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## The Discrete Voronoi game in $\mathbb{R}^2$ ;<sup>2</sup>

### Abstract

In this paper we study the last round of the discrete Voronoi game in  $\mathbb{R}^2$ , a problem which is also of independent interest in competitive facility location. The game consists of two players P1 and P2, and a finite set  $U$  of users in the plane. The players have already placed two disjoint sets of facilities  $F$  and  $S$ , respectively, in the plane. The game begins with P1 placing a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs. In this paper we propose polynomial time algorithms for determining the optimal strategies of both the players for arbitrarily located existing facilities  $F$  and  $S$ . We show that in the  $L_1$  and the  $L_\infty$  metrics, the optimal strategy of P2, given any placement of P1, can be found in  $O(n \log n)$  time, and the optimal strategy of P1 can be found in  $O(n^5 \log n)$  time. In the  $L_2$  metric, the optimal strategies of P2 and P1 can be obtained in  $O(n^2)$  and  $O(n^8)$  times, respectively.

### Keywords

competitive facility location, geometric depth, Voronoi diagram

### Disciplines

Business | Business Analytics | Statistics and Probability

# The Discrete Voronoi Game in $\mathbb{R}^{2*}$

Aritra Banik<sup>†</sup>    Bhaswar B. Bhattacharya<sup>‡</sup>    Sandip Das<sup>§</sup>    Satyaki Mukherjee<sup>¶</sup>

## Abstract

In this paper we study the last round of the discrete Voronoi game in  $\mathbb{R}^2$ , a problem which is also of independent interest in competitive facility location. The game consists of two players P1 and P2, and a finite set  $U$  of users in the plane. The players have already placed two disjoint sets of facilities  $F$  and  $S$ , respectively, in the plane. The game begins with P1 placing a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs. In this paper we propose polynomial time algorithms for determining the optimal strategies of both the players for arbitrarily located existing facilities  $F$  and  $S$ . We show that in the  $L_1$  and the  $L_\infty$  metrics, the optimal strategy of P2, given any placement of P1, can be found in  $O(n \log n)$  time, and the optimal strategy of P1 can be found in  $O(n^5 \log n)$  time. In the  $L_2$  metric, the optimal strategies of P2 and P1 can be obtained in  $O(n^2)$  and  $O(n^8)$  times, respectively.

## 1 Introduction

The main objective in any facility location problem is to place a set of facilities serving a set of users such that certain optimality criteria are satisfied. Facilities and users are generally modeled as points in the plane. The set of users (demands) is either *discrete*, consisting of finitely many points, or *continuous*, that is, a region where every point is considered to be a user. We assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. Consequently, each facility has its *service zone*, consisting of the set of users that are served by it. (Refer to the book by Drezner and Hamacher [14] for a comprehensive discussion on facility location problems and their manifold many generalizations.)

Competitive facility location is concerned with the favorable placement of facilities by competing market players [16, 17]. In general, the users choose the facilities based on the nearest-neighbor rule, and the optimization criteria is to maximize the cardinality or the area of the service zone depending on whether the demand region is discrete or continuous, respectively. For a recent survey on the applications of competitive facility location in economics and operations research, refer to [13].

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In this paper, we study the *discrete Voronoi game in  $\mathbb{R}^2$  in the presence of existing facilities*, which is the game-theoretic variant of a competitive facility location problem for discrete demands in  $\mathbb{R}^2$ . The game consists of two players P1 and P2, and a finite set  $U$  of users in the plane. The players have already placed two sets of facilities  $F$  and  $S$ , respectively, in the plane. To begin with, P1 places a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs, where the payoff of P1/P2 is the cardinality of the set of points in  $U$  which are closer to a facility owned by P1/P2 than to every facility owned by P2/P1. Apart from being the  $(m + 1)$ -th round of discrete Voronoi game in  $\mathbb{R}^2$ , when  $|F| = |S| = m$ , this problem is also of independent interest in competitive facility location: Imagine two competing companies are providing service to a set of users in a city. Suppose both these companies already have their respective service centers located in different parts of the city. If each of them now wishes to open a new service center while attempting to maximize its total payoff, then the problem is an instance of the Voronoi game described above.

In this paper, we develop algorithms and provide geometric characterizations for the optimal strategies of the two players for the discrete Voronoi game in  $\mathbb{R}^2$  in the presence of existing facilities.

**1.1 Related Work** The rich history of competitive facility location problems goes back to the 1929 seminal paper by Hotelling [20] that considers the competitive facility location problem where the users are located uniformly on a line segment. Dehne et al. [12] studied a competitive facility location problem for continuous demand regions, where the problem is to find a new point  $q$  amidst a set of  $n$  existing points  $\mathcal{F}$  such that the Voronoi region of  $q$  is maximized. They showed that when the points in  $\mathcal{F}$  are in convex position, the area function has only a single local maximum inside the region where the set of Voronoi neighbors do not change. For the same problem, Cheong et al. [10] gave a near-linear time algorithm that determines the location of the new optimal point approximately, when the points in  $\mathcal{F}$  are in general position. A variation of this problem, involving maximization of the area of Voronoi regions of a set of points placed inside a circle, was considered by Bhattacharya [7]. In the discrete user case, the analogous problem is to place a set of new facilities amidst a set of existing ones such that the number of users served by the new facilities is maximized [8, 9].

A game-theoretic analogue of such competitive problems for continuous demand regions is a situation where two players alternately place two disjoint set of facilities in the demand region. In this case, the payoff of player P1/P2 is the area of the region that is closer to the facilities owned by P1/P2 than to the other player, and the player which finally owns the larger area is the winner of the game. Ahn et al. [1] studied a one-dimensional Voronoi game, where the demand region is a line segment. They showed that when the players place one facility each for  $m$  rounds, the second player always has a winning strategy that guarantees a payoff of  $1/2 + \varepsilon$ , with  $\varepsilon > 0$ . However, the first player can force  $\varepsilon$  to be arbitrarily small. On the other hand, in the one-round game, where the players alternately place  $m$  facilities simultaneously, the first player always has a winning strategy. The one-round Voronoi game in  $\mathbb{R}^2$  was studied by Cheong et al. [11], for a square-shaped demand region. They proved that for any placement  $W$  of the first player, with  $|W| = m$ , there is a placement  $B$  of the second player  $|B| = m$  such that the payoff of the second player is at least  $1/2 + \alpha$ , where  $\alpha > 0$  is an absolute constant and  $m$  large enough. Fekete and Meijer [18] studied the two-dimensional one-round game played on a rectangular demand region with aspect ratio  $\rho$ . Recently, variants of these games when the demand region is a graph equipped with the shortest-path distance [3], and the demand region is a simple polygon equipped with the

geodesic distance [6], have been studied.

A natural variant of this game can be played on a graph equipped with the shortest-path distance. As before, the players alternately chose nodes (facilities) from the graph, and all vertices (customers) are then assigned to their closest facilities based on the graph distance. The payoff of a player is the number of customers assigned to it. Dürr and Thang [15] showed that deciding the existence of a Nash equilibrium for a given graph is NP-hard. Teramoto et al. [25] studied the same problem in a restricted case: the game arena is an arbitrary graph, the first player occupies just one vertex which is predetermined, and the second player occupies  $m$  vertices in any way. They proved that even in this strongly case it is NP-hard to decide whether the second player has a winning strategy. They also proved that for a given graph  $G$  and the number  $r$  of rounds, determining whether the first player has a winning strategy on  $G$  is PSPACE-complete. Recently, Gerbner et al. [19] derived bounds on the payoff of the players for many graphs, and showed that there are graphs for which the second player gets almost all vertices.

Banik et al. [4] introduced the discrete Voronoi game in  $\mathbb{R}$ , where the demand region is a finite set of points on the line, and users avail the services of facilities closest to them, in Euclidean distance. Given a set of users  $U \subset \mathbb{R}$ , with  $|U| = n$ , player Player 1 (P1) chooses a set of  $m$  facilities, following which Player 2 (P2) chooses another disjoint set of  $m$  facilities, and the objective of both the players is to maximize their respective payoffs. The authors showed that if the sorted order of the points in  $U$  along the line is known, then the optimal strategy of P2, given any placement of facilities by P1, can be computed in  $O(n)$  time, and the optimal strategy of P1 can be computed in  $O(n^{m-\lambda_m})$  time, where  $0 < \lambda_m < 1$ , is a constant depending only on  $m$ . Recently, using connections to  $\epsilon$ -nets, Banik et al. [5] obtained approximation algorithms for a version of the discrete Voronoi game in  $\mathbb{R}^2$ .

**1.2 Summary of Results** In this paper we initiate the study of the discrete Voronoi game when the users are a finite set of points in  $\mathbb{R}^2$ , equipped with the  $L_1, L_2$ , or  $L_\infty$  metrics. To this end, for a finite set  $U$  of users and a set  $\mathcal{F}$  of facilities, define for every  $f \in \mathcal{F}$ ,

$$U(f, \mathcal{F}) = \{u_a \in U : d(u_a, f) < d(u_a, h), \forall h \in \mathcal{F} \setminus \{f\}\}, \quad (1.1)$$

where the distance  $d(\cdot, \cdot)$  is measured in the  $L_1, L_2$ , or  $L_\infty$  metrics (to be denoted by  $d_1(\cdot, \cdot)$ ,  $d_2(\cdot, \cdot)$ , and  $d_\infty(\cdot, \cdot)$ , respectively).

Now, consider a set  $U$  of users in the plane and two players P1 and P2. Throughout the paper, we assume that two facilities are not allowed to be placed in the same location. For any placement of facilities  $A$  and  $B$  by P1 and P2, respectively, the payoff of P2, to be denoted by  $\mathcal{P}_2(A, B)$ , is defined as the cardinality of the set of points in  $U$  which are closer to a facility owned by P2 than to every facility owned by P1, that is,  $\mathcal{P}_2(A, B) = |\bigcup_{f \in B} U(f, A \cup B)|$ . Similarly, the payoff of P1,  $\mathcal{P}_1(A, B) = |U| - \mathcal{P}_2(A, B)$ . Note that this definition implies that if an user is equidistant from a facility in  $A$  and another facility in  $B$ , then it contributes to the payoff of P1, that is, ties are broken in favor of P1.

The problem studied in this paper can now be formally stated as follows:

**One Round Discrete Voronoi Game in  $\mathbb{R}^2$  in Presence of Existing Facilities:** Let  $U$  be a set of  $n$  users in the plane, and  $F$  and  $S$  be two sets of facilities owned by two competing players P1 and P2, respectively. To begin with, P1 chooses a facility  $f_1$  following which P2 chooses another facility  $f_2$  such that

- (a)  $\max_{f'_2 \in \mathbb{R}^2} \mathcal{P}_2(F \cup \{f_1\}, S \cup \{f'_2\})$  is attained at the point  $f_2$ .  
(b)  $\max_{f \in \mathbb{R}^2} \nu(f)$  is attained at the point  $f_1$ , where

$$\nu(f) = n - \max_{f'_2 \in \mathbb{R}^2} \mathcal{P}_2(F \cup \{f\}, S \cup \{f'_2\}). \quad (1.2)$$

The quantity  $\nu(f_1)$  is called the optimal payoff of P1 and  $f_1$  is the optimal strategy of P1. Hereafter, we shall refer to this game as  $G_n(F, S)$ .<sup>1</sup>

When  $|F| = |S| = m$ , the optimal strategies of the game  $G_n(F, S)$  is the last round of the  $(m + 1)$ -round discrete Voronoi game in  $\mathbb{R}^2$ . To the best of our knowledge, the  $G_n(F, S)$  game has never been studied before in the generality described above. However, few special cases are known. For example, when both  $F$  and  $S$  are empty, then it is a well-known fact that optimal strategy of P1 in the  $G_n(F, S)$  game is at the halfspace median of  $U$  [22], which can be computed in  $O(n \log^3 n)$  time [21]. However, when the sets  $F$  and  $S$  are non-empty the problem becomes much more complicated, and answering questions regarding the strategy of P1 is often very difficult. In this paper, we initiate the study of this game, for general placements of the existing facilities  $F$  and  $S$ , and propose polynomial time algorithms for the optimal strategies of both the players and provide geometric characterizations of the solution space.

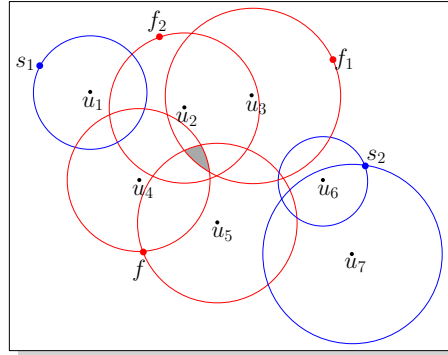


Figure 1: Optimal strategy of P2 in the  $L_2$  metric: The users served by  $F \cup \{f\}$  correspond to the centers of the red circles (the users  $u_2, u_3, u_4$ , and  $u_5$ ). The centers of the blue circles (the users  $u_1, u_6$ , and  $u_7$ ) correspond to the users served by  $S$ . The optimal strategy of P2 is to place a new facility in the region intersected by the maximum number of red circles (the shaded region).

We begin with the optimal strategy of P2. It is easy to see that the optimal strategy of P2, given any placement of P1, follows from the results of Cabello et al. [9]. Suppose we are given a set of users  $U$ , existing facilities  $F$  and  $S$ , and any placement of a new facility  $f$  by P1. Let  $U_1 \subseteq U$  denote the subset of users that are served by P1, in presence of  $F$ ,  $S$ , and  $f$ . For every point  $u \in U_1$ , consider the *nearest-facility disk*  $C_u$  centered at  $u$  and passing through the facility in  $F \cup \{f\}$  which is closest to  $u$ . Note that a new facility  $s$  placed by P2 will serve any user  $u \in U_1$  if and only if  $s \in C_u$ . If  $\mathcal{C} = \{C_u | u \in U_1\}$ , the optimal strategy for P2, given any placement  $f$  of P1, is to place the new facility at a point where maximum number of disks in  $\mathcal{C}$  overlap (Figure 1).

<sup>1</sup>A related problem was considered by Plastria [24], where the goal was to place a new facility among a set of existing friendly facilities  $F$  and competitive facilities  $C$ , with the objective to maximize the total payoff of the friendly facilities, while taking into account the possibility that the competing facilities can raise their current quality in order to gain back users.

Therefore, in the  $L_2$  metric, this is the problem of finding the maximum depth in an arrangement of  $n$  disks, and can be computed in  $O(n^2)$  time [2]. In the  $L_1/L_\infty$  metric this becomes the problem of finding the maximum depth in an arrangement of squares, which can be done in  $O(n \log n)$  time [2].

To obtain the optimal strategy of P1 each cell in the arrangement of the nearest-facility squares/discs have to be partitioned further into finer cells inside which the payoff of P1 remains fixed. To this end, we have the following theorems:

**Theorem 1.1.** *In the  $L_1$  and  $L_\infty$  metrics the optimal strategy of P1 in the  $G_n(F, S)$  game can be found in  $O(n^5 \log n)$  time.*

**Theorem 1.2.** *In the  $L_2$  metric the optimal strategy of P1 in the  $G_n(F, S)$  game can be found in  $O(n^8)$  time.*

The above theorems achieve more than just the optimal strategy of P1. In fact, our algorithm computes the locus of all points which attains the maximum payoff of P1. More generally, it computes the level sets  $\mathcal{L}(r) = \{f \in \mathbb{R}^2 : \nu(f) \geq r\}$ , with  $\nu(f)$  as in (1.2). Note that if  $f_1$  is an optimal location of P1, then  $\mathcal{L}(\nu(f_1))$  is the set of all points which maximizes the payoff of P1.

## 2 Preliminaries

Let  $U = \{u_1, u_2, \dots, u_n\}$  be a set of  $n$  users in the plane and  $F$  and  $S$  be the sets of existing facilities of two competing players P1 and P2, respectively. The set of facilities  $F$  and  $S$ , will divide the set of users  $U$  into two groups  $U_F$  and  $U_S$ , where  $U_F$  is the set of users served by the facilities placed by P1 and  $U_S$  is the set of users served by the facilities placed by P2. Let  $f$  be any new placement by P1. Denote by  $U_{FS}(f)$ , the set of users that are served by  $f$ , that is,

$$U_{FS}(f) = \{u_a \in U : d(u_a, f) \leq d(u_a, h), \forall h \in F \cup S\}, \quad (2.1)$$

where the distance  $d(\cdot, \cdot)$  is measured in the  $L_1, L_2$ , or  $L_\infty$  metrics. The set of users that are served by the set of facilities  $F$  and  $S$  after the placement of  $f$ , will be denoted by  $U_{F \setminus f}$  and  $U_{S \setminus f}$  respectively. More formally,

$$U_{F \setminus f} = \bigcup_{h \in F} U(h, F \cup S \cup \{f\}) \text{ and } U_{S \setminus f} = \bigcup_{h \in S} U(h, F \cup S \cup \{f\}). \quad (2.2)$$

Hence, any facility  $f$  by P1 will divide the set of users into three disjoint sets  $U_{FS}(f)$ ,  $U_{F \setminus f}$  and  $U_{S \setminus f}$  (see Figure 2(a)). Now, any new placement  $s$  by P2 can serve a subset of users from all these three sets. Let  $U_f(s) \subseteq U_{FS}(f)$  be the subset of users that  $s$  steals from  $f$ , that is,

$$U_f(s) = \{u_a \in U_{FS}(f) | d(u_a, s) < d(u_a, f)\}.$$

Similarly, define the set of users

$$U_{F \setminus f}(s) = \{u_a \in U_{F \setminus f} | d(u_a, s) < d(u_a, f_k), \forall f_k \in F\} \quad (2.3)$$

(see Figure 2(b)). This is the subset of users that  $s$  steals from  $F \setminus \{f\}$ .

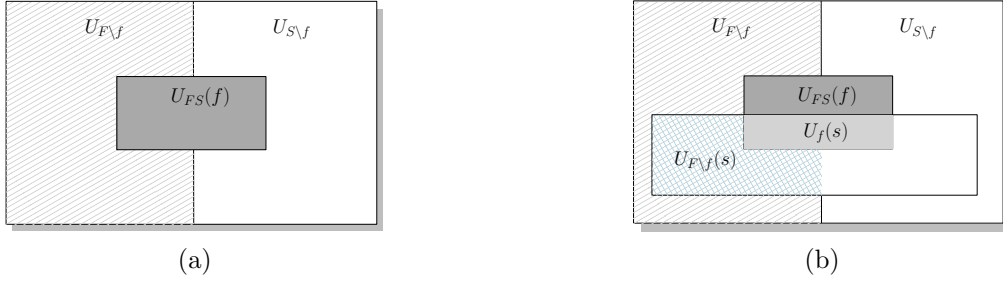


Figure 2: Distribution of users among facilities in  $F$  and  $S$ : (a) after placing  $f$ , (b) after placing both  $f$  and  $s$ .

Observe for any placement  $f$  and  $s$  by P1 and P2 respectively, the payoff of P2 is

$$\mathcal{P}_2(F \cup \{f\}, S \cup \{s\}) = |U_{S \setminus f}| + |U_f(s)| + |U_{F \setminus f}(s)|$$

Note that given any placement of the facility  $f$ ,  $U_{S \setminus f}$  does not depend on  $s$ . Thus, for any given placement of facility  $f$  by P1, optimal placement by P2 corresponds to the point  $s \in \mathbb{R}^2$  which maximizes  $|U_f(s)| + |U_{F \setminus f}(s)|$ . For any placement of facility  $f$  by P1 define *effective depth* of  $f$ , denoted by  $\delta(f)$ , as

$$\delta(f) = \max_{s \in \mathbb{R}^2} (|U_{S \setminus f}| + |U_f(s)| + |U_{F \setminus f}(s)|) = |U_{S \setminus f}| + \max_{s \in \mathbb{R}^2} (|U_f(s)| + |U_{F \setminus f}(s)|). \quad (2.4)$$

The optimal strategy of P1 is to find the point  $f$  with the minimum effective depth.

### 3 Optimal Placement of P1 in the $L_1$ metric

In this section we consider the optimal strategy of P1 in the  $L_1$  metric. The analogous problem in the  $L_\infty$  metric can be dealt with similarly by rotating the axes by  $45^\circ$ .

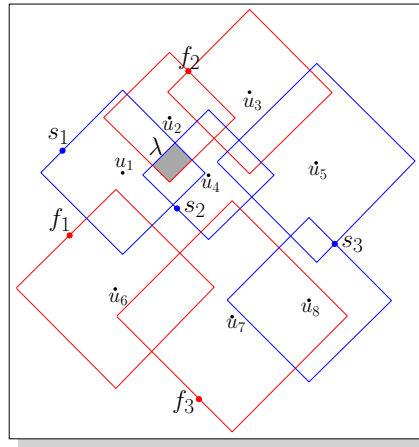


Figure 3: Distribution of users among the facilities  $F$  and  $S$ .

For any two points  $x, y \in \mathbb{R}^2$ , denote by  $d_1(x, y)$  the  $L_1$  distance between  $x$  and  $y$ . For any user  $u_a \in U$ , let  $R_a$  be the open  $L_1$  ball centered at  $u_a$  with radius  $d_1(u_a, h)$ , where  $h \in F \cup S$  is the



facility closest to  $u_a$  (in  $L_1$  distance) among the set of facilities  $F \cup S$ . Note that the boundary of  $R_a$  is a square centered at  $u_a$  and diagonals coinciding with the  $x$ -axis and  $y$ -axis and one of the sides touching  $h$ . Let  $\mathcal{R}_{FS} = \{R_a | u_a \in U\}$  be the collection of these nearest-facility squares, which will tessellate  $\mathbb{R}^2$  into a set of regions (see Figure 3). Denote this tessellation by  $\mathcal{T}(F, S)$ .

For each cell  $\lambda$  in this tessellation  $\mathcal{T}(F, S)$ , and any placement of a facility  $f$  by P1 in that cell, the sets  $U_{S \setminus f}$ ,  $U_{FS}(f)$  and  $U_{F \setminus f}$  remain unchanged. For notational brevity, for any fixed cell  $\lambda \in \mathcal{T}(F, S)$  and for all points  $f \in \lambda$ , the three sets  $U_{S \setminus f}$ ,  $U_{FS}(f)$  and  $U_{F \setminus f}$  will be denoted by  $U_{S \setminus \lambda}$ ,  $U_{FS}(\lambda)$  and  $U_{F \setminus \lambda}$ , respectively. For example, in Figure 3 for the cell  $\lambda$ ,

$$U_{S \setminus \lambda} = \{u_5, u_8\}, \quad U_{FS}(\lambda) = \{u_1, u_2, u_4\}, \quad U_{F \setminus \lambda} = \{u_3, u_6, u_7\}.$$

Our goal is to tessellate  $\lambda$  further such that for all points in each finer the effective depth remains fixed. To this end, for any placement of facility  $x$  by P1 and for any user  $u_a$ , let  $R_a(x)$  be the open square centered at  $u_a$  and passing through the facility closest (in  $L_1$  distance) to  $u_a$  from the set of facilities  $F \cup S \cup \{x\}$ . Recall that  $\delta(x)$  denotes the effective depth of  $x$  (2.4). The following lemma shows that the effective depth of a point  $x \in \mathbb{R}^2$  can be determined by the intersection properties of pairwise nearest-facility squares.

**Lemma 3.1.** *If points  $x, y$  belong to the same cell  $\lambda$  of  $\mathcal{T}(F, S)$  with  $\delta(x) \neq \delta(y)$ , then there exist two users  $u_a, u_b \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$  such that  $R_a(x) \cap R_b(x) \neq \emptyset$  and  $R_a(y) \cap R_b(y) = \emptyset$  or vice versa.*

*Proof.* For any placement  $x$  in the cell  $\lambda$  of  $\mathcal{T}(F, S)$ , recall that  $\delta(x)$  is maximum depth of the collection  $\mathcal{R}(x) := \{R_a(x) : u_a \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}\}$  of squares. Let  $\mathcal{P}(x)$  be the subset of  $\mathcal{R}(x)$  which attains this maximum depth, that is, the largest subset of  $\mathcal{R}(x)$  for which  $\bigcap_{R \in \mathcal{P}(x)} R \neq \emptyset$ .

Now, suppose there are two points  $x, y \in \lambda$  such that,  $\delta(x) \neq \delta(y)$ , and for all  $u_a, u_b \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ ,  $R_a(x) \cap R_b(x) \neq \emptyset$  if and only if  $R_a(y) \cap R_b(y) \neq \emptyset$ . Without loss of generality, assume  $\delta(x) > \delta(y)$ .

By definition, for each pair  $R_a(x), R_b(x) \in \mathcal{P}(x)$ ,  $R_a(x) \cap R_b(x) \neq \emptyset$ . Therefore, by assumption,  $R_a(y) \cap R_b(y) \neq \emptyset$ , for each pair  $R_a(y), R_b(y) \in \mathcal{P}(y) := \{R_a(y) : R_a(x) \in \mathcal{P}(x)\}$ . Therefore, by Helly's theorem for axis-parallel squares [23, Corollary 1.5],  $\bigcap_{R \in \mathcal{P}(y)} R \neq \emptyset$ , which implies that

$$\delta(y) \geq |\mathcal{P}(y)| = |\mathcal{P}(x)| = \delta(x),$$

which is a contradiction. This completes the proof of the result.  $\square$

In light of this lemma we define, for each pair of users  $u_a, u_b \in U$ , and any placement of facility  $x \in \mathbb{R}^2$  by P1, the indicator variable,

$$T_{ab}(x) = \begin{cases} 1 & \text{if } R_a(x) \cap R_b(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Let  $\mathbf{T}(x) = ((T_{ab}(x)))_{1 \leq a, b \leq |U|}$  be the 2 dimensional array of size  $|U| \times |U|$  where each entry  $T_{ab}(x)$  is defined as above. From Lemma 3.1 and the above definition, the following observation is immediate.

**Observation 3.1.** *If the points  $x, y$  belong to the same cell of  $\mathcal{T}(F, S)$  and the two arrays  $\mathbf{T}(x) = \mathbf{T}(y)$ , in every coordinate, then  $\delta(x) = \delta(y)$ .*

Hence, the goal is to tessellate cells of  $\mathcal{T}(F, S)$  into finer set of cells such that for any two points  $x$  and  $y$  in the same cell,  $\mathbf{T}(x) = \mathbf{T}(y)$ . Observation 3.1 would then imply that for all points in a finer cell, the effective depth remains constant. Hence, by checking each cell once we can find out the point with minimum effective depth. To this end, let  $u_a, u_b \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , and consider

$$\mathcal{T}(u_a, u_b) = \left\{ f \in \mathbb{R}^2 : R_a(f) \cap R_b(f) = \emptyset \right\}. \quad (3.1)$$

Now, denote by  $f(u_a)$  the facility in  $F \cup \{f\}$  closest to  $u_a \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ . Then

$$\mathcal{T}(u_a, u_b) = \mathcal{T}_1(u_a, u_b) \cup \mathcal{T}_2(u_a, u_b) \cup \mathcal{T}_2(u_b, u_a) \cup \mathcal{T}_3(u_a, u_b),$$

where

$$\begin{aligned} \mathcal{T}_1(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \cap R_b(f) = \emptyset \text{ and } f(u_a), f(u_b) \in F \right\}, \\ \mathcal{T}_2(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \cap R_b(f) = \emptyset \text{ and } f(u_a) \in F, f(u_b) = f \right\}, \\ \mathcal{T}_2(u_b, u_a) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \cap R_b(f) = \emptyset \text{ and } f(u_b) \in F, f(u_a) = f \right\}, \\ \mathcal{T}_3(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \cap R_b(f) = \emptyset \text{ and } f(u_a) = f(u_b) = f \right\}. \end{aligned} \quad (3.2)$$

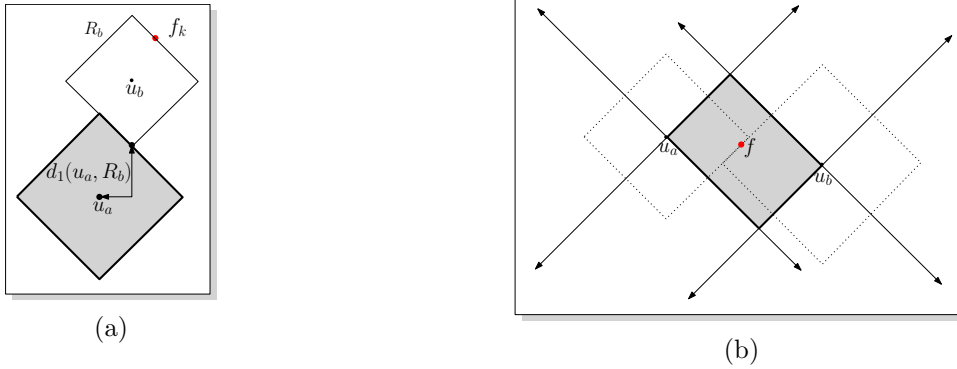


Figure 4: (a) The shaded region denotes the set  $\mathcal{T}_2(u_a, u_b)$ . (b) The shaded region denotes the set  $\mathcal{T}_3(u_a, u_b)$ .

**Lemma 3.2.** *For every pair of users  $u_a, u_b \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , the sets  $\mathcal{T}_1(u_a, u_b)$ ,  $\mathcal{T}_2(u_a, u_b)$ ,  $\mathcal{T}_2(u_b, u_a)$ , and  $\mathcal{T}_3(u_a, u_b)$  can be found in  $O(1)$  time.*

*Proof.* Recall that for  $u_a \in U$ , the square  $R_a$  is the open  $L_1$  ball centered at  $u_a$  with radius  $d_1(u_a, h)$ , where  $h \in F \cup S$  is the facility closest to  $u_a$  (in  $L_1$  distance) among the set of facilities  $F \cup S$ . Then it is easy to see that  $\mathcal{T}_1(u_a, u_b) = (R_a \cup R_b)^c$ .

To compute  $\mathcal{T}_2(u_a, u_b)$ , without loss of generality, assume that  $f(u_b) = f_k \in F$ . Let the minimum  $L_1$  distance between  $u_a$  and  $R_b$  be  $d := d_1(u_a, R_b)$  (see Figure 4(a)). Denote by  $B_1(u_a, d)$  the closed  $L_1$  ball with center at  $u_a$  and distance  $d$ . Observe that for any new facility  $f \in B_1(u_a, d)$ ,  $R_a(f) \cap R_b(f) = \emptyset$ , which implies  $\mathcal{T}_2(u_a, u_b) = B_1(u_a, d) \cap R_b \setminus R_a$ . The set  $\mathcal{T}_2(u_b, u_a)$  can be obtained similarly. Both these sets can be computed in  $O(1)$  time.

Next, we consider the set  $\mathcal{T}_3(u_a, u_b)$ . Consider the four lines making  $45^\circ$  and  $135^\circ$  angle with the  $x$ -axis and passing through the points  $u_a$  and  $u_b$  (see Figure 4(b)). These four lines will divide the plane into nine regions, one bounded and the other eight are unbounded. Observe that except for the bounded region, for any facility  $f$  placed in one of the unbounded regions,  $R_a(f) \cap R_b(f)$  will either contain  $u_a$  or  $u_b$ , that is,  $R_a(f) \cap R_b(f) \neq \emptyset$ . On the other hand, for any point  $f$  in the bounded region, the closures of  $R_a(f)$  and  $R_b(f)$  will share a common edge, but their interiors will not intersect. Therefore,  $\mathcal{T}_3(u_a, u_b)$  is the shaded region in Figure 4(b) intersected with  $R_a \cap R_b$ . This can be computed in  $O(1)$  time as well.  $\square$

**Completing the Proof of Theorem 1.1:** Consider the arrangement in  $\mathbb{R}^2$  generated by the collection of sets

$$\left\{ \mathcal{T}(u_a, u_b) : u_a, u_b \in U_{FS}(\lambda) \cup U_{F \setminus \lambda} \text{ and } \lambda \in \mathcal{T}(F, S) \right\}. \quad (3.3)$$

Note that this arrangement consists of  $O(n^2)$  rectangles which can be computed in  $O(n^2)$  time by Lemma 3.2 above. These  $O(n^2)$  rectangles intersect to generate  $O(n^4)$  cells, and by Observation 3.1, all points in a particular cell have the same the effective depth. Note that the effective depth of a cell can be computed in  $O(n \log n)$  time, by finding the maximum depth of the arrangement of squares  $\{R_a(f) : u_a \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}\}$ , for any point  $f$  in the cell [9, Theorem 8]. Therefore, the optimal strategy of P1 in the  $L_1$  metric can be computed in  $O(n^5 \log n)$  time. Note that since we search over all cells of the tessellation (3.3), the level sets  $L(r) = \{f \in \mathbb{R}^2 : \nu(f) \geq r\}$ , where  $\nu(f)$  is the payoff P1 when placed at the point  $f \in \mathbb{R}^2$  (1.2), can be computed in same running time.

## 4 Optimal Placement of P1 in the $L_2$ metric

For  $u_a \in U$ , denote by  $C_a$  the disc centered at  $u_a$  and passing through the facility closest to  $u_a$  among the set of facilities  $F \cup S$ . Let  $\mathcal{C}_{FS} = \{C_a : u_a \in U\}$  be the collection of all such discs, which will tessellate  $\mathbb{R}^2$  into a set of regions (see Figure 1). As in the case of the  $L_1$  metric, denote this tessellation by  $\mathcal{T}(F, S)$ .

For any placement of facility  $x \in \mathbb{R}^2$  by P1 and any user  $u_a \in U$ , let  $C_a(x)$  be the open disc centered at  $u_a$  and passing through the facility closest to  $u_a$  among the facilities in  $F \cup S \cup \{x\}$ . As before, for any fixed cell  $\lambda \in \mathcal{T}(F, S)$  and for all points  $f \in \lambda$ , the three sets  $U_{S \setminus f}$ ,  $U_{FS}(f)$  and  $U_{F \setminus f}$  will be denoted by  $U_{S \setminus \lambda}$ ,  $U_{FS}(\lambda)$  and  $U_{F \setminus \lambda}$ , respectively. Then, similar to Lemma 3.1, we have the following:

**Lemma 4.1.** *If the points  $x, y$  belong to some cell  $\lambda$  of  $\mathcal{T}(F, S)$  with  $\delta(x) \neq \delta(y)$ , then there exist three users  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$  such that either  $C_a(x) \cap C_b(x) \cap C_c(x) \neq \emptyset$  and  $C_a(y) \cap C_b(y) \cap C_c(y) = \emptyset$  or vice versa.*

*Proof.* As in Lemma 3.1, we shall prove the result by contradiction. To this end, assume  $\delta(x) > \delta(y)$ , and for every three users  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , if  $C_a(x) \cap C_b(x) \cap C_c(x) \neq \emptyset$  then  $C_a(y) \cap C_b(y) \cap C_c(y) \neq \emptyset$ .

For any placement  $x$  in the cell  $\lambda$ ,  $\delta(x)$  is maximum depth of the collection  $\mathcal{C}(x) := \{C_a(x) : u_a \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}\}$  of squares. Let  $\mathcal{D}(x)$  be the subset of  $\mathcal{C}(x)$  which attains this maximum depth, that is, the largest subset of  $\mathcal{C}(x)$  for which  $\bigcap_{C \in \mathcal{D}(x)} C \neq \emptyset$ . By definition, for each triple  $C_a(x), C_b(x), C_c(x) \in \mathcal{D}(x)$ ,  $C_a(x) \cap C_b(x) \cap C_c(x) \neq \emptyset$ . Therefore, by assumption,

$C_a(y) \cap C_b(y) \cap C_c(y) \neq \emptyset$ , for each triple  $C_a(y), C_b(y), C_c(y) \in \mathcal{D}(y) := \{C_a(y) : C_a(x) \in \mathcal{D}(x)\}$ . Therefore, by the Helly's theorem for discs [22, Theorem 1.3.2],  $\bigcap_{C \in \mathcal{D}(y)} C \neq \emptyset$ , which implies that

$$\delta(y) \geq |\mathcal{D}(y)| = |\mathcal{D}(x)| = \delta(x),$$

which is a contradiction. This completes the proof of the result.  $\square$

In light of this lemma, we define, for each triplet of users  $u_a, u_b, u_c \in U$ , and any placement of facility  $x \in \mathbb{R}^2$  by P1, the indicator variable,

$$T_{abc}(x) = \begin{cases} 1 & \text{if } C_a(x) \cap C_b(x) \cap C_c(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Let  $\mathbf{T}(x) = ((T_{abc}(x)))_{1 \leq a, b, c \leq |U|}$  be the 3-dimensional array with cardinality  $|U| \times |U| \times |U|$ , where each cell  $T_{abc}(x)$  is defined as above. Then by Lemma 4.1, the following observation is immediate.

**Observation 4.1.** *If the points  $x, y$  belong to the same cell of  $\mathcal{T}(F, S)$  and the two arrays  $\mathbf{T}(x) = \mathbf{T}(y)$  in every coordinate, then  $\delta(x) = \delta(y)$ .*

As in Section 3, our goal is to tessellate  $\mathcal{T}(F, S)$  in to finer set of cells such that for any two points  $x$  and  $y$  in the same cell,  $\mathbf{T}(x) = \mathbf{T}(y)$ . As in (3.1), define, for  $u_a, u_b, u_c \in U$ ,

$$\mathcal{T}(u_a, u_b, u_c) = \left\{ x \in \mathbb{R}^2 : C_a(x) \cap C_b(x) \cap C_c(x) = \emptyset \right\}. \quad (4.1)$$

**Definition 4.1.** Given any placement  $x$  by P1 and a user  $u_a \in U_{FS}(x) \cup U_{F \setminus x}$ , the disc  $C(x)$  is called an *old disc* if it is centered at  $u_a$  and passes through some facility  $f_j \in F$ , where  $f_j$  is the facility closest to  $u_a$  among the set of facilities  $F \cup \{x\}$ , that is,  $u_a \in U_{F \setminus x}$  (recall (2.2)). The disc  $C(x)$  is called a *new disc* if it is centered at  $u_a$  and passes through  $x$ , that is  $u \in U_{FS}(x)$  (recall (2.1)).

Let  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , for some cell  $\lambda$  of  $\mathcal{T}(F, S)$ . For  $S \subseteq \{a, b, c\}$ , define the following sets:

$$\mathcal{T}_S(u_a, u_b, u_c) = \{x \in \mathcal{T}(u_a, u_b, u_c) : C_s(x) \text{ is new, if } s \in S, \text{ and } C_s(x) \text{ is old, if } s \notin S\},$$

where new/old are as defined in Definition 4.1. Note that

$$\mathcal{T}(u_a, u_b, u_c) = \bigcup_{S \subseteq \{a, b, c\}} \mathcal{T}_S(u_a, u_b, u_c). \quad (4.2)$$

Therefore, to compute the set  $\mathcal{T}(u_a, u_b, u_c)$ , it suffices to understand the sets  $\mathcal{T}_S(u_a, u_b, u_c)$ , for  $S \subseteq \{a, b, c\}$ . To this end, we have the following lemma:

**Lemma 4.2.** *Let  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , for some cell  $\lambda$  of  $\mathcal{T}(F, S)$ . Then*

- (a)  $\mathcal{T}_\emptyset(u_a, u_b, u_c) = (C_a \cup C_b \cup C_c)^c$ .
- (b)  $\mathcal{T}_{\{a, b, c\}}(u_a, u_b, u_c) = \Delta(u_a, u_b, u_c) \cap (C_a \cap C_b \cap C_c)$ , where  $\Delta(u_a, u_b, u_c)$  is the triangle formed by  $u_a, u_b$ , and  $u_c$ .

(c)  $\mathcal{T}_{\{c\}}(u_a, u_b, u_c) = D_{\{c\}}(a, b) \cap C_c \setminus (C_a \cup C_b)$ , where  $D_{\{c\}}(a, b)$  is the closed disc centered at  $u_c$  and passing through the point in  $C_a \cap C_b$  closest to  $u_c$  in the  $L_2$  metric. The sets  $\mathcal{T}_{\{a\}}(u_a, u_b, u_c)$ , and  $\mathcal{T}_{\{b\}}(u_a, u_b, u_c)$  are defined similarly.

*Proof.* The sets  $\mathcal{T}_{\emptyset}(u_a, u_b, u_c)$  and  $\mathcal{T}_{\{a,b,c\}}(u_a, u_b, u_c)$  can be derived easily from the definitions.

To characterize  $\mathcal{T}_{\{c\}}(u_a, u_b, u_c)$ , let  $p_0$  be the point in  $C_a \cap C_b$  which is closest to  $u_c$  in the  $L_2$  metric. Note that  $D_{\{c\}}(a, b)$  is the closed disc centered at  $u_c$  and passing through  $p_0$ . Note that if  $x \in D_{\{c\}}(a, b) \cap C_c \setminus (C_a \cup C_b)$ , then  $C_a(x) \cap C_b(x) \cap C_c(x) = \emptyset$ , and if  $x \notin (D_{\{c\}}(a, b))^c \cap C_c \setminus (C_a \cup C_b)$ , then  $p_0 \in C_a(x) \cap C_b(x) \cap C_c(x)$ . This implies,  $\mathcal{T}_{\{c\}}(u_a, u_b, u_c) = D_{\{c\}}(a, b) \cap C_c \setminus (C_a \cup C_b)$ .  $\square$

The above result shows that the sets  $\mathcal{T}_S(u_a, u_b, u_c)$ , for  $|S| \neq 2$ , can be easily computed in  $O(1)$  time. Thus, it remains to compute  $\mathcal{T}_S(u_a, u_b, u_c)$ , for  $|S| = 2$ . In this case, the sets are more complicated, and it is difficult to explicitly describe the structure of these sets geometrically. The following lemma shows that these sets can also be computed in  $O(1)$  time, and through the proof of this lemma the precise geometry of these sets can be described.

**Lemma 4.3.** *Let  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , for some cell  $\lambda$  of  $\mathcal{T}(F, S)$ . Then the boundary of  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  is made up of  $O(1)$  circular arcs, and can be computed in  $O(1)$  time.*

**4.1 Proof of Lemma 4.3:** For a set  $A \subseteq \mathbb{R}^2$  denote by  $\overline{A}$  and  $\partial A$ , the closure and the boundary of  $A$ , respectively. For example,  $\overline{C_a(x)}$  is the closed disc centered at  $u_a$  and passing through the facility closest to  $u_a$  among the facilities in  $F \cup S \cup \{x\}$ . We begin with the following simple observation:

**Observation 4.2.** *Let  $u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}$ , for some cell  $\lambda$  of  $\mathcal{T}(F, S)$ . Then*

$$\partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) = \left\{ p \in (C_a \cap C_b) \setminus C_c : \left| \overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c(p)} \right| = 1 \right\}.$$

Moreover, the set  $\partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  is symmetric about the line joining  $u_a, u_b$ , that is, if  $p \in \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  then its reflection about the line joining  $u_a, u_b$   $p^\perp$  also belongs to  $\partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ .

*Proof.* Suppose that  $p \in (C_a \cap C_b) \setminus C_c$  is such that  $\overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c(p)} = \{q\}$ , where  $q$  is a point on the boundary of  $\overline{C_a(p)} \cap \overline{C_b(p)}$ . Next, note that for any point  $x$  in the interior of  $\overline{C_a(p)} \cap \overline{C_b(p)}$ ,  $\overline{C_a(x)} \cap \overline{C_b(x)}$  is a proper subset of  $\overline{C_a(p)} \cap \overline{C_b(p)}$ , and  $C_a(x) \cap C_b(x) \cap C_c(x) = \emptyset$ , since  $C_c(x) = C_c$  is an old circle. Therefore, every point in the interior of  $\overline{C_a(p)} \cap \overline{C_b(p)}$  belongs to  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ . Similarly, if  $x$  lies outside  $\overline{C_a(p)} \cup \overline{C_b(p)}$ , then  $q$  is in the interior of  $\overline{C_a(x)} \cap \overline{C_b(x)}$ , and  $C_a(x) \cap C_b(x) \cap C_c(x) \neq \emptyset$ . Therefore, every open ball centered at  $p$  intersects both  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  and the complement  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)^c$ , that is,  $p \in \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ .

Similarly, it can be shown that if  $\overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c(p)}$  contains zero points or more than 1 point, then the point  $p$  does not belong to the boundary of  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ .

Finally, to prove the symmetry, observe that for any point  $p \in (C_a \cup C_b) \setminus C_c$ , the boundaries of the discs  $C_a(p)$  and  $C_b(p)$  intersect at points  $p$  and  $p^\perp$ . Hence, if for any point  $p$ ,  $\overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c(p)}$  is a singleton set then  $\overline{C_a(p^\perp)} \cap \overline{C_b(p^\perp)} \cap \overline{C_c(p^\perp)}$  is also singleton, since  $C_c(p) = C_c(p^\perp) = C_c$  is an old circle. This completes the proof.  $\square$

Note that for  $p \in \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ , the circle  $C_c(p)$  is an old circle, which does not depend on  $p$ . Hereafter, we will drop the dependence on  $p$  and denote this circle by  $C_c$ . Now, define sets

$$\begin{aligned} W_a &:= \left\{ p \in \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_b(p)} \cap \overline{C_c} \right| = 1 \right\}, \\ W_b &:= \left\{ p \in \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_a(p)} \cap \overline{C_c} \right| = 1 \right\}, \\ W_c &:= \left\{ p \in \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_b(p)} \cap \overline{C_a(p)} \right| = 1 \right\}. \end{aligned} \quad (4.3)$$

The sets  $W_a, W_b, W_c$  can be obtained using the above observation:

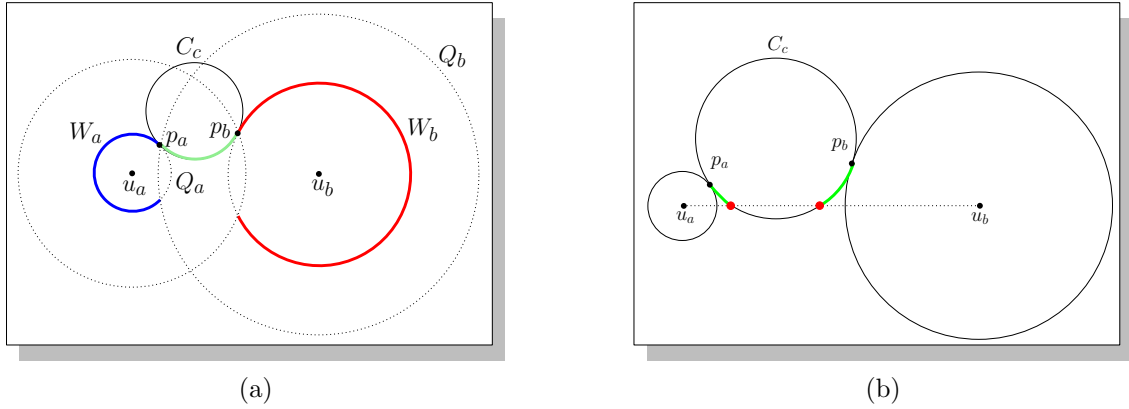


Figure 5: (a) The sets  $W_a$  and  $W_b$ , and (b) the set  $W_c$ , as defined in (4.3).

$W_a/W_b$ : Let the point closest to  $u_a$  in  $\overline{C_c}$  be  $p_a$ , and let  $Q_a$  and  $Q_b$  be the circles with centers at  $u_a$  and  $u_b$  and radii  $d_2(u_a, p_a)$  and  $d_2(u_a, p_b)$ , respectively. Then for all points  $p \in Q_a \setminus Q_b$   $\overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c} = \{p_a\}$ , that is,  $W_a = (Q_a \setminus Q_b) \cap (C_a \cap C_b) \setminus C_c$  (blue curve in Figure 5(a)). The set  $W_b$  can be obtained similarly (red curve in Figure 5(a)).

$W_c$ : In this case, it is easy to see that  $W_c = [u_a, u_b] \cap \partial C_c$ . These are the two red points in Figure 5(b). This set is empty in Figure 5(a).

Finally, let

$$W_0 := \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) \setminus (W_a \cup W_b \cup W_c). \quad (4.4)$$

This is the set of all points  $p$  such that  $\overline{C_a(p)} \cap \overline{C_b(p)} \cap \overline{C_c}$  is a singleton set, but the intersection of no two of them is a singleton.

$W_0$ : Recall that  $p_a$  is the point closest to  $u_a$  in  $\overline{C_c}$ . Let  $p_b$  be the point closest to  $u_b$  in  $\overline{C_c}$ . If the line joining  $u_a, u_b$  does intersect  $C_c$ , then  $W_0$  is the arc between  $p_a$  and  $p_b$  (arc colored in green in Figure 5(a)) and its reflection on the line joining  $u_a$  and  $u_b$  (see Observation 4.2), intersected with  $(C_a \cap C_b) \setminus C_c$ . Otherwise,  $u_a, u_b$  does intersect  $C_c$ , and the segment of the arc between  $p_a$  and  $p_b$  above the line joining  $u_a, u_b$  (green arcs in Figure 5(b)) and its reflection on the line joining  $u_a$  and  $u_b$  gives the set  $W_0$ , when intersected with  $(C_a \cap C_b) \setminus C_c$ .

The sets  $W_a$ ,  $W_b$ ,  $W_c$ , and  $W_0$  together make up the boundary of the set  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ . Putting these together, we get the shapes in Figure 6, depending on whether or not the line joining  $u_a$  and  $u_b$  intersects the disc  $C_c$ . The set  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  can be obtained when the shapes (the regions bounded by the blue curves) in Figure 6 are intersected with  $C_a \cap C_b \setminus C_c$ . This shows that the boundary of  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  is made up of  $O(1)$  circular-arcs. Therefore, the set  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  can be computed in  $O(1)$  time, which completes the proof of Lemma 4.3.

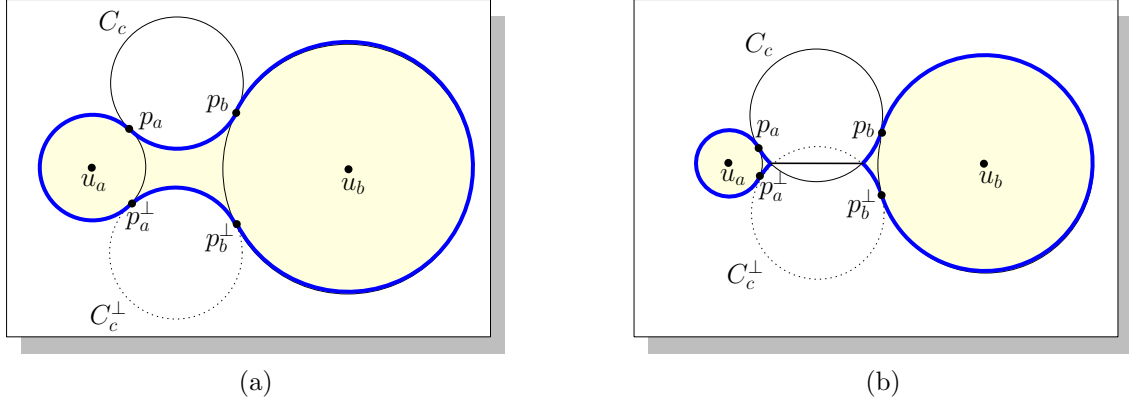


Figure 6: The set  $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$  (a) when the line joining  $u_a$  and  $u_b$  does not intersect  $C_c$ , and (b) when line joining  $u_a$  and  $u_b$  intersects  $C_c$ .

**4.2 Completing the Proof of Theorem 1.2:** Consider the arrangement in  $\mathbb{R}^2$  generated by the collection of sets  $\{\mathcal{T}(u_a, u_b, u_c) : u_a, u_b, u_c \in U_{FS}(\lambda) \cup U_{F \setminus \lambda} \text{ and } \lambda \in \mathcal{T}(F, S)\}$ . Lemmas 4.2 and 4.3 show that the arrangement is made up of  $O(n^3)$  discs, circular arcs, and line segments, which can be computed in  $O(n^3)$  time. These intersect to generate  $O(n^6)$  cells in  $\mathcal{T}(F, S)$ , and by Observation 4.1, all points in a particular cell have the same effective depth. Note that effective depth of a cell can be computed in  $O(n^2)$  time, by finding the maximum depth of the arrangement  $\{C_a(x) : u_a \in U_{FS}(\lambda) \cup U_{F \setminus \lambda}\}$ , for any point  $x$  in the cell [9, Theorem 1]. Therefore, the optimal strategy of P1 in the  $L_2$  metric can be computed in  $O(n^8)$  time. The corresponding level sets can be computed in the same running time.

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