The distance between a system and the set of uncontrollable systems

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Abstract. A controllability measure in terms of the distance between a system and the set of uncontrollable systems is developed. Some properties of a minimal disturbance, rendering a system noncontrollable, are given.

Keywords: controllability, controllability measure, distance between a system and the set of uncontrollable systems, disturbance.

1. Introduction

In [1] Paige argues that the traditional methods for testing the controllability of a system (A,B) where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are not satisfactory in the sense thay they may provide the wrong answer and furthermore they do not give an answer to the question: "is a system well/badly controllable?"

Using unitary state space transformations one can obtain a block Hessenberg form for a system (A,B). This Hessenberg form allows one to give a reliable answer to the question concerning controllability of (A,B). However, this algorithm only provides a "yes/no answer". From a practical point of view it is more important to have some quantitative answer to the issue of controllability of a system in the sense that one would like to know how close a given system is to an uncontrollable system. This is important whenever (A,B) has been obtained on the basis of measurements.

Paige proposes in [1] to use "distance to an uncontrollable system" as a controllability measure. In fact he proposes "minimum $\|(\delta A,\delta B)\|_2$ such that $(A + \delta A,\ B + \delta B)$ is uncontrollable" as such a measure ($\|\cdot\|_2$ is the spectral norm).

This paper is also concerned with controllability measures. We will provide an answer to

(1) "What is the distance between a system (A,B) and the set of uncontrollable systems?"

Of course the set of uncontrollable systems consists of systems with the same dimension (n) as (A,B).

The distance Δ between (A,B) and $(A + \delta A,\ B + \delta B)$ is defined as

$$\Delta^2 = d_A \| \delta A \|^2 + d_B \| \delta B \|^2 \ . \tag{2}$$

The weighting factors d_A and d_B, both positive, have been introduced in order to be able to deal with cases where for instance the confidence in the A-measurements differs from the confidence in the B-measurements.
We use the Frobenius norm $\| \cdot \|$: $\| M \|^2$ = trace $M^H M$ for a (possibly complex) matrix M. Here M^H denotes the conjugate transpose of M. In the first part of this paper the Frobenius norm coincides with the spectral norm: $\| M \|_2^2$ = maximum eigenvalue of $M^H M$. We will use the following characterization of controllability of (A,B)

$$\operatorname{rank}[\lambda I - A, B] = n \quad \text{for all } \lambda \in \mathbb{C} \ . \tag{3}$$

2. Main results

The answer to (1) is provided by the solution of the following minimization problem (for a given system (A,B)).

$$\text{minimize } d_A \| \delta A \|^2 + d_B \| \delta B \|^2 \tag{4}$$

with respect to $(\delta A, \delta B)$ and such that $(A + \delta A,\ B + \delta B)$ is uncontrollable.

The minimum distance in (4) will denoted as d{(A,B),UNCO} (here UNCO denotes the set of uncontrollable systems). We use Δ^2 instead of Δ because Δ^2 is easier to handle.
Direct minimization of (4) is almost always impossible so we have to exploit the information which is in the uncontrollability of $(A + \delta A,\ B + \delta B)$ for any disturbance $(\delta A, \delta B)$ of the system (A,B). If $(A + \delta A,\ B + \delta B)$ is not controllable then there exists a nonzero, possibly complex, row vector x^H

such that

$$x^H(A + \delta A) = \lambda x^H \ , \ x^H(B + \delta B) = 0 \ . \tag{5}$$

Here x^H denotes the conjugate transpose of a vector x and λ is an eigenvalue of $A + \delta A$.

We will now take a different point of view with respect to (5):

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x \in \mathbb{C}^n$, $x \neq 0$. Determine $\delta A \in \mathbb{C}^{n\times n}$, $\delta B \in \mathbb{C}^{n\times m}$ such that (5) is satisfied for some $\lambda \in \mathbb{C}$.

Observe that we allow complex disturbances here. Later on we will deal with the strictly real case.

The fact that we will allow complex disturbances may seem to be unattractive but in a number of cases, for instance if we use our measure of controllability as a condition number for pole assignability, this may not be unreasonable. Let $x \in \mathbb{C}^n$, $A \in \mathbb{R}^{n\times n}$. Then we have

$$x^H A = \frac{x^H A x}{x^H x} x^H + \left(x^H A - \frac{x^H A x}{x^H x} x^H \right) = \lambda_x x^H + x_o^H \ . \tag{6}$$

Observe that x^H and x_o^H are orthogonal (we use $x^H y$ as the inner product (x,y) in $\mathbb{C}^n$). Because we want (5) to hold we have to satisfy

$$x^H(A + \delta A) = \lambda_x x^H + x_o^H + x^H \delta A = \lambda x^H \tag{7}$$

for some $\lambda \in \mathbb{C}$. Thus we must have

$$x^H \delta A = -x_o^H + \rho x^H \tag{8}$$

for some $\rho \in \mathbb{C}$.

The smallest δA ($\|\delta A\|^2$ minimal), satisfying (8), is (see [2])

$$\delta A = x(-x_o^H + \rho x^H) \ / \ x^H x$$

with

$$\|\delta A\|^2 = x_o^H x_o \ / \ x^H x + |\rho|^2 \ .$$

Because we still may choose ρ we take $\rho = 0$ in order to minimize $\|\delta A\|^2$.

The disturbance δB on B has to be taken such that

$$x^H(B + \delta B) = 0 \ .$$

Therefore the smallest δB ($\|\delta B\|^2$ minimal), satisfying (5), is

$$\delta B = -x\,x^H B \ / \ x^H x$$

with

$$\|\delta B\|^2 = x^H B\,B^T x \ / \ x^H x \ .$$

Here $.^T$ stands for transposition.

Now we have obtained

<u>THEOREM</u>. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x \in \mathbb{C}^n$, $x \neq 0$, $d_A > 0$, $d_B > 0$. Then the (possibly complex) disturbance $(\delta A, \delta B)$ such that

(i) $x^H(A + \delta A) = \lambda x^H$ for some $\lambda \in \mathbb{C}$, $x^H(B + \delta B) = 0$

(ii) $d_A\|\delta A\|^2 + d_B\|\delta B\|^2$ is minimal

is given by

$$\delta A = x\left(-x^H A + \frac{x^H A x}{x^H x}\, x^H\right) \ / \ x^H x$$

$$\delta B = -xx^H B \ / \ x^H x \ ,$$

the minimal value in (ii) is

$$(9) \qquad d_A\, x^H A\left(I - \frac{xx^H}{x^H x}\right) A^T x \;/\; x^H x + d_B\, x^H B\, B^T x \;/\; x^H x$$

and

$$\lambda = x^H A x \;/\; x^H x \ .$$

<u>PROOF</u>. Using the arguments above and the fact that we may minimize $\|\delta A\|^2$ and $\|\delta B\|^2$ separately, because both d_A and d_B are positive, the proof is complete. □

From now on we will suppose that $x^H x = 1$ because δA and δB do not depend on $\|x\|$ (as was to be expected).
Using this theorem it is immediately clear how to compute the distance $d\{(A,B),\mathrm{UNCO}\}$ between (A,B) and the set of uncontrollable systems.

$$(10) \qquad d\{(A,B),\mathrm{UNCO}\}^2 = \underset{x\in\mathbb{C}^n,\|x\|=1}{\text{minimum}} \quad d_A\, x^H A(I - xx^H)A^T x + d_B\, x^H B\, B^T x \ .$$

Observe that the gradient and the Hessian of the object function in (10) can easily be derived. This is advantageous for the actual computation of $d\{(A,B),\mathrm{UNCO}\}$. Of course we can also compute $d\{(A,B),\mathrm{UNCO}\}$ using unconstrained minimization of (9). However, this gives rise to a number of problems because (9) does not depend on $\|x\|$. (Up to now the spectral norm and the Frobenius norm coincide because the disturbances turned out to be rank-one matrices.)

Next we consider the case of real disturbances $(\delta A,\delta B)$. We also start with (8)

$$x^H \delta A = -x_o^H + \rho x^H \ .$$

Let $x = x_r + ix_i$, $x_o = x_{or} + ix_{oi}$ where x_r, x_i, x_{or}, x_{oi} are real vectors. Let $\rho = \sigma + i\mu$. Because we want δA to be real we must have

$$(11) \qquad \begin{bmatrix} x_r^T \\ \\ -x_i^T \end{bmatrix} \delta A = \begin{bmatrix} -x_{or}^T \\ \\ x_{oi}^T \end{bmatrix} + \begin{bmatrix} \sigma x_r^T + \mu x_i^T \\ \\ \mu x_r^T - \sigma x_i^T \end{bmatrix} = \begin{bmatrix} -\tilde{x}_{or}^T \\ \\ \tilde{x}_{oi}^T \end{bmatrix} .$$

Using the Moore Penrose inverse we obtain as minimum norm solution for δA (the Moore Penrose inverse of a matrix M is denoted as M^+)

$$(12) \qquad \delta A = \begin{bmatrix} x_r^T \\ \\ -x_i^T \end{bmatrix}^+ \begin{bmatrix} -\tilde{x}_{or}^T \\ \\ \tilde{x}_{oi}^T \end{bmatrix} .$$

Straightforward calculation of $\| \delta A \|^2$ gives

$$(13) \qquad \| \delta A \|^2 = \frac{x_i^T x_i \tilde{x}_{or}^T \tilde{x}_{or} - 2x_r^T x_i \tilde{x}_{oi}^T \tilde{x}_{or} + x_r^T x_r \tilde{x}_{oi}^T \tilde{x}_{oi}}{x_r^T x_r x_i^T x_i - x_r^T x_i x_i^T x_r}$$

whenever $x_r^T x_r x_i^T x_i - x_r^T x_i x_i^T x_r = \det \neq 0$.

Observe that if $\det \neq 0$

$$\begin{bmatrix} x_r^T \\ \\ -x_i^T \end{bmatrix}^+ = [x_r, -x_i] \begin{bmatrix} x_r^T x_r & -x_r^T x_i \\ \\ -x_i^T x_r & x_i^T x_i \end{bmatrix}^{-1} .$$

If $\det = 0$ then x_r and x_i are dependent. Suppose that $x_r = tx_i$. Then we also have $x_{or} = tx_{oi}$. In order to be able to satisfy (8) with a real δA we must have that $\rho \in \mathbb{R}$. A minimum norm solution (12) now satisfies

$$\text{(14)} \qquad \|\delta A\|^2 = x_{or}^T x_{or} \,/\, x_r^T x_r + \rho^2 \qquad (x_r \neq 0)$$

$$\text{(15)} \qquad \|\delta A\|^2 = x_{oi}^T x_{oi} \,/\, x_i^T x_i + \rho^2 \qquad (x_i \neq 0) \; .$$

If $x_i \neq 0$ and $x_r \neq 0$ and $x_r = tx_i$ $(t \in \mathbb{R})$ then (14) and (15) are the same because $x_{or} = tx_{oi}$.

We still have to choose ρ in (8) such that δA becomes a minimal disturbance satisfying

$$x^H(A + \delta A) = \lambda x^H \qquad \lambda \in \mathbb{C} \; .$$

This is obtained by taking $\rho = 0$ in (14) and (15) and by minimizing (13) with respect to σ and μ.

A straightforward calculation of σ and μ, minimizing (13), gives

$$\text{(16)} \qquad \begin{aligned} \sigma &= \frac{x_i^T x_i x_{or}^T x_r + x_r^T x_r x_{oi}^T x_i - x_r^T x_i x_i^T x_{or} - x_r^T x_i x_{oi}^T x_r}{2 \det} \\ \mu &= \frac{x_i^T x_i x_{or}^T x_i + x_r^T x_i x_{or}^T x_r - x_r^T x_i x_{oi}^T x_i - x_r^T x_r x_{oi}^T x_r}{(x_i^T x_i)^2 + (x_r^T x_r)^2 + 2(x_r^T x_i)^2} \; . \end{aligned}$$

for the disturbance δB we have

$$\begin{bmatrix} x_r^T \\ \\ -x_i^T \end{bmatrix} \delta B = \begin{bmatrix} -x_r^T \\ \\ x_i^T \end{bmatrix} B \; .$$

The minimum norm solution for δB is

$$\delta B = \begin{bmatrix} x_r^T \\ \\ -x_i^T \end{bmatrix}^+ \begin{bmatrix} -x_r^T \\ \\ x_i^T \end{bmatrix} B$$

with

$$(17) \qquad \|\delta B\|^2 = \frac{x_i^T x_i x_r^T BB^T x_r - 2x_r^T x_i x_i^T BB^T x_r + x_r^T x_r x_i^T BB^T x_i}{\det}$$

whenever det $\neq 0$.

The analogues of (14) and (15) are

$$(18) \qquad \|\delta B\|^2 = x_r^T BB^T x_r \;/\; x_r^T x_r$$

$$(19) \qquad \|\delta B\|^2 = x_i^T BB^T x_i \;/\; x_i^T x_i \;.$$

Now we could minimize (for $x_r^T x_r + x_i^T x_i = 1$)

$$(20) \qquad d_A \|\delta A\|^2 + d_B \|\delta B\|^2$$

in order to find a minimal real disturbance $(\delta A, \delta B)$ where $\|\delta A\|^2$ is given by (13), (14) or (15) depending upon det being zero or not. In (13) we have to substitute (16) and in (14) and (15) we have to take $\rho = 0$. For $\|\delta B\|^2$ we take (17), (18) or (19) whichever appropriate.

However, $\|\delta A\|^2$ is not a continuous function when x_r and x_i tend to dependency. This can be seen as follows. Suppose that $x_i \neq 0$ ($x_r \neq 0$ can be handled analogously). Let $x_r = tx_i + \varepsilon p$ where $p \neq 0$ and $x_i^T p = 0$. Then we let ε tend to zero and (13) together with (16) generally will not tend to (15) with $\rho = 0$. This can be seen as follows.

We substitute $x_r^T = tx_i^T + \varepsilon p^T$ in (11) and (16) and we obtain

$$\tilde{x}_{or}^T = t\tilde{x}_{oi}^T + \varepsilon(-p^T A + up^T + \frac{v}{\varepsilon}(1+t^2)x_i^T + vtp^T) = t\tilde{x}_{oi}^T + \varepsilon q^T \;.$$

Here

$$u = \alpha + \sigma, \; v = \beta + \mu, \; \alpha + i\beta = \frac{x^H A x}{x^H x} \;.$$

Now (13) becomes

$$\|\delta A\|^2 = \frac{q^T q}{p^T p} + \frac{\tilde{x}_{oi}^T \tilde{x}_{oi}}{x_i^T x_i} .$$

Using

$$\lim_{\varepsilon\to 0} u = \frac{p^T A p}{2p^T p} + \frac{x_i^T A x_i}{2x_i^T x_i} = u_0$$

$$\lim_{\varepsilon\to 0} \frac{v}{\varepsilon} = \frac{p^T A x_i}{(1+t^2) x_i^T x_i} = v_0 \ / \ (1+t^2)$$

$$\lim_{\varepsilon\to 0} \sigma = \frac{p^T A p}{2p^T p} - \frac{x_i^T A x_i}{2x_i^T x_i} = \sigma_0$$

we have

$$\lim_{\varepsilon\to 0} q^T = -p^T A + u_0 p^T + v_0 x_i^T = q_0^T$$

$$\lim_{\varepsilon\to 0} \tilde{x}_{oi}^T = \tilde{x}_i^T A - \frac{x_i^T A x_i}{x_i^T x_i} x_i^T - \sigma_0 x_i^T = x_{oi}^T - \sigma_0 x_i^T .$$

Therefore we obtain

$$\lim_{\varepsilon\to 0} \|\delta A\|^2 = \frac{q_0^T q_0}{p^T p} + \sigma_0^2 + \frac{x_{oi}^T x_{oi}}{x_i^T x_i} .$$

Thereby proving the possible discontinuity of $\|\delta A\|^2$ if x_r an x_i tend to dependency.

It is easily seen that $\| \delta B \|^2$ is a continuous function for x_r and x_i tending to dependency.

Because δA is not continuous we have that the noncontrollable eigenvalue of $(A + \delta A, B + \delta B)$ is also not continuous. This can be seen as follows. For $(\delta A, \delta B)$ computed as above (based on x) we have

$$x^H(A + \delta A) = (u + iv)x^H \ , \ x^H(B + \delta B) = 0 \ .$$

In the open region $x_r^T x_r x_i^T x_i - x_i^T x_r x_r^T x_i \neq 0$ $u + iv$ is the noncontrollable eigenvalue. If x_r and x_i tend to dependency $u + iv$ tends to u_0 whereas for $x_r = tx_i$ we have $x_i^T A x_i \ / \ x_i^T x_i$ as the noncontrollable eigenvalue.

This possible discontinuity in the, to be minimized, object function (20) may present serious problems.

Therefore (10) is to be preferred whenever possible.

3. Comparison of real and complex controllability measures

Up to now we have obtained two controllability measures characterized by

(c) complex disturbances are allowed

(r) only real disturbances are allowed.

Obviously, we have that case (c) generally gives a smaller controllability measure than case (r) (the distance between a system (A,B) and the set of uncontrollable systems UNCO, measured using only real disturbances, generally is larger than the distance between (A,B) and UNCO measured in terms of complex disturbances). The difference between case (c) and case (r) may be investigated as follows. The vectors x_r and x_i may be taken to be orthogonal because instead of the vector x in (5) we may take $e^{i\varphi}x$. Such a factor $e^{i\varphi}$ does not affect (the norm of) the disturbances δA and δB. Furthermore it can easily be seen that φ may be taken such that the real part and the imaginary part of $e^{i\varphi}x$ are orthogonal vectors. Suppose that $x_i \neq 0$, $x_r \neq 0$. Then $\|\delta A\|^2$ is given by (13). We have

$$\frac{x_i^T x_i \tilde{x}_{or}^T \tilde{x}_{or} + x_r^T x_r \tilde{x}_{oi}^T \tilde{x}_{oi}}{x_r^T x_r x_i^T x_i} \geq \frac{\tilde{x}_{or}^T \tilde{x}_{or} + \tilde{x}_{oi}^T \tilde{x}_{oi}}{x_r^T x_r + x_i^T x_i} = \frac{\tilde{x}_o^H \tilde{x}_o}{x^H x} .$$

Because $\tilde{x}_o = x_o - (\sigma - i\mu)x$ (see (11)) we have

$$\frac{\tilde{x}_o^H \tilde{x}_o}{x^H x} = \frac{x_o^H x_o}{x^H x} + \sigma^2 + \mu^2 .$$

Thereby showing that the real disturbance on A generally is larger than the corresponding complex disturbance. An analogous result can be proven for (14), (15), (17), (18), (19). Thus we have shown that in order to measure the distance between (A,B) and UNCO we generally find that case (c) gives a smaller distance than case (r).

4. Special cases

In this section we consider two special cases

(a) only disturbances on A are allowed ($\delta B = 0$)

(b) only disturbances on B are allowed ($\delta A = 0$) .

Case (a) can be handled easily because we only have to restrict x in (10) such that $x^H B = 0$.

Case (b) forces one to compute the eigenvectors of A. Then the minimal disturbance δB on B, such that $(A, B + \delta B)$ is controllable, is

$$\delta B = -vv^H B \; / \; v^H v$$

where v is an eigenvector of A such that

$$\|\delta B\|^2 = v^H B B^T v \; / \; v^H v$$

is minimal.

If computation of the eigenvectors of A presents problems one might approximate case (b) by taking d_A and d_B in (10) such that $d_A \; / \; d_B$ is "large".

5. Discussion

A method to compute the distance between a system and the set of uncontrollable systems has been described. A disadvantage of this method is that one still needs to minimize a function of 2n variables (where n is the dimension of the system) on the hypersphere in $\mathbb{C}^n$. The extra freedom, which exists for this minimization problem because $e^{i\varphi}x$ corresponds to a minimizing vector of (10) for any $\varphi \in \mathbb{R}$ whenever x is such a vector, may be delt with by requiring that the real and imaginary parts of x are orthogonal vectors. A vector $e^{i\varphi}x$ gives rise to the same disturbance $(\delta A, \delta B)$ for any $\varphi \in \mathbb{R}$.

If one considers the case of real disturbances, then a serious difficulty appears because the, to be minimized, object function is not continuous anymore . Therefore computation of the controllability measure using complex disturbances seems to be more attractive.

A nice property of a minimal complex disturbance $(\delta A, \delta B)$ is that both δA and δB are rank one matrices.
While controllability of a system is neither affected by state space transformation nor by feedback we generally have for the controllability measure as described in this paper

$$d\{(A,B),UNCO\} \neq d\{(TAT^{-1},TB),UNCO\}$$

$$d\{(A,B),UNCO\} \neq d\{(A+BF,B),UNCO\} \ .$$

If T is unitary then the controllability measure is the same for (A,B) and (TAT^{-1},TB).

It is easily seen from examples that feedback may enlarge the distance between a system and UNCO but that it also may reduce this distance considerably. In order to illustrate this we consider the following situation.

Let (A,B) be a controllable system. Let σ_{min} be the smallest singular value of B. If $m < n$ we may take σ_{min} to be zero because we may add columns (to B) consisting only of zeroes.

Consider a sequence $(x_k,\ k = 0,1,2,\ldots)$ such that

$$x_k \in \mathbb{C}^n \qquad k = 0,1,2,\ldots$$

$$\|x_k\| = 1$$

$$x_k^H B \neq 0$$

$$\lim_{k\to\infty} x_k^H BB^T x_k = \sigma_{min}^2 \ .$$

Let the sequence of $m \times n$-matrices $(F_k,\ k=0,1,2,\ldots)$ be defined by

$$F_k = B^T x_k (\lambda_k x_k^H - x_k^H A) \ / \ x_k^H B B^T x_k$$

where $(\lambda_k,\ k=0,1,2,\ldots)$ is a sequence of (complex) numbers.
Then

$$d\{(A + BF_k,\ B), \mathrm{UNCO}\} \le x_k^H B B^T x_k \quad ; \quad k = 0,1,2,\ldots$$

because

$$x_k^H (A + BF_k) = \lambda_k x_k^H \ .$$

This shows that the distance to UNCO may be reduced considerably using feedback. If $\sigma_{min} \neq 0$ then the minimal distance which can be obtained in this sense is σ_{min}^2. If $\sigma_{min} = 0$ (for instance if $m < n$) then UNCO may be approached arbitrarily close using feedback matrices (whose norms tend to infinity).

The controllability measure as described in this paper is not directly related to the singular values of the controllability matrix

$$[B,\ AB,\ \ldots\ ,\ A^{n-1}B] \ .$$

In order to show this we consider systems (A_n, B_n) with dimension n where

$$(A_n, B_n) = \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} , \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

for n = 1,2,3,4,5,10,15,20.

The singular values of the controllability matrix of (A_n, B_n) all are 1 for each n.

However, $d\{(A_n, B_n), \mathrm{UNCO}\}$ (with $d_A = d_B = 1$) depends on n as is shown in the following table

n	$d\{(A_n, B_n), \mathrm{UNCO}\}$
1	1.00
2	0.75
3	0.50
4	0.35
5	0.25
10	0.08
15	0.04
20	0.02

controllability measures of (A_n, B_n)

This table indicates that large systems tend to be close to UNCO. This holds if one permits disturbances on each element of a system.

Consider also the system

$$(A_\alpha, B) = \left(\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Again the singular values of the controllability matrix all are 1 for any α.

On the other hand

$$(A_\alpha, B + \delta B) = \left(\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{\alpha} \\ 1 \end{bmatrix} \right)$$

is not controllable. Therefore (A_α, B) is close to UNCO if α is large.
Often a system has many fixed zeroes and / or ones. The method in this paper does not allow for fixed elements in the A-matrix or the B-matrix. Therefore a different method has to be used in order to compute the controllability measure for such cases.

6. References

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