Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 8855987, 8 pages
https://doi.org/10.1155/2020/8855987

Research Article

The Distance Laplacian Spectral Radius of Clique Trees

Xiaoling Zhang and Jiajia Zhou

School of Mathematics and Information Sciences, Yantai University, Yantai, Shandong 264005, China

Correspondence should be addressed to Xiaoling Zhang; zhangxling04@163.com

Received 7 August 2020; Revised 11 September 2020; Accepted 4 October 2020; Published 9 December 2020

Academic Editor: M Javaid

The distance Laplacian matrix of a connected graph G is defined as $\mathscr{L}(G) = \mathrm{Tr}(G) - D(G)$, where $D(G)$ is the distance matrix of G and $\mathrm{Tr}(G)$ is the diagonal matrix of vertex transmissions of G. The largest eigenvalue of $\mathscr{L}(G)$ is called the distance Laplacian spectral radius of G. In this paper, we determine the graphs with maximum and minimum distance Laplacian spectral radius among all clique trees with n vertices and k cliques. Moreover, we obtain$n$ vertices and k cliques.

1. Introduction

In this paper, we consider simple connected graphs [1]. A graph G is represented by $G = (V(G), E(G))$, in which the set $V(G) = \{v_1, v_2, \ldots, v_n\}$ represents its vertex set and $E(G)$ is the edge set connecting pairs of distinct vertices. The number $n = |V(G)|$ is referred to as the order of G. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_G(u, v))_{u,v \in V(G)}$, where $d_G(u, v)$ denotes the distance between vertices u and v in G, i.e., the length of a shortest path from u to v in G. For $u \in V(G)$, the transmission of u in G, denoted by $\mathrm{Tr}_G(u)$, is defined as the sum of distances from u to all other vertices of G. Let $Tr(G)$ be the diagonal matrix of vertex transmissions of G. In 2013, Aouchiche and Hansen [2] first gave the definition of distance Laplacian matrix: for a connected graph G, $\mathscr{L}(G) = \mathrm{Tr}(G) - D(G)$, where $\mathscr{L}(G)$ denotes the distance Laplacian matrix. Obviously, $\mathscr{L}(G)$ is a positive semidefinite, symmetric, and singular matrix. The distance Laplacian eigenvalues of G, denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$, are the eigenvalues of $\mathscr{L}(G)$. Especially, the largest eigenvalue $\lambda_1(G)$ is the distance Laplacian spectral radius of G. The positive unit eigenvector, i.e., all components of the eigenvector are positive, corresponding to $\lambda_1(G)$ is called the Perron eigenvector of $\mathscr{L}(G)$.

For a graph G, two vertices are called adjacent if they are connected by an edge and two edges are called incident if they share a common vertex. The set of vertices that are adjacent to a vertex $v \in V(G)$ is called the neighborhood of v and is presented by $N_G(v)$. As usual, let K_n, $K_{1,n-1}$, and P_n denote the complete graph, the star, and the path with order n, respectively. G is a connected graph, $X \in V(G)$, $G - X$ is not connected, and then X is a cut-vertex set. If X has only vertex v, then v is a cut-vertex. A block of G is a maximal connected subgraph of G that has no cut-vertex. A block is a clique if the block is a complete graph. A graph G is a clique tree if each block of G is a clique. We call $\mathscr{P}_{n_1,n_2,\ldots,n_k}$ a clique path if we replace each edge of P_{k+1} by a clique K_{n_i} such that $V(K_{n_i}) \cap V(K_{n_{i+1}}) = v_i$ for $i = 1, 2, \ldots, k-2$ and $V(K_{n_i}) \cap V(K_{n_j}) = \varnothing$ for $j \neq i-1, i+1$ and $2 \leq i \leq k-1$. We call $K_{u,n_1,n_2,\ldots,n_k}$ a clique star if we replace each edge of the star $K_{1,k}$ with a clique K_{n_i} such that $V(K_{n_i}) \cap V(K_{n_j}) = u$ for $i \neq j$ and $i, j = 1, 2, \ldots, k$ (see Figure 1).

Recently, Xing and Zhou [3] characterized the unique graph with minimum distance Laplacian spectral radius among all the bicyclic graphs with fixed number of vertices; Aouchiche and Hansen [4] showed that the star $K_{1,n}$ is the unique tree with the minimum distance Laplacian spectral radius among all trees; Lin et al. [5, 6] determined the unique

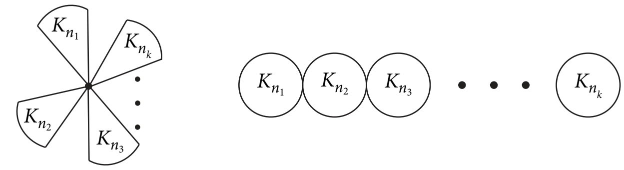

Figure 1: A clique star and a clique path.

graph with minimum distance Laplacian spectral radius among all the trees with fixed bipartition, nonstar-like trees, noncaterpillar trees, nonstar-like noncaterpillar trees, and the graph with fixed edge connectivity at most half of the order, respectively; Niu et al. [7] determined the unique graph with minimum distance Laplacian spectral radius among all the bipartite graphs with fixed matching number and fixed vertex connectivity, respectively; Fan et al. [8] determined the graph with minimum distance Laplacian spectral radius among all the unicyclic and bicyclic graphs with fixed numbers of vertices, respectively; Lin and Zhou [9] determined the unique graph with maximum distance Laplacian spectral radius among all the unicyclic graphs with fixed numbers of vertices.

In 2019, Cui et al. [10] investigated a convex combination of $\mathrm{Tr}(G)$ and $D(G)$ in the form of $D_\alpha(G) = \alpha\mathrm{Tr}(G) + (1-\alpha)D(G)$, $0 \le \alpha \le 1$, which is called the generalized distance matrix. Alhevaz et al. [11] gave some new upper and lower bounds for the generalized distance energy of graphs which are established based on parameters including the Wiener index and the transmission degrees and found that the complete graph has the minimum generalized distance energy among all connected graphs; Lin and Drury et al. [12] established some bounds for the generalized distance Gaussian Estrada index of a connected graph, involving the different graph parameters, including the order, the Wiener index, the transmission degrees, and the parameter $\alpha \in [0,1]$, and characterized the extremal graphs attaining these bounds; Alhevaz et al. [13] obtained some bounds for the generalized distance spectral radius of graphs using graph parameters like the diameter, the order, the minimum degree, the second minimum degree, the transmission degree, and the second transmission degree and characterized the extremal graphs; Alhevaz et al. [14] studied the generalized distance spectrum of join of two regular graphs and join of a regular graph with the union of two different regular graphs; Shang [15] established better lower and upper bounds to the distance Estrada index for almost all graphs.

The distance Laplacian energy is defined as $\mathrm{DLE}(G) = (1/n)\sum_{i=1}^{n} |\lambda_i(G) - t(G)|$, where $t(G)$ is the average transmission of G and is defined by $t(G) = (1/n)\sum_{i=1}^{n} \mathrm{Tr}_G(v_i)$. Although there has been extensive work done on the distance Laplacian spectral radius of graphs, relatively little is known in regard to distance Laplacian energy. The distance Laplacian energy was first introduced in [16], where several lower and upper bounds were obtained; Das et al. [17] gave some lower bounds on distance Laplacian energy in terms of n for graphs and trees and characterized the extremal graphs and trees. In this paper, first, we not only get the distance Laplacian eigenvalues of all clique stars $K_{u,n_1,n_2,\ldots,n_k}$ but also get their distance Laplacian energies; second, we prove all clique stars $K_{u,n_1,n_2,\ldots,n_k}$ are the graphs with minimum distance Laplacian spectral radius among all clique trees with n vertices and k cliques. Then, we show that the clique path $\mathscr{P}_{m,2,\ldots,2,n-m-k+3}$ for $m \ge 3$ is the graph with maximum distance Laplacian spectral radius among all clique trees with n vertices and k cliques.

2. Preliminaries

Let $G = (V, E)$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. A column vector $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, 2, \ldots, n$. Then,

$$x^T \mathscr{L}(G)x = \sum_{\{u,v\}\subseteq V(G)} \mathrm{d}_G(u,v)\left(x_u - x_v\right)^2, \tag{1}$$

and λ is a distance Laplacian eigenvalue with corresponding eigenvector x if and only if $x \ne 0$, for each $u \in V(G)$,

$$(\lambda - \mathrm{Tr}_G(u))x_u = -\sum_{v\in V(G)} \mathrm{d}_G(u,v)x_v, \tag{2}$$

or equivalently

$$\lambda x_u = \sum_{v\in V(G)} \mathrm{d}_G(u,v)\left(x_u - x_v\right). \tag{3}$$

The above equation is called the eigenequation of G at u.

Note that $1_n = (\underbrace{1, 1, \ldots, 1})^T$ is an eigenvector of $\mathscr{L}(G)$ corresponding to $\lambda_n(G) \ne 0$. For $n \ge 2$, if x is an eigenvector of $\mathscr{L}(G)$ corresponding to $\lambda_1(G)$, we have $x^T 1_n = 0$.

For a unit column vector $x \in \mathbb{R}^n$, by Rayleigh's principle, we have $\lambda(G) \ge x^T \mathscr{L}(G)x$ with equality if and only if x is an eigenvector of $\mathscr{L}(G)$ corresponding to $\lambda(G)$.

The following is the well-known Cauchy interlacing theorem.

Lemma 1 (Cauchy interlace theorem) (see [1]). *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ and B be one of its principal submatrices. Let B have eigenvalues $\mu_1 \ge \cdots \ge \mu_m$. Then, the inequalities $\lambda_{n-m+i} \le \mu_i \le \lambda_i$ ($i = 1, \ldots, m$) hold.*

Lemma 2 (see [6]). *Let G be a connected graph with three induced subgraphs G_1, G_2, and G_3 such that $|V(G_i)| \ge 2$ for $i = 1, 2, 3$ and $V(G_i) \cap V(G_j) = \{u\}$ for $1 \le i < j \le 3$ and $\cup_{i=1}^{3} V(G_i) = V(G)$ (see Figure 2). For $v \in V(G_2)\backslash\{u\}$ and $y \in V(G_1)\backslash\{u\}$, let $G^1 = G - \left\{u\omega\colon \omega \in N_{G_3}(u)\right\} + \{v\omega\colon \omega \in$*

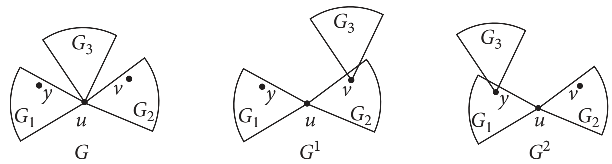

Figure 2: A graph transformation from G to G^1 and G^2.

$N_{G_3}(u)\}$ *and* $G^2 = G - \{u\omega\colon \omega \in N_{G_3}(u)\} + \{y\omega\colon \omega \in N_{G_3}(u)\}$. *If* $N_G(u) = \{y, v\} \cup N_{G_3}(u)$, *then* $\lambda_1(G) < \lambda_1(G^1)$ *or* $\lambda_1(G) < \lambda_1(G^2)$.

3. Minimum Distance Laplacian Spectral Radius of Clique Trees

The diameter of a graph is the maximum distance between any pair of vertices.

Lemma 3. *Let S be a clique tree with n vertices and k cliques. If diam$(S) \geq 3$, then $\lambda_1(S) > 2n - 1$.*

Proof. For convenience, let diam$(S) = d$ and $\mathscr{P}_{n_1,n_2,\ldots,n_d}$ be a clique path of S. Denote the cliques of $\mathscr{P}_{n_1,n_2,\ldots,n_d}$ by $K_{n_1}, K_{n_2}, \ldots, K_{n_d}$. Let $V(K_{n_i}) \cap V(K_{n_{i+1}}) = v_i$ for $i = 1, 2, \ldots, d-1$. Let $\{v_0\} \in V(K_{n_1})\backslash\{v_1\}$ and $\{v_d\} \in V(K_{n_d})\backslash\{v_{d-1}\}$. Then, $v_0 v_1 \ldots v_d$ is a diameter path of S. We can easily get

$$\begin{aligned}\mathrm{Tr}_S(v_0) &\geq (n_1 - 2) + 2(n_2 - 2) + \cdots + d(n_d - 2) + 1 + 2 + \cdots + d \\ &\quad + 2[n - n_1 - n_2 - \cdots - n_d + (d-1)], \\ \mathrm{Tr}_S(v_d) &\geq (n_d - 2) + 2(n_{d-1} - 2) + \cdots + d(n_1 - 2) + 1 + 2 + \cdots + d \\ &\quad + 2[n - n_1 - n_2 - \cdots - n_d + (d-1)].\end{aligned} \tag{4}$$

Then, we have

$$\begin{aligned}\mathrm{Tr}_S(v_0) + Tr_S(v_d) &\geq (d+1)[(n_1 - 2) + (n_2 - 2) + \cdots + (n_d - 2)] + d(d+1) + 4n \\ &\quad - 4[(n_1 - 2) + (n_2 - 2) + \cdots + (n_d - 2)] - 8d + 4(d-1) \\ &= (d + 1 - 4)[(n_1 - 2) + (n_2 - 2) + \cdots + (n_d - 2)] + d(d+1) + 4n - 8d + 4(d-1) \\ &= (d-3)[(n_1 - 2) + (n_2 - 2) + \cdots + (n_d - 2)] + d^2 + 4n - 3d - 4 > 4n + d^2 - 3d - 8.\end{aligned} \tag{5}$$

Let M be the principal submatrix of $\mathscr{L}(S)$ indexed by v_0 and v_d. Then,

$$\begin{aligned}M &= \begin{pmatrix} \mathrm{Tr}_S(v_0) & -d \\ -d & \mathrm{Tr}_S(v_d) \end{pmatrix}, \\ |M - \lambda E| &== \begin{vmatrix} \mathrm{Tr}_S(v_0) - \lambda & -d \\ -d & \mathrm{Tr}_S(v_d) - \lambda \end{vmatrix} \\ &= \lambda^2 - (\mathrm{Tr}_S(v_0) + \mathrm{Tr}_S(v_d))\lambda + \mathrm{Tr}_S(v_0)\mathrm{Tr}_S(v_d) - d^2,\end{aligned} \tag{6}$$

and thus

$$\begin{aligned}\lambda_1(M) &= \frac{\mathrm{Tr}_S(v_0) + \mathrm{Tr}_S(v_d) + \sqrt{(\mathrm{Tr}_S(v_0) - \mathrm{Tr}_S(v_d))^2 + 4d^2}}{2} \\ &\geq \frac{\mathrm{Tr}_S(v_0) + \mathrm{Tr}_S(v_d) + 2d}{2} > \frac{4n + d^2 - d - 8}{2} \\ &\geq \frac{4n + 3^2 - 3 - 8}{2} = 2n - 1.\end{aligned} \tag{7}$$

By Lemma 2, we have $\lambda_1(S) \geq \lambda_1(M) > 2n - 1$. □

Theorem 1. *Let $K_{u,n_1,n_2,\ldots,n_k}$ be an arbitrary clique star with n vertices and k cliques. Then, $\lambda_1(K_{u,n_1,n_2,\ldots,n_k}) = 2n - 1$.*

Proof. Obviously, we have $n_1 + n_2 + n_3 + \cdots + n_k = n + k - 1$. Let x be a Perron eigenvector of $\mathscr{L}(K_{u,n_1,n_2,\ldots,n_k})$ corresponding to $\lambda_1(K_{u,n_1,n_2,\ldots,n_k})$. By symmetry, we may assume $x_v = x_i$ for any $v \in V(K_{n_i})\backslash\{u\}$, $i = 1, 2, \ldots, k$. Let $x_0 = x_u$, then we have

$$\begin{cases} \lambda x_0 = (n_1-1)(x_0-x_1)+(n_2-1)(x_0-x_2)+(n_3-1)(x_0-x_3)+\cdots+(n_k-1)(x_0-x_k), \\ \lambda x_1 = (x_1-x_0)+2(n_2-1)(x_1-x_2)+2(n_3-1)(x_1-x_3)+\cdots+2(n_k-1)(x_1-x_k), \\ \lambda x_2 = (x_2-x_0)+2(n_1-1)(x_2-x_1)+2(n_3-1)(x_2-x_3)+\cdots+2(n_k-1)(x_2-x_k), \\ \lambda x_3 = (x_3-x_0)+2(n_1-1)(x_3-x_1)+2(n_2-1)(x_3-x_2)+\cdots+2(n_k-1)(x_3-x_k), \\ \cdots \\ \lambda x_k = (x_k-x_0)+2(n_1-1)(x_k-x_1)+2(n_2-1)(x_k-x_2)+\cdots+2(n_{k-1}-1)(x_k-x_{k-1}). \end{cases} \tag{8}$$

Thus, λ_1 is the largest root of the equation $f_{n_1,n_2,\ldots,n_k}(t)=0$, where $\beta=\sum_{i=1}^{k} n_i - k$ and

$$\begin{aligned} f_{n_1,n_2,\ldots,n_k}(t) &= \begin{vmatrix} \beta-t & 1-n_1 & 1-n_2 & \cdots & 1-n_k \\ -1 & 2(\beta-n_1)+3-t & 2-2n_2 & \cdots & 2-2n_k \\ -1 & 2-2n_1 & 2(\beta-n_2)+3-t & \cdots & 2-2n_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 2-2n_1 & 2-2n_2 & \cdots & 2(\beta-n_k)+3-t \end{vmatrix} \\ &= \begin{vmatrix} -t & 1-n_1 & 1-n_2 & \cdots & 1-n_k \\ -t & 2(\beta-n_1)+3-t & 2-2n_2 & \cdots & 2-2n_k \\ -t & 2-2n_1 & 2(\beta-n_2)+3-t & \cdots & 2-2n_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -t & 2-2n_1 & 2-2n_2 & \cdots & 2(\beta-n_k)+3-t \end{vmatrix} \\ &= \begin{vmatrix} -t & 1-n_1 & 1-n_2 & 1-n_3 & \cdots & 1-n_k \\ -t & 2(\beta-n_1)+3-t & 2-2n_2 & 2-2n_3 & \cdots & 2-2n_k \\ 0 & 1-2n+t & 2n-1-t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1-2n+t & 0 & 0 & \cdots & 2n-1-t \end{vmatrix} \\ &= \begin{vmatrix} -t & 1-n-t & 1-n_2 & 1-n_3 & \cdots & 1-n_k \\ -t & 1-2t & 2-2n_2 & 2-2n_3 & \cdots & 2-2n_k \\ 0 & 0 & 2n-1-t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2n-1-t \end{vmatrix} \\ &= (-t)(-1)^2(2n-1-t)^{k-1}(1-2t)+(-t)(-1)^3(2n-1-t)^{k-1}(1-n-t) \\ &= (2n-1-t)^{k-1}[(1-2t)(-t)+(1-n-t)t] \\ &= t(2n-1-t)^{k-1}(t-n). \end{aligned} \tag{9}$$

Therefore, we have $\lambda_1(K_{u,n_1,n_2,\ldots,n_k})=2n-1$ n and 0 are also distance Laplacian eigenvalues of $K_{u,n_1,\ldots\ldots,u_k K_{\ldots\ldots}}$.

Combining Lemma 3 and Theorem 1, we have the following result. □

Theorem 2. *Among all clique trees with n vertices and k cliques, the graphs attaining the minimum distance Laplacian spectral radius are clique stars $K_{u,n_1,n_2,\ldots,n_k}$.*

Let I be the identity matrix of order n. The characteristic polynomial of $\mathscr{L}(G)$ can be written as $\psi(G\colon\lambda)=\det(\lambda I-\mathscr{L}(G))$. Let us label the vertices of $K_{u,n_1,n_2,\ldots,n_k}$ such that u is the first vertices, and the first n_1 vertices are from $V(K_{n_1})$, the following n_2-1 vertices are from $V(K_{n_2})\backslash\{u\}$, …, and the last n_k-1 are from $V(K_{n_k})\backslash\{u\}$. Let $\det(\lambda I-\mathscr{L}(K_{u,n_1,n_2,\ldots,n_k}))=0$. Combining Theorem 1, by direct calculations, we get the following result.

Corollary 1. *The distance Laplacian eigenvalues of $K_{u,n_1,n_2,\ldots,n_k}$ are $2n-1$ of multiplicities $k-1$, $2n-n_i$ of multiplicities $n_i-2\,(1\le i\le k)$, n, and 0.*

Theorem 3. *Let $K_{u,n_1,n_2,\ldots,n_k}$ be an arbitrary clique star with n vertices and k cliques. Then, we have $DLE(K_{u,n_1,n_2,\ldots,n_k}) = (2/n)[2n-1+(1/n)(k-1-\sum_{i=1}^{k} n_i^2)]$.*

Proof. Obviously, we have $n_1+n_2+n_3+\cdots+n_k=n+k-1$. For convenience, let $G=K_{u,n_1,n_2,\ldots,n_k}$. For any $v,w\in V(K_{n_i})\backslash\{u\}$, we have $\mathrm{Tr}_G(v)=\mathrm{Tr}_G(w)$. Let $v_i\in V(K_{n_i})\backslash\{u\}$, $1\le i\le k$. Then, we have $\mathrm{Tr}_G(v_i)=2n-n_i-1$ and $t(G)=(1/n)\sum_{i=1}^{n}\mathrm{Tr}_G(v_i)=(([\sum_{i=1}^{k}((n_i-1)(2n-n_i-1)]+n-1)/n)=((2n\sum_{i=}1^k(n_i-1)-\sum_{i=1}^{k}(n_i-1)(n_i+1)+n-1)/\ n)=((2n(n-1)-\sum_{i=1}^{k}n_i^2+k+n-1)/n)=2n-1+((k-1-\sum_{i=}1^k n_i^2)/n)$. By Cauchy–Schwarz inequality, we have $\sum_{i=1}^{k}n_i^2\ge(n+k-1)^2>n^2$. So, we get $t(G)<2n-1+((k-1-n^2)/n)=n-1+(k-1/n)<n$. By Corollary 1, we know $\lambda_i>n>t(G)$ for $i=1,2,\ldots,n-1$, and $\lambda_n=0$. $\mathrm{DLE}(K_{u,n_1,n_2,\ldots,n_k})=(1/n)\sum_{i=1}^{n}|\lambda_i(G)-t(G)|=((\sum_{i=}\ 1^{n-1}[\lambda_i(G)-t(G)]+t(G)\ )/n)=(\sum_{i=1}^{n}\lambda_i(G)+(2-n)t(G)/n)=(2t\ (G)/\ n)$ since $\sum_{i=1}^{n}\lambda_i(G)$ is equal to the trail of $\mathscr{L}(G)$, i.e., $\sum_{i=1}^{n}\lambda_i(G)=\sum_{i=1}^{n}\mathrm{Tr}_G(v_i)$. So, we get $\mathrm{DLE}(K_{u,n_1,n_2,\ldots,n_k})=(2/n)[2n-1+(1/n)(k-1-\sum_{i=}1^k n_i^2)]$. □

4. Maximum Distance Laplacian Spectral Radius of Clique Trees

Lemma 4. *Let H be a connected graph and S be a clique tree with $diam(S)=d$. Suppose $\mathscr{P}_{n_1,n_2,\ldots,n_d}$ is a clique path of S with cliques $K_{n_1},K_{n_2},\ldots,K_{n_d}$ and $V(K_{n_i})\cap V(K_{n_{i+1}})=v_i$ for $i=1,2,\ldots,d-1$. Let H_t be the graph obtained by identifying a vertex v of H and a vertex u of K_{n_t}, where $2\le t\le d-1$. Then, $\lambda_1(H_d)>\lambda_1(H_t)$ or $\lambda_1(H_1)>\lambda_1(H_t)$.*

Proof. By Lemma 2, we may assume $u\ne v_{t-1}$ or $u\ne v_t$ for $2\le t\le d-1$. Denote the component of $S-v_{t-1}$ which contains vertex v_{t-2} by $\mathscr{S}_1$ and the component of $S-v_t$ which contains vertex v_{t+1} by $\mathscr{S}_2$. Let $S_1=V(\mathscr{S}_1)$, $S_2=V(\mathscr{S}_2)$, and $S_3=V(S)\backslash(V(\mathscr{S}_1)\cup V(\mathscr{S}_2))$. Suppose x is a Perron eigenvector of $\mathscr{L}(H_t)$ corresponding to $\lambda_1(H_t)$. In the following, we will first prove $\lambda_1(H_{t+1})>\lambda_1(H_t)$ or $\lambda_1(H_{t-1})>\lambda_1(H_t)$.

Case 1: $\sum_{h\in V(H)\backslash\ \{v\}}\sum_{\omega\in S_1}(x_\omega-x_h)^2\ge\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_2}(x_\omega-x_h)^2$. From H_t to H_{t+1}, we have

$$d_{H_t}(\omega,h)-d_{H_{t+1}}(\omega,h)=\begin{cases}-1, & \omega\in S_1\cup S_3\backslash\{v_t\},\ h\in V(H)\backslash\{v\},\\ 1, & \omega\in S_2,\ h\in V(H)\backslash\{v\},\\ 0, & \text{otherwise},\end{cases}$$

$$\begin{aligned}\lambda_1(H_{t+1})-\lambda_1(H_t)&\ge x^T(\mathscr{L}(H_{t+1})-\mathscr{L}(H_t))x\\ &=\sum_{h\in V(H)\backslash\{v\}}\left[\sum_{\omega\in S_1\cup S_3\backslash\{v_t\}}(x_\omega-x_h)^2-\sum_{\omega\in S_2}(x_\omega-x_h)^2\right]\\ &=\sum_{h\in V(H)\backslash\{v\}}\left[\sum_{\omega\in S_1}(x_\omega-x_h)^2+\sum_{\omega\in S_3\backslash\{v_t\}}(x_\omega-x_h)^2-\sum_{\omega\in S_2}(x_\omega-x_h)^2\right]\\ &\ge\sum_{h\in V(H)\backslash\{v\}}\left[\sum_{\omega\in S_1}(x_\omega-x_h)^2-\sum_{\omega\in S_2}(x_\omega-x_h)^2\right]\\ &\ge 0.\end{aligned}\tag{10}$$

Thus, $\lambda_1(H_{t+1})\ge\lambda_1(H_t)$.

In the following, we will prove $\lambda_1(H_{t+1})>\lambda_1(H_t)$. If $\lambda_1(H_{t+1})=\lambda_1(H_t)$, then $\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_3\backslash\{v_t\}}(x_\omega-x_h)^2=0$, which implies $x_\omega=x_h$ for any $\omega\in S_3\backslash\{v_t\}$, $h\in V(H)\backslash\{v\}$, and x is also a Perron eigenvector of $\mathscr{L}(H_{t+1})$ corresponding to $\lambda_1(H_{t+1})$. For arbitrary $\omega_1\in S_1$, from the eigenequations of H_{t+1} and H_t at ω_1, we have

$$\begin{aligned}\lambda_1(H_{t+1})x_{\omega_1}&=\sum_{h\in V(H_{t+1})}\mathrm{d}_{H_{t+1}}(\omega_1,h)\left(x_{\omega_1}-x_h\right)\\ &=\sum_{h\in V(H_t)}\mathrm{d}_{H_t}(\omega_1,h)\left(x_{\omega_1}-x_h\right)\\ &\quad+\sum_{h\in V(H)\backslash\{v\}}\left(x_{\omega_1}-x_h\right)\\ &=\lambda_1(H_t)x_{\omega_1}+\sum_{h\in V(H)\backslash\{v\}}\left(x_{\omega_1}-x_h\right).\end{aligned}\tag{11}$$

So, we have $\sum_{h\in V(H)\backslash\{v\}}(x_{\omega_1}-x_h)=0$. Similarly, for arbitrary $\omega_2\in S_2$ and $\omega_3\in S_3\backslash\{v_t\}$, we have $\sum_{h\in V(H)\backslash\{v\}}(x_{\omega_2}-x_h)=0$ and $\sum_{h\in V(H)\backslash\{v\}}(x_{\omega_3}-x_h)=0$. Then, we have $x_\omega=x_w$ for any $\omega,w\in V(H_t)\backslash\{v_t\}$. Since $x^T 1_{|V(H_t)|}=0$, we have $(|V(H_t)|-1)x_{v_1}+x_{v_t}=0$, which implies $x_{v_1}\neq 0$ and $x_{v_t}\neq 0$.

From the eigenequation of H_t at v_1 and v_2, we have $0=\lambda_1(H_t)x_{v_1}-\lambda_1(H_t)x_{v_2}=\sum_{h\in V(H)\backslash\{v\}}(x_{v_1}-x_h)+x_{v_t}=x_{v_t}$, which is a contradiction.

Up to now, we have proved $\lambda_1(H_{t+1})>\lambda_1(H_t)$.

Case 2: $\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_1}(x_\omega-x_h)^2<\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_2}(x_\omega-x_h)^2$.

From H_t to H_{t-1}, we have

$$d_{H_t}(\omega,h)-d_{H_{t-1}}(\omega,h)=\begin{cases}-1, & \omega\in S_2\cup S_3\backslash\{v_{t-1}\},\ h\in V(H)\backslash\{v\},\\ 1, & \omega\in S_1, h\in V(H)\backslash\{v\},\\ 0, & \text{otherwise.}\end{cases}\tag{12}$$

Then, we have

$$\begin{aligned}\lambda_1(H_{t-1})-\lambda_1(H_t)&\geq x^T(\mathscr{L}(H_{t-1})-\mathscr{L}(H_t))x\\ &=\sum_{h\in V(H)\backslash\{v\}}\left[\sum_{\omega\in S_2}(x_\omega-x_h)^2+\sum_{\omega\in S_3\backslash\{v_{t-1}\}}(x_\omega-x_h)^2-\sum_{\omega\in S_1}(x_\omega-x_h)^2\right]\\ &\geq\sum_{h\in V(H)\backslash\{v\}}\left[\sum_{\omega\in S_2}(x_\omega-x_h)^2-\sum_{\omega\in S_1}(x_\omega-x_h)^2\right]\\ &>0.\end{aligned}\tag{13}$$

Thus, $\lambda_1(H_{t-1})>\lambda_1(H_t)$.

In the following, we will prove $\lambda_1(H_d)>\lambda_1(H_t)$ or $\lambda_1(H_1)>\lambda_1(H_t)$.

If $\lambda_1(H_{t+1})>\lambda_1(H_t)$, we may denote the component of $S-v_t$ which contains vertex v_{t-1} by $\mathcal{S}_1'$ and the component of $S-v_{t+1}$ which contains vertex v_{t+2} by $\mathcal{S}_2'$. Let $\mathcal{S}_1'=V(\mathcal{S}_1')$, $\mathcal{S}_2'=V(\mathcal{S}_2')$, and $\mathcal{S}_3'=V(S)\backslash(V(\mathcal{S}_1')\cup V(\mathcal{S}_2'))$. Let x' be a Perron eigenvector of $\mathscr{L}(H_{t+1})$ corresponding to $\lambda_1(H_{t+1})$. If $\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_1'}(x_\omega'-x_h')^2<\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_2'}(x_\omega'-x_h')^2$, then $\lambda_1(H_t)-\lambda_1(H_{t+1})\geq\sum_{h\in V(H)\backslash\{v\}}[\sum_{\omega\in S_2'}(x_\omega'-x_h')^2-\sum_{\omega\in S_1'}(x_\omega'-x_h')^2]$, and we can get $\lambda_1(H_t)>\lambda_1(H_{t+1})$, which is a contradiction. So, we have $\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_1'}(x_\omega'-x_h')^2\geq\sum_{h\in V(H)\backslash\{v\}}\sum_{\omega\in S_2'}(x_\omega'-x_h')^2$. Then, we have $\lambda_1(H_{t+2})-\lambda_1(H_{t+1})\geq\sum_{h\in V(H)\backslash\{v\}}[\sum_{\omega\in S_1'}(x_\omega'-x_h')^2-\sum_{\omega\in S_2'}(x_\omega'-x_h')^2]\geq 0$, similar to case 1, and we can get the equal sign in the above inequality does not hold. So, we have $\lambda_1(H_{t+2})>\lambda_1(H_{t+1})$. Repeating the above procedure, we can get $\lambda_1(H_d)>\cdots>\lambda_1(H_{t+2})>\lambda_1(H_{t+1})>\lambda_1(H_t)$.

Similarly, if $\lambda_1(H_{t-1})>\lambda_1(H_t)$, we can prove $\lambda_1(H_1)>\cdots>\lambda_1(H_{t-2})>\lambda_1(H_{t-1})>\lambda_1(H_t)$. □

Theorem 4. *Among all clique trees with n vertices and k cliques, the graph attaining the maximum distance Laplacian spectral radius is $\mathscr{P}_{m,2,\ldots,2,n-m-k+3}$ for some $m\geq 3$.*

Proof. Let G be the graph with maximum distance Laplacian spectral radius among all clique trees with n vertices and k cliques. By Lemma 4, we get $G=\mathscr{P}_{n_1,n_2,\ldots,n_k}$. Let $V(K_{n_i})\cap V(K_{n_{i+1}})=v_i$ for $i=1,2,\ldots,k-1$. If $k\leq 2$, the result holds. Next, we may assume $k\geq 3$. Suppose there exists some $2\leq t\leq k$ such that $n_t\geq 3$. Denote the component of $G-v_{t-1}$ which contains vertex v_{t-2} by G_1 and the component of $G-v_t$ which contains vertex v_{t+1} by G_2. Let $S_1=V(G_1)$, $S_2=V(G_2)$, and $S_3=V(G)\backslash(V(G_1)\cup V(G_2))$, i.e., $S_3=V(K_{n_t})$. Let

$$\begin{aligned}G^{t-1}&=G-\left\{vv_t|v\in V\left(K_{n_t}\right)\backslash\{v_{t-1},v_t\}\right\}+\left\{uv|u\in V\left(K_{n_{t-1}}\right)\backslash\{v_{t-1}\},v\in V\left(K_{n_t}\right)\backslash\{v_{t-1},v_t\}\right\},\\ G^{t+1}&=G-\left\{v_{t-1}|v\in V\left(K_{n_t}\right)\backslash\{v_{t-1},v_t\}\right\}+\left\{uv|u\in V\left(K_{n_{t-1}}\right)\backslash\{v_t\},v\in V\left(K_{n_t}\right)\backslash\{v_{t-1},v_t\}\right\},\end{aligned}\tag{14}$$

i.e., $G^{t-1}=\mathscr{P}_{n_1,\ldots,n_{t-2},n_{t-1}+n_t-2,2,n_{t+1},\ldots,n_k}$ and $G^{t+1}=\mathscr{P}_{n_1,\ldots,n_{t-1},2,n_{t+1}+n_t-2,n_{t+2},\ldots,n_k}$. Suppose x is a Perron eigenvector of $\mathscr{L}(G)$ corresponding to $\lambda_1(G)$. In the following, we will first prove $\lambda_1(G^{t-1})>\lambda_1(G)$ or $\lambda_1(G^{t+1})>\lambda_1(G)$.

Case 1: $\sum_{\omega\in S_2}\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}(x_\omega-x_h)^2\geq\sum_{\omega\in S_1}\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}(x_\omega-x_h)^2$.

From G to G^{t-1}, we have

$$d_G(\omega,h)-d_{G^{t-1}}(\omega,h)=\begin{pmatrix} -1, & \omega\in S_2\cup\{v_t\},\ h\in S_3\backslash\{v_{t-1},v_t\}, \\ 1, & \omega\in S_1,\ h\in S_3\backslash\{v_{t-1},v_t\}, \\ 0, & otherwise. \end{pmatrix}$$

$$\begin{aligned}\lambda_1\left(G^{t-1}\right)-\lambda_1(G)&\geq x^T\left(\mathscr{L}\left(G^{t-1}\right)-\mathscr{L}(G)\right)x\\ &=\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_2\cup\{v_t\}}\left(x_\omega-x_h\right)^2-\sum_{\omega\in S_1}\left(x_\omega-x_h\right)^2\right]\\ &=\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_2}\left(x_\omega-x_h\right)^2+\left(x_{v_t}-x_h\right)^2-\sum_{\omega\in S_1}\left(x_\omega-x_h\right)^2\right]\\ &\geq\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_2}\left(x_\omega-x_h\right)^2-\sum_{\omega\in S_1}\left(x_\omega-x_h\right)^2\right]\\ &\geq 0,\end{aligned}\tag{15}$$

which implies $\lambda_1(G^{t-1})\geq\lambda_1(G)$. Similar to Case 1 of Lemma 4, we can get the equal sign in the above inequality does not hold. So, we have $\lambda_1(G^{t-1})\geq\lambda_1(G)$.

Case 2: $\sum_{\omega\in S_2}\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}(x_\omega-x_h)^2<\sum_{\omega\in S_1}\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}(x_\omega-x_h)^2$.

Then, we have

$$\begin{aligned}\lambda_1\left(G^{t+1}\right)-\lambda_1(G)&\geq x^T\left(\mathscr{L}\left(G^{t+1}\right)-\mathscr{L}(G)\right)x\\ &=\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_1\cup\{v_{t-1}\}}\left(x_\omega-x_h\right)^2-\sum_{\omega\in S_2}\left(x_\omega-x_h\right)^2\right]\\ &=\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_1}\left(x_\omega-x_h\right)^2+\left(x_{v_{t-1}}-x_h\right)^2-\sum_{\omega\in S_2}\left(x_\omega-x_h\right)^2\right]\\ &\geq\sum_{h\in S_3\backslash\{v_{t-1},v_t\}}\left[\sum_{\omega\in S_1}\left(x_\omega-x_h\right)^2-\sum_{\omega\in S_2}\left(x_\omega-x_h\right)^2\right]\\ &>0.\end{aligned}\tag{16}$$

Thus, we have $\lambda_1(G^{t+1})>\lambda_1(G)$.

Doing the above graph transformations until $n_2=n_3=\cdots=n_{k-1}=2$, we get G as $\mathscr{P}_{m,2,\ldots,2,n-m-k+3}$ for some $m\geq 3$. □

5. Conclusion

This paper mainly determines the extremal graphs with maximum and minimum distance Laplacian spectral radius among all clique trees with n vertices and k cliques. Moreover, we get the distance Laplacian energies of all the clique stars with n vertices and k cliques. Based on our results, we conjecture that the line graphs of S_n^+ and $K_{i_n,3}$ are the unique graphs with minimum and maximum distance Laplacian spectral radius among all the line graphs of unicyclic graphs, respectively, where S_n^+ is the graph obtained by adding an edge to the star $K_{1,n-1}$ of order n and $K_{i_n,3}$ is the graph obtained by adding an edge between a vertex of a triangle and a terminal vertex of a path on $n-3$ vertices. Moreover, we can study the distance Laplacian spectral radius of diclique trees in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the NSFC (grant nos. 11501491, 11671347, and 61771019).

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