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Research Letter

The Distribution of Path Losses for Uniformly Distributed Nodes in a Circle

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When simulating a wireless network, users/nodes are usually assumed to be distributed uniformly in space. Path losses between nodes in a simulated network are generally calculated by determining the distance between every pair of nodes and applying a suitable path loss model as a function of this distance (power of distance with an environment-specific path loss exponent) and adding a random component to represent the log-normal shadowing. A network with N nodes consists of $N(N-1)/2$ path loss values. In order to generate statistically significant results for system-level simulations, Monte Carlo simulations must be performed where the nodes are randomly distributed at the start of every run. This is a time-consuming operation which need not be carried out if the distribution of path losses between the nodes is known. The probability density function (pdf) of the path loss between the centre of a circle and a node distributed uniformly within a the circle is derived in this work.

1. Introduction

System-level computer simulations of wireless, mobile networks are commonplace in current research in the wireless communications field. Users are oftentimes uniformly distributed in hexagonal cells in a system simulation. In order to find the signal-to-interference-and-noise ratio (SINR) on any link, path loss information must be known between the two communicating entities as well as the entity in question and all other potential interfering entities. Various empirically obtained path loss models exist which simulate varying propagation environments as shown in [1–4]. In [5], the authors derive the probability density function (pdf) of the distance between two nodes within communication range of one another (based on the path loss between them). This is then extended to calculating the number of communicable nodes in the vicinity of the node in question. However, to the best of the authors' knowledge, no work has been done to analytically model the distribution of path losses between uniformly distributed nodes in a network (irrespective of whether they can sustain communication between one another or not).

Calculating the path loss between the nodes in a cellular system is usually done by actually distributing the nodes uniformly, calculating the distances between them and applying an appropriate path loss model to these distances. This must be repeated many times in order to get a statistically significant result. This operation is computationally expensive and simulation runtimes can be drastically reduced if this step could be skipped. In order to do so, knowledge of the distribution of path losses between uniformly distributed nodes in a hexagon is essential.

In this letter, the pdf of the distribution of path losses between the centre of a circle and uniformly distributed nodes within it is derived. This novel derivation can be used to find the distribution of a whole class of path loss models. A circular scenario is preferred over a hexagonal one because the derivation of the aforementioned pdf is straightforward for the circular case. Furthermore, it is also shown in this paper that the theory derived for a circular scenario agrees with the simulations for the hexagonal case very closely and is therefore a very good approximation.

The derivation of the pdf is shown in Section 2. The comparisons between theory and simulation and discussions thereof are shown in Section 3. Section 4 contains concluding remarks.

2. Distribution of Path Losses for Uniformly Distributed Users in a Circle

Path loss is generally represented as some power of distance, γ, plus a random variation about this power law due to shadowing [1]. Beyond some close-in distance d_0, the path loss (in dB) can be written as

$$L = a + 10\gamma \log_{10}\left(\frac{d}{d_0}\right) + \xi; \quad d \geq d_0, \tag{1}$$

where a is an intercept which is the free-space path loss at distance d_0, d is the distance between the two points, and ξ is the shadow fading variation about the linear relationship in the log domain and is a zero-mean, normally distributed random variable with standard deviation σ, that is, $\mathcal{N}(0, \sigma^2)$. For the sake of convenience, in (1), we make the substitutions $b = 10\gamma$ and $X = d/d_0$. Thus, the path loss between two points separated by Xd_0 meters is written as

$$L = a + b\ \log_{10}(X) + \xi. \tag{2}$$

For a circle of radius R having uniform node distribution, it is known that the pdf of the distance of any point from the centre, x, is [6]

$$f_X(x) = \frac{2x}{R^2}, \quad x \in [0, R]. \tag{3}$$

The angular distribution is uniform on $[0, 2\pi]$. Deriving the pdf of L involves the addition of two random variables and a constant. First we find the pdf of the random variable $Y = b\log_{10}X$ using the transformation of random variables. Thus,

$$f_Y(y) = \frac{2(\ln 10)10^{2y/b}}{bR^2}, \quad y \in (-\infty, b\log_{10}R]. \tag{4}$$

Next, we calculate the pdf of

$$Z = \overbrace{b\log_{10}X}^{Y} + \overbrace{\mathcal{N}(0, \sigma^2)}^{W}. \tag{5}$$

The pdf of W is known to be

$$f_W(w) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{w^2}{2\sigma^2}\right). \tag{6}$$

Since this is an addition of two independent random variables (Y and W), we can use the convolution integral to find the pdf of Z, that is, $f_Z(z) = f_Y(y) \star f_W(w)$ (where $\star$ is the convolution operator):

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_W(w) f_Y(z-w)\mathrm{d}w \\ &= \frac{2(\ln 10)}{\sqrt{2\pi}\sigma bR^2}\int_A^{\infty} \exp\left(-\frac{w^2}{2\sigma^2}\right) 10^{2(z-w)/b}\,\mathrm{d}w \\ &= \frac{2(\ln 10)}{\sqrt{2\pi}\sigma bR^2} \exp\left\{\frac{2b(\ln 10)z + 2\sigma^2(\ln 10)^2}{b^2}\right\} \\ &\quad \times \int_A^{\infty} \exp\left\{-\frac{(w + (2\sigma^2(\ln 10))/b)^2}{2\sigma^2}\right\}\mathrm{d}w, \end{aligned} \tag{7}$$

where $A = z - b\log_{10}R$. Let $k = w + (2\sigma^2(\ln 10))/b$. Using this in (7), then

$$\begin{aligned} f_Z(z) &= \frac{2(\ln 10)}{\sqrt{2\pi}\sigma bR^2} \exp\left\{\frac{2b(\ln 10)z + 2\sigma^2(\ln 10)^2}{b^2}\right\} \\ &\quad \times \int_B^{\infty} \exp\left\{-\frac{k^2}{2\sigma^2}\right\}\mathrm{d}k, \end{aligned} \tag{8}$$

where $B = z - b\log_{10}R + (2\sigma^2(\ln 10))/b$. Let $l = k/\sqrt{2}\sigma$. Decomposing the integral into the standard form of the error function, then

$$\begin{aligned} f_Z(z) &= \frac{2(\ln 10)}{\sqrt{2\pi}\sigma bR^2} \exp\left\{\frac{2b(\ln 10)z + 2\sigma^2(\ln 10)^2}{b^2}\right\} \\ &\quad \times \int_C^{\infty} \exp\{-l^2\}\sqrt{2}\sigma\,\mathrm{d}l, \end{aligned} \tag{9}$$

where

$$C = \frac{z}{\sqrt{2}\sigma} - \frac{b\log_{10}R}{\sqrt{2}\sigma} + \frac{\sqrt{2}\sigma(\ln 10)}{b}. \tag{10}$$

Using the definition of the error function as follows:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t, \tag{11}$$

we get

$$\begin{aligned} f_Z(z) &= \frac{(\ln 10)}{bR} \exp\left\{\frac{2b(\ln 10)z + 2\sigma^2(\ln 10)^2}{b^2}\right\} \\ &\quad \times \{1 - \mathrm{erf}(C)\}. \end{aligned} \tag{12}$$

Finally, we solve for the pdf of

$$L = a + \overbrace{b\log_{10}X + \mathcal{N}(0, \sigma^2)}^{Z}. \tag{13}$$

This final step translates to a shift and the pdf of L is given as

$$\begin{aligned} f_L(\ell) &= \frac{(\ln 10)}{bR^2} \exp\left\{\frac{2b(\ln 10)(\ell - a) + 2\sigma^2(\ln 10)^2}{b^2}\right\} \\ &\quad \times \{1 - \mathrm{erf}(D)\}; \quad \forall \ell, \end{aligned} \tag{14}$$

where

$$D = \frac{\ell\, b - a\, b - b^2 \log_{10} R + 2\sigma^2 (\ln 10)}{\sqrt{2} b \sigma}. \tag{15}$$

Equation (14) gives the pdf of path loss distribution in a circular scenario of radius R as seen from the centre of the circle. Using an alternate expression for the error function as shown in [7], (14) can be written as

$$f_L(\ell) = \frac{(\ln 10)}{bR^2} \exp\left\{ \frac{2b(\ln 10)(\ell - a) + 2\sigma^2 (\ln 10)^2}{b^2} \right\} \times \left\{ 1 - \left(1 - \frac{2}{\pi} \int_0^{\pi/2} \exp\left[-\frac{2D^2}{2\sin^2\theta} \right] d\theta \right) \operatorname{sgn}(D) \right\}; \quad \forall \ell, \tag{16}$$

where D holds the same meaning as defined above and $\operatorname{sgn}(\cdot)$ is the signum function.

3. Results

The parameters specified in the UMTS path loss model [8] are used for the simulations (intercept, $a = 37$ and $b = 30$). The match between theory and simulations is investigated for varying circle radii and standard deviations for the log-normal shadowing component. Simulations are also carried out with the nodes distributed uniformly in hexagons having the same major radii as the circles to demonstrate the practicality of the derived theory.

The simulation is run by uniformly distributing 100,000 nodes in a circle or hexagon of the specified radius and then calculating the distance between the centre and each point. The histogram of these distances is then compared against theory. Figure 1(a) shows the match between the pdfs obtained through simulations and those derived in Section 2 for varying scenario radii and log-normal shadowing standard deviation values. Good agreement between theory and simulation is seen in all cases. Furthermore, it is observed that the theory for uniformly distributed nodes in a circle very closely approximates the case when the nodes are uniformly distributed in a hexagon having the same major radius as the circle. As expected, the pdf for the hexagonal case shifts to the left. This is because the major radius of the hexagon is the same as the radius of the circle, causing the hexagon to fit inside the circle. Therefore, there are a few areas not covered by nodes in the hexagonal case which causes the shift. The situation would be reversed if the minor radius of the hexagon were made equal to the radius of the circle. Figure 1(b) shows the cumulative distribution functions (cdfs) associated with the pdfs depicted in Figure 1(a).

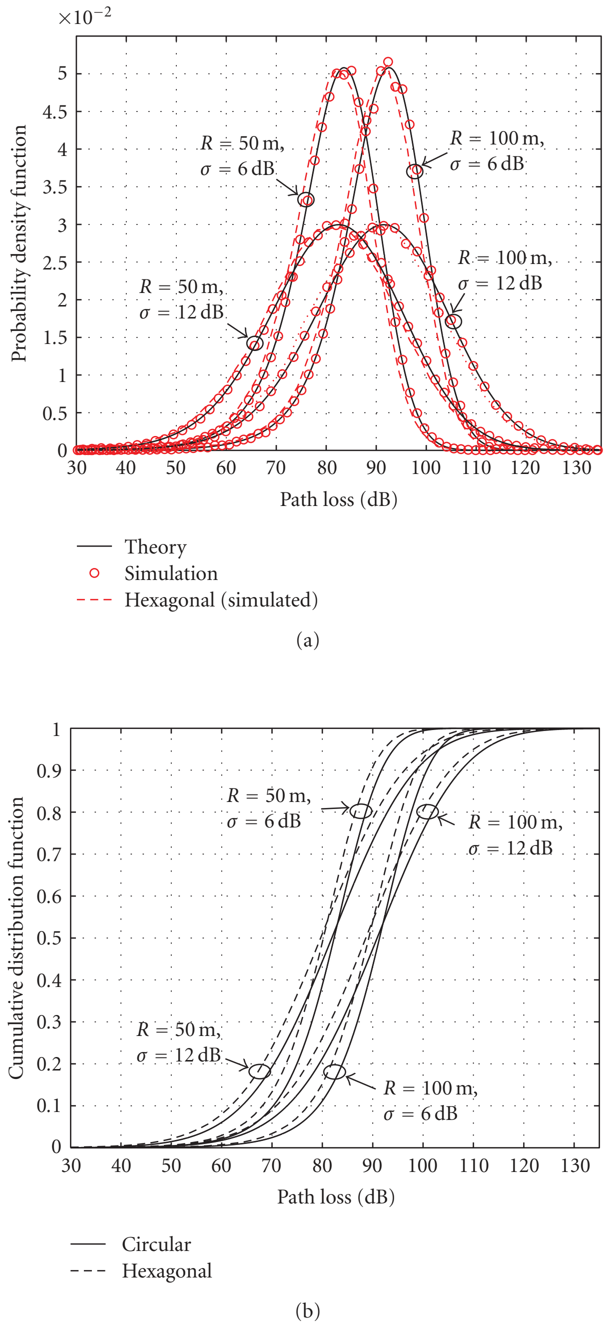

Figure 1: (a) Pdfs of path loss distribution for uniformly distributed nodes in circles and hexagons having the same major radii. Simulation and theory are in good agreement as scenario radii and standard deviation of the log-normal component are varied. (b) The corresponding cdfs.

4. Conclusion

This paper has presented an analytical derivation of the pdf of path loss distribution between the centre of a circle and uniformly distributed nodes within it.

The derived pdf assists in calculating the carrier-to-interference ratios, interference thresholds and exclusion regions in *ad hoc* and sensor networks. In addition, it is envisaged that the result can be used to develop and assess routing and scheduling algorithms for such networks without running time-consuming Monte Carlo simulations.

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