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THE DISCRETE TIME H_∞ CONTROL PROBLEM WITH MEASUREMENT FEEDBACK*

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Abstract. This paper is concerned with the discrete time H_∞ control problem with measurement feedback. It follows that, as in the continuous time case, the existence of an internally stabilizing controller that makes the H_∞ norm strictly less than 1 is related to the existence of stabilizing solutions to two algebraic Riccati equations. However, in the discrete time case, the solutions of these algebraic Riccati equations must satisfy extra conditions.

Key words. H_∞ control, discrete time, algebraic Riccati equation, measurement feedback.

AMS(MOS) subject classifications. 93C05, 93C35, 93C45, 93C55, 49B99, 49C05

1. Introduction. The H_∞ control problem with measurement feedback has been thoroughly investigated (see, e.g., [5], [6], [7], [13], [14], [21], [22], [24], [28]). However, all of these papers discuss the continuous time case. In this paper, in contrast with the above papers, we discuss the discrete time case.

In practical applications, most people are mainly concerned with discrete time systems. One major reason is that to control a continuous time system we often apply a digital computer on which we can only implement a discrete time controller. One possible approach is to derive a continuous time H_∞ controller and then to discretize the controller so that a computer may be used.

Discretizing the system first and then using H_∞ control designed for discrete time systems might be a more useful approach. This comparison can, only be made, however, after the discrete time H_∞ control problem has been solved. Taking the effects of discretization into account is another possibility, see [3], [4].

Also, certain systems are in themselves inherently discrete, and certainly for these systems it is useful to have results available for H_∞ control problems.

One approach is to apply a transformation in the frequency-domain that transforms discrete time systems to continuous time systems. The transformation we have in mind is discussed, for instance, in [8, App. 1]. With this transformation, discrete time H_∞ functions are mapped isometrically onto continuous time H_∞ functions. We can then use the results available for continuous time systems and afterward apply the inverse transformation on the controller thus obtained.

This transformation, however, is not always attractive. It maps systems with a pole in 1 into nonproper systems. Also this transformation is such that it clouds the understanding of specific features of discrete time H_∞ control because of this complex transformation. If it is possible to derive results for discrete time systems, why not apply these results directly instead of performing this unnatural transformation?

In the papers on H_∞ control with continuous time, several methods were used to solve the H_∞ control problem. Recently, a paper solving the discrete time H_∞ control problem using frequency domain techniques has appeared (see [12]). Also the polynomial approach has been applied to discrete time systems (see [11]). Derivation of the results for the discrete time H_∞ control problem could probably also be based on the work of [26]. In addition, several papers have appeared using a time-domain approach

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(see [1], [16], [27]). However, [16] does not contain any proof of the results obtained, and [1], [27] make a number of extra assumptions on the system under consideration. In [1], [16], [27] the authors first investigate the finite horizon problem and then derive a solution of the infinite horizon problem by considering it as a kind of limiting case as the endpoint tends to infinity.

In contrast, this paper directly investigates the infinite horizon case. We use time-domain techniques that have many familiarities with those used in [22], [24], which deal with the continuous time case. The method used in this paper was derived independently from [1], [16], [27]. The current paper is an extension of [23], which deals with the full-information case. However, contrary to the latter paper, here we give detailed proofs of all our results.

We assume that two particular transfer matrices are left- and right-invertible, respectively. The only other assumption we must make is that two subsystems have no invariant zeros on the unit circle. Our assumptions are exactly the discrete time analogues of the assumptions in [9]. The assumptions we make are weaker than the assumptions in [12], [27], and the same as the ones made in [16].

As in the continuous time case, the necessary and sufficient conditions for the existence of suitable controllers involve positive semidefinite stabilizing solutions of two algebraic Riccati equations. As in the continuous time case, the quadratic term in these algebraic Riccati equations is indefinite. However, compared to the continuous time case, the solutions of these equations must satisfy another assumption: matrices depending on these solutions should be positive definite.

The outline of this paper is as follows. In §2 we will formulate the problem and give our main results. In §3 we will derive the existence of a stabilizing solution of the first algebraic Riccati equation starting from the assumption that there exists an internally stabilizing feedback that makes the H_∞ norm of the closed loop system less than 1. In §4 we will show the existence of a stabilizing solution of the second algebraic Riccati equation and complete the proof that our conditions are necessary. This is done by transforming the original system into a new system with the property that a controller “works” for the new system if and only if it “works” for the original system. In §5 it is shown that our conditions are also sufficient. It follows that the system transformation of §4 repeated in a dual form exactly gives the desired results. We will end with some concluding remarks in §6.

2. Problem formulation and main results. By \mathcal{N} and \mathcal{R} we denote the natural numbers and the real numbers, respectively. Moreover, by σ we denote the shift $(\sigma x)(k) := x(k+1)$ for all $k \in \mathcal{N}$. At a certain stage, we also need a backward difference equation of the form $\sigma^{-1}x = Ax + Bu$. We define the solution x to be a mapping from $\mathcal{N} \cup \{-1\}$ to \mathcal{R}^n given by

$$\begin{cases} x|_{\mathcal{N}} = \sigma A(x|_{\mathcal{N}}) + \sigma Bu, \\ x(-1) = Ax(0) + Bu(0). \end{cases}$$

It will follow that extending this function from \mathcal{N} to $\mathcal{N} \cup \{-1\}$ is a useful definition.

We consider the following time-invariant system:

$$(2.1) \quad \Sigma : \begin{cases} \sigma x = Ax + Bu + Ew, \\ y = C_1x + \quad \quad + D_{12}w, \\ z = C_2x + D_{21}u + D_{22}w, \end{cases}$$

where for all $k \in \mathcal{N}$, $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^l$ is the measurement, $w(k) \in \mathcal{R}^9$ is the unknown disturbance, and $z(k) \in \mathcal{R}^p$ is the

output to be controlled. $A, B, E, C_1, C_2, D_{12}, D_{21}$, and D_{22} are matrices of appropriate dimension.

If we apply a dynamic feedback law $u = Fy$ to Σ , then the closed loop system with zero initial conditions defines a convolution operator $\Sigma_{cl,F}$ from w to y . We seek a feedback law $u = Fy$ that is internally stabilizing and that minimizes the ℓ_2 -induced operator norm of $\Sigma_{cl,F}$ over all internally stabilizing feedback laws. We will investigate dynamic feedback laws of the form

$$(2.2) \quad \Sigma_F : \begin{cases} \sigma p = Kp + Ly, \\ u = Mp + Ny. \end{cases}$$

We will say that the dynamic compensator Σ_F , given by (2.2), is internally stabilizing when applied to the system Σ , described by (2.1), if the following matrix is asymptotically stable:

$$(2.3) \quad \begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix};$$

i.e., all its eigenvalues lie in the open unit disc. Denote by G_F the closed loop transfer matrix. The ℓ_2 -induced operator norm of the convolution operator $\Sigma_{cl,F}$ is equal to the H_∞ norm of the transfer matrix G_F and is given by

$$\begin{aligned} \|G_F\|_\infty &:= \sup_{\theta \in [0, 2\pi]} \|G_F(e^{i\theta})\| \\ &= \sup_w \left\{ \frac{\|z\|_2}{\|w\|_2} \mid w \in \ell_2^l, w \neq 0 \right\}, \end{aligned}$$

where the ℓ_2 -norm is given by

$$\|p\|_2 := \left(\sum_{k=0}^{\infty} p^T(k)p(k) \right)^{1/2}$$

and where $\|\cdot\|$ denotes the largest singular value. We refer to this norm as the H_∞ norm of the closed loop system.

In this paper we will derive necessary and sufficient conditions for the existence of a dynamic compensator Σ_F that is internally stabilizing and which is such that the closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$. Furthermore, if a stabilizing Σ_F exists, which makes the H_∞ norm of the closed loop system less than 1, then we derive an explicit formula for one particular Σ_F satisfying these requirements.

By scaling the plant, we can thus, in principle, find the infimum of the H_∞ norm of the closed loop system over all stabilizing controllers. This will involve a search procedure.

In the formulation of our main result we will need the concept of invariant zero: z_0 is called an *invariant zero* of the system (A, B, C, D) if

$$\text{rank}_{\mathcal{R}} \begin{pmatrix} z_0 I - A & -B \\ C & D \end{pmatrix} < \text{rank}_{\mathcal{R}(z)} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix},$$

where $\text{rank}_{\mathcal{K}}$ denotes the rank as a matrix with entries in the field \mathcal{K} . By $\mathcal{R}(z)$ we denote the field of real rational functions. The system (A, B, C, D) is called *left- (right-) invertible* if the transfer matrix $C(zI - A)^{-1}B + D$ is left- (right-) invertible

as a matrix with entries in the field of real rational functions. We can now formulate our main result.

THEOREM 2.1. *Consider system (2.1). Assume that the system (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that the system (A, E, C_1, D_{12}) has no invariant zeros on the unit circle and is right invertible. The following statements are equivalent:*

- (i) *There exists a dynamic compensator Σ_F of the form (2.2) such that the resulting closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$ and the closed loop system is internally stable.*
- (ii) *There exist symmetric matrices $P \geq 0$ and $Y \geq 0$ such that*
- (a) *We have*

$$(2.4) \quad V > 0, \quad R > 0,$$

where

$$\begin{aligned} V &:= B^T P B + D_{21}^T D_{21}, \\ R &:= I - D_{22}^T D_{22} - E^T P E \\ &\quad + (E^T P B + D_{22}^T D_{21}) V^{-1} (B^T P E + D_{21}^T D_{22}). \end{aligned}$$

This implies that the matrix $G(P)$ is invertible, where

$$(2.5) \quad G(P) := \begin{pmatrix} D_{21}^T D_{21} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} - I \end{pmatrix} + \begin{pmatrix} B^T \\ E^T \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}.$$

- (b) *P satisfies the discrete algebraic Riccati equation*

$$(2.6) \quad P = A^T P A + C_2^T C_2 - \begin{pmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{pmatrix}^T G(P)^{-1} \begin{pmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{pmatrix}.$$

- (c) *The matrix $A_{cl,P}$ is asymptotically stable, where*

$$(2.7) \quad A_{cl,P} := A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{pmatrix}.$$

Moreover, if, given the matrix P satisfying (a)–(c), we define the following matrices:

$$\begin{aligned} H &:= E^T P A + D_{22}^T C_2 - [E^T P B + D_{22}^T D_{21}] V^{-1} [B^T P A + D_{21}^T C_2], \\ A_P &:= A + E R^{-1} H, \\ E_P &:= E R^{-1/2}, \\ C_{1,P} &:= C_1 + D_{12} R^{-1} H, \\ C_{2,P} &:= V^{-1/2} (B^T P A + D_{21}^T C_2) + V^{-1/2} [B^T P E + D_{21}^T D_{22}] R^{-1} H, \\ D_{12,P} &:= D_{12} R^{-1/2}, \\ D_{21,P} &:= V^{1/2}, \\ D_{22,P} &:= V^{-1/2} (B^T P E + D_{21}^T D_{22}) R^{-1/2}, \end{aligned}$$

then the matrix Y should satisfy

(d) *We have*

$$(2.8) \quad W > 0, \quad S > 0,$$

where

$$\begin{aligned} W &:= D_{12,P} D_{12,P}^T + C_{1,P} Y C_{1,P}^T, \\ S &:= I - D_{22,P} D_{22,P}^T - C_{2,P} Y C_{2,P}^T \\ &\quad + (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) W^{-1} (C_{1,P} Y C_{2,P}^T + D_{12,P} D_{22,P}^T). \end{aligned}$$

This implies that the matrix $H_P(Y)$ is invertible, where

$$(2.9) \quad H_P(Y) := \begin{pmatrix} D_{12,P} D_{12,P}^T & D_{12,P} D_{22,P}^T \\ D_{22,P} D_{12,P}^T & D_{22,P} D_{22,P}^T - I \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}^T.$$

(e) *Y satisfies the following discrete algebraic Riccati equation:*

$$(2.10) \quad Y = A_P Y A_P^T + E_P E_P^T - \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}^T H_P(Y)^{-1} \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}.$$

(f) *The matrix $A_{cl,P,Y}$ is asymptotically stable, where*

$$(2.11) \quad A_{cl,P,Y} := A_P - \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}^T H_P(Y)^{-1} \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}.$$

In the case where there exist $P \geq 0$ and $Y \geq 0$ satisfying (ii), then a controller of the form (2.2) satisfying the requirements in (i) is given by

$$\begin{aligned} N &:= -D_{21,P}^{-1} (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) W^{-1}, \\ M &:= -(D_{21,P}^{-1} C_{2,P} + N C_{1,P}), \\ L &:= B N + (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W^{-1}, \\ K &:= A_{cl,P} - L C_{1,P}. \end{aligned}$$

Remarks.

(i) Necessary and sufficient conditions for the existence of an internally stabilizing feedback compensator, which makes the H_∞ norm of the closed loop system less than some, a priori given, upper bound $\gamma > 0$, can be easily derived from Theorem 2.1 by scaling.

(ii) If we compare these conditions with the conditions for the continuous time case (see [6], [22]) we note that conditions (2.4) and (2.8) are this time depending on the solutions of the two Riccati equations. A simple example showing that the assumption $G(P)$ invertible is not sufficient is given by the system

$$\begin{cases} \sigma x &= & u &+ & 2w, \\ y &= & \begin{pmatrix} 1 \\ 0 \end{pmatrix} x &+ & \begin{pmatrix} 0 \\ 1 \end{pmatrix} w, \\ z &= & \begin{pmatrix} 1 \\ 0 \end{pmatrix} x &+ & \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \end{cases}$$

There does not exist a dynamic compensator satisfying the requirements of part (i) of Theorem 2.1, but there does exist a positive semidefinite matrix P satisfying (2.6) such that matrix (2.7) is asymptotically stable, namely $P = 1$. However, for this P we have $R = -1$. Therefore matrices like E_P are ill-defined and we cannot even look for a matrix Y satisfying (2.8)–(2.11).

(iii) Since our starting point of the proof of (i) \Rightarrow (ii) will not be part (i) of theorem 2.1 but Condition 3.2, it can be shown that we cannot make the H_∞ norm less by allowing more general, possibly even nonlinear, causal feedbacks.

The proof of the existence of a stabilizing solution of the Riccati equation will be reminiscent of the proof given in [24] for the continuous time case. However due to our weaker assumptions and conditions (2.4) and (2.8), there are quite a number of extra intricacies. The remainder of the proof is based on [22].

Another interesting case was discussed in [23]. However, the latter reference only gives the general outline of the proof. In contrast, the present paper will give much more detail. Reference [23] discusses the so-called full-information case, shown below.

Full information case. $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

In this case, we have $y_1 = x$ and $y_2 = w$; i.e., we know both the state and the disturbance of the system at time k . However, we cannot apply Theorem 2.1 to this case since system (A, E, C_1, D_{12}) is not right-invertible. Nevertheless, following the proof for this special case, it can be shown that there exists a feedback satisfying part (i) of Theorem 2.1 if and only if there exists a symmetric matrix $P \geq 0$ satisfying conditions (a)–(c) of part (ii) of Theorem 2.1. Moreover, in that case we can find static output feedbacks $u = F_1x + F_2w$ with the desired properties. One particular choice for $F = (F_1, F_2)$ is given by

$$(2.12) \quad F_1 := -(D_{21}^T D_{21} + B^T P B)^{-1} (B^T P A + D_{21}^T C_2),$$

$$(2.13) \quad F_2 := -(D_{21}^T D_{21} + B^T P B)^{-1} (B^T P E + D_{21}^T D_{22}).$$

3. Existence of stabilizing solutions of the Riccati equations. In this section we assume that part (i) of Theorem 2.1 is satisfied. We will show that the existence of P satisfying conditions parts (a)–(c) in (ii) is necessary.

Consider system (2.1). For given disturbance w and control input u let $x_{u,w,\xi}$ and $z_{u,w,\xi}$ denote, respectively, the resulting state and output for initial state $x(0) = \xi$. If $\xi = 0$ we will simply write $x_{u,w}$ and $z_{u,w}$. We first give a definition.

DEFINITION 3.1. An operator $f : \ell_2 \rightarrow \ell_2$, $w \rightarrow f(w)$ is called causal if for any $w_1, w_2 \in \ell_2$, and $k \in \mathcal{N}$:

$$w_1|_{[0,k]} = w_2|_{[0,k]} \Rightarrow f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}.$$

f is called strictly causal if for any $w_1, w_2 \in \ell_2$, and $k \in \mathcal{N}$ we have

$$w_1|_{[0,k-1]} = w_2|_{[0,k-1]} \Rightarrow f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}.$$

A controller of the form (2.2) always defines a causal operator. In the case where $N = 0$, this operator is strictly causal. We will label the following condition.

CONDITION 3.2. (A, B) stabilizable and for system (2.1) there exists causal $f : \ell_2^l \rightarrow \ell_2^m$ and $\delta < 1$ such that for all $w \in \ell_2^l$ with $u = f(w)$ we have $x_{u,w} \in \ell_2^n$ and $\|z_{u,w}\|_2 \leq \delta \|w\|_2$.

If there exists a dynamic compensator Σ_F such that $\|G_F\|_\infty < 1$ and such that the closed loop system is internally stable, then Condition 3.2 is satisfied. Hence if the

requirements of part (i) of Theorem 2.1 are satisfied, then Condition 3.2 holds. Note that Condition 3.2 is equivalent to the requirement that there exists a causal operator f such that the feedback $u = f(x, w)$ satisfies Condition 3.2. This follows from the fact that, after applying the feedback, there exists a causal operator g mapping w to x and, therefore, we could have started with the causal operator $u = f(g(w), w)$ in the first place. Conversely, if we have the feedback $u = f(w)$, then we define $f_1(x, w) := f(w)$, which then satisfies the requirements of the reformulated Condition 3.2.

Finally, we would like to remark that besides the obvious condition that (A, B) should be stabilizable, there is a more implicit extra condition $x_{u,w} \in \ell_2^n$, which is also related to stability. Intuitively, these two conditions imply that we cannot only find a controller that is input-output stabilizing (i.e., the closed loop transfer matrix is in H_∞) and that makes the H_∞ of the closed loop system less than 1, but even an internally stabilizing controller with the same property. This is only true for the full-information case (see [23]). For the more general measurement-feedback case, we need extra stability conditions related to detectability.

We will show that the existence of such causal f and $\delta < 1$ satisfying Condition 3.2 already implies that there exists a positive semidefinite solution of the discrete algebraic Riccati equation (2.6) such that (2.7) is asymptotically stable and (2.4) is satisfied. *We will assume for the time being that*

$$(3.1) \quad D_{21}^T [C_2 \quad D_{22}] = 0,$$

and we will derive the more general statement later. To prove the existence of the desired P , we will investigate the following sup-inf problem:

$$(3.2) \quad C^*(\xi) := \sup_{w \in \ell_2^m} \inf_u \{ \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2 \mid u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \}$$

for arbitrary initial state ξ . We will prove that Condition 3.2 implies that $C^*(\xi)$ is finite for every ξ . Moreover, it will be shown that there exists a $P \geq 0$ such that $C^*(\xi) = \xi^T P \xi$. At the end of this section, we then prove that this P exactly satisfies conditions (a)–(c) of Theorem 2.1. We first infimize, for given $w \in \ell_2$ and $\xi \in \mathcal{R}^n$, the function $\|z_{u,w,\xi}\|_2^2 - \|w\|_2^2$ over all $u \in \ell_2$ for which $x_{u,w,\xi} \in \ell_2$. After that, we maximize over $w \in \ell_2$.

Our proof is based on Pontryagin’s maximum principle. This only gives necessary conditions for optimality. However, in [15] a sufficient condition for optimality is derived over a finite horizon. We will use the ideas from [15], together with our stability requirement, $x_{u,w,\xi} \in \ell_2$, to adapt the proof to the infinite horizon case.

We start by constructing a solution of the adjoint Hamilton–Jacobi equation, which is a natural starting point if we use Pontryagin’s maximum principle.

Let L be such that $D_{21}^T D_{21} + B^T L B$ is invertible and such that L is the positive semidefinite solution of the following discrete algebraic Riccati equation:

$$(3.3) \quad L = A^T L A + C_2^T C_2 - A^T L B (D_{21}^T D_{21} + B^T L B)^{-1} B^T L A$$

for which

$$(3.4) \quad A_L := A - B (D_{21}^T D_{21} + B^T L B)^{-1} B^T L A$$

is asymptotically stable. The existence of such L is guaranteed under the assumption that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible and,

moreover, that (A, B) is stabilizable (see [20]). We define

$$(3.5) \quad r(k) := - \sum_{i=k}^{\infty} [X_1 A^T]^{i-k} X_1 (LEw(i) + C_2^T D_{22}w(i+1)),$$

where

$$(3.6) \quad X_1 := I - LB(D_{21}^T D_{21} + B^T LB)^{-1} B^T.$$

Note that r is well defined since the matrix $A_L = X_1^T A$ is asymptotically stable, which implies that $X_1 A^T$ is asymptotically stable. Next, we define the functions y, \tilde{x} , and η by

$$(3.7) \quad y := M^{-1} B^T [A^T \sigma r - LEw - C_2^T D_{22} \sigma w],$$

$$(3.8) \quad \sigma \tilde{x} = A_L \tilde{x} + B y + E w, \quad \tilde{x}(0) = \xi,$$

$$(3.9) \quad \eta := -X_1 L A \tilde{x} + r,$$

where $M := D_{21}^T D_{21} + B^T L B$. Since $X_1 A^T$ is asymptotically stable, it can be checked straightforwardly that, given $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$, we have $r, \tilde{x}, \eta \in \ell_2$.

After some standard calculations, we find the following lemma.

LEMMA 3.3. *Let $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$ be given. The function $\eta \in \ell_2^n$ is a solution of the following backward difference equation:*

$$(3.10) \quad \sigma^{-1} \eta = A^T \eta - C_2^T C_2 \tilde{x} - C_2^T D_{22} w, \quad \lim_{k \rightarrow \infty} \eta(k) = 0.$$

Here η is extended to a function from $\mathcal{N} \cup \{-1\}$ to \mathcal{R}^n by choosing $\eta(-1)$ such that (3.10) is satisfied.

In the statement of Pontryagin's maximum principle, this equation is the so-called "adjoint Hamilton-Jacobi equation," and η is called the "adjoint state variable." We have constructed a solution to this equation and we show that this η indeed yields a minimizing u . Note the difference with the continuous time case (see [24]), where we could derive a differential equation forward in time, while in discrete time we can only derive a difference equation forward in time when A is invertible. To prevent these kinds of difficulties, it is assumed in [12] that A is invertible. The following lemma states that η yields a minimizing u .

LEMMA 3.4. *Let system (2.1) be given. Moreover, let w and ξ be fixed. Then*

$$\begin{aligned} \tilde{u} &:= -(D_{21}^T D_{21} + B^T L B)^{-1} B^T L A \tilde{x} + y \\ &= \arg \inf_u \{ \|z_{u,w,\xi}\|_2 \mid u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \}. \end{aligned}$$

Proof. It can be easily checked that $\tilde{x} = x_{\tilde{u},w,\xi}$. Define

$$\mathcal{J}_T(u) := \sum_{i=0}^T \|C_2 x_{u,w,\xi}(i) + D_{21} u(i) + D_{22} w(i)\|^2.$$

Let $u \in \ell_2^m$ be an arbitrary control input such that $x_{u,w,\xi} \in \ell_2^n$. We find that

$$\begin{aligned} &\mathcal{J}_T(u) - \mathcal{J}_{T-1}(u) - 2\eta^T(T)x(T+1) + 2\eta^T(T-1)x(T) \\ &\|C_2 x(T)\|^2 + [D_{21}^T D_{21} u(T) - 2B^T \eta(T)]^T u(T) - 2\eta^T(T)Ew(T) - 2x^T(T)C_2^T C_2 \tilde{x}(T). \end{aligned}$$

We also find that

$$\begin{aligned} \mathcal{J}_T(\tilde{u}) - \mathcal{J}_{T-1}(\tilde{u}) - 2\eta^\top(T)\tilde{x}(T+1) + 2\eta^\top(T-1)\tilde{x}(T) \\ = -\|C_2\tilde{x}(T)\|^2 + [D_{21}^\top D_{21}\tilde{u}(T) - 2B^\top\eta(T)]^\top \tilde{u}(T) - 2\eta^\top(T)Ew(T). \end{aligned}$$

Hence if we sum the last two equations from zero to infinity and subtract from each other we find that

$$\begin{aligned} \|z_{\tilde{u},w,\xi}\|_2^2 - \|z_{u,w,\xi}\|_2^2 &= \sum_{i=0}^{\infty} -\|C_2(x(i) - \tilde{x}(i))\|^2 \\ &+ \sum_{i=0}^{\infty} [D_{21}^\top D_{21}\tilde{u}(i) - 2B^\top\eta(i)]^\top \tilde{u}(i) - [D_{21}^\top D_{21}u(i) - 2B^\top\eta(i)]^\top u(i). \end{aligned}$$

It can easily be checked that $B^\top\eta(i) = D_{21}^\top D_{21}\tilde{u}(i)$ for all i . Therefore, for every i we have

$$[D_{21}^\top D_{21}\tilde{u}(i) - 2B^\top\eta(i)]^\top \tilde{u}(i) = \inf_u [D_{21}^\top D_{21}u - 2B^\top\eta(i)]^\top u.$$

Together, the last two equations imply that $\|z_{\tilde{u},w,\xi}\|_2 \leq \|z_{u,w,\xi}\|_2$, which is exactly what we had to prove. Since (A, B, C_2, D_{21}) is left-invertible, it can easily be shown that the minimizing u is unique. \square

We now maximize over $w \in \ell_2^l$. This will then yield $\mathcal{C}^*(\xi)$. Define $\mathcal{F}(\xi, w) := (\tilde{x}, \tilde{u}, \eta)$ and $\mathcal{G}(\xi, w) := z_{\tilde{u},w,\xi} = C_2\tilde{x} + D_{21}\tilde{u} + D_{22}w$. It is clear from the previous lemma that \mathcal{F} and \mathcal{G} are bounded linear operators. Define

$$\begin{aligned} \mathcal{C}(\xi, w) &:= \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2, \\ \|w\|_C &:= (-\mathcal{C}(0, w))^{1/2}. \end{aligned}$$

It can be easily shown that $\|\cdot\|_C$ defines a norm on ℓ_2^l . Using Condition 3.2, it can be shown straightforwardly that

$$(3.11) \quad \|w\|_2 \geq \|w\|_C \geq \rho\|w\|_2,$$

where $\rho > 0$ is such that $\rho^2 = 1 - \delta^2$ and δ is such that Condition 3.2 is satisfied. Hence $\|\cdot\|_C$ and $\|\cdot\|_2$ are equivalent norms.

Note that Lemma 3.4 still holds if Condition 3.2 does not hold. However, the result that $\|\cdot\|_C$ is a norm and that even $\|\cdot\|_C$ and $\|\cdot\|_2$ are equivalent norms is the essential property, which is implied by Condition 3.2 and which is the key to our derivation.

We have

$$(3.12) \quad \mathcal{C}^*(\xi) = \sup_{w \in \ell_2^l} \mathcal{C}(\xi, w).$$

We can derive the following properties of \mathcal{C}^* .

LEMMA 3.5.

(i) For all $\xi \in \mathcal{R}^n$ we have

$$(3.13) \quad 0 \leq \xi^\top L\xi \leq \mathcal{C}^*(\xi) \leq \frac{\xi^\top L\xi}{1 - \delta^2},$$

where δ is such that Condition 3.2 is satisfied.

(ii) For all $\xi \in \mathcal{R}^n$ there exists an unique $w_* \in \ell_2^l$ such that $\mathcal{C}^*(\xi) = \mathcal{C}(\xi, w_*)$.

Proof. Part (i). It is well known that L , as the stabilizing solution of the discrete time algebraic Riccati equation (3.3), is the cost of the discrete time, linear quadratic problem with internal stability (see [20]). Hence $\|\mathcal{G}(\xi, 0)\|_2^2 = \mathcal{C}(\xi, 0) = \xi^T L \xi$. Therefore we have $0 \leq \xi^T L \xi \leq \mathcal{C}^*(\xi)$. Moreover,

$$\begin{aligned} \mathcal{C}(\xi, w) &= \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2 \\ &\leq (\|\mathcal{G}(\xi, 0)\|_2 + \|\mathcal{G}(0, w)\|_2)^2 - \|w\|_2^2 \\ &\leq \left(\sqrt{\xi^T L \xi} + \delta \|w\|_2 \right)^2 - \|w\|_2^2 \\ &\leq \frac{\xi^T L \xi}{1 - \delta^2}. \end{aligned}$$

Part (ii) can be proved in the same way as in [24]. First, show that $\|\cdot\|_C$ satisfies

$$(3.14) \quad -\|w_\alpha - w_\beta\|_C^2 = 2\mathcal{C}(\xi, w_\alpha) + 2\mathcal{C}(\xi, w_\beta) - 4\mathcal{C}\left(\xi, \frac{1}{2}(w_\alpha + w_\beta)\right)$$

for arbitrary $\xi \in \mathcal{R}^n$. Then it can be shown that a maximizing sequence of $\mathcal{C}(\xi, w)$ is a Cauchy sequence with respect to the $\|\cdot\|_C$ -norm and hence, since $\|\cdot\|_C$ and $\|\cdot\|_2$ are equivalent norms, there exists a maximizing ℓ_2 function w_* . It is easy to show uniqueness using (3.14). \square

Define $\mathcal{H} : \mathcal{R}^n \rightarrow \ell_2^l$, $\xi \rightarrow w_*$. Unlike the explicit expression for \tilde{u} we can only derive an implicit formula for w_* . However, we show with the following lemma that w_* is the unique solution of a linear equation.

LEMMA 3.6. *Let $\xi \in \mathcal{R}^n$ be given. Then $w_* = \mathcal{H}\xi$ is the unique ℓ_2 -function w satisfying:*

$$(3.15) \quad (I - D_{22}^T D_{22})w = -E^T \eta + D_{22}^T C_2 x,$$

where $(x, u, \eta) = \mathcal{F}(\xi, w)$.

Proof. Define $(x_*, u_*, \eta_*) = \mathcal{F}(\xi, w_*)$. Moreover, define $w_0 := -E^T \eta(w_*) + D_{22}^T D_{22} w_* + D_{22}^T C_2 x_*$ and $(x_0, u_0, \eta_0) := \mathcal{F}(\xi, w_0)$. We find that

$$(3.16) \quad \begin{aligned} &\|z_{u_0, w_0, \xi}(T)\|^2 - \|w_0(T)\|^2 - 2\eta_*^T(T)x_0(T+1) + 2\eta_*^T(T-1)x_0(T) \\ &= \|z_{u_*, w_*, \xi}(T) - z_{u_0, w_0, \xi}(T)\|^2 - \|z_{u_*, w_*, \xi}(T)\|^2 + \|w_0(T)\|^2 \end{aligned}$$

Here we use the fact that $D_{21}^T D_{21} u_*(i) = B^T \eta_*(i)$ for all i . We also find that

$$(3.17) \quad \begin{aligned} &\|z_{u_*, w_*, \xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^T(T)x_*(T+1) + 2\eta_*^T(T-1)x_*(T) \\ &= 2w_0^T(T)w_*(T) - \|z_{u_*, w_*, \xi}(T)\|^2 - \|w_*(T)\|^2 \end{aligned}$$

Summing (3.16) and (3.17) from zero to infinity and subtracting from each other gives us

$$(3.18) \quad \mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_0) - \|w_0 - w_*\|_2^2 - \|z_{u_0, w_0, \xi} - z_{u_*, w_*, \xi}\|_2^2.$$

Since w_* maximizes $\mathcal{C}(\xi, w)$ over all w , this implies $w_0 = w_*$.

That w_* is the unique solution of the equation (3.15) can be shown in a similar way. Assume that, apart from w_* , w_1 also satisfies (3.15). Let $(x_1, u_1, \eta_1) := \mathcal{F}(\xi, w_1)$. We find from (3.17) that

$$(3.19) \quad \begin{aligned} &\|z_{u_*, w_*, \xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^T(T)x_*(T+1) + 2\eta_*^T(T-1)x_*(T) \\ &= \|w_*(T)\|^2 - \|z_{u_*, w_*, \xi}(T)\|^2 \end{aligned}$$

We also find that

$$(3.20) \quad \begin{aligned} & \|z_{u_1, w_1, \xi}(T)\|^2 - \|w_1(T)\|^2 - 2\eta_*^T(T)x_1(T+1) + 2\eta_*^T(T-1)x_1(T) \\ &= \|z_{u_1, w_1, \xi}(T)\|^2 - \|w_1(T)\|^2 + 2w_*^T(T)w_1(T) - 2z_{u_*, w_*, \xi}^T(T)z_{u_1, w_1, \xi}(T). \end{aligned}$$

Summing (3.19) and (3.20) from 0 to ∞ and subtracting from each other gives us

$$(3.21) \quad \mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_1) - \|w_* - w_1\|_C^2.$$

Since w_* was maximizing, we find that $\|w_* - w_1\|_C = 0$ and hence $w_* = w_1$. \square

Next, we show that $\mathcal{C}^*(\xi) = \xi^T P \xi$ for some matrix P . To do that we first show that u_* , η_* , and w_* are linear functions of x_* in the result below.

LEMMA 3.7. *There exist constant matrices K_1 , K_2 , and K_3 such that*

$$(3.22) \quad u_* = K_1 x_*,$$

$$(3.23) \quad \eta_* = K_2 x_*,$$

$$(3.24) \quad w_* = K_3 x_*.$$

Proof. First we look at time 0. By Lemma 3.6 it is easily seen that $\mathcal{H} : \xi \rightarrow w_*$ is linear. Hence the mapping from ξ to $w_*(0)$ is also linear. This implies the existence of a matrix K_3 such that $w_*(0) = K_3 \xi$. From (3.10) and Lemma 3.4, it is easily seen that u_* and η_* are linear functions of ξ and w_* . This implies, since w_* is a linear function of ξ , that $u_*(0)$ and $\eta_*(0)$ are linear functions of ξ , and hence there exist K_1 and K_2 such that $u_*(0) = K_1 \xi$ and $\eta_*(0) = K_2 \xi$.

We now look at time t . The sup-inf problem starting at time t with initial value $x(t)$ can now be solved. Due to time invariance, we see that w_* restricted to $[t, \infty)$ satisfies (3.15), and hence for this problem the optimal x and η are x_* and η_* . However, since t is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find (3.22)–(3.24) at time t with the same matrices K_1 , K_2 , and K_3 as at time 0. Since t was arbitrary this completes the proof. \square

LEMMA 3.8. *There exists a matrix P such that $\sigma^{-1}\eta_* = -P x_*$. Moreover, for this P we find that*

$$(3.25) \quad \mathcal{C}^*(\xi) = \xi^T P \xi.$$

Proof. We have

$$\begin{aligned} \sigma^{-1}\eta_* &= [A^T \eta_* - C_2^T C_2 x_* - C_2^T D_{22} w_*] \\ &= (A^T K_2 - C_2^T C_2 - C_2^T D_{22} K_3) x_*. \end{aligned}$$

We define $P := -(A^T K_2 - C_2^T C_2 - C_2^T D_{22} K_3)$ using the matrices defined in lemma 3.7. We prove that this P satisfies (3.25). We can derive the following equation:

$$\begin{aligned} & \|z_{u_*, w_*, \xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^T(T)x_*(T+1) + 2\eta_*^T(T-1)x_*(T) \\ &= \|w_*(T)\|^2 - \|z_{u_*, w_*, \xi}(T)\|^2 \end{aligned}$$

We sum this equation from zero to infinity. Since $\lim_{T \rightarrow \infty} \eta_*(T) = 0$ and $\lim_{T \rightarrow \infty} x_*(T) = 0$, we find that

$$\mathcal{C}(\xi, w_*) + 2\eta_*^T(-1)x_*(0) = -\mathcal{C}(\xi, w_*).$$

Since $\mathcal{C}(\xi, w_\star) = \mathcal{C}^\star(\xi)$ and $\eta_\star(-1) = -P\xi$, we find (3.25). \square

Next, we show that this matrix P satisfies conditions (a)–(c) of theorem 2.1. We first show part (a). Since we do not yet know if P is symmetric, we must be careful. This essential step in our derivation is new compared to the method for the continuous time as used in [24].

LEMMA 3.9. *Let P be given by Lemma 3.8. The matrices V and R as defined in part (ii) of theorem 2.1, condition (a), satisfy $V + V^\top > 0$, $R + R^\top > 0$.*

Proof. By Lemmas 3.5 and 3.8, we know that $(P + P^\top)/2 \geq L$, and therefore we find that $(V + V^\top)/2 \geq D_{21}^\top D_{21} + B^\top L B$. The latter matrix is positive definite, and hence $(V + V^\top)/2$ is positive definite, i.e., $V + V^\top > 0$.

We now look at the following “sup-inf-sup-inf” problem for initial condition 0:

$$(3.26) \quad \mathcal{J}(0) := \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u,w}\|_2^2 - \|w\|_2^2,$$

where $w^+ := w|_{[1,\infty)}$ and $u^+ := u|_{[1,\infty)}$. We will always add the constraint that u^+ is such that the resulting state x is in ℓ_2 .

We know that there exists a causal operator f satisfying Condition 3.2, and hence this function makes the ℓ_2 -induced operator norm strictly less than 1 under the constraint $x \in \ell_2^n$. In (3.26) we set $u = f(w)$. This is possible since by causality we know that $u(0)$ only depends on $w(0)$ and u^+ depends on the whole function w . Thus we get

$$(3.27) \quad \begin{aligned} \mathcal{J}(0) &= \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u,w}\|_2^2 - \|w\|_2^2 \\ &\leq \sup_w \|z_{f(w),w}\|_2^2 - \|w\|_2^2 \\ &\leq 0. \end{aligned}$$

Since, by Lemma 3.8, we have

$$(3.28) \quad \sup_{w^+} \inf_{u^+} \|z_{u^+,w^+,x(1)}\|_2^2 - \|w^+\|_2^2 = x^\top(1)P x(1),$$

we can reduce (3.26) to the following “sup-inf” problem:

$$\sup_{w(0)} \inf_{u(0)} \begin{pmatrix} u(0) \\ w(0) \end{pmatrix}^\top \begin{pmatrix} V & B^\top P E + D_{21}^\top D_{22} \\ E^\top P B + D_{22}^\top D_{21} & E^\top P E + D_{22}^\top D_{22} - I \end{pmatrix} \begin{pmatrix} u(0) \\ w(0) \end{pmatrix}.$$

When we define

$$\tilde{u}(0) = u(0) - (E^\top P B + D_{22}^\top D_{21}) V^{-1} w(0),$$

we get

$$(3.29) \quad \mathcal{J}(0) = \sup_{w(0)} \inf_{\tilde{u}(0)} \begin{pmatrix} \tilde{u}(0) \\ w(0) \end{pmatrix}^\top \begin{pmatrix} V & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} \tilde{u}(0) \\ w(0) \end{pmatrix}.$$

Since, by (3.27), $\mathcal{J}(0)$ is finite we immediately find that a necessary condition is $R + R^\top \geq 0$.

Assume that $R + R^\top$ is not invertible. Then there exists a $v \neq 0$ such that $v^\top R v = 0$. Let $w^+(u(0))$ be the ℓ_2 -function that attains the optimum in the optimization (3.28) with initial state $x(1) = B u(0) + E v$. We define the function w by

$$(3.30) \quad [w(u(0))](t) := \begin{cases} v & \text{if } t = 0, \\ w^+(u(0))(t) & \text{otherwise.} \end{cases}$$

Assume that δ and f are such that Condition 3.2 is satisfied. Define u by

$$(3.31) \quad u = f[w(u(0))].$$

Since the map from u to w defined by (3.30) is strictly causal and since f is causal, u is uniquely defined by (3.31). To prove this, note that $u(0)$ only depends on $w(u(0))(0) = v$, and hence w^+ as a function of $u(0)$ is uniquely defined, which in turn yields u . Denote u and w obtained in this way by u_1 and w_1 . By (3.27) and (3.29), we find that, for this particular choice of w_1 and u_1 , we have

$$(3.32) \quad \|z_{u_1, w_1}\|_2^2 - \|w_1\|_2^2 \geq 0.$$

On the other hand, using Condition 3.2 we find that

$$\|z_{u_1, w_1}\|_2^2 - \|w_1\|_2^2 < (\delta^2 - 1) \|w_1\|_2^2 < 0$$

since $w_1(0) = v \neq 0$. Therefore we have a contradiction, and hence our assumption that $R + R^T$ is not invertible was incorrect. Together with $R + R^T \geq 0$, this yields $R + R^T > 0$. \square

LEMMA 3.10. *Assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that $D_{21}^T[C_2 \ D_{22}] = 0$. If the condition in part (i) of Theorem 2.1 is satisfied, then there exists a symmetric matrix $P \geq 0$ satisfying (a)–(c) of part (ii) of Theorem 2.1.*

Proof. We define the matrices

$$M := D_{21}^T D_{21} + B^T L B > 0, \\ Z := I - D_{22}^T D_{22} - E^T X_1 L E.$$

We know that $-(R + R^T)/2$ is the Schur complement of $(V + V^T)/2$ in $G((P + P^T)/2)$. By Lemma 3.9, we know that $R + R^T > 0$ and $V + V^T > 0$. Therefore $G((P + P^T)/2)$ has m eigenvalues on the positive real axis and l eigenvalues on the negative real axis. We know that $G((P + P^T)/2) - G(L) \geq 0$ since $(P + P^T)/2 \geq L$. An easy consequence of the theorem of Courant–Fischer (see [2]) then tells us that $G(L)$ has at least l eigenvalues on the negative real axis. Since $-Z$ is the Schur complement of $M > 0$ in $G(L)$, this implies that $Z > 0$.

By Lemma 3.8, we have $\eta_* = -\sigma P x_*$. By combining Lemmas 3.4 and 3.6 and rewriting the equations, we find that u_* and w_* satisfy the following equations:

$$w_* = Z^{-1} \{E^T X_1 (P - L) \sigma x_* + (D_{22}^T C_2 + E^T X_1 L A) x_*\}, \\ u_* = -M^{-1} B^T \{(P - L) \sigma x_* + L A x_* + L E w_*\}.$$

Thus we get

$$(3.33) \quad \{I + [B M^{-1} B^T - X_1^T E Z^{-1} E^T X_1] (P - L)\} x_*(k + 1) \\ = X_1^T \{A + E Z^{-1} E^T X_1 L A + E Z^{-1} D_{22}^T C_2\} x_*(k)$$

Since, by Lemma 3.9, R as defined in Theorem 2.1 is invertible, it can be shown that the matrix on the left is invertible, and hence (3.33) uniquely defines $x_*(k + 1)$ as a function of $x_*(k)$. It follows that (3.33) can be rewritten in the form $\sigma x_* = A_{cl} x_*$, with A_{cl} as defined by (2.7). Since $x_* \in \ell_2^n$ for every initial state ξ , we know that A_{cl} is asymptotically stable. Next, we show that P satisfies the discrete algebraic

Riccati equation (2.6). From the backward difference equation in (3.10) combined with Lemma 3.8 and the formula given above for w_* , we find that

$$P = A^T P A_{cl} + C_2^T C_2 + C_2^T D_{22} Z^{-1} \{E^T X_1 (P - L) A_{cl} + D_{22}^T C_2 + E^T X_1 L A\}$$

By some extensive calculations this equation turns out to be equivalent to the discrete algebraic Riccati equation (2.6). Next we show that P is symmetric. Note that both P and P^T satisfy the discrete algebraic Riccati equation. Using this we find that $(P - P^T) = A_{cl}^T (P - P^T) A_{cl}$. Since A_{cl} is asymptotically stable, this implies that $P = P^T$. P can be shown to be positive semidefinite by combining Lemma 3.5 and (3.25). It remains to be shown that P satisfies (2.4). Since P is symmetric, we know that V and R are symmetric. Equation (2.4) is then an immediate consequence of lemma 3.9. \square

We extend this result in the following corollary to systems that do not satisfy (3.1).

COROLLARY 3.11. *Assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left invertible. If part (i) of Theorem 2.1 is satisfied, then there exists a symmetric matrix $P \geq 0$ satisfying (a)–(c) of part (ii) of Theorem 2.1.*

Proof. We first apply a preliminary feedback $u = \tilde{F}_1 x + \tilde{F}_2 w + v$ such that $D_{21}^T (C_2 + D_{21} \tilde{F}_1) = 0$ and $D_{21}^T (D_{22} + D_{21} \tilde{F}_2) = 0$. Denote the new A, C_2, D_{22} , and E by $\tilde{A}, \tilde{C}_2, \tilde{D}_{22}$, and \tilde{E} . For this new system, Condition 3.2 is satisfied. We also know that by applying a preliminary state feedback, the invariant zeros of a system do not change. Therefore the new subsystem $(\tilde{A}, B, \tilde{C}_2, D_{21})$ does not have invariant zeros on the imaginary axis. Hence, since for this new system $D_{21}^T [\tilde{C}_2 \quad \tilde{D}_{22}] = 0$, we find conditions in terms of the new parameters by applying Lemma 3.10. Rewriting in terms of the original parameters gives the desired conditions (a)–(c) as given in part (ii) of Theorem 2.1. \square

4. A first system transformation. To proceed with the proof of Theorem 2.1, (i) \Rightarrow (ii), in this section we will transform our original system (2.1) into a new system. The problem of finding an internally stabilizing feedback that makes the H_∞ norm of the closed loop system less than 1 for the original system is equivalent to the problem of finding an internally stabilizing feedback that makes the H_∞ norm of the closed loop system less than 1 for the new transformed system. However, this new system has some very desirable properties, which makes working with it much easier. In particular, for this new system the disturbance decoupling problem with measurement feedback is solvable. We will perform the transformation in two steps.

First we will perform a transformation related to the full-information H_∞ problem, and next a transformation related to the filtering problem.

We assume that we have a positive semidefinite matrix P satisfying conditions (a)–(c) of Theorem 2.1. By the results of the previous section, this matrix exists in the case where part (i) of Theorem 2.1 is satisfied. We define the following system:

$$(4.1) \quad \Sigma_P : \begin{cases} \sigma x_P = A_P x_P + B u_P + E_P w_P, \\ y_P = C_{1,P} x_P + \quad \quad \quad + D_{12,P} w_P, \\ z_P = C_{2,P} x_P + D_{21,P} u_P + D_{22,P} w_P, \end{cases}$$

where the matrices are as defined in the statement of Theorem 2.1. Furthermore, we define the following system:

$$(4.2) \quad \Sigma_U : \begin{cases} \sigma x_U = A_U x_U + B_U u_U + E_U w, \\ y_U = C_{1,U} x_U + \quad \quad \quad + D_{12,U} w, \\ z_U = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w, \end{cases}$$

where

$$\begin{aligned}
 A_U &:= A - BV^{-1}(B^T P A + D_{21}^T C_2), \\
 B_U &:= BV^{-1/2}, \\
 E_U &:= E - BV^{-1}(B^T P E + D_{21}^T D_{22}), \\
 C_{1,U} &:= -R^{-1/2} H, \\
 C_{2,U} &:= C_2 - D_{21} V^{-1}(B^T P A + D_{21}^T C_2), \\
 D_{12,U} &:= R^{1/2}, \\
 D_{21,U} &:= D_{21} V^{-1/2}, \\
 D_{22,U} &:= D_{22} - D_{21} V^{-1}(B^T P E + D_{21}^T D_{22}),
 \end{aligned}$$

and V , R , and H are as defined in Theorem 2.1. We will show that Σ_U has a very nice property. To do this, we will first give a definition and some results we will need in the sequel. A system is called *inner* if the system is internally stable, square (i.e., the number of inputs is equal to the number of outputs) and the transfer matrix of the system, denoted by G , satisfies

$$(4.3) \quad G(z)G^T(z^{-1}) = I$$

We will now formulate a generalization of [12, Lemma 5] to the case where $G(z)$ may have poles in zero. The proof is slightly more complicated than the one given in [12], since if G has a pole in zero then $G^T(z^{-1})$ is no longer proper. Nevertheless, a proof can be given by simply writing out (4.3).

LEMMA 4.1. *Assume that we have a square system*

$$(4.4) \quad \Sigma_{st} : \begin{cases} \sigma x = Ax + Bu, \\ z = Cx + Du. \end{cases}$$

Assume that A is asymptotically stable. The system Σ_{st} is inner if there exists a matrix X satisfying

1. $X = A^T X A + C^T C$,
2. $D^T C + B^T X A = 0$,
3. $D^T D + B^T X B = I$.

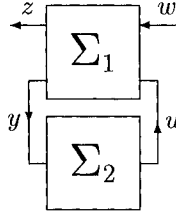
Remarks.

(i) If (A, B) is controllable, the reverse of the above implication is also true. However, in general, the reverse does not hold. A simple counterexample is given by $\Sigma_{st} := (0.5, 0, 1, 1)$, which is inner but for which (ii) does not hold for any choice of X .

(ii) Note that if a matrix X satisfies part (1) of Lemma 4.1, then it is equal to the observability gramian of (C, A) . We know, for instance, that $X > 0$ if and only if (C, A) is observable. In general, we only have $X \geq 0$.

We have the following important property of inner systems (see [17], [22]).

LEMMA 4.2. *Suppose that we have the following interconnection of two systems Σ_1 and Σ_2 , both described by some state-space representation:*



(4.5)

Assume that Σ_1 is inner. Denote its transfer matrix from (w, u) to (z, y) by L . Moreover, assume that if we decompose L compatible with the sizes of $w, u, z,$ and y :

$$(4.6) \quad L \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix},$$

we have $L_{21}^{-1} \in H_\infty$, and L_{22} is strictly proper. Then the following two statements are equivalent:

(i) The closed loop system (4.5) is internally stable and its closed loop transfer matrix has H_∞ norm less than 1.

(ii) The system Σ_2 is internally stable and its transfer matrix has H_∞ norm less than 1.

LEMMA 4.3. The system Σ_U as defined by (4.2) is inner. Denote the transfer matrix of Σ_U by U . We decompose U compatible with the sizes of $w, u_U, z_U,$ and y_U :

$$U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}.$$

Then U_{21} is invertible and its inverse is in H_∞ . Moreover, U_{22} is strictly proper.

Proof. It can be easily checked that P as defined by Theorem 2.1 (a)–(c) satisfies conditions (i)–(iii) of Lemma 4.1. Part (i) of Lemma 4.1 turns out to be equal to the discrete algebraic Riccati equation (2.6). Parts (ii) and (iii) follow by simply writing out the equations in terms of the original system parameters of system (2.1).

Next, we show that A_U is asymptotically stable. We know $P \geq 0$ and

$$(4.7) \quad P = A_U^T P A_U + \begin{pmatrix} C_{1,U}^T & C_{2,U}^T \end{pmatrix} \begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix}.$$

It can be easily checked that $x \neq 0, A_U x = \lambda x, C_{1,U} x = 0,$ and $C_{2,U} x = 0$ imply that $A_{cl,P} x = \lambda x$, where $A_{cl,P}$ is defined by (2.7). Since $A_{cl,P}$ is asymptotically stable, we have $\text{Re } \lambda < 0$. Hence the realization (4.2) is detectable. By standard Lyapunov theory, the existence of a positive semidefinite solution of (4.7), together with detectability, guarantees asymptotic stability of A_U .

We can immediately write down the following realization for U_{21}^{-1} :

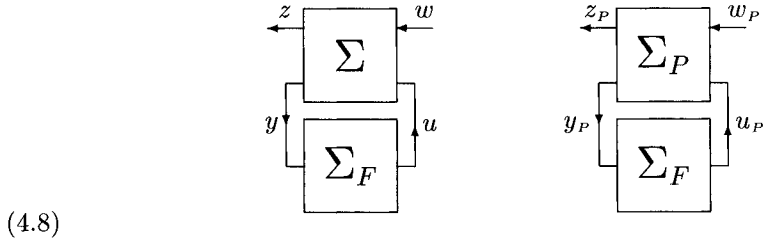
$$\Sigma_{U_{21}^{-1}} : \begin{cases} \sigma x_U = (A_U - E_U D_{12,U}^{-1} C_{1,U}) x_U + E_U D_{12,U}^{-1} w, \\ y_U = -D_{12,U}^{-1} C_{1,U} x_U + D_{12,U}^{-1} w. \end{cases}$$

Since $A_{cl,P} = A_U - E_U D_{12,U}^{-1} C_{1,U}$ we know that U_{21}^{-1} is an H_∞ function.

Finally, the claim that U_{22} is strictly proper is trivial to check. This completes the proof. \square

We will now formulate our key lemma, below.

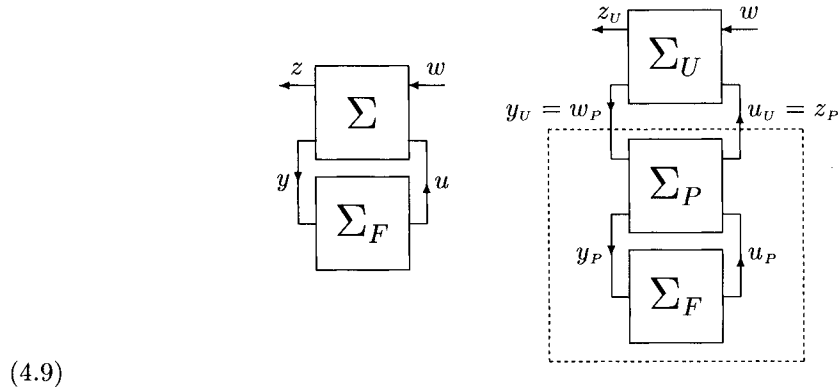
LEMMA 4.4. *Let P satisfy Theorem 2.1 part (ii), (a)–(c). Moreover, let Σ_F be an arbitrary linear time-invariant finite-dimensional compensator in the form (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2), and the system on the right is the interconnection of (4.1) and (2.2):*



Then the following statements are equivalent

- (i) The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.
- (ii) The system on the right is internally stable and its transfer matrix from w_P to z_P has H_∞ norm less than 1.

Proof. We investigate the following systems:



The system on the left is the same as the system on the left in (4.8), and the system on the right is described by system (4.2) interconnected with the system on the right in (4.8). A realization for the system on the right is given by

$$\sigma \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} = \begin{pmatrix} A_{cl,P} & 0 & 0 \\ * & A + BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E + BND_{12} \\ LD_{12} \end{pmatrix} w,$$

$$z_U = \begin{pmatrix} * & C_2 + D_{21}NC_1 & D_{21}M \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + (D_{22} + D_{21}ND_{12}) w$$

where $A_{cl,P}$ is defined by (2.7). The *'s denote matrices that are unimportant for this argument. The system on the right is internally stable if and only if the system

described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (4.9), we immediately see that, since $A_{cl,P}$ is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input w then we have $z = z_U$; i.e., the input-output behaviour of both systems are equivalent. Hence the system on the left has H_∞ norm less than 1 if and only if the system on the right has H_∞ norm less than 1.

By Lemma 4.3, we may apply Lemma 4.2 to the system on the right in (4.9), and hence we find that the closed loop system is internally stable and has H_∞ norm less than 1 if and only if the dashed system is internally stable and has H_∞ norm less than 1.

Since the dashed system is exactly the system on the right in (4.8) and the system on the left in (4.9) is exactly equal to the system on the left in (4.8), we have completed the proof. \square

Using the previous lemma, we know that we only have to investigate the system Σ_P . This new system has some very nice properties, which we will use. First, we will look at the Riccati equation for the system Σ_P . It can be checked immediately that $X = 0$ satisfies (a)-(c) of Theorem 2.1 for the system Σ_P .

We now dualize Σ_P . We know that (A, E, C_1, D_{12}) is right-invertible and has no invariant zeros on the unit circle. It can be easily checked that this implies that $(A_P, E, C_{1,P}, D_{12})$ is right-invertible and has no invariant zeros on the unit circle. Hence for the dual of Σ_P we know that $(A_P^T, C_{1,P}^T, E^T, D_{21}^T)$ is left-invertible and has no invariant zeros on the unit circle. If there exists an internally stabilizing feedback for the system Σ , which makes the H_∞ norm of the closed loop system less than 1, then the same feedback is internally stabilizing and makes the H_∞ norm of the closed loop system less than 1 for the system Σ_P . If we dualize this feedback and apply it to the dual of Σ_P , then it is again internally stabilizing and again it makes the H_∞ norm of the closed loop system less than 1. We can now apply the dual version of Corollary 3.11, which exactly guarantees the existence of a matrix Y satisfying conditions (d)-(f) of Theorem 2.1. Thus we derived the following lemma, which gives the necessity part of Theorem 2.1.

LEMMA 4.5. *Let system (2.1) be given with zero initial state. Assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that (A, E, C_1, D_{12}) has no invariant zeros on the unit circle and is right invertible. If part (i) of theorem 2.1 is satisfied, then there exist matrices P and Y satisfying (a)-(f) of part (ii) of Theorem 2.1.*

This completes the proof (i) \Rightarrow (ii). In the next section we will prove the reverse implication. Moreover, in the case where the desired feedback exists, we will derive an explicit formula for one choice for Σ_F that satisfies all requirements.

5. The transformation into a disturbance decoupling problem with measurement feedback. In this section we will assume that there exist matrices P and Y satisfying part (ii) of Theorem 2.1 for system (2.1). We will transform our original system Σ into another system $\Sigma_{P,Y}$. We will show that a compensator is internally stabilizing and makes the H_∞ norm of the closed loop system less than 1 for the system Σ if and only if the same compensator is internally stabilizing and makes the H_∞ norm of the closed loop system less than 1 for our transformed system $\Sigma_{P,Y}$. After that, we will show that $\Sigma_{P,Y}$ has the following very special property (see [19]):

There exists an internally stabilizing compensator that makes the closed loop transfer matrix equal to zero; i.e., w does not have any effect on the output of the system z . This property of $\Sigma_{P,Y}$ has a special name: “the disturbance decoupling problem with measurement feedback and internal stability (DDPMS).”

We first define $\Sigma_{P,Y}$. We start by transforming Σ into Σ_P . Then we apply the dual transformation on Σ_P to obtain $\Sigma_{P,Y}$:

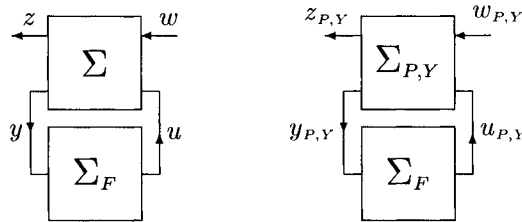
$$(5.1) \quad \Sigma_{P,Y} : \begin{cases} \sigma x_{P,Y} = A_{P,Y} x_{P,Y} + B_{P,Y} u_{P,Y} + E_{P,Y} w_{P,Y}, \\ y_{P,Y} = C_{1,P} x_{P,Y} + D_{12,P} w_{P,Y}, \\ z_{P,Y} = C_{2,P} x_{P,Y} + D_{21,P} u_{P,Y} + D_{22,P} w_{P,Y}, \end{cases}$$

where

$$\begin{aligned} \tilde{H} &:= A_P Y C_{2,P}^T + E_P D_{22,P}^T - (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W^{-1} \times \\ &\quad (C_{1,P} Y C_{2,P}^T + D_{12,P} D_{22,P}^T), \\ A_{P,Y} &:= A_P + \tilde{H} S^{-1} C_{2,P}, \\ C_{2,P,Y} &:= S^{-1/2} C_{2,P}, \\ B_{P,Y} &:= B + \tilde{H} S^{-1} D_{21,P}, \\ E_{P,Y} &:= (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W^{-1/2} + \tilde{H} S^{-1} (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) W^{-1/2}, \\ D_{12,P,Y} &:= W^{1/2}, \\ D_{21,P,Y} &:= S^{-1/2} D_{21,P}, \\ D_{22,P,Y} &:= S^{-1/2} (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) W^{-1/2}. \end{aligned}$$

When we first apply Lemma 4.4 on the transformation from Σ to Σ_P and then the dual of Lemma 4.4 on the transformation from Σ_P to $\Sigma_{P,Y}$, we find the following result.

LEMMA 5.1. *Let P satisfy Theorem 2.1, part (ii) (a)–(c). Moreover, let an arbitrary linear time-invariant finite-dimensional compensator Σ_F be given, described by (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2), and the system on the right is the interconnection of (5.1) and (2.2):*



Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ has H_∞ norm less than 1.

It remains to be shown that for $\Sigma_{P,Y}$ the DDPMS is solvable.

LEMMA 5.2. Let Σ_F be given by

$$(5.2) \quad \Sigma_F : \begin{cases} \sigma p &= K_{P,Y}p + L_{P,Y}y_{P,Y}, \\ u_{P,Y} &= M_{P,Y}p + N_{P,Y}y_{P,Y}, \end{cases}$$

where

$$\begin{aligned} N_{P,Y} &:= -D_{21,P,Y}^{-1} D_{22,P,Y} D_{12,P,Y}^{-1}, \\ M_{P,Y} &:= -\left(D_{21,P,Y}^{-1} C_{2,P,Y} + N_{P,Y} C_{1,P} \right), \\ L_{P,Y} &:= B_{P,Y} N_{P,Y} + E_{P,Y} D_{12,P,Y}^{-1}, \\ K_{P,Y} &:= A_{P,Y} + B_{P,Y} M_{P,Y} - E_{P,Y} D_{12,P,Y}^{-1} C_{1,P}. \end{aligned}$$

The interconnection of Σ_F and $\Sigma_{P,Y}$ is internally stable, and the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero.

Proof. We can write out the formulas for a state-space representation of the interconnection of $\Sigma_{P,Y}$ and Σ_F . We then apply the following basis transformation:

$$\begin{pmatrix} x_{P,Y} - p \\ p \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{P,Y} \\ p \end{pmatrix}.$$

After this transformation we immediately see that the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. Moreover, the system matrix (2.3) after this transformation is given by:

$$\begin{pmatrix} A_{cl,P,Y} & 0 \\ L_{P,Y} C_{1,P} & A_{cl,P} \end{pmatrix}$$

Since $A_{cl,P,Y}$ and $A_{cl,P}$ are asymptotically stable matrices, this implies that, indeed, Σ_F is internally stabilizing. \square

This controller is the same as the one described in the statement of Theorem 2.1. We know that Σ_F is internally stabilizing, and the resulting closed loop system has H_∞ norm less than 1 for the system $\Sigma_{P,Y}$. Hence, by applying Lemma 5.1, we find that Σ_F satisfies part (i) of Theorem 2.1. This completes the proof of (ii) \Rightarrow (i) of Theorem 2.1. We have already shown the reverse implication and hence the proof of Theorem 2.1 is complete.

6. Conclusions. In this paper we have solved the discrete time H_∞ problem with measurement feedback. It is shown that the techniques for the continuous time case can be applied to the discrete time case. Unfortunately, the formulas are much more complex, but it is still possible to give an explicit formula for one controller satisfying all requirements. It would, however, be interesting to generalize this work and find a characterization of all controllers satisfying the requirements. Another interesting open problem is to derive recursive formulas to calculate the solutions to these algebraic Riccati equations. It would also be interesting to find two dual Riccati equations and a coupling condition, as in [9]. Nevertheless, the results presented in this paper show that it is very possible to solve discrete time H_∞ problems directly, instead of transforming them to continuous time. The assumption of left-invertibility is not very restrictive. It implies that there are several inputs that have the same effect on the output and this nonuniqueness can be factored out (see [18] for a continuous time treatment). The assumption of right-invertibility can be removed by dualizing this reasoning. However, at this moment it is unclear as to how to remove the assumptions concerning zeros on the unit circle.

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