$$\begin{aligned} &\int_0^t x_M^n(s)\,ds + \int_0^t \int_0^s p^n(\tau)\,d\tau\,ds \\ &\qquad \leqslant 3\varepsilon + \int_0^t \int_0^s \left(\sum_{T_3} j + 2M \sum_{T_1}\right) a_{j,k} c_j^n(\tau)\, c_k^n(\tau)\,d\tau\,ds \end{aligned} \tag{2.15}$$

Now

$$\begin{aligned} \sum_{T_3} j a_{j,k} c_j^n c_k^n &\leqslant K \sum_{j=1}^{M-1} j c_j^n \sum_{k=M}^{n-j} (j+k) c_k^n \\ &\leqslant 2K \sum_{j=1}^{M-1} j c_j^n \sum_{k=M}^{n-j} k c_k^n \\ &\leqslant 2K \|c_0\|\, x_M^n \end{aligned} \tag{2.16}$$

and

$$2M \sum_{T_1} a_{j,k} c_j^n c_k^n \leqslant 2KM \sum_{j=M}^{n} \sum_{k=M}^{n} (j+k) c_j^n c_k^n \leqslant 4K \|c_0\|\, x_M^n \tag{2.17}$$

Using (2.16), (2.17) in (2.15) and applying Gronwall's inequality, it follows that for all $t \in [0, T]$, $n \geqslant N_0$,

$$\int_0^t x_M^n(s)\,ds + \int_0^t \int_0^s p^n(\tau)\,d\tau\,ds \leqslant K_1 \varepsilon \tag{2.18}$$

where K_1 is a constant depending only on K, T, and $\|c_0\|$. Since

$$\int_0^t \int_0^s p^n(\tau)\,d\tau\,ds = \int_0^t (t-s)\, p^n(s)\,ds$$

it follows from (2.18) that for all $n \geqslant N_0$

$$\int_0^{T/2} x_M^n(s)\,ds + \frac{T}{2}\int_0^{T/2} p^n(s)\,ds \leqslant K_1\varepsilon$$

With an appropriate rescaling of T, ε, this gives (2.5).

From the pointwise convergence of $c_j^n(t)$ and (2.5) we deduce that

$$\int_0^T \left[\sum_{j=1}^{M-1}\sum_{k=2M}^{\infty} jb_{j,k}c_{j+k}^n(s) + \sum_{j=M}^{\infty} jc_j(s)\right] ds \leqslant \varepsilon \tag{2.19}$$

In particular, since $M > J$,

$$\int_0^T \sum_{k=1}^{\infty} a_{J,k}c_k^n(s)\,ds < \infty, \qquad \int_0^T \sum_{k=1}^{\infty} b_{J,k}c_{J+k}^n(s)\,ds < \infty \tag{2.20}$$

For any $l > 2M$, $n > J + l$, $t \in [0, T]$ we have from (2.1), (2.5) that

$$\left| c_J^n(t) - c_J(0) - \int_0^t \left[\frac{1}{2}\sum_{k=1}^{J-1} W_{J-k,k}(c^n(s)) - \sum_{k=1}^{l} W_{J,k}(c(s))\right] ds \right| \leqslant (2K\|c_0\| + 1)\varepsilon \tag{2.21}$$

Letting $n \to \infty$ and then $l \to \infty$ in (2.21), we deduce by (2.20) and the arbitrariness of ε that

$$c_J(t) = c_J(0) + \int_0^t \left[\frac{1}{2}\sum_{k=1}^{J-1} W_{J-k,k}(c(s)) - \sum_{k=1}^{\infty} W_{J,k}(c(s))\right] ds$$

for all $t \geqslant 0$ and each J, as required. In particular, each c_J is continuous on $[0, \infty)$. ▮

In general, even if $a_{j,k} \leqslant K(j+k)$, solutions of (1.1) do not conserve density. The next result shows that for $a_{j,k} \leqslant K(j+k)$, the solution constructed in Theorem 2.4 conserves density.

Theorem 2.5. Assume that $a_{j,k} \leqslant K(j+k)$ and that c is the solution constructed in Theorem 2.4. Then for all $t \geqslant 0$,

$$\sum_{j=1}^{\infty} jc_j(t) = \sum_{j=1}^{\infty} jc_j(0) \tag{2.22}$$

Proof. Let c^n be the solution to (2.1) with $c^n(0) = c_j(0)$, $1 \leqslant j \leqslant n$. Then, writing n for the subsequence n_k, we have that $c_j^n(t) \to c_j(t)$ as $n \to \infty$ for each j and all $t \geqslant 0$. We use various relations derived in the proof of Theorem 2.4. In particular, from (2.7)–(2.10), (2.16) and (2.17) we obtain for $t \geqslant 0$

$$x_m^n(t) \leqslant x_m^n(0) + q_m^n(t) + K_2 \int_0^t x_m^n(s)\, ds \tag{2.23}$$

where K_2 is a constant. Fix $t \geqslant 0$ and let $\varepsilon > 0$. By (2.18) and

$$\lim_{n \to \infty} q_m^n(t) = q_m(t), \qquad \lim_{m \to \infty} q_m(t) = 0, \qquad x_m^n(0) = \sum_{j=m}^{n} jc_j(0), \quad c(0) \in X^+$$

there exist M and $N_1 > M$, depending on t, such that

$$|q_M^n(t)| < \varepsilon, \qquad |x_M^n(0)| < \varepsilon, \qquad \int_0^t x_M^n(s)\, ds < \varepsilon \qquad \text{for all} \quad n \geqslant N_1 \tag{2.24}$$

Using this in (2.23) gives

$$x_M^n(t) \leqslant K_3 \varepsilon \qquad \text{for} \quad n \geqslant N_1 \tag{2.25}$$

where K_3 is a constant. Thus, also

$$\sum_{j=M}^{\infty} jc_j(t) \leqslant K_3 \varepsilon \tag{2.26}$$

Writing $c_j^n(t) = 0$ for $j > n$, we deduce from (2.25)–(2.26) that for $n \geqslant N_1$

$$\left| \sum_{j=1}^{\infty} j[c_j^n(t) - c_j(t)] \right| \leqslant \left| \sum_{j=1}^{M-1} j[c_j^n(t) - c_j(t)] \right| + 2K_3 \varepsilon \tag{2.27}$$

Since $\sum_{j=1}^{\infty} jc_j^n(t) = \sum_{j=1}^{\infty} jc_j(0)$, letting $n \to \infty$ in (2.27) gives

$$\left| \sum_{j=1}^{\infty} j[c_j(0) - c_j(t)] \right| \leqslant 2K_3 \varepsilon$$

and the result follows. ∎

Corollary 2.6. Let the hypotheses of Theorem 2.5 hold, and denote by c^{n_k} the corresponding pointwise convergent subsequence of solutions to (2.1). Then $c^{n_k}(t) \to c(t)$ in X uniformly on compact subsets of $[0, \infty)$.

Proof. We again write n for the subsequence n_k. We first prove that, for each j, $c_j^n(t) \to c_j(t)$ uniformly on compact subsets of $[0, \infty)$. For this it is clearly sufficient to show that for each $m > 1$,

$$y_m^n(t) \stackrel{\text{def}}{=} e^{-t} \left[\rho^n(0) - \sum_{j=1}^{m-1} jc_j^n(t) + 4mK\rho^n(0)^2 \right]$$

converges to

$$y_m(t) \stackrel{\text{def}}{=} e^{-t}\left[\rho(0) - \sum_{j=1}^{m-1} jc_j(t) + 4mK\rho(0)^2\right]$$

uniformly on compact subsets of $[0, T)$, where $\rho(0) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} jc_j(0)$. But this follows from the pointwise convergence of $y_m^n(t)$ to the continuous function $y_m(t)$ and the fact that by Lemmas 2.2, 2.3,

$$\frac{d}{dt} y_m^n(t) \leqslant 0, \qquad t \in [0, T), \quad n \geqslant m$$

Let $I \subset [0, \infty)$ be compact and $t_n \to t$ in I. By Lemma 2.2 and Theorem 2.5

$$\lim_{n\to\infty} \|c^n(t_n)\| = \lim_{n\to\infty} \|c(t_n)\| = \|c(t)\|$$

Applying Lemma 2.7 below, we deduce that as $n \to \infty$, $c^n(t_n) \to c(t)$, $c(t_n) \to c(t)$ in X. Hence $c^n \to c$ in $C(I, X)$, as required. ∎

Lemma 2.7. (Cf. ref. 2, Lemma 3.3). If $y^n \stackrel{*}{\rightharpoonup} y$ in X and $\|y^n\| \to \|y\|$, then $y^n \to y$ in X.

3. CONDITIONS UNDER WHICH ALL SOLUTIONS CONSERVE DENSITY

We first give an easily proved identity valid for solutions of (1.1) similar to the identity given in Lemma 2.1. The reader is encouraged to sketch the analogue of Fig. 1 for the regions R_k.

Lemma 3.1. Let c be a solution of (1.1) on $[0, T)$ and let (g_j) be a sequence. Then for $1 \leqslant m \leqslant n$ and $0 \leqslant t_1 < t_2 < T$,

$$\sum_{j=m}^{n} g_j[c_j(t_2) - c_j(t_1)] = \int_{t_1}^{t_2} \left[\frac{1}{2}\sum_{R_1} (g_{j+k} - g_j - g_k) + \frac{1}{2}\sum_{R_2} g_{j+k} + \sum_{R_3} (g_{j+k} - g_k) - \sum_{R_4} g_j\right] W_{j,k}(c(s))\, ds \tag{3.1}$$

where

$$R_1 = \{(j, k) \colon j, k \geqslant m, j+k \leqslant n\}$$

$$R_2 = \{(j, k) \colon m \leqslant j+k \leqslant n, \quad j, k < m\}$$

$$R_3 = \{(j, k) \colon 1 \leqslant j \leqslant m-1, k \geqslant m, j+k \leqslant n\}$$

$$R_4 = \{(j, k) \colon m \leqslant j \leqslant n, j+k > n\}$$

with the sums equal to zero if the associated region is empty. (Note that R_2 and R_3 are empty if $m=1$.)

Setting $g_j = j$ and $m=1$ in Lemma 3.1, to prove that $\sum_{j=1}^{\infty} jc_j(t)$ is a conserved quantity, it is sufficient to prove that

$$\lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} j \sum_{k=n-j+1}^{\infty} W_{j,k}(c(s))\, ds = 0 \tag{3.2}$$

Considering, for example, the case $a_{j,k} = j+k$, $b_{j,k}=0$, it is clear that we will require more information on the solution in order to prove (3.2). The basic plan is as follows.

(a) Taking $g_j = 1$ in Lemma 3.1 and letting $n \to \infty$, obtain an identity of the form

$$\sum_{j=m}^{\infty} [c_j(t_2) - c_j(t_1)] = \int_{t_1}^{t_2} D_m(c(s))\, ds \tag{3.3}$$

(b) Since $c \in X^+$, for each t, $m \sum_{j=m}^{\infty} c_j(t) \leqslant \sum_{j=m}^{\infty} jc_j(t) \to 0$ as $m \to \infty$. Thus, from (3.3),

$$\lim_{m\to\infty} m \int_{t_1}^{t_2} D_m(c(s))\, ds = 0 \tag{3.4}$$

The idea is to exploit the extra information contained in (3.4) to prove (3.2).

Taking $g_j = 1$ in Lemma 3.1 and letting $n \to \infty$, we formally obtain

$$\sum_{j=m}^{\infty} [c_j(t_2) - c_j(t_1)] = \frac{1}{2} \int_{t_1}^{t_2} \left[-\sum_{j=m}^{\infty} \sum_{k=m}^{\infty} W_{j,k}(c(s)) + \sum_{j=1}^{m-1} \sum_{k=m-j}^{m-1} W_{j,k}(c(s)) \right] ds \tag{3.5}$$

In order to execute our plan, we will need to let $m \to \infty$ in (3.5) and to be able to manipulate the resulting double series. Thus, we will require that

$$\int_{t_1}^{t_2} \sum_{j,k=1}^{\infty} a_{j,k} c_j(s)\, c_k(s)\, ds < \infty \tag{3.6}$$

$$\int_{t_1}^{t_2} \sum_{j,k=1}^{\infty} b_{j,k} c_{j+k}(s)\, ds < \infty \tag{3.7}$$

This need not be the case even if (3.4) holds.

Example 3.2. Let $a_{j,k}=(jk)^3\,(j+k)^{-2}$, $b_{j,k}=j+k$, $c_j=j^{-3}$. Then $c=(c_j)\in X^+$ and $W_{j,k}(c)=0$ for all j, k. Also,

$$\sum_{k=1}^{\infty} a_{j,k}c_jc_k=\sum_{k=1}^{\infty} b_{j,k}c_{j+k}=\sum_{k=1}^{\infty} (j+k)^{-2}<\infty$$

so that c is a solution of (1.1). In this case $D_m=0$. However, the double sums in (3.6)–(3.7) are infinite.

Lemma 3.3. Let c be a solution of (1.1) on $[0, T)$ and let $0\leqslant t_1<t_2<T$. Suppose that either (3.6) or (3.7) holds. Then (3.5)–(3.7) hold and

$$\lim_{m\to\infty} m\int_{t_1}^{t_2}\left(\sum_{j=m}^{\infty}\sum_{k=m}^{\infty}-\sum_{j=1}^{m-1}\sum_{k=m-j}^{m-1}\right) W_{j,k}(c(s))\,ds=0 \tag{3.8}$$

Proof. Setting $g_j=1$ and $m=1$ in Lemma 3.1 gives

$$\sum_{j=1}^{n}\left[c_j(t_2)-c_j(t_1)\right]=-\int_{t_1}^{t_2}\left(\frac{1}{2}\sum_{j+k\leqslant n}+\sum_{j=1}^{n}\sum_{k=n-j+1}^{\infty}\right) W_{j,k}(c(s))\,ds \tag{3.9}$$

Since $c\in X^+$, the right-hand side of (3.9) is bounded independently of n. Hence, if either (3.6) or (3.7) holds, so does the other. Letting $n\to\infty$ in (3.9) and using the dominated convergence theorem gives (3.5) for $m=1$; the case of general m follows from adding on a finite sum. Finally, since $c\in X^+$, for fixed t we have that

$$m\sum_{j=m}^{\infty} c_j(t)\leqslant\sum_{j=m}^{\infty} jc_j(t)\to 0 \qquad \text{as} \quad m\to\infty \tag{3.10}$$

Combining this with (3.5) proves (3.8). ∎

The following easily proved proposition gives some examples of kinetic coefficients which satisfy either (3.6) or (3.7).

Proposition 3.4. Let c be a solution of (1.1) on $[0, T)$. Let (r_j) be a nonnegative sequence and let $\alpha_{j,k}$ satisfy $\alpha_{j,k}\geqslant 0$ for all j,k and $\alpha_{j,k}\leqslant K(j+k)$ for $j\geqslant n_0$ and $k\geqslant n_0$, where K, n_0 are constants. Then (3.6) and (3.7) hold in the following cases:

(i) $a_{j,k}=r_j+r_k+\alpha_{j,k}$.

(ii) $a_{j,k}=r_jr_k+\alpha_{j,k}$.

(iii) $b_{j,k}\equiv 0$ for all j, k with $j\geqslant n_0$ and $k\geqslant n_0$.

We use below the following notation: if r is an integer, then $h(r)=[(r+1)/2]$, the integer part of $(r+1)/2$. This notation is used in sums over j, k space with $j+k=r$ up to the diagonal $j=k$. In practice the reader can think of $h(r)$ as equalling $r/2$.

We show that all solutions of (1.1) conserve density under the following conditions (here K and n_0 are constants):

(H1) $a_{j,k}=r_j+r_k+\alpha_{j,k}$, where (r_j) is a nonnegative sequence and $\alpha_{j,k}\geqslant 0$ for all j, k, and $\alpha_{j,k}\leqslant K(j+k)$ for all $j, k\geqslant n_0$.

(H2) $\sum_{j=n_0}^{h(r)} jb_{r-j,j}\leqslant Kr$ for all $r\geqslant 2n_0$.

Note that if $a_{j,k}=b_{j,k}=0$ when both $j, k\geqslant n_0$, then (H1)–(H2) are trivially satisfied.

To prove conservation, we combine (3.8) with various estimates on sums of coagulation and fragmentation terms.

Lemma 3.5. Assume (H1)–(H2). Let c be a solution of (1.1) on $[0, T)$ and let $t\in[0, T)$. Then if $\bar{W}_{j,k}(c(s))=a_{j,k}c_j(s)\,c_k(s)$ or $\bar{W}_{j,k}(c(s))=b_{j,k}c_{j+k}(s)$,

$$\lim_{n\to\infty}\int_0^t\sum_{j=1}^{n}\sum_{k=h(n)}^{\infty} j\bar{W}_{j,k}(c(s))\,ds=0 \tag{3.11}$$

$$\lim_{n\to\infty} n\int_0^t\sum_{j=n}^{\infty}\sum_{k=n}^{\infty}\bar{W}_{j,k}(c(s))\,ds=0 \tag{3.12}$$

$$\lim_{n\to\infty}\int_0^t\sum_{T}(n-j)\,\bar{W}_{j,k}(c(s))\,ds=0 \tag{3.13}$$

where

$$T=\{(j, k): j+k\geqslant n,\ k\leqslant h(n),\ j\leqslant n-1\}$$

Proof. We first make some observations which restrict the regions of summation in (3.11) and (3.13). From the definition of a solution,

$$\lim_{n\to\infty}\int_0^t\sum_{j=1}^{n_0-1}\sum_{k=h(n)}^{\infty}\bar{W}_{j,k}(c(s))\,ds=0$$

Thus, to prove (3.11), we need only consider $j\geqslant n_0$. Also, set $T'=\{(j, k)\in T: k\leqslant n_0-1\}$. Then

$$\int_0^t\sum_{T'}(n-j)\,\bar{W}_{j,k}(c(s))\,ds\leqslant n_0\int_0^t\sum_{j=n-n_0}^{n}\sum_{k=1}^{n_0-1}\bar{W}_{j,k}(c(s))\,ds\to 0$$

as $n\to\infty$, by the definition of a solution. Thus, to prove (3.13), we need only sum over the region $R=\{(j, k)\in T: k\geqslant n_0\}$.

We estimate the coagulation and fragmentation terms in (3.11)–(3.13) separately. Let

$$u_n = \sum_{j=n_0}^{n} \sum_{k=h(n)}^{\infty} j a_{j,k} c_j c_k, \qquad v_n = n \sum_{j=n}^{\infty} \sum_{k=n}^{\infty} a_{j,k} c_j c_k$$

Using the assumptions on $a_{j,k}$, for n sufficiently large,

$$u_n \leqslant \sum_{j=n_0}^{\infty} j c_j \left(3K \sum_{k=h(n)}^{\infty} k c_k + \sum_{k=h(n)}^{\infty} r_k c_k \right) + 2 \sum_{j=n_0}^{n} r_j c_j \sum_{k=h(n)}^{\infty} k c_k \tag{3.14}$$

$$v_n \leqslant 2 \sum_{k=n}^{\infty} k c_k \left(\sum_{j=n}^{\infty} r_j c_j + K \sum_{j=n}^{\infty} j c_j \right) \tag{3.15}$$

From the definition of a solution, $\sup_{s \in [0,t]} \|c(s)\| < \infty$ and $\int_0^t \sum_{j=1}^{\infty} r_j c_j(s)\, ds < \infty$, so that as $n \to \infty$,

$$\int_0^t \sum_{j=n}^{\infty} j c_j(s)\, ds, \int_0^t \sum_{j=n}^{\infty} r_j c_j(s)\, ds,$$
$$\int_0^t \left[\sum_{j=n}^{\infty} j c_j(s) \sum_{k=1}^{\infty} r_k c_k(s)) \right] ds \to 0 \tag{3.16}$$

Using (3.14)–(3.16), $\int_0^t u_n(s)\, ds$, $\int_0^t v_n(s)\, ds \to 0$ as $n \to \infty$, which proves (3.11)–(3.12) for the coagulation terms. Now

$$w_n = \sum_R (n-j) a_{j,k} c_j c_k$$
$$\leqslant \sum_{j=h(n)}^{n-1} \sum_{k=n-j}^{h(n)} (n-j)[K(j+k) + r_j + r_k] c_j c_k$$

In the above sum, $n - j \leqslant j$ and $n - j \leqslant k$. Thus,

$$w_n \leqslant \sum_{j=h(n)}^{n} j c_j \left(2K \sum_{k=1}^{h(n)} k c_k + \sum_{k=1}^{h(n)} r_k c_k \right) + \sum_{j=h(n)}^{n} r_j c_j \sum_{k=1}^{h(n)} k c_k$$

so that by (3.16), $\int_0^t w_n(s)\, ds \to 0$ as $n \to \infty$. This proves (3.13) for the coagulation terms.

We now estimate the fragmentation terms. Let

$$\bar{v}_n = n \sum_{j=n}^{\infty} \sum_{k=n}^{\infty} b_{j,k} c_{j+k} = n \sum_{r=2n}^{\infty} c_r \sum_{j=n}^{r-n} b_{r-j,j} \tag{3.17}$$

For $r \geqslant 2n$,

$$n \sum_{j=n}^{r-n} b_{r-j,j} \leqslant 2n \sum_{j=n}^{h(r)} b_{r-j,j} \leqslant 2 \sum_{j=n}^{h(r)} j b_{r-j,j}$$

so that by (H2), $\bar{v}_n \leqslant 2K \sum_{r=2n}^{\infty} r c_r$ for $n \geqslant n_0$, which proves (3.12).

Using (3.12) with n replaced by $h(n)$, we see that to prove (3.11) we need only estimate

$$\sum_{j=n_0}^{h(n)} \sum_{k=h(n)}^{\infty} j b_{j,k} c_{j+k} = \sum_{r=h(n)+n_0}^{\infty} c_r \sum_{j=n_0}^{q} j b_{r-j,j} \tag{3.18}$$

where $q = \min(h(n), r-h(n)) \leqslant h(r)$. Integrating (3.18) and using (H2), we obtain (3.11).

Finally,

$$\sum_R (n-j) b_{j,k} c_{j+k} = \sum_{r=n}^{n-1+h(n)} c_r \sum_{k=s}^{h(n)} (k+n-r) b_{r-k,k} \tag{3.19}$$

with $s = \max(n_0, r-n+1)$. Since $r \geqslant n$,

$$\sum_{k=s}^{h(n)} (k+n-r) b_{r-k,k} \leqslant \sum_{k=n_0}^{h(r)} k b_{r-k,k} \leqslant Kr$$

and (3.13) follows. ∎

Theorem 3.6. Assume (H1) and (H2), and let c be a solution of (1.1) on $[0, T)$ with $\rho_0 = \sum_{j=1}^{\infty} j c_j(0)$. Then

$$\sum_{j=1}^{\infty} j c_j(t) = \rho_0 \qquad \text{for all} \quad t \in [0, T)$$

Proof. By (3.2), to prove the result, we have to show that

$$\lim_{n \to \infty} \int_0^t \sum_{j=1}^{n-1} j \sum_{k=n-j}^{\infty} W_{j,k}(c(s))\, ds = 0$$

Set $T = \{(j, k): j+k \geqslant n,\ k \leqslant h(n),\ j \leqslant n-1\}$. By (3.11) it is sufficient to show that

$$\lim_{n \to \infty} \int_0^t \sum_T j W_{j,k}(c(s))\, ds = 0 \tag{3.20}$$

By (3.8) and (3.12),

$$\lim_{n \to \infty} n \int_0^t \sum_{j=1}^{n-1} \sum_{k=n-j}^{n-1} W_{j,k}(c(s))\, ds = 0 \tag{3.21}$$

Using the symmetry of $W_{j,k}$ and (3.12) with n replaced by $h(n)$, it follows that

$$\lim_{n \to \infty} n \int_0^t \sum_T W_{j,k}(c(s))\, ds = 0 \tag{3.22}$$

Finally, by writing $jW_{j,k} = (j-n) W_{j,k} + n W_{j,k}$ and using (3.13) and (3.22), we prove (3.20). ▮

For future applications, it is useful to generalize (3.2).

Theorem 3.7. Assume (H1) and (H2). Let (g_j) be a sequence with $|g_j - g_k| \leqslant K_1 |j-k|$ for all j, k, where K_1 is a constant. Then if c is a solution of (1.1) on $[0, T)$, for all $t \in [0, T)$,

$$\lim_{n \to \infty} \int_0^t \sum_{j=1}^{n} g_j \sum_{k=n-j+1}^{\infty} W_{j,k}(c(s))\, ds = 0 \tag{3.23}$$

The above result is proved in exactly the same way as Theorem 3.6.

4. UNIQUENESS

As noted in the introduction, in general, solutions of (1.1) need not be unique (see also Section 6). However, by imposing growth conditions on the kinetic coefficients, we are able to prove uniqueness.

Theorem 4.1. Let $K > 0$ and $0 \leqslant \alpha \leqslant 1/2$ and assume the following:

(i) $a_{j,k} \leqslant K(jk)^{\alpha}$ for all j, k.

(ii) $\sum_{j=1}^{h(r)} j^{1-\alpha} b_{r-j,j} \leqslant K r^{1-\alpha}$ for all $r \geqslant 2$.

Let $c_0 \in X^+$ and $T > 0$. Then there is exactly one solution c of (1.1) on $[0, T)$ satisfying $c(0) = c_0$.

Proof. Let c, d be two solutions of (1.1) on $[0, T)$ satisfying $c(0) = d(0) = c_0$ and set $x = c - d$. Let $\beta = 1 - \alpha$ and

$$\vartheta(t) = \sum_{j=1}^{\infty} j^{\beta} |x_j(t)|$$

We show that

$$\vartheta(t) \leqslant \text{const.} \int_0^t \vartheta(s)\, ds, \qquad t \in [0, T) \tag{4.1}$$

so that, by Gronwall's lemma, $\vartheta(t) = 0$ and $c = d$.

For $\lambda \in \mathbb{R}$, define sgn λ to equal 1, 0, or -1 according as $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. Note that if $\varphi(\cdot)$ is an absolutely continuous function of t, then so is $t \mapsto |\varphi(t)|$, and

$$\frac{d}{dt}|\varphi(t)| = \operatorname{sgn} \varphi(t) \frac{d\varphi}{dt}(t) \quad \text{a.e.}$$

For $t \in [0, T)$, it thus follows from the same calculation leading to Lemma 3.1 that

$$\sum_{j=1}^{n} j^{\beta} |x_j(t)| = \int_0^t [U_n(s) + V_n(s)]\, ds \tag{4.2}$$

where

$$U_n = \frac{1}{2} \sum_{j+k \leqslant n} (g_{j+k} - g_j - g_k) M_{j,k}, \qquad V_n = -\sum_{j=1}^{n} g_j \sum_{k=n-j+1}^{\infty} M_{j,k}$$

$$g_j = j^{\beta} \operatorname{sgn}(x_j)$$

$$M_{j,k} = W_{j,k}(c) - W_{j,k}(d) = a_{j,k}(c_j x_k + d_k x_j) - b_{j,k} x_{j+k}$$

We first estimate U_n. Now

$$\begin{aligned}
&[(j+k)^{\beta} \operatorname{sgn}(x_{j+k}) - j^{\beta} \operatorname{sgn}(x_j) - k^{\beta} \operatorname{sgn}(x_k)] x_k \\
&\quad = [(j+k)^{\beta} \operatorname{sgn}(x_{j+k} x_k) - j^{\beta} \operatorname{sgn}(x_k x_j) - k^{\beta}] |x_k| \\
&\quad \leqslant [(j+k)^{\beta} + j^{\beta} - k^{\beta}] |x_k| \leqslant 2j^{\beta} |x_k|
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2} \sum_{j+k \leqslant n} (g_{j+k} - g_j - g_k)\, a_{j,k} c_j |x_k| \\
&\quad \leqslant K \sum_{j+k \leqslant n} jk^{\alpha} c_j |x_k| \\
&\quad \leqslant K \sum_{j=1}^{\infty} jc_j \sum_{k=1}^{\infty} k^{\alpha} |x_k| \leqslant \text{const.}\, \vartheta
\end{aligned} \tag{4.3}$$

since $\alpha \leqslant 1 - \alpha$. We get a similar estimate for the terms involving $d_k x_j$. To estimate the fragmentation terms in U_n, note that

$$\begin{aligned}
&-[(j+k)^{\beta} \operatorname{sgn}(x_{j+k}) - j^{\beta} \operatorname{sgn}(x_j) - k^{\beta} \operatorname{sgn}(x_k)] x_{j+k} \\
&\quad \leqslant [j^{\beta} + k^{\beta} - (j+k)^{\beta}] |x_{j+k}|
\end{aligned}$$

Thus,

$$
\begin{aligned}
-\sum_{j+k\leqslant n} &(g_{j+k}-g_j-g_k)b_{j,k}x_{j+k}\\
&\leqslant \sum_{j+k\leqslant n}[j^\beta+k^\beta-(j+k)^\beta]b_{j,k}|x_{j+k}|\\
&\leqslant \sum_{r=2}^{n}|x_r|\sum_{j=1}^{r-1}\alpha_{r-j,j}b_{r-j,j} \qquad (4.4)
\end{aligned}
$$

with $\alpha_{r-j,j}=(r-j)^\beta+j^\beta-r^\beta\leqslant j^\beta$. Also,

$$\sum_{j=1}^{r-1}\alpha_{r-j,j}b_{r-j,j}\leqslant 2\sum_{j=1}^{h(r)}\alpha_{r-j,j}b_{r-j,j}\leqslant 2\sum_{j=1}^{h(r)}j^\beta b_{r-j,j}\leqslant 2Kr^\beta$$

so that, by (4.4),

$$-\sum_{j+k\leqslant n}(g_{j+k}-g_j-g_k)b_{j,k}c_{j+k}\leqslant \text{const. }\vartheta \qquad (4.5)$$

Combining (4.3), (4.5), for all n,

$$\int_0^t U_n(s)\,ds\leqslant \text{const.}\int_0^t \vartheta(s)\,ds \qquad (4.6)$$

Next we show that

$$\lim_{n\to\infty}\int_0^t V_n(s)\,ds=0 \qquad (4.7)$$

It is easy to check that the assumptions on the kinetic coefficients imply that (H1) and (H2) hold. Thus, from (3.11),

$$
\begin{aligned}
\lim_{n\to\infty}&\int_0^t\left|\sum_S g_jM_{j,k}\right|ds\\
&\leqslant \lim_{n\to\infty}\int_0^t\sum_S j[|W_{j,k}(c)|+|W_{j,k}(d)|]\,ds=0 \qquad (4.8)
\end{aligned}
$$

where $S=\{(j,k): k\geqslant h(n),\ 1\leqslant j\leqslant n\}$. Hence, to prove (4.7), it suffices to show that

$$\lim_{n\to\infty}\int_0^t\sum_{S'} g_jM_{j,k}\,ds=0 \qquad (4.9)$$

where $S' = \{(j, k): j+k \geqslant n,\ k \leqslant h(n),\ h(n) \leqslant j \leqslant n\}$. Now

$$\sum_{S'} j^\beta a_{j,k} c_j c_k = \sum_{j=h(n)}^{n-1} \sum_{k=n-j}^{h(n)} a_{j,k} c_j c_k \leqslant K \sum_{j=h(n)}^{n} jc_j \sum_{k=1}^{\infty} k^\alpha c_k$$

Hence,

$$\lim_{n\to\infty} \int_0^t \sum_{S'} j^\beta a_{j,k} c_j c_k \, ds = 0 \tag{4.10}$$

Applying Theorem 3.7 with $g_j = j^\beta$ and using (4.8) gives

$$\lim_{n\to\infty} \int_0^t \sum_{S'} j^\beta W_{j,k}(c(s)) \, ds = 0 \tag{4.11}$$

Combining (4.10) and (4.11), we deduce that

$$\lim_{n\to\infty} \int_0^t \sum_{S'} j^\beta |W_{j,k}(c(s))| \, ds = 0 \tag{4.12}$$

Then (4.9) follows immediately, completing the proof. ∎

We can also prove uniqueness of density-conserving solutions at the expense of making very strong assumptions about the coagulation coefficients.

Theorem 4.2. Suppose that $a_{j,k} \leqslant K$ for all j, k. Let $c_0 \in X^+$ and $T > 0$. Then there is at most one density-conserving solution c of (1.1) on $[0, T)$ with $c(0) = c_0$.

Proof. The proof is very similar to that given for the previous result, so we only give the main steps. Let c, d be density-conserving solutions with $c(0) = d(0) = c_0$ and let $\psi(t) = \sum_{j=1}^{\infty} j|x_j(t)|$. From the proof of the previous result, (4.2) holds with $\beta = 1$. Using $a_{j,k} \leqslant K$, it is easy to show that if $g_j = j \operatorname{sgn}(x_j)$, then

$$\sum_{j+k \leqslant n} (g_{j+k} - g_j - g_k)(c_j x_k + x_j d_k) a_{j,k} \leqslant \text{const.}\ \psi$$

Also,

$$-[(j+k)\operatorname{sgn}(x_{j+k}) - j \operatorname{sgn}(x_j) - k \operatorname{sgn}(x_k)]\, x_{j+k}$$
$$\leqslant [j+k-(j+k)]\,|x_{j+k}| = 0$$

Thus,

$$\int_0^t U_n(s)\, ds \leqslant \text{const.}\ \psi(t)$$

Finally, we have to show that

$$\lim_{n\to\infty}\int_0^t \sum_U g_j M_{j,k}\,ds=0 \tag{4.13}$$

where $U=\{(j,k): j\leqslant n,\ j+k>n\}$. Since c and d conserve density, by applying Lemma 3.1 to $\sum_{j=1}^{n} j(c_j-d_j)$, we obtain

$$\lim_{n\to\infty}\int_0^t \sum_U jM_{j,k}\,ds=0 \tag{4.14}$$

where $M_{j,k}=W_{j,k}(c)-W_{j,k}(d)$. Using the bounds on $a_{j,k}$, it is easy to show that

$$\lim_{n\to\infty}\int_0^t \sum_U ja_{j,k}c_jc_k\,ds=0 \tag{4.15}$$

Combining (4.14) and (4.15) proves (4.13) and the result follows. ∎

5. CONTINUOUS DEPENDENCE

Solutions are not unique in general, so by gluing together various solutions, we can demonstrate unpleasant analytic phenomena. For example, let $a_{j,k}=0$, $b_{j,k}=1$. Then it is shown in Example 6.2(b) that there is a solution $\bar{c}$ of (1.1) with $\bar{c}(0)=0$ and density $\sum_{r=1}^{\infty} r\bar{c}_r(t)=1$ for $t>0$. Let c be the solution of (1.1) defined by

$$c(t)=\begin{cases}0, & 0<t<T\\ \bar{c}(t-T), & t\geqslant T\end{cases}$$

Then c is a solution on $[0,\infty)$ with $\sum_{r=1}^{\infty} rc_r(t)$ discontinuous at $t=T$.

To obtain good analytic information on solutions, we need to filter out these spurious solutions. Thus, in what follows we usually assume the conditions of density conservation.

Theorem 5.1. Let c be a solution of (1.1) on $[0,T)$, $T\leqslant\infty$. Assume that c conserves density on $[0,T)$. Then $c\colon[0,T)\to X^+$ is continuous, and the series $\sum_{r=1}^{\infty} rc_r(t)$ is uniformly convergent on compact intervals of $[0,T)$.

Proof. This follows from applying Dini's theorem to $\sum_{r=1}^{n} rc_r(t)$. ∎

Theorem 5.2. Let c be a solution of (1.1) on $[0,T)$, $T\leqslant\infty$. Assume $a_{j,k}\leqslant K(j+k)$, $b_{j,k}\leqslant Kjk$, and that c conserves density on $[0,T)$. Then each c_j is continuously differentiable on $[0,T)$.

Proof. The growth assumptions on the kinetic coefficients and Theorem 5.1 imply that the right-hand side of (1.1) is continuous in t, and the result follows. ∎

Definition. A *generalized flow* G on a metric space Y is a family of continuous mappings ϕ: $[0, \infty) \to Y$ with the following properties:

(i) If $\phi \in G$ and $\tau \geqslant 0$, then $\phi_\tau \in G$, where

$$\phi_\tau(t) \stackrel{\text{def}}{=} \phi(t+\tau), \qquad t \in [0, \infty)$$

(ii) If $y \in Y$, there exists at least one $\phi \in G$ with $\phi(0) = y$.

(iii) If $\phi_j \in G$ with $\phi_j(0)$ convergent in Y as $j \to \infty$, then there exist a subsequence ϕ_{j_k} of ϕ_j and an element $\phi \in G$ such that $\phi_{j_k}(t) \to \phi(t)$ in Y uniformly for t in compact intervals of $[0, \infty)$.

The following, which are analogues of results in Section 2, will be used in the proof of the upper semicontinuity property (iii) for the set of density-conserving solutions of (1.1).

Lemma 5.3. Assume $a_{j,k} \leqslant K(j+k)$ and let c be a density-conserving solution of (1.1) on $[0, \infty)$. Then for a.e. $t \geqslant 0$,

$$\dot{x}_m = \frac{1}{2} \sum_{Z_2} (j+k)\, W_{j,k}(c) + \sum_{Z_3} j W_{j,k}(c) \tag{5.1}$$

$$\dot{q}_m = \frac{1}{2} \sum_{Z_1} \mu_{j,k} W_{j,k}(c) + \frac{1}{2} \sum_{Z_2} (j+k)\, W_{j,k}(c) + \sum_{Z_3} \lambda_{j,k} W_{j,k}(c) \tag{5.2}$$

and

$$\frac{d}{dt} [e^{-t}(x_m + 2Km\rho^2)] \leqslant 0 \tag{5.3}$$

where

$$x_m = \sum_{r=m}^{\infty} rc_r, \qquad q_m = \sum_{r=m}^{2m} rc_r + 2m \sum_{r=2m+1}^{\infty} c_r$$

$$Z_1 = \{(j, k)\colon j, k \geqslant m\}, \qquad Z_2 = \{(j, k)\colon j+k \geqslant m,\ \ j, k < m\}$$

$$Z_3 = \{(j, k)\colon 1 \leqslant j \leqslant m-1, k \geqslant m\}, \qquad \rho = \sum_{r=1}^{\infty} rc_r(0)$$

and $\lambda_{j,k}$, $\mu_{j,k}$ are as defined in the proof of Theorem 2.4.

Proof. Applying Lemma 3.1 to $x_m = \rho - \sum_{r=1}^{m-1} rc_r(t)$ gives $\dot{x}_m = \sum_Z jW_{j,k}$, where $Z = \{(j, k): 1 \leqslant j \leqslant m-1,\ j+k \geqslant m\}$. Since Z is the disjoint union of Z_2 and Z_3,

$$\dot{x}_m = \sum_{Z_2} jW_{j,k} + \sum_{Z_3} jW_{j,k}$$

Then (5.1) follows from the symmetry of Z_2. Using Lemma 3.2 and Proposition 3.3, we obtain a formula for the derivative of the second term in q_m, and (5.2) follows by combining this with the derivative of the first term in q_m. Finally,

$$\dot{x}_m = \sum_Z jW_{j,k} \leqslant Km \sum_Z (j+k) c_j c_k \leqslant 2Km\rho^2$$

from which we get (5.3). ▮

Theorem 5.4. Assume $a_{j,k} \leqslant K(j+k)$ for all j, k. Let G denote the set of all density-conserving solutions c of (1.1) on $[0, \infty)$. Then G is a generalized flow on the closed metric subspace X^+ of X.

Proof. By Theorem 5.1, if $c \in G$, then c: $[0, T) \to X^+$ is continuous. The semigroup property (i) follows from (2.1), while property (ii) follows from Theorem 2.4. To check property (iii), let $c^{(n)}$ be a sequence of solutions of (1.1) on $[0, \infty)$ with each $c^{(n)} \in G$ and $c^{(n)}(0) \to c_0$ in X as $n \to \infty$. We repeat the proofs of Theorem 2.4 and Corollary 2.6 with $c^{(n)}$ playing the role of the approximating solutions. Since the details are very similar, we only outline the changes required. Set

$$\vartheta_m^n(t) = e^{-t}[x_m^n(t) + 2Km(\rho^n)^2]$$

where ρ^n is the density of c^n and $x_m^n = \sum_{r=m}^{\infty} rc_r^n$. Lemma 5.3 ensures that we can apply Helly's theorem to ϑ_m^n. By using Lemma 5.3, we derive the analogue of (2.10):

$$x_m^n(t) = x_m^n(0) + q_m^n(t) - q_m^n(0) + \int_0^t \left[\sum_{Z_3} (j - \lambda_{j,k}) - \sum_{Z_1} \mu_{j,k} \right] W_{j,k}(c^n(s))\, ds$$

where

$$q_m^n = \sum_{r=m}^{2m} rc_r^n + 2m \sum_{r=2m+1}^{\infty} c_r^n$$

Control of $x_m^n(0)$ follows from the strong convergence of $c^n(0)$. Finally,

$$\sum_{Z_3} ja_{j,k} c_j^n c_k^n \leqslant K \sum_{j=1}^{m-1} jc_j^n \sum_{k=m}^{\infty} (j+k) c_k^n \leqslant 2K \|c^n\| x_m^n$$

with a similar estimate for the sum over Z_1. ▮

We also consider the continuous dependence of solutions with respect to weak* convergence in X. Recall that y^j converges in the weak* sense to $y \in X$ (symbolically $y_r^j \xrightarrow{*} y_r$) if

(i) $\sup_j \|y^j\| < \infty$.

(ii) $y_r^j \to y_r$ as $j \to \infty$ for each $r = 1, 2,...$.

For $\rho > 0$ let $B_\rho = \{y \in X: \|y\| \leqslant \rho\}$. Then (B_ρ, d) is a metric space with metric $d(y, z) = \sum_{r=1}^{\infty} |y_r - z_r|$. Clearly, a sequence $\{y^j\} \subset B_\rho$ converges in the weak* sense to $y \in X$ if and only if $y \in B_r$ and $d(y^j, y) \to 0$ as $j \to \infty$. For $\rho > 0$ set $B_\rho^+ = B_\rho \cap X^+$; then B_ρ^+ is a closed metric subspace of B_ρ.

Theorem 5.5. Let $g_j \geqslant 0$ with $g_j = o(j)$ as $j \to \infty$. Assume that

$$a_{j,k} \leqslant g_j + g_k, \qquad b_{j,k} \leqslant g_j g_k$$

for all j, k. For $\rho > 0$ let G_ρ denote the set of all solutions c of (1.1) on $[0, \infty)$ with $c(0) \in B_\rho^+$. Then G_ρ is a generalized flow on B_ρ^+.

The proof of the above theorem is a simple application of the Arzela–Ascoli theorem, so we omit it.

6. EXAMPLES

In this section we give a number of results which highlight the role of the assumptions on the kinetic coefficients. We begin with nonexistence results. Let $b_{i,j} = 0$ and $a_{i,j} = r_i + r_j$. A formal calculation in ref. 8 showed that solutions with time-dependent densities do not exist for these kernels (cf. Theorem 3.6 for a rigorous proof of this). For the case $r_j = j^\alpha$, $\alpha > 1$, it is formally argued in ref. 8 that solutions do not exist globally in time. We give two rigorous nonexistence results for kernels of this type. The first shows that for any initial data, the corresponding solution only exists for a finite time; the second shows that for a class of initial data, there are no solutions even on a short time interval.

Theorem 6.1. Let $b_{j,k} = 0$ and $a_{j,k} = r_j + r_k + \alpha_{j,k}$, where $0 \leqslant \alpha_{j,k} \leqslant K_1(j+k)$ and $r_j \geqslant 0$.

(i) Suppose $r_j \geqslant K_2 j^\alpha$, where $K_2 > 0$ and $\alpha > 1$. If c is a solution of (1.1) on $[0, T)$ with $c(0) \neq 0$, then $T < \infty$.

(ii) Suppose $j^{-1} r_j \to \infty$ as $j \to \infty$. Let $c_0 = (c_{0j}) \in X^+$ be such that $\exp(\gamma_m \delta) \sum_{j=m+1}^{\infty} j c_{0j}$ does not tend to zero as $m \to \infty$ for all $\delta > 0$, where $\gamma_m \stackrel{\text{def}}{=} \min_{j \geqslant m} j^{-1} r_j$. Then there is no solution c of (1.1), defined on any interval $[0, T)$, $T > 0$, and with initial data $c(0) = c_0$.

Proof. (i) Suppose for contradiction that c is a solution of (1.1) on $[0, \infty)$ with $c(0) \neq 0$. Set $h(t) = \sum_{j=1}^{\infty} c_j(t)$. By Lemma 3.3, Proposition 3.4, and (3.5), h is absolutely continuous and

$$h(t) - h(0) = -\int_0^t Q(s)\, ds \tag{6.1}$$

where

$$Q = \frac{1}{2} \sum_{j,k=1}^{\infty} a_{j,k} c_j c_k \geqslant \left(\sum_{j=1}^{\infty} r_j c_j \right) \left(\sum_{k=1}^{\infty} c_k \right) \tag{6.2}$$

By Hölder's inequality,

$$\left(\sum_{j=1}^{\infty} jc_j \right)^{\alpha} \left(\sum_{j=1}^{\infty} c_j \right)^{1-\alpha} \leqslant \sum_{j=1}^{\infty} j^{\alpha} c_j$$

Using this and the conservation of density (Theorem 3.6) in (6.2) gives $Q \geqslant K_3 h^{2-\alpha}$, where $K_3 > 0$ is a constant. Hence, from (6.1),

$$\dot{h}(t) \leqslant -K_3 h(t)^{2-\alpha} \qquad \text{a.e.} \quad t \in [0, \infty)$$

It follows easily from this that $h(t_0) = 0$ for some $t_0 > 0$, which contradicts the positiveness of the density.

(ii) Let c be a solution on $[0, T)$ with $c(0) = c_0$, where $T < \infty$. Set $u_m(t) = \sum_{j=m+1}^{\infty} jc_j(t)$ for $t \in [0, T)$. From Theorem 3.6 and Lemma 3.1

$$u_m(t) - u_m(0) = \int_0^t p_m(s)\, ds$$

where

$$p_m = \sum_{j=1}^{m} \sum_{k=m-j+1}^{\infty} ja_{j,k} c_j c_k \geqslant \sum_{j=1}^{m} jc_j \sum_{k=m+1}^{\infty} r_k c_k$$

By Theorem 5.1, there exists $M > 0$ such that $\sum_{j=1}^{m} jc_j(t) \geqslant \rho/2$ for all $m > M$, $t \in [0, T)$ where ρ is the density. Thus, for $m > M$, $p_m \geqslant (\rho/2)\gamma_m u_m$, so that

$$u_m(t) - u_m(0) \geqslant (\rho/2)\gamma_m \int_0^t u_m(s)\, ds$$

Hence $u_m(t) \geqslant \exp[(\rho/2)\gamma_m t]\, u_m(0)$, which contradicts $u_m(t) \to 0$ as $m \to \infty$. ∎

There are always $c_0 \in X^+$ satisfying condition (ii) of the above theorem (see ref. 2, p. 670).

Next we give an example which shows both that the density need not be conserved by all solutions of (1.1) and that in general we do not have continuous dependence of solutions with respect to weak* convergence for the set of density-conserving solutions.

Example 6.2. Let $a_{i,j}=0$, $b_{i,j}=(i+j)^\beta$, so that (1.1) takes the form

$$\dot{c}_j = \sum_{k=j+1}^{\infty} k^\beta c_k - \frac{1}{2} c_j (j-1) j^\beta \tag{6.3}$$

(a) For $\lambda > 0$, $\beta > -1$, define a sequence $y_j(\lambda)$ by $y_1(\lambda)=1$ and

$$\alpha_j y_j - \alpha_{j+1} y_{j+1} = (j+1)^\beta y_{j+1}, \qquad j \geqslant 1 \tag{6.4}$$

where $\alpha_j = \lambda + \frac{1}{2}(j-1) j^\beta$. Define $c_j(t) = e^{\lambda t} y_j(\lambda)$. Writing $y_j = j^{-(\beta+3)} z_j$, it is not hard to prove that $z_{j+1} = (1+\delta_j) z_j$, where $\delta_j = O(j^{-\gamma})$ and $\gamma = \min(3, 2+\beta)$. It follows that $0 \leqslant y_j(\lambda) \leqslant \text{const.}\, j^{-(\beta+3)}$ for all j. In particular, $(y_j(\lambda)) \in X^+$ and

$$\sum_{k=j+1}^{\infty} k^\beta c_k = \sum_{k=j}^{\infty} (\alpha_j c_j - \alpha_{j+1} c_{j+1}) = \alpha_j c_j$$

so that $c=(c_j)$ satisfies (6.3). Thus, for $\beta > -1$ we have a solution c as defined in Section 2 with density $Ke^{\lambda t}$, K a constant. If, on the other hand, $\beta \leqslant -1$, then from Theorem 3.6 we see that any solution of (6.3) with finite density must conserve density.

(b) Consider the special case $\beta = 0$, i.e., $a_{i,j}=0$, $b_{i,j}=1$, so that (6.3) becomes

$$\dot{c}_j = \sum_{k=j+1}^{\infty} c_k - \frac{1}{2}(j-1) c_j \tag{6.5}$$

For $r = 1, 2, \ldots$, let $c_0^r \in X^+$ be given by $c_0^r = (r^{-1} \delta_{r,j})$, so that $\|c_0^r\| = 1$ for all r and c_0^r converges in the weak* sense to the zero sequence as $r \to \infty$. The unique density-conserving solution $c^r(t)$ of (5.4) with initial data c_0^r is given by

$$c_j^r(t) = r^{-1}(e^{-t/2})^{j-1}[2(1-e^{-t/2}) + (1-e^{-t/2})^2 (r-j-1)] \qquad \text{if} \quad j<r$$

$$c_j^j(t) = j^{-1}(e^{-t/2})^{j-1}$$

$$c_j^r(t) = 0 \qquad \text{for} \quad j>r$$

Then, as $r \to \infty$, $c^r \xrightarrow{*} \bar{c} = (\bar{c}_j)$, where $\bar{c}_j(t) = (e^{-t/2})^{j-1}(1-e^{-t/2})^2$. It is easy to check that $\bar{c}$ is a solution of (6.5) with $\bar{c}(0) = 0$ and $\|\bar{c}(t)\| = 1$ for all $t > 0$.

By taking linear combinations of the solutions in Example 6.2, it is seen that solutions of (6.3) are nonunique for any initial data in the case $\beta > -1$. Clearly, these are nonphysical solutions. Theorems 2.9 and 4.2 show that if we define an admissible solution of (1.1) to be a solution of (1.1) which is a limit of the truncated system (2.1), then admissible solutions are unique and conserve density.

Example 6.3. If $a_{j,k} = j + k$, $b_{j,k} = 0$, then the conclusion of Theorem 5.5 is false. To see this, we note that by using Theorems 5.1 and 5.2, we can differentiate the relation in (3.5) to get

$$\dot{M}(t) = -\rho M(t) \tag{6.6}$$

where

$$M(t) = \sum_{j=1}^{\infty} c_j(t), \qquad \rho = \sum_{j=1}^{\infty} j c_j(t)$$

Also,

$$\dot{c}_1(t) = -[\rho + M(t)]\, c_1(t) \tag{6.7}$$

Solving (6.6)–(6.7) gives

$$c_1(t) = c_1(0) \exp\{\rho^{-1} M(0) [\exp(-\rho t) - 1] - \rho t\}$$

Therefore, if $c^r(0) \xrightarrow{*} c_0$ with

$$\lim_{r \to \infty} \sum_{j=1}^{\infty} j c_j^r(0) = \bar{\rho} > \rho = \sum_{j=1}^{\infty} j c_{0j}$$

and $c \neq 0$, then

$$\lim_{r \to \infty} \sum_{j=1}^{\infty} c_j^r(0) = \sum_{j=1}^{\infty} c_{0j} \stackrel{\text{def}}{=} \alpha$$

and

$$\lim_{r \to \infty} c_1^r(t) = c_{01} \exp\{(\bar{\rho})^{-1} \alpha [\exp(-\bar{\rho} t) - 1] - \bar{\rho} t\}$$

$$\not\equiv c_{01} \exp\{\rho^{-1} \alpha [\exp(-\rho t) - 1] - \rho t\}$$

For applications to phase transitions in a quenched binary alloy, one set of conditions suggested by O. Penrose on the coagulation and fragmentation rates is that $a_{j,k}=O(j^{1/3}+k^{1/3})$ and that $b_{j,k}=a_{j,k}Q_j Q_k(Q_{j+k})^{-1}$, where $Q_j \sim z_s^{-j}\exp(-\lambda j^{1/3})$ with λ, z_s positive constants. Note that in this case we may have $b_{j,k}\sim j^{1/3}$ for j large and k bounded, while for j and k large with $j-k$ bounded, $b_{j,k}$ is small. The physical motivation here is that surface area considerations show that it is unlikely that a large cluster of size $j+k$ will split into two large clusters of size j and k (and hence increase the surface energy by a large amount).

Proposition 6.4. (a) The hypotheses of Theorem 3.6 (conservation of density) are satisfied if $a_{j,k}\leqslant K(j+k)$, $b_{j,k}\leqslant K(j+k)\exp\{\lambda[(j+k)^p-j^p-k^p]\}$, where $K, \lambda>0$ and $0<p<1$.

(b) The hypotheses of Theorem 4.1 (uniqueness) are satisfied if $a_{j,k}\leqslant K(jk)^\alpha$, $b_{j,k}\leqslant K(j+k)^{1-\alpha}\exp\{\lambda[(j+k)^p-j^p-k^p]\}$, where $K, \lambda>0$, $0\leqslant\alpha\leqslant 1/2$, $0<p<1$.

Proof. We need only check the conditions on $b_{j,k}$. Fix B with $0<B<2-2^p$. Then it is easy to show that there exists r_0 such that for $1\leqslant j\leqslant h(r)$ and $r\geqslant r_0$,

$$r^p-(r-j)^p-j^p\leqslant -Bj^p \tag{6.8}$$

Thus, in case (a), for $r\geqslant r_0$ and $1\leqslant j\leqslant h(r)$, using (6.8), we have that $b_{r-j,j}\leqslant Kr\exp(-Bj^p)$, with a similar inequality for case (b). ∎

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