The dissipation inequality and the algebraic Riccati equation

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1 Introduction

Undoubtedly one of the most important concepts in linear systems and control, both from a theoretical as well as from a practical point of view, is the algebraic Riccati equation. Since its introduction in control theory by Kalman [16] the beginning of the sixties, the algebraic Riccati equation has known an impressive range of applications, such as linear quadratic optimal control, stability theory, stochastic filtering and stochastic control, stochastic realization theory, synthesis of linear passive networks, differential games and, most recently, H_∞ optimal control and robust stabilization. The purpose of the present paper is to give an expository survey of the main concepts, results and applications related to the algebraic Riccati equation.

In our opinion, the most appealing framework for studying the Riccati equation is the framework of dissipative systems. In this framework, the Riccati equation emerges in a natural way as a consequense of the dissipation inequality, which expresses the fact that the system under consideration is dissipative. In this framework, real symmetric solutions of the algebraic Riccati correspond to storage functions, i.e. functions on the state space of the system that measure the amount of supply that is stored inside the system.

In this paper we intend to review and slightly extend the existing theory of dissipative systems. We will start with a treatment of dissipativeness for a very general class of systems. Contrary to most of the existing literature on dissipative systems, where the property of dissipativeness is described in terms of the internal (state space) properties of the system, we will consider dissipativeness as a property of the external behavior of the system. The question whether the system is internally dissipative then becomes one of finding a suitable state space representation and a suitable storage function. This expresses the property that, internally, the system can only store a finite amount of supply.

We will show that for linear, time-invariant, finite-dimensional systems with quadratic supply rates, the property of internal dissipativeness leads to solvability of what is called the linear matrix inequality, which in turn leads to solvabilty of the associated algebraic Riccati equation. As mentioned before, real symmetric solutions of the Riccati equation correspond to storage functions of the system. In this vein, it turns out that the

Riccati equation has a positive semi-definite solution if and only if there exists a storage function that attains its minimum at the origin. We will explain how the question of existence of positive semi-definite solutions leads to the Kalman-Yakubovich-Popov lemma (sometimes called the Positive Real Lemma) and the Bounded Real Lemma.

A major part of this paper is devoted to a discussion of the main applications of the ideas developed here. Without any attempt to be complete, we will discuss some of the most important problems in the context of linear quadratic optimal control. We will also briefly outline how the concept of storage function can be used to construct Lyapunov functions and how, in this way, the Positive Real Lemma and the Bounded Real Lemma can be used to obtain results on the stability of feedback systems. Next, we will outline how the Positive Real Lemma can be used as an important tool in the synthesis of linear passive networks and in the state space representation theory of stationary Gaussian random processes (the problem of covariance generation). Finally, as a most recent application, we will discuss the application of the Bounded Real Lemma to the problem of H_∞ optimal control.

2 Dissipative systems

In this section we will review the definition of dynamical system as proposed in [43]. For most of the definitions and basic facts we refer to [43,44]. Furthermore, we will give a definition of the concept of dissipative dynamical system.

Definition 2.1 *A* dynamical system Σ *is defined as a triple*

$$\Sigma = (T, W, \mathcal{B})$$

with $T \subseteq \mathcal{R}$ *the* time axis*;* W *a set, called the* signal alphabet*; and* $\mathcal{B} \subseteq W^T$ *the* behavior.

Let $T \subseteq \mathcal{R}$ and let W be a set. If w_1, w_2 : $T \to W$ and if $t \in T$ then we define the *concatenation at* t of w_1 and w_2, $w_1 \underset{t}{\wedge} w_2$, as the mapping from T to W defined by

$$(w_1 \underset{t}{\wedge} w_2)(\tau) = \begin{cases} w_1(\tau) & \text{for } \tau < t \\ w_2(\tau) & \text{for } \tau \geq t \end{cases}$$

In this paper we shall restrict ourselves to continuous time systems with time axis T equal to $\mathcal{R}$. Furthermore, we shall assume throughout that the systems Σ under consideration are *time-invariant*, i.e. $\sigma^t \mathcal{B} = \mathcal{B}$ for all $t \in \mathcal{R}$. Here, σ^t denotes the *t-shift*: $(\sigma^t f)(\tau) := f(\tau + t)$.

Let $\Sigma = (\mathcal{R}, W, \mathcal{B})$ be a time-invariant dynamical system. If $w \in \mathcal{B}$ then we define

$$\mathcal{B}^+(w) := \{\tilde{w} : [0, \infty) \to W \mid w \underset{0}{\wedge} \tilde{w} \in \mathcal{B}\}$$

In other words, $\mathcal{B}^+(w)$ consists of all time functions on the positive real half line with the property that their concatenation with the past of w is an element of the behavior $\mathcal{B}$. Stated differently, $\mathcal{B}^+(w)$ contains all possible future continuations of the past of

w. In a similar fashion, for $w \in \mathcal{B}$ we define $\mathcal{B}^-(w)$ to be the set of all possible past continuations of the future of w:

$$\mathcal{B}^-(w) := \{\tilde{w} : (-\infty, 0) \to W \mid \tilde{w} \underset{0}{\wedge} w \in \mathcal{B}\}$$

We shall now define the notion of dissipative dynamical system. Assume $\Sigma = (\mathcal{R}, W, \mathcal{B})$ is a time-invariant dynamical system. Let s be a real valued function defined on W. This function will be called the *supply rate*. We assume that s is such that for all $w \in \mathcal{B}$ the function $s(w(.))$ is locally integrable, i.e., $\int_{t_0}^{t_1} |s(w(t))|dt < \infty$ for all $(t_0, t_1) \in \mathcal{R}_2^+ := \{(\tau_0, \tau_1) \in \mathcal{R}^2 | \tau_0 \le \tau_1\}$. The notion of dissipativeness requires a condition on periodic signals in $\mathcal{B}$. If $T > 0$ then $w \in \mathcal{B}$ is called *periodic with period T* or *T-periodic* if for all $t \in \mathcal{R}$ we have $w(t) = w(t+T)$.

Definition 2.2 *The pair (Σ, s) with $\Sigma = (\mathcal{R}, W, \mathcal{B})$ a time-invariant dynamical system and $s : W \to \mathcal{R}$ the supply rate is called* dissipative *if for all for all periodic $w \in \mathcal{B}$ we have*

$$\int_0^T s(w(t))dt \ge 0 \tag{2.1}$$

where T is the period of $w \in \mathcal{B}$.

The idea is that the function $s(w(.))$ is the rate at which supply (for example, the power, the rate at which energy is supplied) flows into the system if the system produces the signal $w(.)$. Thus, for $T \ge 0$ $\int_0^T s(w(t))dt$ is equal to the the total supply that flows into the system over the time-interval $[0, T]$. The above definition formalizes the idea that along periodic signals supply flows net *into* the system. The above definition of dissipativeness is taken from [37], where however a somewhat different terminology was used.

In addition to the concept of dissipativeness in terms of periodic signals, we will give the following definition in terms of continuations of signals $w \in \mathcal{B}$.

Definition 2.3 *The pair (Σ, s) is called* forward dissipative *if for all $w \in \mathcal{B}$ and for all $T \ge 0$ such that $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$ we have $\int_0^T s(w(t))dt \ge 0$.*

The interpretation of this definition is as follows: the system is called forward dissipative if along any signal with the property that its future continuations at time $t = 0$ and $t = T$ coincide, the supply flow over the time-interval $[0, T]$ is net *into* the system. It is also possible to give a backward version of this concept:

Definition 2.4 *The pair (Σ, s) is called* backward dissipative *if for all $w \in \mathcal{B}$ and for all $T \ge 0$ such that $\mathcal{B}^-(w) = \mathcal{B}^-(\sigma^T w)$ we have $\int_0^T s(w(t))dt \ge 0$.*

It turns out that the properties of forward dissipativeness and backward dissipativeness both imply dissipativeness:

Proposition 2.5 *Let Σ be a time-invariant dynamical system and let s be a supply rate. Then we have*

(a) *If (Σ, s) is forward dissipative then (Σ, s) is dissipative,*

(b) *If (Σ, s) is backward dissipative then (Σ, s) is dissipative.*

Proof If $T > 0$ and w is T-periodic then we trivially have $\sigma^T w = w$. This implies $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$ and $\mathcal{B}^-(w) = \mathcal{B}^-(\sigma^T w)$. □

We will now show that if the dynamical system is *complete*, then the three notions of dissipativeness introduced above are equivalent. If $(t_0, t_1) \in \mathcal{R}_2^+$ then we define

$$\mathcal{B}\,|_{[t_0,t_1]} := \{w : [t_0, t_1] \to W | w \in \mathcal{B}\}$$

The dynamical system $\Sigma = (\mathcal{R}, W, \mathcal{B})$ is called *complete* if for all $w : \mathcal{R} \to W$ the following implication holds:

$$\{\forall (t_0, t_1) \in \mathcal{R}_2^+ \quad w\mid_{[t_0,t_1]} \in \mathcal{B}\mid_{[t_0,t_1]}\} \Rightarrow \{w \in \mathcal{B}\}.$$

In other words, a system is complete if for any function $w : \mathcal{R} \to W$, in order to check whether it is an element of the behavior of the system, it is sufficient to check whether its restrictions to finite intervals are elements of the behavior restricted to these intervals. For any $w : [0, T) \to W$ we define the *periodic continuation of w to $\mathcal{R}$* as the T-periodic function $\bar{w} : \mathcal{R} \to W$ whose restriction to $[0, T)$ is equal to w. The following lemma states that if Σ is time-invariant and complete, if $T > 0$ and if w is an element in the behavior of Σ with the property that its future continuations at time $t = 0$ and time $t = T$ coincide, then the T-periodic continuation of $w\mid_{[0,T)}$ is also an element of the behavior of Σ:

Lemma 2.6 *Let $\Sigma = (\mathcal{R}, W, \mathcal{B})$ be time-invariant and complete. Let $w \in \mathcal{B}$ and $T > 0$ be such that $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$. Then the T-periodic continuation of $w\mid_{[0,T)}$ is an element of $\mathcal{B}$.*

Proof Clearly, $w\mid_{[0,\infty)} \in \mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$. This implies that $\sigma^T w \underset{0}{\wedge} w \in \mathcal{B}$. Define

$$w_1 := \sigma^{-T}(\sigma^T w \underset{0}{\wedge} w).$$

Then by time-invariance $w_1 \in \mathcal{B}$. Furthermore, $w_1(t) = w(t)$ on $[0, T)$ and $w_1(t + T) = w_1(t)$ for all $t \in [0, T)$. Also, $w_1\mid_{(-\infty,0]} = w\mid_{(-\infty,0]}$. The latter implies that $w_1\mid_{[0,\infty)} \in \mathcal{B}^+(w)$. Since $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$ we find that $\sigma^T w \underset{0}{\wedge} w_1 \in \mathcal{B}$. Define

$$w_2 := \sigma^{-T}(\sigma^T w \underset{0}{\wedge} w_1).$$

Then $w_2 \in \mathcal{B}$. Also, $w_2(t) = w(t)$ on $[0, T)$, $w_2(t + T) = w_2(t)$ for all $t \in [0, 2T)$ and $w_2\mid_{(-\infty,0]} = w\mid_{(-\infty,0]}$. Carrying on inductively we can for each n construct $w_n \in \mathcal{B}$ with the following properties: (i) $w_n(t) = w(t)$ for all $t \in [0, T)$ and (ii) $w_n(t+T) = w_n(t)$ for all $t \in [0, nT)$. Now define $\bar{w}_n := \sigma^{nT} w_{2n}$. Then we have $\bar{w}_n(t) = w(t)$ for all $t \in [0, T)$ and $\bar{w}_n(t + T) = \bar{w}_n(t)$ for all $t \in [-nT, nT)$. Let $\bar{w}$ be the T-periodic continuation of $w\mid_{[0,T)}$. Let $(t_0, t_1) \in \mathcal{R}_2^+$. For n sufficiently large we have $\bar{w}\mid_{[t_0,t_1]} = \bar{w}_n\mid_{[t_0,t_1]} \in \mathcal{B}\mid_{[t_0,t_1]}$. Completeness then implies that $\bar{w} \in \mathcal{B}$. □

Using the previous lemma it is easy to show that if a time-invariant complete system is dissipative then it is also forward dissipative:

Proposition 2.7 *Let $\Sigma = (\mathcal{R}, W, \mathcal{B})$ be time-invariant and complete and let $s : W \to \mathcal{R}$ be a supply rate. Then (Σ, s) is dissipative if and only if (Σ, s) is forward dissipative.*

Proof Let $T > 0$ and $w \in \mathcal{B}$ be such that $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$. Let $\bar{w} \in \mathcal{B}$ be the T-periodic continuation of $w\mid_{[0,T)}$. By dissipativeness we then have $\int_0^T s(w(t))dt = \int_0^T s(\bar{w}(t)dt \geq 0$. □

Completely analogously it can be shown that for time-invariant complete systems dissipativeness implies backward dissipativeness. Thus we arrive at the following :

Theorem 2.8 *Let $\Sigma = (\mathcal{R}, W, \mathcal{B})$ be time-invariant and complete and let $s : W \to \mathcal{R}$ be a supply rate. Then the following statements are equivalent:*

(a) *(Σ, s) is dissipative,*

(b) *(Σ, s) is forward dissipative,*

(c) *(Σ, s) is backward dissipative.*

□

3 Internally dissipative systems

The definition of dissipativeness that we gave in the previous section was given completely in terms of the external behavior of the system and is independent of any consideration with respect to state space representations. In the present section we will discuss the notion of state space representation and give a definition of internal dissipativeness of a dynamical system in state space form. We will then study the relations between (external) dissipativeness and internal dissipativeness.

Definition 3.1 *A* dynamical system in state space form *is defined as a quadruple* $\Sigma_s = (T, W, X, \mathcal{B}_s)$ *with* T *and* W *as in Def. 2.1,* X *the* state space *and where the* state behavior $\mathcal{B}_s \subseteq (W \times X)^T$ *satisfies the* axiom of state*:*

$$\forall t \in T : \{(w_1, x_1), (w_2, x_2) \in \mathcal{B}_s \text{ and } x_1(t) = x_2(t)\}$$

$$\Rightarrow \{(w_1, x_1) \underset{t}{\wedge} (w_2, x_2) \in \mathcal{B}_s\}$$

We note that if $(T, W, X, \mathcal{B}_s)$ is a dynamical system in state space form then the triple $(T, W \times X, \mathcal{B}_s)$ defines a dynamical system in the sense of Def 2.1. We denote by P_W the projection of $W \times X$ onto W along X, i.e., $P_W(w, x) := w$. Likewise, P_X denotes the projection of $W \times X$ onto X along W. If $(T, W, X, \mathcal{B}_s)$ is a dynamical system in state space form then $P_W\mathcal{B}_s$ will be called the *external behavior* of Σ_s and the triple $(T, W, P_W\mathcal{B}_s)$ will be called the *system induced by* Σ_s. Conversely, we will call $(T, W, X, \mathcal{B}_s)$ a *state space representation* of $(T, W, P_W\mathcal{B}_s)$. Finally, we will call $P_X\mathcal{B}_s$ the *state behavior* of Σ_s. A dynamical system in state space form Σ_s is called time-invariant if the associated dynamical system with signal alphabet $W \times X$ is time-invariant.

Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a system in state space form. We will say that Σ_s is *observable* if the state trajectory is uniquely determined by the external signal, in the sense that the following implication holds:

$$\{(w, x_1), (w, x_2) \in \mathcal{B}_s\} \Rightarrow \{x_1 = x_2\}$$

Obviously, this condition is equivalent with the existence of a mapping $F : P_W\mathcal{B}_s \to P_X\mathcal{B}_s$ such that $(w, x) \in \mathcal{B}_s$ if and only if $x = F(w)$. It is easy to show that if Σ_s is observable and *time-invariant*, then the mapping F commutes with the shift σ^t for all t.

Assume that Σ_s is time-invariant. We will say that Σ_s is *connected* if for any two elements x_0 and x_1 in X there exists a state trajectory $x \in P_X\mathcal{B}_s$ and $t \geq 0$ such that $x(0) = x_0$ and $x(t) = x_1$, i.e., if any two points in the state space of Σ_s can be connected by means of a suitable state trajectory.

If $\Sigma = (\mathcal{R}, W, \mathcal{B})$ is a dynamical system then there always exists a state space representation of Σ, i.e., a dynamical system in state space form $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ such that $P_W\mathcal{B}_s = \mathcal{B}$. If Σ is time-invariant then Σ_s can also be chosen to time-invariant. Two particular state space representations of a given dynamical system will play an important role in the sequel. Let $\Sigma = (\mathcal{R}, W, \mathcal{B})$ be a dynamical system. Two signals w_1 and w_2 in $\mathcal{B}$ are said to be *past equivalent* if $\mathcal{B}^+(w_1) = \mathcal{B}^+(w_2)$. In this case we will write $w_1 \overset{-}{\sim} w_2$. Clearly, the relation $\overset{-}{\sim}$ defines an equivalence relation on $\mathcal{B}$. Define $X^{past} := \mathcal{B}(mod \overset{-}{\sim})$ and let

$$\mathcal{B}_s^{past} := \{(w,x) : \mathcal{R} \to X^{past} | w \in \mathcal{B}, x(t) = (\sigma^t w)(mod \overset{-}{\sim})\}.$$

It can be shown that $\Sigma_s^{past} := (\mathcal{R}, W, X^{past}, \mathcal{B}_s^{past})$ defines a state space representation of Σ. This state space representation is called *the canonical past induced state space representation of* Σ. Obviously, if Σ is time-invariant then also Σ_s^{past} is time-invariant. In a similar way, two signals w_1 and w_2 in $\mathcal{B}$ are called *future equivalent* if $\mathcal{B}^-(w_1) = \mathcal{B}^-(w_2)$. Again this defines an equivalence relation on $\mathcal{B}$, denoted by $\overset{+}{\sim}$. We define $X^{fut} := \mathcal{B}(mod \overset{+}{\sim})$ and

$$\mathcal{B}_s^{fut} := \{(w,x) : \mathcal{R} \to X^{fut} | w \in \mathcal{B}, x(t) = (\sigma^t w)(mod \overset{+}{\sim})\}.$$

It can be shown fairly easily that also $\Sigma_s^{fut} := (\mathcal{R}, W \times X^{fut}, \mathcal{B}_s^{fut})$ defines a state space representation of Σ (note that this is not trivial, since the state axiom is not really time-symmetric). This particular state space representation of Σ is called *the canonical future induced state space representation of* Σ. Again, Σ_s^{fut} is time-invariant whenever Σ is time-invariant.

Definition 3.2 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant dynamical system in state space form, let $s : W \to \mathcal{R}$ be a supply rate and let $V : X \to \mathcal{R}$ be a function. The triple (Σ_s, s, V) is called* internally dissipative *if for all $(t_0, t_1) \in \mathcal{R}_2^+$ and for all $(w, x) \in \mathcal{B}_s$ we have*

$$V(x(t_0)) + \int_{t_0}^{t_1} s(w(t))dt \geq V(x(t_1)). \tag{3.1}$$

Any function $V : X \to \mathcal{R}$ that satisfies the inequality 3.1 is called a storage function of (Σ_s, s). *Let us single out a special point $x^\star \in X$. If V is a storage function of (Σ_s, s) with the property that $V(x^\star) = 0$, then V is called a* normalized storage function (at $x^\star$).

The inequality 3.1 is called the *dissipation inequality.* Obviously, if (Σ_s, s, V) is internally dissipative then for each $x^\star \in X$ there exists a normalized storage function $\tilde{V}$, i.e., a storage function with the property that $\tilde{V}(x^\star) = 0$. Indeed for any storage function V and for any real constant C the function $V + C$ is a storage function as well. Thus we can obtain a normalized storage function by defining $\tilde{V}(x) := V(x) - V(x^\star)$.

The idea is that the function $s(w(.))$ is the rate at which supply flows into the system if the system produces the particular external signal $w(.)$. Thus, if $(t_0, t_1) \in \mathcal{R}_2^+$, then $\int_{t_0}^{t_1} s(w(t))dt$ is equal to the amount of supply that flows into the system over the time interval $[t_0, t_1]$. After choosing a reference point $x^\star$, a normalized storage function V is meant to measure the amount of supply that is stored inside the system: if $x_0 \in X$ then $V(x_0)$ is equal to the amount of supply of the system if it is in the particular state x_0. Of course, by normalization the amount of supply in the reference point $x^\star$ has been agreed upon to be equal to zero. For a given point $x_0 \in X$ the statement $V(x_0) < 0$ thus means that the amount of supply of the system in x_0 is less that in $x^\star$, while the statement $V(x_0) > 0$ means that the amount of supply in x_0 is larger than in $x^\star$. The dissipation inequality expresses the property that if a system produces the signal $(w(.), x(.))$ and if $(t_0, t_1) \in \mathcal{R}_2^+$, then the total supply of the system at time t_1

cannot exceed the sum of the supply of the system at time t_0 and the amount that was supplied to the system through the external channels during the time interval $[t_0, t_1]$. Physically, this describes the property that somewhere inside the system supply must have been dissipated, for example in the form of heat.

We will now show that if a given time-invariant observable system in state space form is internally dissipative, then its externally induced system is dissipative:

Proposition 3.3 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant state space representation of the time-invariant system $\Sigma = (\mathcal{R}, W, \mathcal{B})$. Assume that Σ_s is observable. Let s be a supply rate. If there exists $V : X \to \mathcal{R}$ such that (Σ_s, s, V) is internally dissipative, then (Σ, s) is dissipative.*

Proof Since Σ_s is observable, there exists a mapping $F : \mathcal{B} \to P_X\mathcal{B}_s$ such that $(w, x) \in \mathcal{B}_s$ if and only if $x = F(w)$. Define $\pi : P_X\mathcal{B}_s \to X$ by $\pi x := x(0)$. Let $w \in \mathcal{B}$ be a T-periodic signal and let $x = F(w)$. Since $\sigma^T w = w$ we have

$$x(0) = \pi x = (\pi \circ F)(w) = (\pi \circ F \circ \sigma^T)(w)$$

$$= (\pi \circ \sigma^T \circ F)(w) = (\pi \circ \sigma^T)(x) = x(T).$$

From this it immediately follows that $V(x(0)) = V(x(T))$ and hence, from the dissipation inequality, that $\int_0^T s(w(t))dt \geq 0$. Since the latter holds for any T-periodic signal $w \in \mathcal{B}$ we conclude that (Σ, s) is dissipative. □

Next we will study the following question: Given a time-invariant dissipative dynamical system, does there exist an internally dissipative state space representation of this system? It will turn out that if the canonical past induced state space representation of the system is connected, then forward dissipativeness of the system is equivalent with internal dissipativeness of the canonical past induced state space representation. A similar result holds for the canonical future induced state space representation (see [37]):

Theorem 3.4 *Let Σ be a time-invariant dynamical system and let s be a supply rate. Let Σ_s^{past} and Σ_s^{fut} be the canonical past induced and future induced state space representations of Σ, respectively. Then we have:*

(a) *If Σ_s^{past} is connected then (Σ, s) is forward dissipative if and only if there exists $V : X^{past} \to \mathcal{R}$ such that (Σ_s^{past}, s, V) is internally dissipative.*

(b) *If Σ_s^{fut} is connected then (Σ, s) is backward dissipative if and only if there exists $V : X^{fut} \to \mathcal{R}$ such that (Σ_s^{fut}, s, V) is internally dissipative.* □

Recalling that for time-invariant and complete systems the notions of dissipativeness, forward dissipativeness and backward dissipativeness are equivalent, we thus obtain the following result:

Corollary 3.5 *Let Σ be a time-invariant complete dynamical system and let s be a supply rate. Let Σ_s^{past} and Σ_s^{fut} be the canonical past induced and future induced state space representations of Σ, respectively. Then we have:*

(a) *If Σ_s^{past} is connected then (Σ_s, s) is dissipative if and only if there exists $V : X^{past} \to \mathcal{R}$ such that (Σ_s^{past}, s, V) is internally dissipative.*

(b) *If Σ_s^{fut} is connected then (Σ_s, s) is dissipative if and only if there exists $V : X^{fut} \to \mathcal{R}$ such that (Σ_s^{fut}, s, V) is internally dissipative.* □

Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant system in state space form and let s be a supply rate. Assuming the existence of a function V such that (Σ_s, s, V) is internally dissipative, one would like to obtain general properties of the set of all possible storage functions of (Σ_s, s). As noted before, if V is a storage function then for any real constant C the function $V + C$ is a storage function as well. Thus, instead of making general statements on the set of all possible storage functions, it is more reasonable to choose an arbitrary but fixed element $x^\star \in X$ and to restrict oneself to those storage functions V with the property that $V(x^\star) = 0$, i.e., to the set of normalized storage functions. The set of normalized storage functions associated with the system Σ_s, the supply rate s and the reference point $x^\star$ will be denoted by

$$\mathcal{V}(x^\star) := \{V : X \to \mathcal{R} \,|\, (\Sigma_s, s, V) \text{ is internally dissipative and } V(x^\star) = 0\}.$$

It turns out to be possible to identify a smallest element and a largest element in the set of normalized storage functions associated with a given internally dissipative system and reference point $x^\star$. For a given connected system $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ and supply rate s we define functions $V_a : X \to \mathcal{R} \cup \{\infty\}$ and $V_r : X \to \mathcal{R} \cup \{-\infty\}$ by:

$$V_a(x) := \sup\{-\int_0^{t_1} s(w(t))dt \,|\, t_1 \geq 0, (w, x) \in \mathcal{B}_s, x(0) = x, x(t_1) = x^\star\}$$

and

$$V_r(x) := \inf\{-\int_{t_{-1}}^{0} s(w(t))dt \,|\, t_{-1} \leq 0, (w, x) \in \mathcal{B}_s, x(0) = x, x(t_{-1}) = x^\star\}.$$

The function V_a is called the *available storage* of the dynamical system Σ_s: the quantity $V_a(x)$ is the maximum amount of supply that can be extracted from the system over all state trajectories connecting x to the reference point $x^\star$. The function V_r is called the *required supply* of the system Σ_s. The quantity $V_r(x)$ is equal to the minimum amount of supply that has to be delivered to the system in order to connect the reference point $x^\star$ to the point x. The following important theorem then holds:

Theorem 3.6 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant system in state space form and let s be a supply rate. Assume that Σ_s is connected. Let $x^\star \in X$. Then the following statements are equivalent:*

(a) *There exists $V : X \to \mathcal{R}$ such that (Σ_s, s, V) is internally dissipative,*

(b) *$V_a(x) < \infty$ for all $x \in X$,*

(c) *$V_r(x) > -\infty$ for all $x \in X$.*

If one of these statements holds then both $V_a \in \mathcal{V}(x^\star)$ as well as $V_r \in \mathcal{V}(x^\star)$. In addition, for all $V \in \mathcal{V}(x^\star)$ we have $V_a \leq V \leq V_r$.

Proof (a) $\Rightarrow$ (b) Let V be a storage function for (Σ_s, s) such that $V(x^\star) = 0$. Let $x_0 \in X$. By the dissipation inequality, for any $(w, x) \in \mathcal{B}_s$ such that $x(0) = x_0$ and $x(t_1) = x^\star$ we have $V(x_0) + \int_0^{t_1} s(w(t))dt \geq 0$. This yields $-\int_0^{t_1} s(w(t))dt \leq V(x_0)$. By taking the supremum on the left in the latter inequality we obtain $V_a(x_0) \leq V(x_0)$. (a) $\Rightarrow$ (c) Again let V be a storage function such that $V(x^\star) = 0$. Let $x_0 \in X$. For any $(w, x) \in \mathcal{B}_s$ such that $x(t_{-1}) = x^\star$ and $x(0) = x_0$ we have $\int_{t_{-1}}^{0} s(w(t))dt \geq V(x_0)$. This yields $V_r(x_0) \geq V(x_0)$. The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) follow from the fact that if V_a and V_r define real (finite) valued function on X then these functions satisfy the dissipation inequality. Consequently, V_a and V_r are storage functions so (Σ_s, s, V_a) and (Σ_s, s, V_r) are internally dissipative. We will now show that $V_a(x^\star) = 0$. The fact that $V_a(x^\star) \geq 0$ follows immediately from the definition (take $t_1 = 0$). To prove the converse inequality, let $(w, x) \in \mathcal{B}_s$ such that $x(0) = x^\star$ and $x(t_1) = x^\star$ Let V be a normalized storage function. It follows from the dissipation inequality that $-\int_0^{t_1} s(w(t))dt \leq 0$. Taking the supremum on the left in this inequality then yields $V_a(x^\star) \leq 0$. A proof that V_r is normalized can be given in a similar way. □

To summarize the above, we have shown that any possible normalized storage function of an internally dissipative system satisfies the *a priori* sharp inequality

$$V_a \leq V \leq V_r.$$

This can be interpreted by saying that an internally dissipative system can never supply to the outside more than what it has stored and can never store more than what has been supplied to it. Of course, not every function that satisfies the above inequality will be a storage function. We will state one more interesting property of the set of all normalized storage functions. It turns out that this set is *convex*:

Theorem 3.7 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant system in state space form and let s be a supply rate. Assume that there exists a function $V : X \to \mathcal{R}$ such that (Σ_s, s, V) is internally dissipative. Let $x^\star \in X$. Then for any pair of storage functions $V_1, V_2 \in \mathcal{V}(x^\star)$ and any $\alpha \in [0,1]$ we have $\alpha V_1 + (1-\alpha)V_2 \in \mathcal{V}(x^\star)$. Hence, if Σ_s is connected then $\alpha V_a + (1-\alpha)V_r \in \mathcal{V}(x^\star)$ for all $\alpha \in [0,1]$.* □

We shall now turn to the question whether for a given time-invariant system in state space form there exist storage functions that only take *non-negative values*, i.e., storage functions V with the property that $V(x_0) \geq 0$ for all $x_0 \in X$. Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant system in state space form and let s be a supply rate. First observe that there exists a storage function $V \geq 0$ if and only if there exists a storage function $\tilde{V}$ that is bounded from below, i.e., a storage function $\tilde{V}$ for which there exists a real number M such that $\tilde{V}(x_0) \geq M$ for all $x_0 \in X$. Indeed this follows immediately from the fact that if $\tilde{V}$ is a storage function and if $C \in \mathcal{R}$, then also $\tilde{V} + C$ is a storage function. In particular, if there exists a storage function $\tilde{V}$ with the property that there exists $x^\star \in X$ such that $\tilde{V}(x^\star) = \min_{x \in X} \tilde{V}(x)$ (i.e., $x^\star$ is a point of *minimal storage*) then there exists a storage function $V \geq 0$.

An important quantity in connection with the existence of non-negative storage functions is the *free-endpoint available storage*, which is defined by:

$$V_{a,f}(x_0) := \sup\{-\int_0^{t_1} s(w(t))dt \mid t_1 \geq 0, (w, x) \in \mathcal{B}_s, x(0) = x_0\}$$

Note that $V_{a,f}(x_0) \in [0, \infty) \cup \{\infty\}$ for all $x_0 \in X$. It turns out that the free-endpoint available storage is finite if and only if there exists a non-negative storage function. In fact, in this case the free endpoint available storage is the *smallest* non-negative storage function:

Theorem 3.8 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a time-invariant system in state space form and let s be a supply rate. There exists a storage function $V \geq 0$ for (Σ_s, s) if and only if $V_{a,f}(x_0) < \infty$ for all $x_0 \in X$. If this is the case then $V_{a,f}$ itself is a storage function and $0 \leq V_{a,f} \leq V$ for every storage function $V \geq 0$.*

Proof Let $V \geq 0$ be a storage function. For all $(w, x) \in \mathcal{B}_s$ such that $x(0) = x_0$ and for all $t_1 \geq 0$ we have

$$V(x_0) + \int_0^{t_1} s(w(t))dt \geq V(x(t_1)) \geq 0,$$

which implies

$$-\int_0^{t_1} s(w(t))dt \leq V(x_0).$$

Taking the supremum on the left in this inequality yields $V_{a,f}(x_0) \leq V(x_0)$. Conversely, if $V_{a,f}$ is a real valued function then it can easily be shown to satisfy the dissipation inequality. □

If there exists a non-negative storage function and if $x^\star \in X$ then we would like to know whether there also exists a *normalized* non-negative storage function, i.e., a storage function $V \geq 0$ with the property that $V(x^\star) = 0$. The set of normalized non-negative storage functions associated with the system in state space form Σ_s, supply rate s and reference point $x^\star \in X$ will be denoted by $\mathcal{V}_+(x^\star)$. We will show that there exists a normalized non-negative storage function if and only if there exists a (arbitrary) storage function for which $x^\star$ is a point of minimal storage. Furthermore, in that case the free endpoint available storage is the *smallest* normalized non-negative storage function and the required supply is the *largest* normalized non-negative storage function:

Theorem 3.9 *Let $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ be a connected time-invariant system in state space form and let s be a supply rate. Let $x^\star \in X$. Then the following statements are equivalent:*

(a) $\mathcal{V}_+(x^\star) \neq \emptyset$,

(b) *there exists a storage function V such that $V(x^\star) = \min_{x \in X} V(x)$,*

(c) $V_r(x) \geq 0$ *for all $x \in X$,*

(d) $V_{a,f}(x) < \infty$ *for all $x \in X$ and $V_{a,f}(x^\star) = 0$.*

If one of the above statements hold then we have $V_{a,f} \in \mathcal{V}_+(x^\star)$ and $V_r \in \mathcal{V}_+(x^\star)$. Furthermore, for any $V \in \mathcal{V}_+(x^\star)$ we have:

$$V_a \leq V_{a,f} \leq V \leq V_r.$$

Here, V_a and V_r denote the available storage and the required supply, respectively, taken with respect to the reference point $x^\star$.

Proof The implication (a) ⇒(b) is obvious. (b) ⇒(a) Define a new storage function $\tilde{V} \in \mathcal{V}(x^\star)$ by $\tilde{V}(x) := V(x) - V(x^\star)$. (a) ⇒(d) By th. 2.8 we have $V_{a,f} \leq V$ for any storage function $V \geq 0$. Thus, if there exists such V with the property that $V(x^\star) = 0$ then $V_{a,f}(x^\star) \leq 0$. Since the converse inequality always holds, we conclude that $V_{a,f}$ is normalized. (d) ⇒(a) Is obvious. (a) ⇒(c) By Th. 3.6, for every normalized storage function we have $V \leq V_r$. Thus, if there exists a non-negative normalized storage function then we have $V_r \geq 0$. (c) ⇒(a) If $V_r \geq 0$ then it is a non-negative storage function. It follows immediately from the definition that $V_r(x^\star) \leq 0$ and hence $V_r(x^\star) = 0$. □

Example 3.10 Consider a simple (nonlinear) series RLC-circuit, containing inductance, capacitance and resistance. The circuit interacts with its environment through the external signal (I, V), where I denotes the current into the network and where V is the voltage across the external terminals. Let V_R, I_R denote the voltage across and the current into the resistor, respectively. The characteristics of the resistor are described by

$$V_R = R(I_R)I_R, \tag{3.2}$$

where $R(.) \geq 0$ is a given smooth function. Likewise, let V_C, I_C be the voltage across and the current into the capacitor, respectively. If Q_C is the charge on the capacitor, then we have $Q_C = C(V_C)$ for some smooth fuction $C(.)$ with $C'(.) > 0$. Hence the characteristics of this element are given by

$$I_C = C'(V_C)\frac{dV_C}{dt}. \tag{3.3}$$

For the inductor, let the voltage and the current be denoted by V_L and I_L. If Φ_L is the flux, then we have $\Phi_L = L(I_L)$ for some smooth function $L(.)$ with $L'(.) > 0$. Hence the characteristics of the inductor are given by

$$V_L = L'(I_L)\frac{dI_L}{dt}. \tag{3.4}$$

Furthermore, the behavior of the network is governed by Kirchoff's laws:

$$I = I_R = I_C = I_L \tag{3.5}$$

$$V = V_R + V_C + V_L \tag{3.6}$$

Thus we see that the electrical network can be modelled as a dynamical system $\Sigma = (T, W, \mathcal{B})$ with $T = \mathcal{R}$, $W = \mathcal{R}^2$ and behavior $\mathcal{B}$ given by

$$\mathcal{B} = ((I,V) : \mathcal{R} \to \mathcal{R}^2 | \exists (I_R, V_R, I_C, V_C, I_L, V_L) : \mathcal{R} \to \mathcal{R}^6$$
$$\text{satisfying equations 3,4,5,6,7 }\}.$$

A state space representation of this dynamical system is described as follows. For the state variables we take I_L and V_C and for the state space we take $X = \mathcal{R}^2$. Define a function $f : \mathcal{R}^3 \to \mathcal{R}^2$ by

$$f(I_L, V_C, V) = \left(\frac{-V_C - R(I_L)I_L + V}{L'(I_L)}, \frac{I_L}{C'(V_C)}\right).$$

It is easily verified that $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$, with

$$\mathcal{B}_s = \{((I,V),(I_L,V_C)) : \mathcal{R} \to \mathcal{R}^4 | I = I_L \text{ and } (\dot{I}_L, \dot{V}_C) = f(I_L, V_L, V)\}$$

is a state space representation of Σ. The rate at which electrical energy flows into the network is given by the function $I(.)V(.)$. Thus it is reasonable to define a supply rate $s : \mathcal{R}^2 \to \mathcal{R}$ by

$$s(I,V) = IV.$$

The electrical energy in the capacitor in the presence of a voltage V_C is equal to

$$E_C(V_C) = \int_0^{V_C} vC'(v)dv,$$

and the electrical energy in the inductor in the presence of a current I_L is equal to

$$E_L(I_L) = \int_0^{I_L} iL'(i)di.$$

Denote the total amount of electrical energy in the network by

$$E(I_L, V_C) = E_C(V_C) + E_L(I_L).$$

It can easily be shown that (Σ_s, s, E) is internally dissipative, i.e., that $E(.,.)$ is a storage function.

4 Linear systems with quadratic supply rate

In the previous sections we have studied the concept of dissipativeness on a rather high level of generality. Most of our results were concerned with general time-invariant dynamical systems, while sometimes in addition we assumed the system to be complete. In the present section we shall make the additional assumption that the systems under consideration are *linear*. Furthermore, we shall be concerned with supply rates that are *quadratic functions*. We will study time-invariant dynamical systems $\Sigma = (\mathcal{R}, W, \mathcal{B})$ for which the signal alphabet W is equal to $\mathcal{R}^q$, with q a given positive integer. Such a dynamical system is called *linear* if its behavior $\mathcal{B}$ is a linear subspace of the real linear space $(\mathcal{R}^q)^{\mathcal{R}}$. The state space representations of the systems that will be considered in this section will all be finite-dimensional. A system in state space form Σ_s is called *finite-dimensional* if its state space X is equal to $\mathcal{R}^n$ for some positive integer n. A finite-dimensional system in state space form $\Sigma_s = (\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_s)$ is called linear if $\mathcal{B}_s$ is a linear subspace of $(\mathcal{R}^q \times \mathcal{R}^n)^{\mathcal{R}}$.

An important class of linear, time-invariant and complete dynamical systems is the class of systems for which there exists a state space representation in the form of a *driving variable representation.* A system in state space form $\Sigma_s = (\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_s)$ is said to have a driving variable representation if there exist a non-negative integer m and matrices $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{q \times n}$ and $D \in \mathcal{R}^{q \times m}$ such that the behavior $\mathcal{B}_s$ is equal to

$$\mathcal{B}_{DV}(A, B, C, D) := \{(w, x) : \mathcal{R} \to \mathcal{R}^q \times \mathcal{R}^n \mid \exists\, v \in L_{2,loc}(\mathcal{R}^+)$$

$$\text{such that } \dot{x} = Ax + Bv, w = Cx + Dv\}$$

It is easily seen that the quadruple $(\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_{DV}(A, B, C, D))$ indeed defines a linear, finite-dimensional, time-invariant dynamical system in state space form. This system will be denoted by $\Sigma_{DV}(A, B, C, D)$ or sometimes simply by Σ_{DV}. It is easy to see that the externally induced system of Σ_{DV} is linear, time-invariant and complete. In this section we shall restrict ourselves to linear, finite-dimensional and complete dynamical systems $\Sigma = (\mathcal{R}, W, \mathcal{B})$ for which there exists a state space representation $\Sigma_{DV} = (\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_{DV}(A, B, C, D))$ in driving variable representation. The driving variable state space representation Σ_{DV} of Σ is called *minimal* if both the integer m (i.e., the dimension of the linear space in which the driving variable v takes its values) as well as n (the dimension of the state space) are minimal (over the class of all driving variable representations of the given system Σ). It can be shown that if Σ_{DV} is minimal, then it is observable (in the sense of section 2, i.e., the state trajectory is uniquely determined by the external signal w). Furthermore, it is easily seen that $\Sigma_{DV}(A, B, C, D)$ is connected if and only if the pair (A, B) is controllable.

Throughout this section we will assume that the supply rate s is a quadratic function. More specifically, we will assume that $s : \mathcal{R}^q \to \mathcal{R}$ is given by $s(w) = w^T M w$, where $M \in \mathcal{R}^{q \times q}$ is a given symmetric matrix. It turns out that if the externally induced system of a state space system in driving variable representation is dissipative, then there exists at least one *quadratic* storage function. In fact:

Theorem 4.1 *Let $\Sigma_{DV}(A, B, C, D)$ be a state space representation of Σ and assume that Σ_{DV} is connected. Then the following statements are equivalent:*

(a) *There exists $V : \mathcal{R}^n \to \mathcal{R}$ such that (Σ_{DV}, s, V) is internally dissipative,*

(b) *There exists a symmetric matrix $K \in \mathcal{R}^{n \times n}$ such that if we define $V(x) := x^T K x$ then (Σ_{DV}, s, V) is internally dissipative,*

(c) *For all $(w, x) \in \mathcal{B}_{DV}(A, B, C, D)$ and for all $T \geq 0$ such that $x(0) = x(T) = 0$ we have $\int_0^T s(w(t))dt \geq 0$.*

If, in addition, Σ_{DV} is a minimal state space representation of Σ then any of the above statements is equivalent with:

(d) *(Σ, s) is dissipative.*

Proof (a) $\Rightarrow$(c) Let V be any storage function and let $(w, x) \in \mathcal{B}_{DV}$ be such that $x(0) = x(T) = 0$. It then follows immediately from the dissipation inequality that $\int_0^T s(w(t))dt \geq 0$. (c) $\Rightarrow$(b) Assuming that (c) holds, for $x_0 \in X$ define

$$V(x_0) := \sup\{-\int_0^{t_1} s(w(t))dt \mid t_1 \geq 0, (w, x) \in \mathcal{B}_{DV}, x(0) = x_0, x(t_1) = 0\}.$$

In a similar way as in the proof of Th. 3.6 it can be shown that V defines a storage function for (Σ_{DV}, s). Now, we claim that, in fact, V is a quadratic function of x_0. Indeed, this follows from [23] upon noting that s is quadratic so $V(x_0)$ represents the optimal cost of a linear quadratic optimization problem. (b) $\Rightarrow$(a) Is obvious. (d) $\Rightarrow$(c) If $(w, x) \in \mathcal{B}_{DV}$ is such that $x(0) = x(T)$, then clearly $\mathcal{B}^+(w) = \mathcal{B}^+(\sigma^T w)$. By completeness, the system Σ is forward dissipative. It follows that $\int_0^T s(w(t))dt \geq 0$. Finally, the implication (a) $\Rightarrow$(d) follows from Th. 3.3 (use minimality). □

The set of all quadratic storage functions of a given internally dissipative system $\Sigma_{DV}(A, B, C, D)$ can be characterized as the set of real symmetric solutions of a linear matrix inequality involving the system parameters A, B, C and D and the symmetric matrix M defining the supply rate s. Indeed, if $V(x) = x^T K x$ is a quadratic storage function then the dissipation inequality can be reformulated as: for all $(t_0, t_1) \in \mathcal{R}_+^2$ and for all $(w, x) \in \mathcal{B}_{DV}(A, B, C, D)$ we have

$$\int_{t_0}^{t_1} \left(-\frac{d}{dt}(x(t)^T K x(t)) + w(t)^T M w(t) \right) dt \geq 0.$$

Since $(w, x) \in \mathcal{B}_{DV}(A, B, C, D)$ if and only if there exists $v \in L_{2,loc}(\mathcal{R}^+)$ such that $\dot{x} = Ax + Bv$, $w = Cx + Dv$, the latter inequality can be seen to be equivalent to: for all $v \in L_{2,loc}(\mathcal{R}^+)$ and for all x such that $\dot{x} = Ax + Bv$ we have

$$\int_{t_0}^{t_1} \left(x(t)^T, v(t)^T\right) \begin{pmatrix} -A^TK - KA + C^TMC & -KB + C^TMD \\ -B^TK + D^TMC & D^TMD \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} dt \geq 0.$$

It is easily seen that the latter is equivalent with the single requirement that the matrix K satisfies the *linear matrix inequality* (LMI):

$$\begin{pmatrix} -A^TK - KA + C^TMC & -KB + C^TMD \\ -B^TK + D^TMC & D^TMD \end{pmatrix} \geq 0.$$

This leads to the following result:

Theorem 4.2 *Consider the system $\Sigma_{DV}(A, B, C, D)$, together with the quadratic supply rate $s(w) = w^TMw$. Let $K \in \mathcal{R}^{n \times n}$ be a symmetric matrix. Then the following statements are equivalent:*

(a) *$V(x) = x^TKx$ is a storage function for (Σ_{DV}, s),*

(b) *K is a solution of the linear matrix inequality LMI.*

If there exists a function $V : \mathcal{R}^n \to \mathcal{R}$ such that (Σ_{DV}, s, V) is internally dissipative and if Σ_{DV} is connected then there exist symmetric solutions $K^-, K^+ \in \mathcal{R}^{n \times n}$ of the LMI such that for any symmetric solution $K \in \mathcal{R}^{n \times n}$ of the LMI we have

$$K^- \leq K \leq K^+.$$

In fact, for all $x \in \mathcal{R}^n$ we have

$$V_a(x) = x^TK^-x, \; V_r(x) = x^TK^+x.$$

Here, V_a and V_r denote the available storage and the required supply with respect to the reference point $x^* = 0$.

Proof The equivalence of statements (a) and (b) was already proven in the above. Consider the available storage V_a and the required supply V_r. Again, being defined as the optimal costs corresponding to linear quadratic optimization problems, these functions are quadratic. Let K^- and K^+ be real symmetric matrices such that $V_a(x) = x^TK^-x$ and $V_r(x) = x^TK^+x$ for all $x \in \mathcal{R}^n$. By the above, K^- and K^+ are solutions to the LMI. The claim that these solutions are extremal then follows from Th. 3.6 (with $x^* = 0$). □

In addition to the necessary and sufficient conditions that were derived in Th. 4.1, we will now derive a frequency domain condition for internal dissipativeness. In order to be able to explain the idea, assume for the moment that we allow all signals to be complex valued. Consider the system $\Sigma_{DV}(A, B, C, D)$ and define a real rational matrix $G(s)$ by $G(s) := C(Is - A)^{-1}B + D$. We contend that if the complex number $i\omega$ is not an eigenvalue of the matrix A, then for each $v_0 \in \mathcal{C}^m$ the signal

$$(w, x) = (e^{i\omega t}G(i\omega)v_0, e^{i\omega t}(Ii\omega - A)^{-1}Bv_0)$$

is an element of (the complexification of) $\mathcal{B}_{DV}(A, B, C, D)$. Indeed, as driving variable take $v(t) := e^{i\omega t}v_0$ and as initial condition take $x_0 := (Ii\omega - A)^{-1}Bv_0$. It is a matter of straightforward calculation to verify that the signal (w, x) given above indeed satisfies the equations $\dot{x} = Ax + Bv$, $w = Cx + Dv$. Now, let the symbol $*$ denote conjugate transpose. We obviously have

$$v_0^* G(-i\omega)^T M G(i\omega) v_0 = v(t)^* G(-i\omega)^T M G(i\omega) v(t) = w(t)^* M w(t) = s(w(t)).$$

Assume now that Σ_{DV} is internally dissipative and let V be a storage function. Since x is periodic (with period $2\pi/\omega$), by the dissipation inequality we obtain $\int_0^{2\pi/\omega} s(w(t))dt \geq 0$. This immediately implies that $v_0^* G(-i\omega)^T M G(i\omega) v_0 \geq 0$. Thus we find that the hermitian matrix $G(-i\omega)^T M G(i\omega)$ is positive semi-definite. In this way we are lead to the following result:

Theorem 4.3 *Assume that the system $\Sigma_{DV}(A, B, C, D)$ is connected. Then the following statements are equivalent:*

(a) *(Σ_{DV}, s) is internally dissipative,*

(b) *$G(-i\omega)^T M G(i\omega) \geq 0$ for all $\omega \in \mathcal{R}$, $i\omega \notin \sigma(A)$.*

Proof A proof of the implication (a) $\Rightarrow$(b) can be given using the above ideas. We will prove the converse implication (b) $\Rightarrow$(a). Let $(w, x) \in \mathcal{B}_{DV}$ be such that $x(0) = 0$ and $x(T) = 0$. Define $(\bar{w}, \bar{x})$ as the concatenation $(0,0) \underset{0}{\wedge} (w, x) \underset{T}{\wedge} (0,0)$. Then clearly $(\bar{w}, \bar{x}) \in \mathcal{B}_{DV}$. Let v be a corresponding driving variable. We can take $v(t) = 0$ for $t \notin [0, T]$. Let $x(i\omega)$, $v(i\omega)$ and $w(i\omega)$ be the Fourier-transforms of $\bar{x}$, v and $\bar{w}$, respectively. Then for all $i\omega \notin \sigma(A)$ we have $x(i\omega) = (Ii\omega - A)^{-1} Bv(i\omega)$ so $w(i\omega) = G(i\omega)v(i\omega)$. Using Parseval's theorem this yields $\int_0^T s(w(t))dt \geq 0$. The claim then follows from Th. 4.1 $\square$

Summarizing, we obtain the following result on the existence of solutions of the linear matrix inequality LMI:

Corollary 4.4 *Let $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{q \times n}$ and $D \in \mathcal{R}^{q \times m}$. Assume that (A, B) is a controllable pair. Let $M \in \mathcal{R}^{q \times q}$ be symmetric. Define $G(s) := C(Is - A)^{-1}B + D$. Then the following two statements are equivalent:*

(a) *the LMI has a symmetric solution $K \in \mathcal{R}^{n \times n}$,*

(b) *$G(-i\omega)^T M G(i\omega) \geq 0$ for all $\omega \in \mathcal{R}$, $i\omega \notin \sigma(A)$.*

In that case there exist symmetric solutions $K^-, K^+ \in \mathcal{R}^{n \times n}$ such that for any symmetric solution $K \in \mathcal{R}^{n \times n}$ we have $K^- \leq K \leq K^+$. $\square$

We will now study the existence of non-negative storage functions in the context of linear systems with quadratic supply rate. As a first observation, note that for a given linear system $\Sigma_{DV}(A, B, C, D)$ there is a one-to-one correspondence between the set of all non-negative *quadratic* storage functions and the set of real *positive semi-definite* solutions of the linear matrix inequality LMI. We will show that if there exists a non-negative storage function, then there always exists a non-negative *quadratic* storage function and hence a positive semi-definite solution to the LMI. This result will imply that if the linear matrix inequality has a real positive semi-definite solution, then it always has a *smallest* real positive semi-definite solution and a *largest* real positive semi-definite solution.

Theorem 4.5 *Consider the system $\Sigma_{DV}(A, B, C, D)$, together with the quadratic supply rate $s(w) = w^T M w$. Assume that Σ_{DV} is connected. Then the following statements are equivalent:*

(a) *There exists a storage function $V \geq 0$ for (Σ_{DV}, s),*

(b) *There exists* $K \in \mathcal{R}^{n \times n}$, $K \geq 0$ *such that* $V(x) := x^T K x$ *is a storage function for* (Σ_{DV}, s),
(c) *The LMI has a solution* $K \geq 0$,
(d) *For all* $(w, x) \in \mathcal{B}_{DV}$ *such that* $x(0) = 0$ *and* $T \geq 0$ *we have* $\int_0^T s(w(t))dt \geq 0$,
(e) $V_r(x) \geq 0$ *for all* $x \in \mathcal{R}^n$,
(f) $V_{a,f}(x) < \infty$ *for all* $x \in \mathcal{R}^n$.

If one of these statements holds then we have $K^+ \geq 0$ *and there exists a real solution* $K_f \geq 0$ *of the LMI such that for any real solution* $K \geq 0$ *of the LMI we have*

$$K_f \leq K \leq K^+.$$

In fact, for all $x \in \mathcal{R}^n$ *we have* $V_{a,f}(x) = x^T K_f x$ *and* $V_r(x) = x^T K^+ x$. *Here,* V_r *denotes the required supply with respect to the reference point* $x^* = 0$.

Proof The equivalences (a) $\Leftrightarrow$ (f) and (b) $\Leftrightarrow$ (c) follow immediately from Th. 3.9 and Th. 4.2, respectively. The implications (e) $\Rightarrow$(a) and (b) $\Rightarrow$(e) follow from Th. 3.9. The claim (f) $\Rightarrow$(b) follows by noting that $V_{a,f}(x_0)$ is the optimal cost of a linear quadratic optimization problem. Thus, there exists a real matrix $K_f \geq 0$ such that $V(x_0) = x_0^T K_f x_0$. The implication (b) $\Rightarrow$(d) follows by writing out the dissipation inequality for the quadratic storage function $V(x) = x^T K x$. A proof of the implication (d) $\Rightarrow$(f) can be given completely analogously to the proof of Th. 3.4. The remaining statements follow from Th. 3.9 and Th. 4.2. □

The condition appearing in statement (d) of the above theorem is often used as a definition of *passivity* of the system Σ_{DV}: the system Σ_{DV} is called passive if for any signal $(w, x) \in \mathcal{B}_{DV}$ with $x(0) = 0$ we have $\int_0^T s(w(t))dt \geq 0$ for all $T \geq 0$. Thus we see that a connected system is passive if and only if there exists a non-negative storage function.

To conclude this section we shall derive frequency domain conditions for the existence of non-negative storage functions for the linear system $\Sigma_{DV}(A, B, C, D)$. Again, define $G(s) := C(Is - A)^{-1}B + D$ and for the moment, assume that we allow all signals to be complex valued. Let s be a complex number that is not an eigenvalue of A. Then for each $v_0 \in \mathcal{C}^m$ the signal

$$(w, x) = (e^{st}G(s)v_0, e^{st}(Is - A)^{-1}Bv_0)$$

is an element of the complexification of $\mathcal{B}_{DV}(A, B, C, D)$. Indeed, if we take as driving variable $v(t) := e^{st}v_0$ then the equations $\dot{x} = Ax + Bv$, $w = Cx + Dv$ are satisfied. Let $\bar{s}$ denote the complex conjugate of s. It is easily verified that

$$e^{2t\Re e\, s} v_0^* G(\bar{s})^T M G(s) v_0 = v(t)^* G(\bar{s})^T M G(s) v(t) = w(t)^* M w(t) = s(w(t)).$$

Now, assume that there exists a non-negative storage function, say, $V \geq 0$. Then the free endpoint available storage $V_{a,f}(x_0)$ is finite and for all $T \geq 0$ we must have

$$-\int_0^T s(w(t))dt \leq V_{a,f}(x_0)$$

By the above, this implies that for all $s \notin \sigma(A)$, for all $T \geq 0$ we have

$$\int_0^T e^{2t\Re e\, s} dt \,.\, v_0^* G(\bar{s})^T M G(s) v_0 \geq -V_{a,f}(x_0).$$

This can of course only be true if for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$ and $\Re e\ s \geq 0$, we have $v_0^* G(\bar{s})^T M G(s) v_0 \geq 0$. This argument holds for all $v_0 \in \mathcal{C}^m$ and thus we find that the hermitian matrix $G(\bar{s})^T M G(s)$ must be positive semi-definite for all such s. This leads to the following theorem:

Theorem 4.6 *Assume that the system $\Sigma_{DV}(A, B, C, D)$ is connected. Then the following statements are equivalent:*

(a) *There exists a non-negative storage function for (Σ_{DV}, s)*

(b) *$G(\bar{s})^T M G(s) \geq 0$ for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$, $\Re e\ s \geq 0$.* □

Summarizing, we obtain the following results on the existence of positive semi-definite solutions of the LMI:

Corollary 4.7 *Let $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{q \times n}$ and $D \in \mathcal{R}^{q \times m}$. Assume that (A, B) is a controllable pair. Let $M \in \mathcal{R}^{q \times q}$ be symmetric. Define $G(s) := C(Is - A)^{-1}B + D$. Then the following statements are equivalent:*

(a) *the LMI has a real symmetric solution $K \geq 0$,*

(b) *$G(\bar{s})^T M G(s) \geq 0$ for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$, $\Re e\ s \geq 0$.*

In that case the LMI has real symmetric solutions K^-, K_f and K^+ with $K_f \geq 0$ and $K^+ \geq 0$ with the following properties: for any real symmetric solution K of the LMI we have $K^- \leq K \leq K^+$ and for any real symmetric solution $K \geq 0$ we have $K_f \leq K \leq K^+$. □

5 Positive real and bounded real functions

A special case of the class of systems considered in the previous section is the class of linear, finite-dimensional, time-invariant systems in input/state/output form. Such a system is defined as a time-invariant linear system in state space form $\Sigma_s = (\mathcal{R}, W, X, \mathcal{B}_s)$ for which the signal alphabet W is equal to the cartesian product $U \times Y$, with $U = \mathcal{R}^m$ and $Y = \mathcal{R}^p$, the state space X is equal to $\mathcal{R}^n$ and $\mathcal{B}_s$ is equal to

$$\mathcal{B}_{i/s/o}(A, B, C_0, D_0) = \{(\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), x) : \mathcal{R} \to \mathcal{R}^m \times \mathcal{R}^p \times \mathcal{R}^n \mid u \in L_{2,loc}(\mathcal{R}^+)$$

$$\text{and } \dot{x} = Ax + Bu,\ y = C_0 x + D_0 u\}.$$

Here, A, B, C_0 and D_0 are matrices in $\mathcal{R}^{n \times n}$, $\mathcal{R}^{n \times m}$, $\mathcal{R}^{p \times n}$ and $\mathcal{R}^{p \times m}$, respectively. Thus, the external signal w is equal to $\mathrm{col}(u, y)$. The function u is called the *input*, the function y is called the *output*. Likewise, $U = \mathcal{R}^m$ is called the *input alphabet*, while $Y = \mathcal{R}^p$ is called the *output alphabet*. The system $(\mathcal{R}, \mathcal{R}^m \times \mathcal{R}^p, \mathcal{R}^n, \mathcal{B}_{i/s/o}(A, B, C_0, D_0))$ will be denoted by $\Sigma_{i/s/o}(A, B, C_0, D_0)$ or simply by $\Sigma_{i/s/o}$. The *transfer matrix* of $\Sigma_{i/s/o}$ is defined as the real rational matrix $G_0(s) = C_0(Is - A)^{-1}B + D_0$. It is easy to see that any system $\Sigma_{i/s/o}$ can be considered as a system in state space form in driving variable representation Σ_{DV}, with the driving variable equal to the input. Indeed, we always have

$$\Sigma_{i/s/o}(A, B, C_0, D_0) = \Sigma_{DV}(A, B, C, D),$$

with

$$C := \begin{pmatrix} 0 \\ C_0 \end{pmatrix}, D := \begin{pmatrix} I \\ D_0 \end{pmatrix}.$$

Now, let $\Sigma_{i/s/o}(A, B, C_0, D_0)$ be given. Assume that $U = Y$ so, in particular, that $p = m$. By taking the supply rate s defined by $s(u, y) = u^T y$ or, equivalently, $s(w) = w^T M w$ with M given by

$$M := \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

we obtain the following special case of Cor. 4.7:

Corollary 5.1 *Assume that the system $\Sigma_{i/s/o}(A, B, C_0, D_0)$ is connected (equivalently: the pair (A, B) is controllable). Let $G_0(s) = C_0(Is - A)^{-1}B + D_0$ be the transfer matrix of $\Sigma_{i/s/o}$. Define a supply rate by $s(u, y) = u^T y$. Then the following statements are equivalent:*

(a) *There exists a function $V \geq 0$ such that $(\Sigma_{i/s/o}, s, V)$ is internally dissipative,*

(b) *$G_0(\bar{s})^T + G_0(s) \geq 0$ for all $s \in C$ with $s \notin \sigma(A)$ and $\Re e\, s \geq 0$,*

(c) *the linear matrix inequality*

$$\begin{pmatrix} -A^T K - KA & -KB + C_0^T \\ -B^T K + C_0 & D_0^T + D_0 \end{pmatrix} \geq 0 \tag{5.1}$$

has a real symmetric solution $K \geq 0$.

In that case there exist real symmetric solutions $K_f \geq 0$ and $K^+ \geq 0$ such that for any solution $K \geq 0$ we have $K_f \leq K \leq K^+$. □

A transfer function $G_0(s)$ that satisfies the condition in statement (b) of the above corollary is called *positive real.* The result on the equivalence of statements (b) and (c) is known as the *positive real lemma* or the *Kalman-Yakubovich-Popov lemma* and plays an important role in stability theory of control systems (see [46,47,17,1,27], see also section 9). It also plays an important role in the synthesis theory of passive networks (see [4,18], see also section 10) and in the covariance generation problem (see section 11). It can in fact be shown that if $G_0(s)$ is positive real, then *all* real symmetric solutions of the linear matrix inequality 4.1 satisfy the inequality $K_f \leq K \leq K^+$. In particular this implies that if the LMI has a real positive semi-definite solution, then all real symmetric solutions are positive semi-definite and the smallest (overall) real symmetric solution coincides with the smallest positive semi-definite solution, i.e., $K^- = K_f$.

Let us consider one more example. Again consider the system $\Sigma_{i/s/o}(A, B, C_0, D_0)$. Define a supply rate s by $s(u, y) = \|u\|^2 - \|y\|^2$ or, equivalently, by $s(w) = w^T M w$ with M the symmetric matrix given by

$$M := \begin{pmatrix} I_m & 0 \\ 0 & -I_p \end{pmatrix}.$$

Here, I_m and I_p denote the $m \times m$ and $p \times p$ identity matrices, respectively. As a special case of Cor. 4.4 we then obtain:

Corollary 5.2 *Assume that the system $\Sigma_{i/s/o}(A, B, C_0, D_0)$ is connected (equivalently: the pair (A, B) is controllable). Let $G_0(s) = C_0(Is - A)^{-1}B + D_0$ be the transfer matrix of $\Sigma_{i/s/o}$. Define a supply rate by $s(u, y) = \|u\|^2 - \|y\|^2$. Then the following statements are equivalent:*

(a) *There exists a function* $V \geq 0$ *such that* $(\Sigma_{i/s/o}, s, V)$ *is internally dissipative,*
(b) $G_0(\bar{s})^T G(s) \leq I$ *for all* $s \in \mathbb{C}$ *with* $s \notin \sigma(A)$ *and* $\Re e\, s \geq 0$*,*
(c) *the linear matrix inequality*

$$\begin{pmatrix} -A^T K - KA - C_0^T C_0 & KB - C_0^T D_0 \\ B^T K - D_0^T C_0 & I - D_0^T D_0 \end{pmatrix} \geq 0 \tag{5.2}$$

has a real symmetric solution $K \geq 0$.

In that case there exist real symmetric solutions $K_f \geq 0$ *and* $K^+ \geq 0$ *such that for any solution* $K \geq 0$ *we have* $K_f \leq K \leq K^+$. □

A transfer matrix $G_0(s)$ that satisfies the condition in statement (b) of the above corollary is called *bounded real.* The result on the equivalence between (b) and (c) is known as the *bounded real lemma* (see [4]).

6 The dissipation rate

Let $\Sigma_{DV}(A, B, C, D)$ be a linear finite-dimensional system in driving variable representation, let $s(w) = w^T M w$ be a quadratic supply rate and let $V(x) = x^T K x$ be a quadratic storage function. If the system produces the signal (w, x), then the amount of supply that is dissipated in the system during the time-interval $[t_0, t]$ is equal to

$$V(x(t_0)) - V(x(t)) + \int_{t_0}^{t} s(w(t))dt.$$

The rate at which the supply is dissipated is equal to the derivative of this function and is equal to

$$\frac{d}{dt}\left(-x(t)^T K x(t)\right) + w(t)^T M w(t)$$

If v is a driving variable associated with the signal (w, x) (i.e., (x, w, v) satisfies $\dot{x} = Ax + Bv, w = Cx + Dv$) then the latter can be seen to be equal to

$$\left(x(t)^T, v(t)^T\right) F(K) \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}.$$

where

$$F(K) := \begin{pmatrix} -A^T K - KA + C^T M C & -KB + C^T M D \\ -B^T K + D^T M C & D^T M D \end{pmatrix}.$$

Since $V(x) = x^T K x$ is a storage function, we know that the real symmetric matrix K satisfies the linear matrix inequality $F(K) \geq 0$. Thus we can factorize

$$F(K) = \begin{pmatrix} M_K^T \\ N_K^T \end{pmatrix} \begin{pmatrix} M_K & N_K \end{pmatrix}, \tag{6.1}$$

with $(M_K\ N_K)$ a suitable real matrix with $n + m$ columns and, say, r rows. Therefore the rate at which the supply is dissipated 'along' a signal (w, x) with driving variable v is given by

$$\|M_K x(t) + N_K v(t)\|^2.$$

The quadratic function $d(x,v) = \|M_K x + N_K v\|^2$ is called a *dissipation rate* associated with the quadratic storage function $V(x) = x^T K x$. We note that there is no a priori upper bound to the number of rows r of the matrix $(M_K \; N_K)$. Of course, the number of rows of $(M_K \; N_K)$ is equal to the rank of $F(K)$ and, of course, $r = \text{rank}F(K)$ if and only if the matrix $(M_K \; N_K)$ is of full row rank. There is a close connection between the factorization 6.1 of $F(K)$ and certain factorizations of the matrices $G(-s)^T MG(s)$ and, more general, $G(z)^T MG(s)$. Indeed, if we factorize $F(K)$ and define a real rational matrix $W_K(s)$ by

$$W_K(s) := N_K + M_K(Is - A)^{-1}B \tag{6.2}$$

then it can be shown by straightforward calculation that for all $s, z \in \mathcal{C}$ with $s, z \notin \sigma(A)$ we have

$$G(z)^T MG(s) = W_K(z)^T W_K(s) + (s+z)B^T(Iz - A^T)^{-1}K(Is - A)^{-1}B.$$

From this, it immediately follows that

$$G(-s)^T MG(s) = W_K(-s)^T W_K(s) \tag{6.3}$$

(where the latter should be interpreted as an equality between real rational matrices). A factorization 6.3 of the real rational matrix $G(-s)^T MG(s)$ is often called a *spectral factorization* (see [48,1,2,3,4]). The real rational matrix $W_K(s)$ is called the spectral factor corresponding to the solution K of the linear matrix inequality. The spectral factorization equation 6.3 plays an important role in stochastic realization theory and filtering [10,11].

The above can be used to obtain an a priori lower bound to the number r of rows of the matrix $(M_K \; N_K)$ in the factorization of $F(K)$. Let r^* denote the rank of the real rational matrix $G(-s)^T MG(s)$ (i.e., the rank of the latter matrix considered as a matrix with entries in the field of real rational functions) Furthermore, let $r(K)$ denote the rank of the real matrix $F(K)$. The following result states that for every real symmetric solution of the linear matrix inequality we have $r(K) \geq r^*$, while this lower bound is attained for the matrix K if and only if the corresponding rational matrix $W_K(s)$ is right-invertible:

Theorem 6.1 *Let $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{q \times n}$ and $D \in \mathcal{R}^{q \times n}$ and let $M \in \mathcal{R}^{q \times q}$ be symmetric. Assume that the LMI has a symmetric solution, i.e., there exists $K \in \mathcal{R}^{n \times n}$ such that $F(K) \geq 0$. Then for each symmetric solution $K \in \mathcal{R}^{n \times n}$ of the LMI we have $r(K) \geq r^*$. Furthermore, if $K \in \mathcal{R}^{n \times n}$ is a symmetric solution of the LMI and if we factorize $F(K)$ as in 6.1 with $(M_K \; N_K)$ of full row rank then $r(K) = r^*$ if and only if the real rational matrix $W_K(s)$ has full row rank (considered as a matrix with entries in the field of real rational functions).*

Proof Factorize the matrix $F(K)$ as in 6.1, such that $(M_K \; N_K)$ is of full row rank. The number of rows of $(M_K \; , N_K)$ is then equal to $r(K)$. Let $W_K(s)$ be given by 6.2. Since the number of rows of $W_K(s)$ is equal to the number of rows of $(M_K \; N_K)$, the number of rows of $W_K(s)$ is equal to $r(K)$. On the other hand, it follows from 6.3 that the number of rows of $W_K(s)$ is larger that or equal to r^*, with equality if and only if $W_K(s)$ is a full row rank real rational matrix. This completes the proof. □

At this point of course the question arises whether there always *exists* a symmetric solution to the LMI such that the lower bound r^* is actually attained or, equivalently, such that the real rational matrix $W_K(s)$ is right-invertible. It can indeed be shown that if the pair (A, B) is controllable and if the LMI has at least one real symmetric solution, then for the largest solution K^+ as well as for the smallest solution K^- the lower bound r^* is attained, i.e., $r(K^+) = r(K^-) = r^*$. Also, if the LMI has least one positive semi-definite solution then the lower bound is attained by the smallest positive semi-definite solution K_f, i.e., $r(K_f) = r^*$ (see [30,13]). In the following section we shall give a proof of these facts for the special case that the matrix D^TMD is positive definite.

7 The algebraic Riccati equation

Consider again the driving variable system $\Sigma_{DV}(A, B, C, D)$ studied in section 4 and let $s(w) = w^TMw$ be a quadratic supply rate. In the present section we will assume that the following regularity assumption holds:

$$D^TMD > 0.$$

If this assumption holds then a real symmetric matrix K satisfies the linear matrix inequality $F(K) \geq 0$ if and only if it satisfies the inequality $R(K) \geq 0$, where $R(K)$ is defined by

$$R(K) := -A^TK - KA + C^TMC-$$
$$(-KB + C^TMD)(D^TMD)^{-1}(-B^TK + D^TMC).$$

(This follows by noting that $R(K)$ is equal to the Schur-complement of $-A^TK - KA + C^TMC$ in the matrix $F(K)$). The equation

$$R(K) = 0$$

is called the *algebraic Riccati equation* (ARE) associated with $\Sigma_{DV}(A, B, C, D)$ and the quadratic supply rate $s(w) = w^TMw$. Of course, if K is a real symmetric solution of the ARE then it also satifies the linear matrix inequality LMI and hence $V(x) = x^TKx$ defines a quadratic storage function for (Σ_{DV}, s). Thus, if the ARE has a real symmetric solution and if, in addition, the pair (A, B) is controllable (equivalently: the system $\Sigma_{DV}(A, B, C, D)$ is connected), then we have $G(-i\omega)^TMG(i\omega) \geq 0$ for all $i\omega \notin \sigma(A)$. In this section it will be shown that also the converse holds: if the inequality $G(-i\omega)^TMG(i\omega) \geq 0$ holds for all $i\omega \notin \sigma(A)$, and if the pair (A, B) is controllable, then the ARE has a real symmetric solution. Furthermore, it will turn out that in this case the algebraic Riccati equation has a *smallest* real symmetric solution and a *largest* real symmetric solution which exactly coincide with the smallest real symmetric solution K^- and the largest real symmetric solution K^+, respectively, of the linear matrix inequality LMI. In order to prove these claims, first note that under the regularity assumption $D^TMD > 0$ we have the following explicit expression for the rank of the matrix $F(K)$:

$$r(K) = m + \text{rank } R(K). \tag{7.1}$$

From this we immediately see that for every real symmetric matrix K we have $r(K) \geq m$, with equality if and only if K satisfies the algebraic Riccati equation $R(K) = 0$. Now, let $\Sigma_{DV}(A,B,C,D)$ be connected, assume that $G(-i\omega)^T MG(i\omega) \geq 0$ for all $i\omega \notin \sigma(A)$ and let K^- and K^+ denote the smallest and the largest real symmetric solution of the linear matrix inequality, respectively. We claim that both K^- as well as K^+ are solutions to the algebraic Riccati equation. By the above, this is equivalent to saying that $r(K^-) = m$ and $r(K^+) = m$. Following the notation introduced in the previous section, let r^* denote the rank of the real rational matrix $G(-s)^T MG(s)$. It is immediately clear that under the regularity condition $D^T MD > 0$ we have $r^* = m$. Next, recall from the previous section that if K is a real symmetric solution of the LMI and if we factorize $F(K)$ as in 6.1 with $(M_K \ N_K)$ of full row rank, then $r(K) = r^*$ if and only if the real rational matrix $W_K(s)$ defined by 6.2 has full row rank. Thus, we conclude that K^- and K^+ are solutions to the ARE if and only if $W_{K^-}(s)$ and $W_{K^+}(s)$ are full row rank real rational matrices. This indeed follows from the following lemma:

Lemma 7.1 *Let $\Sigma_{DV}(A,B,C,D)$ be a connected system in driving variable representation and assume that D has full column rank. Define a supply rate s by $s(w) := \|w\|^2$ (i.e,. take $M = I$). Let*

$$V_a(x_0) = \sup\{-\int_0^{t_1} \|w(t)\|^2 dt \mid (w,x) \in \mathcal{B}_{DV}(A,B,C,D), t_1 \geq 0,$$
$$x(0) = 0 \text{ and } x(t_1) = 0\}$$

be the available storage. Then we have: if $V_a(x_0) = 0$ for all x_0 then the real rational matrix $G(s) := C(Is - A)^{-1}B + D$ has full row rank.

Proof Let $v \in L_{2,loc}(\mathcal{R}^+)$ denote the driving variable of $\mathcal{B}_{DV}(A,B,C,D)$. For all $x_0 \in \mathcal{R}^n$ we have

$$J^*(x_0) := \inf\{\int_0^{t_1} \|w(t)\|^2 dt \mid v \in L_{2,loc}(\mathcal{R}^+), t_1 \geq 0 \text{ such}$$
$$\text{that } x(0) = x_0, x(t_1) = 0\} = 0.$$

For $T \geq 0$ define

$$J_T^*(x_0) = \inf\{\int_0^{t_1} \|w(t)\|^2 dt \mid v \in L_{2,loc}(\mathcal{R}^+), t_1 \geq T \text{ such that}$$
$$x(0) = x_0, x(t_1) = 0\}.$$

It is easily shown that for all $T \geq 0$ we have $J_T^*(x_0) = J^*(x_0)$ and hence that $J_T^*(x_0) = 0$. In the following, take a fixed $T > 0$. Obviously we have

$$\inf\{\int_0^T \|w(t)\|^2 dt \mid v \in L_{2,loc}(\mathcal{R}^+),\ x(0) = x_0\} \leq J_T^*(x_0) \ = \ 0.$$

Substitute $v = -(D^T D)^{-1} D^T Cx + u$ and denote $\tilde{A} = A - B(D^T D)^{-1} D^T C$ and $\tilde{C} = C - D(D^T D)^{-1} D^T C$. Then our system equations become $\dot{x} = \tilde{A}x + Bu, w = \tilde{C}x + Du$ and we find that

$$\inf\{\int_0^T \|\tilde{C}x(t)\|^2 + \|Du(t)\|^2 dt \mid u \in L_{2,loc}(\mathcal{R}^+),\ x(0) = x_0\} \ = \ 0$$

Now, let $\{u_n\}$ be an infimizing sequence with x_n the corresponding state trajectories with $x_n(0) = x_0$. Then, since D has full column rank, we have $u_n \to 0$ and $\tilde{C}x_n \to 0$ as $n \to \infty$ in $L_2[0,T]$. Denote $x(t) := e^{At}x_0$. Since $u_n \to 0$ in $L_2[0,T]$, we also have $x_n \to x$ in $L_2[0,T]$ and hence $\tilde{C}x_n \to \tilde{C}x$. This implies that $\tilde{C}x$ is identically equal to zero on $[0,T]$. In particular we have $\tilde{C}x_0 = 0$. Since the latter holds for all x_0, we conclude that $\tilde{C} = 0$. Now define a real rational matrix $R(s)$ by

$$R(s) = -(D^TD)^{-1}D^TC(Is - A + B(D^TD)^{-1}D^TC)^{-1}B.$$

It is a easily seen that $R(s)$ is a right inverse of $G(s)$. □

Using the previous lemma we will prove that W_{K^+} is has full row rank. Let $t_1 \geq 0$, let $(w,x) \in \mathcal{B}_{DV}$ be such that $x(0) = x_0$ and $x(t_1) = 0$ and let v be a driving variable associated with (w,x). As in the previous section it can be seen that

$$-\int_0^{t_1} s(w(t))dt = x_0^T K^+ x_0 - \int_0^{t_1} \|M_{K^+}x(t) + N_{K^+}v(t)\|^2 dt.$$

By taking suprema with respect to t_1 and $(w,x) \in \mathcal{B}_{DV}$ such that $x(0) = x_0$ and $x(t_1) = 0$ on both sides in this equality we find that

$$\sup\{-\int_0^{t_1} \|M_{K^+}x(t) + N_{K^+}v(t)\|^2 dt \mid t_1 \geq 0, v \in L_{2,loc}(\mathcal{R}^+),\ x(0) = x_0 \text{ and } x(t_1) = 0\} = 0.$$

The latter statement can be reformulated as: the available storage of the auxiliary system $\Sigma_{DV}(A, B, M_{K^+}, N_{K^+})$ with supply rate given by $M = I$ is equal to zero for every initial condition. According to the previous lemma this immediately implies that $W_{K^+}(s)$ has full row rank (note that N_{K^+} has full column rank). A similar argument can be given to prove that the real rational matrix $W_{K^-}(s)$ has full row rank. Summarizing, we have proven the following theorem:

Theorem 7.2 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{q\times n}$ and $D \in \mathcal{R}^{q\times n}$, let $M \in \mathcal{R}^{q\times q}$ be symmetric and assume that $D^TMD > 0$. Furthermore, assume that the pair (A,B) is controllable. Then the following statements are equivalent:*

(a) *$G(-i\omega)^TMG(i\omega) \geq 0$ for all $\omega \in \mathcal{R}$, $i\omega \notin \sigma(A)$,*

(b) *there exists a symmetric solution $K \in \mathcal{R}^{n\times n}$ of the algebraic Riccati equation ARE.*

Assume that one of these statements holds. Let K^- and K^+ be the smallest and the largest real symmetric solution, respectively, of the linear matrix inequality LMI. Then K^- and K^+ are solutions of the ARE and for any real symmetric solution $K \in \mathcal{R}^{n\times n}$ of the ARE we have $K^- \leq K \leq K^+$ □

Summarizing, we conclude that the set of real symmetric solutions of the ARE is a subset of the set of real symmetric solutions of the LMI. We see that the LMI has a real symmetric solution if and only if the ARE has a real symmetric solution. Furthermore, in that case both for the LMI as well as for the ARE there exist a smallest and a largest real symmetric solution. In addition, these smallest real symmetric solutions of the LMI and the ARE, respectively, coincide and the same holds for the largest real symmetric solution of the LMI and the ARE. Of course, in contrast with the solution set of the LMI, the solution set of the ARE will in general *not be convex* (in fact, generically it is a finite set). In the following we will give a complete characterization of the set of all real symmetric solutions of the algebraic Riccati equation. We introduce the following notation:

$$A_+ := A - B(D^TMD)^{-1}(-B^TK^+ + D^TMC),$$
$$A_- := A - B(D^TMD)^{-1}(-B^TK^- + D^TMC),$$
$$\Delta := K^+ - K^-.$$

Note that we always have $\Delta \geq 0$. For a given matrix M, $\sigma(M)$ will denote its set of eigenvalues. We will also denote $\mathcal{C}^- := \{s \in \mathcal{C} \mid \Re e\ s < 0\}$. Likewise, we will denote $\mathcal{C}^0 := \{s \in \mathcal{C} \mid \Re e\ s = 0\}$ and $\mathcal{C}^+ := \{s \in \mathcal{C} \mid \Re e\ s > 0\}$. Now, we claim that

$$\sigma(A^-) \subset \mathcal{C}^- \cup \mathcal{C}^0, \tag{7.2}$$

$$\sigma(A^+) \subset \mathcal{C}^+ \cup \mathcal{C}^0. \tag{7.3}$$

Indeed, since K^- and K^+ satisfy the ARE it is easily seen that

$$(A^+)^T\Delta + \Delta A^+ = \Delta B(D^TMD)^{-1}B^T\Delta$$

and

$$(A^-)^T\Delta + \Delta A^- = -\Delta B(D^TMD)^{-1}B^T\Delta.$$

Assume now that λ is an eigenvalue of A^+ with corresponding eigenvector x. By pre- and post-multiplying the first of the above two equations with x^* and x, respectively, we obtain

$$2(\Re e\ \lambda)\ x^*\Delta x = x^*\Delta B(D^TMD)^{-1}B^T\Delta x.$$

From this we immediately see that $\Re e\ \lambda \geq 0$. In the same way, the second of the above two equations can be used to show that $\sigma(A^-) \subset \mathcal{C}^- \cup \mathcal{C}^0$. In the sequel, if K is a solution of the ARE, we shall denote

$$A_K := A - B(D^TMD)^{-1}(-B^TK + D^TMC),$$

It was shown in [40] that the extremal solutions of the ARE are in fact uniquely determined by the conditions 7.2 and 7.3:

Theorem 7.3 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{q\times n}$ and $D \in \mathcal{R}^{q\times n}$, let $M \in \mathcal{R}^{q\times q}$ be symmetric and assume that $D^TMD > 0$. Furthermore, assume that the pair (A, B) is controllable. Assume that the ARE has a real symmetric solution. Let K^- and K^+ be the smallest and the largest real symmetric solution of the ARE, respectively. Then we have $\sigma(A^-) \subset \mathcal{C}^- \cup \mathcal{C}^0$ and $\sigma(A^+) \subset \mathcal{C}^+ \cup \mathcal{C}^0$. Furthermore, if K is a solution of the ARE with the property that $\sigma(A_K) \subset \mathcal{C}^- \cup \mathcal{C}^0$ then we have $K = K^-$. If K is a solution of the ARE such that $\sigma(A_K) \subset \mathcal{C}^+ \cup \mathcal{C}^0$ then we have $K = K^+$.* □

In order to be able to give a characterization of all real symmetric solutions of the ARE, we need the concept of *modal subspace* of a given matrix. If M is a real $n \times n$ matrix, then we will denote by $\mathcal{X}^-(M)$ ($\mathcal{X}^0(M)$, $\mathcal{X}^+(M)$) the largest M-invariant subspace $\mathcal{V}$ of $\mathcal{R}^n$ such that $\sigma(M|\mathcal{V}) \subset \mathcal{C}^-$ ($\sigma(M|\mathcal{V}) \subset \mathcal{C}^0$, $\sigma(M|\mathcal{V}) \subset \mathcal{C}^+$). The subspace $\mathcal{X}^-(M)$ is called the modal subspace of M corresponding to $\mathcal{C}^-$. Likewise, $\mathcal{X}^0(M)$ and $\mathcal{X}^+(M)$ are called the modal subspaces of M corresponding to $\mathcal{C}^0$ and $\mathcal{C}^+$, respectively. An important role will be played by the modal subspace $\mathcal{X}^+(A^+)$. Let Ω denote the set of all A^+-invariant subspaces of $\mathcal{X}^+(A^+)$. Let Γ denote the set of all real symmetric solutions of the ARE. It turns out that there is a one-to-one correspondence between the set Γ and the set Ω:

Theorem 7.4 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{q\times n}$ and $D \in \mathcal{R}^{q\times n}$, let $M \in \mathcal{R}^{q\times q}$ be symmetric and assume that $D^TMD > 0$. Furthermore, assume that the pair (A, B) is controllable. Assume that the ARE has a real symmetric solution, i.e., assume that $\Gamma \neq \emptyset$. Let K^- and K^+ be the smallest and the largest real symmetric solution of the ARE, respectively. Then the following holds: if $\mathcal{V}$ is an A^+-invariant subspace of $\mathcal{X}^+(A^+)$ (that is, if $\mathcal{V} \in \Omega$) then $\mathcal{R}^n = \mathcal{V} \oplus \Delta^{-1}\mathcal{V}^\perp$. There exists a bijection $\gamma : \Omega \to \Gamma$ defined by*

$$\gamma(\mathcal{V}) := K^+P_{\mathcal{V}} + K^-(I - P_{\mathcal{V}}),$$

where $P_{\mathcal{V}}$ is the projection onto $\mathcal{V}$ along $\Delta^{-1}\mathcal{V}^\perp$. If $K = \gamma(\mathcal{V})$ then

$$\mathcal{X}^+(A_K) = \mathcal{V},$$

$$\mathcal{X}^0(A_K) = \ker \Delta,$$

$$\mathcal{X}^-(A_K) = \mathcal{X}^-(A^-) \cap \Delta^{-1}\mathcal{V}^\perp.$$

□

We want to stress that $\Delta^{-1}\mathcal{V}^\perp$ denotes the inverse image of $\mathcal{V}^\perp$ under Δ, i.e. the subspace $\{x \in \mathcal{R}^n \mid \Delta x \in \mathcal{V}^\perp\}$. For a proof of this theorem we refer to [40,8] (see also [29,22]). As noted before, the above result states that there exists a one-to-one correpondence between the set of all real symmetric solutions of the ARE and the set of all A^+-invariant subspaces of $\mathcal{X}^+(A^+)$. If K is a real symmetric solution of the ARE then the corresponding subspace $\gamma^{-1}(K)$ is given by $\gamma^{-1}(K) = \mathcal{X}^+(A_K)$. If $K = \gamma(\mathcal{V})$ then we say that the solution K is supported by the subspace $\mathcal{V}$.

We want to conclude this section with a result on the existence of positive semi-definite solutions of the algebraic Riccati equation. A proof of this theorem can be given analogous to theorem 7.2

Theorem 7.5 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{q\times n}$ and $D \in \mathcal{R}^{q\times n}$, let $M \in \mathcal{R}^{q\times q}$ be symmetric and assume that $D^TMD > 0$. Furthermore, assume that the pair (A, B) is controllable. Then the following statements are equivalent:*

(a) *$G(\bar{s})^TMG(s) \geq 0$ for all $s \in \mathcal{C}$, $s \notin \sigma(A)$, $\Re e\ s \geq 0$,*

(b) *there exists a positive semi-definite solution $K \in \mathcal{R}^{n\times n}$ of the algebraic Riccati equation ARE.*

Assume that one of these statements holds. Let K^+ be the largest real symmetric solution of the ARE. Furthermore, let K_f be the smallest positive semi-definite solution of the LMI. Then K_f is a solution of the ARE and for any positive semi-definite solution $K \in \mathcal{R}^{n\times n}$ of the ARE we have $K_f \leq K \leq K^+$.

□

8 Linear quadratic problems

Undoubtedly the best-known application of the linear matrix inequality and the algebraic Riccati equation is linear quadratic optimal control theory. Consider the finite-dimensional linear time-invariant system

$$\dot{x} = Ax + Bu, \ x(0) = x_0. \tag{8.1}$$

where, as usual, x and u are assumed to take their values in $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. We will be dealing with optimization problems of the type

$$\inf \int_0^\infty \omega(x,u)dt, \tag{8.2}$$

where ω is a general real quadratic form on $\mathcal{R}^n \times \mathcal{R}^m$ defined by

$$\omega(x,u) := u^T Ru + 2u^T Sx + x^T Qx.$$

Here, R, S and Q are assumed to be real matrices such that $R = R^T$ and $Q = Q^T$. The infimization in 8.2 should be interpreted as follows. For a given control function $u \in L_{2,loc}(\mathcal{R}^+)$, let $x(x_0, u)$ denote the state trajectory of 8.1 and if $T \geq 0$ let

$$J_T(x_0, u) := \int_0^T \omega(x(x_0,u)(t), u(t))dt. \tag{8.3}$$

Define the following classes of control functions:

$$U(x_0) := \{u \in L_{2,loc}(\mathcal{R}^+) \mid \lim_{T\to\infty} J_T(x_0,u) \text{ exists in } \mathcal{R} \cup \{-\infty, +\infty\}\},$$

$$U_s(x_0) := \{u \in U(x_0) \mid \lim_{t\to\infty} x(x_0,u)(t) = 0\}.$$

Note that if (A, B) is controllable, then $U(x_0) \neq \emptyset$ and $U_s(x_0) \neq \emptyset$ for all $x_0 \in \mathcal{R}^n$. For $u \in U(x_0)$ we now define the cost associated with u by

$$J(x_0, u) := \lim_{T\to\infty} J_T(x_0, u).$$

Note that $J(x_0,u) \in \mathcal{R} \cup \{-\infty, +\infty\}$. We are now ready to define the linear quadratic problems that we want to consider. The first problem that we shall introduce is the *zero-endpoint linear quadratic problem* and consist of the minimization of the functional $J(x_0, u)$ over the class of input functions $U_s(x_0)$. The optimal cost associated with this problem is

$$J^+(x_0) := \inf\{J(x_0,u) \mid u \in U_s(x_0)\}.$$

In addition to the zero-endpoint problem we have the *free-endpoint problem*, which consists of minimizing the functional $J(x_0, u)$ over the (larger) class $U(x_0)$ of input functions. The optimal cost for the latter problem is

$$J_f^+(x_0) := \inf\{J(x_0,u) \mid u \in U_s(x_0)\}.$$

If in any of the the above linear quadratic optimization problems the matrix R appearing in the quadratic form $\omega(x,u)$ is positive definite, then the corresponding optimization problem is called *regular.* If the quadratic form ω itself is positive semi-definite then the corresponding linear quadratic problem is called *positive-semi-definite.*

In most of the existing literature on the linear quadratic problem it is assumed that the problem under consideration is both regular as well as positive semi-definite. In fact, under these assumptions the linear quadratic problem has become quite standard and is treated in many basic textbooks in systems and control [5,6,21,45]. For a treatment of the more general problem in which the quadratic form ω is arbitrary, we refer to [27,40,35,31]. If the problem is not regular, then it is called *singular* (see [7,15,42,12]).

It turns out that it is quite easy to characterize the optimal cost J^+ for the zero-endpoint problem in its most general formulation, while for the free-endpoint optimal cost J_f^+ this is, up to now, only possible for the important special cases that the optimization problem is either regular or positive semi-definite. We will first discuss the zero-endpoint problem. In order to provide some intuition we would like to consider this problem in a context of dissipative systems. To this end we consider the system in driving variable form $\mathcal{B}_{DV}(A, B, C, D)$, with

$$C := \begin{pmatrix} I \\ 0 \end{pmatrix}, \; D := \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{8.4}$$

For this system the signal alphabet W is equal to $X \times U$ so we have $w = (x, u)$. In connection with this system we consider the quadratic supply rate $s(x, u) = -\omega(x, u)$, i.e., the matrix M defining the supply rate is equal to

$$M := -\begin{pmatrix} Q & S^T \\ S & R \end{pmatrix}.$$

Let $V_a(x_0)$ be the associated available storage (with respect to the reference point $x^* = 0$). Under the assumption that (A, B) is controllable, it is quite easy to verify that, in fact, for all x_0 we have $J^+(x_0) = -V_a(x_0)$. Thus we see that the optimal cost $J^+(x_0)$ is finite for every x_0 if and only if the system $\mathcal{B}_{DV}(A, B, C, D)$ with supply rate $-\omega(x, u)$ is internally dissipative. For this particular choice of system we have

$$G(s) := C(Is - A)^{-1} + D = \begin{pmatrix} (Is - A)^{-1} \\ I \end{pmatrix}$$

and hence $G(z)^T M G(s) = H(z, s)$, where

$$H(z,s) :=$$

$$B^T(Iz - A^T)^{-1}Q(Is - A)^{-1}B + S(Is - A)^{-1}B + B^T(Iz - A^T)^{-1}S^T + R$$

Thus we obtain the following:

Theorem 8.1 *Let (A, B) be controllable. Then the following statements are equivalent:*

(a) *$J^+(x_0)$ is finite for all $x_0 \in \mathcal{R}^n$,*

(b) *$H(-i\omega, i\omega) \geq 0$ for all $i\omega \notin \sigma(A)$,*

(c) *there exists a real symmetric solution K to the linear matrix inequality*

$$\begin{pmatrix} -A^TK - KA + Q & -KB + S^T \\ -B^TK + S & R \end{pmatrix} \geq 0. \tag{8.5}$$

If one of these statements holds then we have $J^+(x_0) = x_0^T(-K^-)x_0$, where K^- is the smallest real symmetric solution of the linear matrix inequality 8.5. □

Of course, a real symmetric matrix K is a solution to the linear matrix inequality 8.5 if and only if the matrix $-K$ is a solution to the following linear matrix inequality:

$$\begin{pmatrix} A^TK + KA + Q & KB + S^T \\ B^TK + S & R \end{pmatrix} \geq 0. \tag{8.6}$$

If K^- is the smallest real symmetric solution of 8.5, then $-K^-$ is the largest real symmetric solution of 8.6. Thus we see that the optimal cost $J^+(x_0)$ is equal to $x_0^T\bar{K}^+x_0$, where $\bar{K}^+$ is the *largest* real symmetric solution of the linear matrix inequality 8.6.

We shall now discuss the issue of optimal controls. For a given x_0, an input u^* is called optimal for the zero-endpoint problem if $u^* \in U_s(x_0)$ and if $J(x_0, u^*) = J^+(x_0)$, i.e. the control input attains the optimal value of the cost functional. It is well known that the question of the existence of optimal controls is closely connected to the question whether the optimization problem is regular, i.e., whether the weighting matrix R is *positive definite*. In fact, if this is not the case then optimal controls in general will not exist unless we extend the class of inputs $U_s(x_0)$ to include *distributions*. In this paper we do not want to go into the intricacies of distribution theory and therefore we will assume that $R > 0$. As before, the role of the linear matrix inequality is then taken over by an algebraic Riccati equation, which, in this particular case, is given by

$$-A^TK - KA + Q - (-KB + S^T)R^{-1}(-B^TK + S) = 0. \tag{8.7}$$

Obviously, if K^- is the smallest real symmetric solution of this algebraic Riccati equation, then $\bar{K}^+ := -K^-$ is the largest real symmetric solution of the following algebraic Riccati equation:

$$A^TK + KA + Q - (KB + S^T)R^{-1}(B^TK + S) = 0. \tag{8.8}$$

In the following, let $\bar{\Gamma}$ denote the set of all real symmetric solutions of the algebraic Riccati equation 8.8. If $\bar{\Gamma} \neq \emptyset$, let $\bar{K}^-$ denote the smallest element of $\bar{\Gamma}$. Furthermore, let $\bar{\Delta}$ denote the difference $\bar{K}^+ - \bar{K}^-$ between the largest and the smallest real symmetric solution of the latter ARE. The following result was proven in [40]:

Theorem 8.2 *Let (A, B) be controllable and assume that $R > 0$. Then*

(a) *the following statements are equivalent:*

(i) *$J^+(x_0)$ is finite for all $x_0 \in \mathcal{R}^n$,*

(ii) *$H(-i\omega, i\omega) \geq 0$ for all $i\omega \notin \sigma(A)$,*

(iii) *there exists a real symmetric solution K to the algebraic Riccati equation 8.8, i.e. $\bar{\Gamma} \neq \emptyset$.*

Assume that one of the above statements hold. Then:

(b) *for all $x_0 \in \mathcal{R}^n$ we have $J^+(x_0) = x_0^T\bar{K}^+x_0$,*

(c) *for all $x_0 \in \mathcal{R}^n$ there exists an optimal input u^* if and only if $\bar{\Delta} > 0$,*

(d) *if $\bar{\Delta} > 0$ then for each x_0 there exists exactly one optimal input u^* and, moreover, this input u^* is given by the feedback control law $u^* = -R^{-1}(B^T\bar{K}^+ + S)x$.* □

We will now turn to the second of the linear quadratic optimization problems we shall consider in this section, the free-endpoint problem. In our discussion of this problem, we will restrict ourselves to two important special cases, the case that the optimization problem is positive semi-definite and the case that the problem is regular. First we shall treat the positive semi-definite case, i.e., the case that the quadratic form $\omega(x,u)$ is positive semi-definite. In that case there exist matrices C_0 and D_0 of appropriate dimensions such that

$$\begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} = (C_0\ D_0)^T(C_0\ D_0), \tag{8.9}$$

so the cost functional J is given by

$$J(x_0,u) = \int_0^\infty \|C_0x(t) + D_0u(t)\|^2 dt,$$

and it is clear that, since the latter is a priori bounded from below by zero, finiteness of the optimal cost is no issue in this case. The linear matrix inequality 8.5 in this case reads

$$\begin{pmatrix} -A^TK - KA + C_0^TC_0 & -KB + C_0^TD_0 \\ -B^TK + D_0^TC_0 & D_0^TD_0 \end{pmatrix} \geq 0 \tag{8.10}$$

and clearly this inequality always has a solution (take for example K=0). In order to characterize the optimal cost for the free-endpoint problem it turns out that we have to look at a particular subset of the set of all negative semi-definite solutions of the linear matrix inequality, more specifically, the subset of all negative semi-definite *rank-minimizing* solutions of the linear matrix inequality. Note that the above linear matrix inequality is exactly the linear matrix inequality associated with the system in input/state/output form $\mathcal{B}_{i/s/o}(A,B,C_0,D_0)$ and supply rate given by $s(u,y) := \|y\|^2$. Thus it can be easily seen that for this special case we have

$$G(-s)^TMG(s) = G_0(-s)^TG_0(s),$$

where $G_0(s) = C_0(Is-A)^{-1}B + D_0$. By applying theorem 5.1 we thus find that for each real symmetric solution K of the linear matrix inequality 8.10 we have $r(K) \geq r^*$, where r^* is equal to the rank of the rational matrix $G(-s)^TMG(s)$ which, by the above, is equal to the rank of the rational matrix $G_0(s)$. Now, instead of working with the linear matrix inequality 8.10, we will work with the following linear matrix inequality:

$$\begin{pmatrix} A^TK + KA + C_0^TC_0 & KB + C_0^TD_0 \\ B^TK + D_0^TC_0 & D_0^TD_0 \end{pmatrix} \geq 0, \tag{8.11}$$

Obviously, also for any real symmetric solution K of the latter linear matrix inequality we have $r(K) \geq r^*$. Furthermore, 8.11 always has at least one positive semi-definite solution. Following [12] we introduce the set

$$\bar{\Gamma}^+_{min} := \{K \in \mathcal{R}^{n\times n} \mid K \text{ satisfies the LMI 8.11, } K \geq 0 \text{ and } r(K) = r^*\}$$

of positive semi-definite rank-minimizing solutions of the linear matrix inequality given by 8.11. It was shown in [12] that if (A,B) is controllable then $\bar{\Gamma}^+_{min}$ contains a smallest element, say $\bar{K}_f$, characterized by the following properties:

- $\bar{K}_f \in \bar{\Gamma}^+_{min}$,
- $K \in \bar{\Gamma}^+_{min} \Rightarrow \bar{K}_f \leq K$.

It turns out that the optimal cost for the positive semi-definite free-endpoint problem is determined by the latter solution of the LMI:

Theorem 8.3 *Consider the free-endpoint linear quadratic problem associated with the system (A,B) and the quadratic form $\omega(x,u) := \|C_0x + D_0u\|^2$. Assume that (A,B) is controllable. For all $x_0 \in \mathcal{R}^n$ we have $J_f^+(x_0) = x_0^T\bar{K}_f x_0$, where $\bar{K}_f$ is the smallest element of $\bar{\Gamma}^+_{min}$.* □

It can be shown that if, in addition, we assume that the problem is regular then the optimal cost for the free-endpoint problem is in fact determined by the smallest positive semi-definite solution of the algebraic Riccati equation associated with 8.11 (see [24,42]).

As a last subject of this section we shall consider the *regular* free-endpoint problem. Thus, in the remainder we will drop the assumption that the quadratic form ω is positive semi-definite. Instead, we will assume that the matrix R is positive definite. Again let $\bar{\Gamma}$ be the set of all real symmetric solutions of the algebraic Riccati equation 8.8. If $\bar{\Gamma} \neq \emptyset$, let $\bar{K}^-$ and $\bar{K}^+$ denote the smallest and the largest real symmetric solution of the ARE 8.8, respectively. Denote

$$\bar{A}^+ := A - BR^{-1}(B^T\bar{K}^+ + S),$$

$$\bar{A}^- := A - BR^{-1}(B^T\bar{K}^- + S).$$

Let $\bar{\Omega}$ be the set of all $\bar{A}^-$-invariant subspaces of the subspace $\mathcal{X}^+(\bar{A}^-)$. By applying theorem 6.4 to the system in driving variable form $\mathcal{B}_{DV}(A,B,C,D)$ with C and D given by 8.4, we find that there is a one to one correspondence between the sets $\bar{\Omega}$ and $\bar{\Gamma}$:

Theorem 8.4 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, let $Q \in \mathcal{R}^{n\times n}$ be symmetric, let $S \in \mathcal{R}^{n\times m}$ and let $R \in \mathcal{R}^{m\times m}$ be positive definite. Furthermore, assume that the pair (A,B) is controllable. Assume that the ARE given by 8.8 has a real symmetric solution, i.e., assume that $\bar{\Gamma} \neq \emptyset$. Let $\bar{K}^-$ and $\bar{K}^+$ be the smallest and the largest real symmetric solution of the ARE, respectively. Let $\bar{\Delta} := \bar{K}^+ - \bar{K}^-$. Then the following holds: if $\mathcal{V}$ is an $\bar{A}^-$-invariant subspace of $\mathcal{X}^+(\bar{A}^-)$ (that is, if $\mathcal{V} \in \bar{\Omega}$) then $\mathcal{R}^n = \mathcal{V} \oplus \Delta^{-1}\mathcal{V}^\perp$. There exists a bijection $\bar{\gamma} : \bar{\Omega} \to \bar{\Gamma}$ defined by*

$$\bar{\gamma}(\mathcal{V}) := \bar{K}^-P_\mathcal{V} + \bar{K}^+(I - P_\mathcal{V}),$$

where $P_\mathcal{V}$ is the projection onto $\mathcal{V}$ along $\Delta^{-1}\mathcal{V}^\perp$. □

In the characterization of the free-endpoint optimal cost J_f^+ the following subspace of the state space plays a central role:

$$\mathcal{N} :=< \ker\bar{K}^- \mid \bar{A}^- > \cap\, \mathcal{X}^+(\bar{A}^-) \tag{8.12}$$

Here, for an arbitrary subspace $\mathcal{L}$ and an arbitrary linear map M, $< \mathcal{L} \mid M >$ denotes the smallest M-invariant subspace contained in $\mathcal{L}$. Note that $\mathcal{L}$ is an $\bar{A}^-$-invariant subspace contained in $\mathcal{X}^+(\bar{A}^-)$ and hence an element of $\bar{\Omega}$. Let $\bar{K}_f^+$ be the real symmetric solution of the ARE corresponding to the subspace $\mathcal{N}$, i.e., let $\bar{K}_f^+ = \bar{\gamma}(\mathcal{N})$. It was shown in [35] that the free-endpoint optimal cost is determined by this particular solution of the ARE. For a given x_0, an input u^* is called optimal for the free-endpoint problem if $u^* \in U(x_0)$ and if $J(x_0,u^*) = J_f^+(x_0)$. Let $\bar{\Gamma}_-$ be the set of all negative semi-definite solutions of the ARE 8.8. Then we have:

Theorem 8.5 *Let (A,B) be controllable and assume that $R > 0$. Then*

(a) *the following statements are equivalent:*
  (i) $H(\bar{s}, s) \geq 0$ *for all* $s \in \mathcal{C}$, $s \notin \sigma(A)$, $\Re e\ s \geq 0$,
  (ii) *there exists a negative semi-definite solution* K *to the algebraic Riccati equation 8.8, i.e.* $\bar{\Gamma}_{-} \neq \emptyset$.

Assume that one of the above statements hold. Then:

(b) $J_f^+(x_0)$ *is finite for all* $x_0 \in \mathcal{R}^n$,

(c) *for all* $x_0 \in \mathcal{R}^n$ *we have* $J_f^+(x_0) = x_0^T \bar{K}_f^+ x_0$,

(d) *for all* $x_0 \in \mathcal{R}^n$ *there exists an optimal input* u^* *if and only if* $\ker \bar{\Delta} \subset \ker \bar{K}^-$,

(e) *if* $\ker \bar{\Delta} \subset \ker \bar{K}^-$ *then for each* x_0 *there exists exactly one optimal input* u^* *and, moreover, this input* u^* *is given by the feedback control law* $u^* = -R^{-1}(B^T \bar{K}_f^+ + S)x$. □

9 Stability Theory

In this section we will discuss the application of the ideas in this paper to stability theory. We start by giving the main underlying idea in the context of dissipative systems. Let $\Sigma_s^1 = (\mathcal{R}, W, X_1, \mathcal{B}_s^1)$ and $\Sigma_s^2 = (\mathcal{R}, W, X_2, \mathcal{B}_s^2)$ be two dynamical systems in state space form. Then their *interconnection* $\Sigma_s = \Sigma_s^1 \times \Sigma_s^2$ is defined as $\Sigma_s = (\mathcal{R}, W, X_1 \times X_2, \mathcal{B}_s)$ with

$$\mathcal{B}_s := \{(w, (x_1, x_2)) \mid (w, x_1) \in \mathcal{B}_s^1 \text{ and } (w, x_2) \in \mathcal{B}_s^2\}.$$

Now assumne that (Σ_s^1, s_1, V_1) and (Σ_s^2, s_2, V_2) are both dissipative. It follows immediately that $(\Sigma_s, s_1 + s_2, V_1 + V_2)$ is also dissipative. Now, in many applications $s_1 + s_2 = 0$ which shows that $V_1 + V_2$ will be a Lyapunov function for Σ_s in the sense that $V_1(x_1(t)) + V_2(x_2(t))$ will be non-increasing along elements of $\mathcal{B}_s$.

We will now apply this in order to derive stability conditions for the system described by the differential equation

$$\Sigma : \ \dot{x} = Ax - Bf(Cx, t) \tag{9.1}$$

with $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{p \times n}$, and $f : \mathcal{R}^p \times \mathcal{R} \rightarrow \mathcal{R}^m$. We will derive conditions on (A, B, C) and f such that the solutions of 9.1 are either bounded on $[0, \infty)$ or converge to zero as $t \rightarrow \infty$. We will do this by viewing 9.1 as the (feedback) interconnection of the following two systems:

$$\Sigma_1 : \ \dot{x} = Ax + Bu, \ y = Cx \tag{9.2}$$

$$\Sigma_2 : \ u = -f(y, t) \tag{9.3}$$

Now use as supply rates $s_1(u, y) = u^T y$ and $s_2(u, y) = -u^T y$ to obtain

Theorem 9.1 *Assume in 9.1 that* (A, B, C) *is minimal and* $m = p$. *Let* $G(s) := C(Is - A)^{-1}B$. *There exists an* $M < \infty$ *such that for every solution* $x : \mathcal{R} \rightarrow \mathcal{R}^n$ *and* $t \geq 0$ *there holds*

$$\|x(t)\| \leq M \|x(0)\|$$

if $G(\bar{s})^T + G(s) \geq 0$ *for all* $s \in \mathcal{C}$ *with* $s \notin \sigma(A)$ *and* $\Re e\ s \geq 0$ *and if* $y^T f(y, t) \geq 0$ *for all* $y \in \mathcal{R}^p$.

Proof According to Corollary 5.1 there exists a positive semi-definite solution K to the linear matrix inequality 5.1 with $C_0 = C$ and $D = 0$, or, equivalently, to

$$-A^T K - KA \geq 0, \quad KB = C^T.$$

Using the assumption that (C, A) is observable it is easily shown that, in fact, $K > 0$. Now compute the derivative $\dot{V}_\Sigma$ of $V(x) = x^T K x$ along solutions of 9.1. It follows that

$$\dot{V}_\Sigma(x(t)) \leq -2(Cx(t))^T f(Cx(t), t).$$

From this it follows that

$$x^T(t) K x(t) \leq x^T(0) K x(0)$$

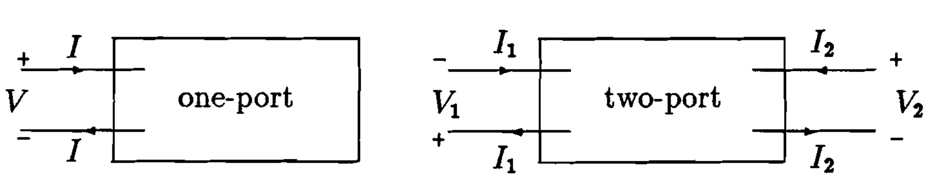
for all $t \geq 0$. This yields the result. □

The above result can be generalized in many different directions:
(1) If we use the supply rates $\|u\|^2 - \|y\|^2$ for 9.2 and $\|y\|^2 - \|u\|^2$ for 9.3 we see that $\|x(t)\| \leq M\|x(0)\|$ can be concluded by assuming $G(\bar{s})^T G(s) \leq I$ for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$ and $\Re e\ s \geq 0$ and $\|f(y,t)\| \leq \|y\|$ for all $y \in \mathcal{R}^p$ and $t \in \mathcal{R}$.
(2) If we assume certain 'observability' properties of the dissipation rate, it can be proven that in addition to $\|x(t)\| \leq M\|x(0)\|$ we will have $\lim_{t\to\infty} x(t) = 0$.
(3) By assuming $G(-i\omega)^T + G(i\omega) \geq 0$ for all $\omega \in \mathcal{R}$ with $i\omega \notin \sigma(A)$, but $G(\bar{s})^T + G(s) \not\geq 0$ for some $s \in C$ with $\Re e\ s \geq 0$, in addition to $y^T f(y,t) \geq 0$, one can prove that there are solutions such that $\lim_{t\to\infty} x(t) \neq 0$. By assuming also certain 'observability' properties of the dissipation rate one can prove that in this case there are solutions which are unbounded on $[0, \infty)$.

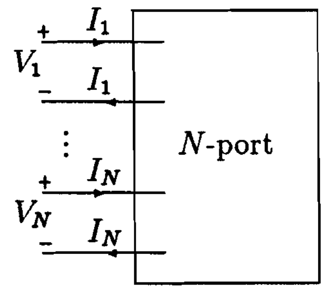
The constuction of the quadratic Lyapunov function $x^T K x$ which lies at the basis of these stability results is identical to the analysis of the linear matrix inequality as in 5.1 and 5.2 (see also [27,46,47,17,41]).

10 Electrical Network Synthesis

A formal definition of a synthesis question may be given as follows: given certain ideal elements which may be interconnected according to certain interconnection laws, what systems can be realized this way and, for a given system, what elements should be used and how should they be interconnected? In linear passive electrical network synthesis, these elements are taken to be linear resistors ($V = RI$, $R > 0$), linear capacitors ($I = C\frac{dV}{dt}$, $C > 0$), linear inductors ($V = L\frac{dI}{dt}$, $L > 0$), transformers ($V_2 = nV_1$, $I_1 = -nI_2$), and gyrators ($I_1 = gV_2$, $I_2 = -gV_1$). Here (V, I) denote the port variables of a one-port, while $((V_1, I_1), (V_2, I_2))$ denote the port variables of a two-port:



The R's, L's, C's, T's and G's are assumed to be interconnected by the usual electrical interconnections obeying Kirchoff's current and voltage laws. Assume that such an interconnection is set up yielding an N-port:



it can be shown that if the N-port contains only linear passive R's, L's, C's, T's and G's, the resulting N-port will allow a hybrid description meaning that there will exist a componentwise partition of the vectors $I = \mathrm{col}(I_1, I_2, \ldots, I_N)$ and $V = \mathrm{col}(V_1, V_2, \ldots, V_N)$ into $I = \mathrm{col}(I^1, I^2)$ and $V = \mathrm{col}(V^1, V^2)$ such that the network is described by a proper transfer function $G(s)$:

$$\begin{pmatrix} V^1 \\ I^2 \end{pmatrix} = G(s) \begin{pmatrix} I^1 \\ V^2 \end{pmatrix}. \tag{10.1}$$

A representation as 10.1 is called a hybrid description. If $V^1 = V$, then we speak about an impedance description and if $V^2 = V$, we speak about an admittance description.

The question arises what properties on $G(s)$ follow from the assumption that the N-port contains only (linear passive) R, L, C, T and G's. In network synthesis we ask for necessary and sufficient conditions, in the sense that we look for conditions on $G(s)$, and a blue-print for synthesizing the network in case these conditions are satisfied. In this section we will describe in broad lines how this problem is solved. We will see how the positive real lemma (Corollary 5.1) enters this procedure in an essential way.

It can be shown that the memoryless multiport

$$\begin{pmatrix} V^1 \\ I^2 \end{pmatrix} = R \begin{pmatrix} I^1 \\ V^2 \end{pmatrix} \tag{10.2}$$

can be synthesized using R's, T's and G's (memoryless elements) if and only if the matrix R satisfies $R + R^T \geq 0$. We will take this result to be our starting point for the synthesis of a dynamic N-port with transfer matrix $G(s)$.

Let (A, B, C, D) be a minimal realization of $G(s)$. Hence $G(s) = D + C(Is - A)^{-1}B$ and (A, B) is controllable and (C, A) is observable. Now assume that $G(s)$ is positive real, that is, assume that $G(\bar{s})^T + G(s) \geq 0$ for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$ and $\Re e\, s \geq 0$. Then there exists a matrix $K = K^T > 0$ such that

$$\begin{pmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -D - D^T \end{pmatrix} \leq 0 \tag{10.3}$$

Now, by using a suitable basis transformation S such that $S^T S = K$ and considering the new realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of $G(s)$ with $\bar{A} = SAS^{-1}$, $\bar{B} = SB$, $\bar{C} = CS^{-1}$, and $\bar{D} = D$, we obtain a realization with

$$\begin{pmatrix} \bar{A}^T + \bar{A} & \bar{B} - \bar{C}^T \\ \bar{B}^T - \bar{C} & -\bar{D} - \bar{D}^T \end{pmatrix} \leq 0. \tag{10.4}$$

Now realize the memoryless multiport

$$\begin{pmatrix} I \\ V^1 \\ I^2 \end{pmatrix} = \begin{pmatrix} -\bar{A} & -\bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} V \\ I^1 \\ V^2 \end{pmatrix} \tag{10.5}$$

using R's, T's and G's. Next, terminate the first n ports of this multiport using unit capacitors. This imposes the condition

$$\dot{V} = -I,$$

yielding the behavioral equations

$$\dot{I} = \bar{A}I + \bar{B}\begin{pmatrix} I^1 \\ V^2 \end{pmatrix}, \quad \begin{pmatrix} V^1 \\ I^2 \end{pmatrix} = \begin{pmatrix} \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} I \\ I^1 \\ V^2 \end{pmatrix}. \tag{10.6}$$

The transfer matrix of 10.6 is $G(s)$, as desired. We summarize this result in:
Theorem 10.1 *The N-port hybrid description 10.1 can be realized using linear passive R's, L's, C's, T's and G's if and only if $G(s)$ is positive real: $G(\bar{s})^T + G(s) \geq 0$ for all $s \in \mathcal{C}$ with $s \notin \sigma(A)$ and $\Re e\ s \geq 0$.* □

Similar synthesis procedures can be obtained for networks without gyrators, without resistors, without L's and G's, without C's and G's, or without R's and G's. For this we refer to [18,36,4].

11 Covariance Generation

Let $y(t)$ $(t \in \mathcal{R}, y(t) \in \mathcal{R}^p)$ be a zero mean stationary Gaussian stochastic vector process defined on a probability space $(\Omega, \mathcal{A}, P)$. Define its autocorrelation function by

$$R(t) = \mathcal{E}\{y(t'+t)y^T(t')\} \tag{11.1}$$

Obviously $R(t) = R^T(-t)$ and, as is easily derived from a direct calculation, $R(.)$ is non-negative definite in the sense that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^T(t')R(t'-t'')v(t'')dt'dt'' \geq 0 \tag{11.2}$$

for all $v(.) : \mathcal{R} \rightarrow \mathcal{R}^p$ for which the above double integral exists.

The stochastic realization problem consists of finding a *Markov representation* of y. By this, we mean a zero-mean, stationary Gauss-Markov process $x(t)$, $(t \in \mathcal{R}, x(t) \in \mathcal{R}^n)$ and a matrix $C \in \mathcal{R}^{p \times n}$ such that $Cx(.)$ has the same autocorrelation function as $y(.)$. Equivalently, such that $Cx(.)$ and $y(.)$ are stochastically equivalent. Two processes $y_1(t), y_2(t)$ $(t \in \mathcal{R},\ y_1(t), y_2(t) \in \mathcal{R}^p)$ are said to be stochastically equivalent if for all $t_1, t_2, \ldots, t_k \in \mathcal{R}$ the vector random variables $(y_1(t_1), y_1(t_2), \ldots, y_1(t_k))$ and $(y_2(t_1), y_2(t_2), \ldots, y_2(t_k))$ have the same distributution.

The Markov process $x(t)$ $(t \in \mathcal{R}, x(t) \in \mathcal{R}^n)$ and the matrix C are called a Markov representation of y. A Markov representation (x, C) is said to be *minimal* if n, the dimension of x, is as small as possible. It can be shown that if (x, C) is minimal, then x is mean-square continuous. We proceed by classifying the mean-square continuous Markov processes.

It is easy to see that a mean square continuous zero-mean stationary Markov process $x(t)$ $(t \in \mathcal{R}, x(t) \in \mathcal{R}^n)$ is, up to stochastic equivalence, completely specified by the matrices $Q \in \mathcal{R}^{n \times n}$ and $A \in \mathcal{R}^{n \times n}$ by

$$\mathcal{E}\{x(t)x^T(0)\} = e^{At}Q \quad t \geq 0. \tag{11.3}$$

Thus $Q = \mathcal{E}\{x(t)x^T(t)\}$ while A is such that $\mathcal{E}\{x(t)|x(0)\} = e^{At}x(0),\ t \geq 0$. Clearly, $Q = Q^T \geq 0$. The question arises what matrices $Q, A \in \mathcal{R}^{n \times n}$ can arise in this way. In fact:

Lemma 11.1 *Let $Q, A \in \mathcal{R}^{n \times n}$. Then there exist a mean square continuous zero- mean stationary Gauss-Markov process $x(t)$ $(t \in \mathcal{R}, x(t) \in \mathcal{R}^n)$ such that 11.3 holds if and only if*

$$Q = Q^T \geq 0 \text{ and } AQ + QA^T \leq 0 \tag{11.4}$$

Proof ($\Rightarrow$) The first condition in 11.4 is obvious. In order to prove the second, calculate $0 \leq \mathcal{E}\{(x(t) - x(0))(x(t) - x(0))^T\}$. This yields $e^{At}Q + Qe^{A^Tt} \leq 2Q$ for $t \geq 0$. Now take the limit as $t \downarrow 0$ to obtain $AQ + QA^T \leq 0$.($\Leftarrow$) Factor $AQ + QA^T$ as $-BB^T$ and consider the solution of the stochastic differential equation $dx = Axdt + Bdw$ with $x(0)$ a zero mean Gaussian random vector with $\mathcal{E}\{x(0)x^T(0)\} = Q$ and $w(t), t \in \mathcal{R}$ a Wiener process independent of $x(0)$. □

We now return to the stochastic realization problem. This question has now been reduced to that of finding an n and matrices $A, Q \in \mathcal{R}^{n \times n}$ and $C \in \mathcal{R}^{p \times n}$ such that 11.4 holds and such that $Ce^{At}QC^T = R(t)$ for $t \geq 0$.

Theorem 11.2 *Let $y(t)$ $(t \in \mathcal{R}, y(t) \in \mathcal{R}^p)$ be a zero mean stationary Gaussian process. Let its autocorrelation be $R(t) = \mathcal{E}\{y(t)y^T(t)\}$. The following are equivalent:*

(a) *There exists a zero-mean stationary finite-dimensional Gauss-Markov process $x(t)$ $(t \in \mathcal{R}, x(t) \in \mathcal{R}^n)$ and a matrix $C \in \mathcal{R}^{p \times n}$ such that (x, C) is a Markov representation of y,*

(b) *There exists $n < \infty$ and matrices $A, Q \in \mathcal{R}^{n \times n}$, $C \in \mathcal{R}^{p \times n}$ such that 11.4 holds and such that $Ce^{At}QC^T = R(t)$ for $t \geq 0$,*

(c) *$R(t)$ is a Bohl function, that is, there exist matrices F, G, H such that $R(t) = He^{At}G$ for $t \geq 0$,*

(d) *The spectral density matrix $S(\omega)$ of $R(t)$, that is, the Fourier transform of R:*

$$S(\omega) = \int_{-\infty}^{\infty} R(t)e^{-i\omega t}dt$$

is rational.

Proof The equivalence of (a) and (b) follows from the pre-amble to the theorem. (b) $\Rightarrow$(c) is immediate and (c) $\Rightarrow$(d) can be seen from

$$S(\omega) = H(Ii\omega - F)^{-1}G + G^T(-Ii\omega - F^T)H^T.$$

We will now prove that (c) $\Rightarrow$(b). Let (F, G, H) be a triple such that $R(t) = He^{Ft}G$. Now observe that 11.2 implies by Corollary 5.1 that there exists a matrix $K = K^T \geq 0$ such that $FK + KF^T \leq 0$ and $KH^T = G$. Now take $A = F$, $Q = K$, and $C = H$. Then $Q = Q^T \geq 0$, $A^TQ + QA \leq 0$, and $Ce^{At}QC^T = He^{At}G = R(t)$ for $t \geq 0$. □

The proof of the above theorem contains an algorithm for computing all minimal Markov representations for y. Let (F, G, H) be a minimal triple such that $R(t) = He^{At}G$ for $t \geq 0$. Then all minimal Markov representations may be obtained by choosing any solution $K = K^T$ such that $FK+KF^T \leq 0$ and $KH^T = G$ (every solution $K = K^T$ will be positive definite and there exist solutions K^- and K^+ such that $K^- \leq K \leq K^+$). Now take $A = F$, $Q = K$ and $C = H$. The only remaining freedom is now a basis choice: $A \rightarrow SAS^{-1}$, $Q \rightarrow SQS^T$ and $C \rightarrow CS^{-1}$.

What we have described above can be called the *covariance generation problem.* What we have done is describe a way of matching by (x, C), the given autocorrelation $R(t)$. Note that the Markov process x can be easily simulated by the stochastic differential equation $dx = Axdt + Bdw$ with B such that $AQ + QA^T = -BB^T$. A related problem is the strong stochastic realization problem in which it is required that x is supported by the given probability space $(\Omega, \mathcal{A}, P)$ which supports y. This version of the stochastic realization problem leads to the question what minimal Markov representations (x, C) are such that $x(t)$ is $y(.)$-measurable. It turns out that this is the case if and only if the solution K to $FK + KF^T \leq 0$, $KH^T = G$ is such that $FK + KF^T$ is of minimal rank. In particular, K^- (in which case $x(t)$ is a function of the past of y) and K^+ (in which case $x(t)$ is a function of the future of y) yield solutions of the strong stochastic realization problem. For details and further references, we refer to the article by Lindquist and Picci elsewhere in this volume (see also [2,10,11,26]).

12 The H_∞ control problem

As one of the most recent applications of the ideas developed in this paper, in this section we will discuss the application to the problem of H_∞ optimal control. We will consider the following linear time-invariant system:

$$\dot{x} = Ax + Bu + Ed, \quad z = Cx + Du. \tag{12.1}$$

In these equations, as usual, x and u are assumed to take their values in $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. The variable d represents an unknown disturbance, which is assumed to take its values in $\mathcal{R}^l$. Finally, z represents the output to be controlled, which is assumed to take its values in $\mathcal{R}^q$. A, B, E, C and D are real matrices of appropriate dimensions. We will restrict ourselves here to the H_∞ control problem with static state feedback.

If F is a real $m \times n$ matrix then the closed system resulting from the state feedback control law $u = Fx$ is given by

$$\dot{x} = (A + BF)x + Ed, \quad z = (C + DF)x. \tag{12.2}$$

The transfer matrix of this system is called the closed loop transfer matrix and is equal to

$$G_F(s) = (C + DF)(Is - A - BF)^{-1}E. \tag{12.3}$$

Obviously, if we put $x(0) = 0$ then the closed loop system defines an operator mapping disturbances d to outputs z. If we restrict ourselves to disturbances $d \in L_2(\mathcal{R}^+)$ and if the closed loop system is asymptotically stable, i.e., if $\sigma(A + BF) \subset \mathcal{C}^-$ then this convolution operator is a bounded operator from $L_2(\mathcal{R}^+)$ to $L_2(\mathcal{R}^+)$. The influence of a disturbance d on the output z can then be measured by the induced norm of this operator. It is well known that this norm is equal to the H_∞ norm of the closed loop transfer matrix, which is denoted by

$$\|G_F\|_\infty := \sup_{\omega \in \mathcal{R}} \rho[G_F(i\omega)].$$

Here, $\rho[M]$ denotes the largest singular value of the complex matrix M. Now, the problem that we shall consider in this section is the following: given a positive real number γ, find a real $m \times n$ matrix F such that $\sigma(A + BF) \subset \mathcal{C}^-$ and such that $\|G_F\|_\infty < \gamma$. It will turn out that if the matrix D has full column rank then the existence of such matrix F is equivalent to the existence of a given solution of a certain algebraic Riccati equation.

Before embarking on the details, first note that under the assumption that D has full column rank, we can assume without loss of generality that $D^TD = I$ and that $D^TC = 0$. Indeed, if these conditions do not hold then we may apply a preliminary feedback of the form $u = -(D^TD)^{-1}D^TCx + (D^TD)^{-1/2}v$ in order to obtain a system for which the conditions do hold. Now, assume that for our system there exists an F such that $\sigma(A + BF) \subset \mathcal{C}^-$ and $\|G_F\|_\infty \leq \gamma$. Then clearly $G_F(-i\omega)^TG_F(i\omega) - \gamma^2 I \leq 0$ for all $\omega \in \mathcal{R}$. By applying Th. 7.2 to the system in driving variable representation

$$\Sigma\left(A + BF,\ E,\ \begin{pmatrix} 0 \\ C + DF \end{pmatrix},\ \begin{pmatrix} I \\ 0 \end{pmatrix}\right)$$

with $w = (d, z)$ and supply rate $s(w) = w^TMw$ with M given by

$$M := \begin{pmatrix} \gamma^2 I & 0 \\ 0 & -I \end{pmatrix},$$

we find that there exists a real symmetric solution Z to the following algebraic Riccati equation:

$$-(A + BF)^TZ - Z(A + BF) - (C + DF)^T(C + DF) - \gamma^{-2}ZEE^TZ = 0. \tag{12.4}$$

Note that strictly speaking one needs controllabilty of the pair $(A+BF, E)$ in order to be able to apply theorem 7.2. However, using the fact that the matrix $A+BF$ is asymptotically stable it can be shown that also without this controllability assumption equation 12.4 has a real symmetric solution. By interpreting the above Riccati equation as a Lyapunov equation it can be shown that, in fact, $Z \geq 0$. Writing out 12.4 we find

$$A^T Z + ZA + C^T C + \gamma^{-2} ZEE^T Z + (B^T Z + F)^T (B^T Z + F) - ZBB^T Z = 0 \tag{12.5}$$

Now, temporarily let us assume that the pair (C, A) is observable. Using 12.5 it can be shown that Z is positive definite: assume $Zx = 0$. Then we find $\|Cx\|^2 + \|(B^T Z + F)x\|^2 = 0$. Obviously, this implies $Cx = 0$ and $Fx = 0$. Thus, again using 12.5, we find that $ZAx = 0$, which contradicts the assumption that (C, A) is observable. Next, define $P := Z^{-1}$. Then from 12.5 we find that P satisfies the inequality

$$-PA^T - AP + BB^T - \gamma^{-2} EE^T - PC^T CP \geq 0$$

or, equivalently,

$$\begin{pmatrix} -PA^T - AP + BB^T - \gamma^{-2} EE^T & PC^T \\ CP & I \end{pmatrix} \geq 0. \tag{12.6}$$

Consider the system in driving variable representation Σ_{DV} given by the equations $\dot{x} = A^T x - C^T v$, $w = (x, v)$. In connection with this system consider the supply rate s given by $s(w) := w^T M w$, with M given by

$$\begin{pmatrix} \gamma^{-2} EE^T - BB^T & 0 \\ 0 & I \end{pmatrix}.$$

The inequality 12.6 then expresses the fact that the pair (Σ_{DV}, s) is internally dissipative. Consequently, by 7.2, also the associated algebraic Riccati equation

$$-YA^T - AY + BB^T - \gamma^{-2} EE^T - YC^T CY = 0$$

has a real symmetric solution Y. We claim that, in fact, there exists a positive definite solution Y^+. Indeed, let Y^+ be the largest real symmetric solution of the latter algebraic Riccati equation. Since Y^+ coincides with the largest real symmetric solution of 12.6 and since $P > 0$ is a solution of 12.6 this proves our claim. Finally, define $K := (Y^+)^{-1}$. Then K satisfies

$$A^T K + KA - K(BB^T + \gamma^{-2} EE^T)K + C^T C = 0. \tag{12.7}$$

Thus, if there exists F such that $A + BF$ is asymptotically stable and the H_∞ norm of G_F is less than or equal to γ, then there exists a positive definite solution K of the algebraic Riccati equation 12.7. If we assume that the *strict* inequality $\|G_F\|_\infty < \gamma$ holds then it can be shown that, in fact, such K can be found with the property that

$$\sigma(A - BB^T K + \gamma^{-2} EE^T K) \subset \mathbb{C}^-. \tag{12.8}$$

It turns out that also the converse holds: if the Riccati equation 12.7 has a positive semi-definite solution K such that condition 12.8 holds, then there exists F with the desired properties. In that case one such F is given by $F = -B^T K$. It can be shown that the condition that (C, A) should be observable can in fact be replaced by the weaker condition that (C, A) should have no unobservable eigenvalues on the imaginary axis, i.e. by the condition

$$\text{rank} \begin{pmatrix} Ii\omega - A \\ C \end{pmatrix} = n \quad \forall \omega \in \mathcal{R}.$$

In the general situation that we do not necessarily have $D^T D = 0$ and $D^T C = 0$, the condition that there should be no unobservable eigenvalues on the imaginary axis should be replaced by the condition that the quadruple (A, B, C, D) should have no *invariant zeros* on the imaginary axis. An invariant zero of (A, B, C, D) is any complex number s_0 with the property that

$$\text{rank} \begin{pmatrix} Is_0 - A & -B \\ C & D \end{pmatrix} < \text{normrank} \begin{pmatrix} Is - A & -B \\ C & D \end{pmatrix}.$$

Here, for a given real rational matrix $R(s)$, normrank R denotes the rank of R considered as a matrix with entries in the field of real rational functions. We can thus obtain the following theorem:

Theorem 12.1 *Consider the system 12.1. Let $\gamma > 0$. Assume that the matrix D has full column rank, and that the quadruple (A, B, C, D) has no invariant zeros on the imaginary axis. Then the following two statements are equivalent:*

(a) *There exists a static state feedback law $u = Fx$ such that $\sigma(A + BF) \subset \mathcal{C}^-$ and $\|G_F\|_\infty < \gamma$,*

(b) *There exists a positive semi-definite solution K of the algebraic Riccati equation*

$$A^T K + KA + \gamma^{-2} KEE^T K + C^T C - (KB + C^T D)(D^T D)^{-1}(B^T K + D^T C) = 0$$

such that

$$\sigma(A + \gamma^{-2} EE^T K - B(D^T D)^{-1}(B^T K + D^T C)) \subset \mathcal{C}^-$$

If one of the above condition hold then a suitable F is given by

$$F = -(D^T D)^{-1}(D^T C + B^T K).$$

□

The first contributions on the application of Riccati equations in the context of H_∞ optimal control theory were given in [19,20,25,49]. The first references in which the result of Th. 12.1 appears in the form as given above are [9,34] (see also [28]).

Of course, if the matrix D is *not of full column rank* then the above algebraic Riccati equation does not exist. It was shown in [32] that in the more general case that D is not necessarily of full column rank, the role of the algebraic Riccati equation is taken over by a *quadratic matrix inequality*. For any real number $\gamma > 0$ and matrix $K \in \mathcal{R}^{n \times n}$ we define a matrix $F_\gamma(K) \in \mathcal{R}^{(n+m)\times(n+m)}$ by

$$F_\gamma(K) := \begin{pmatrix} A^T K + KA + \gamma^{-2} KEE^T K + C^T C & KB + C^T D \\ B^T K + D^T C & D^T D \end{pmatrix}. \tag{12.9}$$

If a real symmetric matrix K satisfies the inequality $F_\gamma(K) \geq 0$ then it is said to satisfy the quadratic matrix inequality. In addition to 12.9, for any $\gamma > 0$ and $K \in \mathcal{R}^{n \times n}$ we define a $n \times (n+m)$ polynomial matrix $L_\gamma(K,s)$ by

$$L_\gamma(K,s) := \begin{pmatrix} Is - A - \gamma^{-2}EE^TK & -B \end{pmatrix}$$

Finally, let $G(s) = D + C(Is - A)^{-1}B$ be the open loop transfer matrix from u to z. The previous theorem can then in fact be shown to be a corrolary of the following more general result:

Theorem 12.2 *Consider the system 12.1. Let $\gamma > 0$. Assume that the quadruple (A, B, C, D) has no invariant zeros on the imaginary axis. Then the following two statements are equivalent:*

(**a**) *There exists a static state feedback law $u = Fx$ such that $\sigma(A + BF) \subset \mathcal{C}^-$ and $\|G_F\|_\infty < \gamma$,*

(**b**) *There exists a positive semi-definite solution K to the quadratic matrix inequality $F_\gamma(K) \geq 0$ such that*

$$\text{rank } L_\gamma(K,s) = \text{normrank } G$$

and

$$\text{rank}\begin{pmatrix} L_\gamma(K,s) \\ F_\gamma(K) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$$

□

A more realistic problem formulation is obtained if, instead of static state feedback, we require the control law to be given by *dynamic measurement feedback.* Conditions for the existence of a dynamic compensator that makes the closed loop system internally stable and which makes the H_∞ norm of the closed loop transfer matrix smaller than an a priori given positive real number can be given in terms of *a pair* of algebraic Riccati equations [9,14]. Again, if due to the singularity of certain system parameters these algebraic Riccati equations do not exist, then these conditions can be reformulated in terms of a pair of quadratic matrix inequalities [33].

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