(9) $$\varphi(m', \lambda^2) \leq F \leq \varphi(m'', \lambda^2).$$

Denote by $\lambda_1'^2, \lambda_1''^2, \lambda_2'^2, \lambda_2''^2$ the roots in λ^2 of the following equations respectively:

$$\varphi(m', \lambda^2) = F_2 ;$$
$$\varphi(m'', \lambda^2) = F_2 ;$$
$$\varphi(m', \lambda^2) = F_1 ; \qquad \varphi(m'', \lambda^2) = F_1 .$$

Since F is monotonically decreasing with increasing λ^2, on account of (7), (8), and (9) we obviously have

$$\lambda_1'^2 \leq \lambda_1^2 \leq \lambda_1''^2$$

and

$$\lambda_2'^2 \leq \lambda_2^2 \leq \lambda_2''^2.$$

The above inequalities give us the required limits.

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THE DISTRIBUTION OF QUADRATIC FORMS IN NON-CENTRAL NORMAL RANDOM VARIABLES

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The following theorem is the algebraic basis of the theorem of R. A. Fisher and W. G. Cochran which states necessary and sufficient conditions that a set of quadratic forms in normally and independently distributed random variables should themselves be independently distributed in χ^2-distributions.[2]

THEOREM I. *If the real quadratic forms* $q_1, \cdots, q_m$, *in* $x_1, \cdots, x_n$, *are such that*

(1) $$\sum_\gamma q_\gamma = \sum_\nu x_\nu^2,$$

and if the rank of q_γ *is* n_γ, *then a necessary and sufficient condition that*

(2) $$q_\gamma = \sum_\alpha z_\alpha^2,$$

[1] The letters i, j, μ, ν will assume all integral values from 1 through n, the letter γ will assume all integral values from 1 through m, $(n \geq m)$, the letter α will assume all integral values from $n_1 + \cdots + n_{\gamma-1} + 1$ through $n_1 + \cdots + n_\gamma$, $(n_0 = 0,\ n_1 + \cdots + n_m = n')$, the letters β, β' will assume all integral values from 1 through n', and the letters r, s will assume all integral values from 1 through $n - 1$.

[2] The references are: W. G. Cochran, "The Distribution of Quadratic Forms in a Normal System, with Applications to the Analysis of Covariance," *Proc. Camb. Phil. Soc.*, Vol. 30 (1934), pp. 178–191, and R. A. Fisher, "Applications of 'Student's' Distribution," *Metron*, Vol. 5 (1926), pp. 90–104.

where the real linear functions z_β *of the* x_ν *are defined by*

$$x_\nu = \sum_\beta c_{\nu\beta} z_\beta \tag{3}$$

is

$$n' = n. \tag{4}$$

Furthermore the system of linear forms (3) *constitute an orthogonal transformation.*

PROOF: *Necessity.* Since the rank of a sum of quadratic forms is less than or equal to the sum of their ranks, it follows that $n' \geq n$. Upon substituting from (3) for the x's in (1), and using (2), it is seen that, for all values of the z's,

$$\sum_\beta z_\beta^2 = \sum_{\beta,\beta'} \left(\sum_\nu c_{\nu\beta} c_{\nu\beta'}\right) z_\beta z_{\beta'}$$

and hence, from (1), it follows that

$$\sum_\nu c_{\nu\beta} c_{\nu\beta'} = \delta_{\beta\beta'} \tag{5}$$

where $\delta_{\beta\beta'} = 0$, if $\beta \neq \beta'$, and $\delta_{\beta\beta'} = 1$ if $\beta = \beta'$. However, since the rank of the system of linear forms (3) is not greater than n, and since the matrix of (5) is the product of the matrix of (3) by its transposed matrix, it follows that (5) can be true only if n' is not greater than n. Consequently $n' = n$. It then is an immediate result of (5) that the transformation (3) is orthogonal.

Sufficiency. We assume that $n' = n$. By a real linear transformation of $x_1, \cdots, x_n$ we obtain linear forms z_ν such that

$$q_\gamma = \sum_\alpha c_\alpha z_\alpha^2,$$

where $c_\alpha = 1$ or -1. The set of linear functions $z_1, \cdots, z_n$ are linearly independent, for if $z_n \not\equiv 0$, and if real numbers $h_1, \cdots, h_{n-1}$ not all zero, exist such that, say,

$$z_n = \sum_r h_r z_r$$

then

$$\sum_\nu z_\nu^2 = \sum_{r,s} H_s z_r z_s.$$

Substituting, we have

$$\sum_\gamma q_\gamma = \sum_\nu c_\nu z_\nu^2 = \sum_{r,s} \sum_{\mu,\nu} H_{rs} c^{r\mu} c^{s\nu} x_\mu x_\nu$$

where $z_\nu = \sum_\mu c^{\nu\mu} x_\mu$. (It is not assumed here that the matrix of the $c^{\mu\nu}$ is the inverse of the matrix of the $c_{\mu\nu}$. That fact is a consequence of this proof.)

Denoting the matrix of $z_1, \cdots, z_{n-1}$ by $\bar{C}_n$ we see that the matrix of $\sum_\gamma q_\gamma$ is $\bar{C}'_m H \bar{C}_n$ where H is the matrix of the H_{rs} and has rank less than or equal to $n - 1$ which contradicts the hypothesis. Hence if C is the matrix having the elements

c_ν in its main diagonal and zeros elsewhere and if C_n is the matrix of $z_1, \cdots, z_n$ it follows that

$$C_n' C C_n = I,$$

where I is the identity matrix, i.e. the matrix having ones in the main diagonal and zeros elsewhere and C_n non-singular. Then $C = C_n^{-1\prime} C_n^{-1}$ and hence C is the identity matrix and C_n is orthogonal.

Among the hypotheses of the Fisher-Cochran theorem is the hypothesis that the mean value of x_μ is 0, and the variance of x_μ is σ^2. However, in connection with his analysis of the distribution of the multiple correlation coefficient,[3] R. A. Fisher derived the distribution of the sum of the squares of n independently distributed random variables $x_1, \cdots, x_n$, the probability density of x_μ being given by

$$p(x_\mu) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(x_\mu - a_\mu)^2\right]. \tag{6}$$

More recently, P. C. Tang,[4] has used the distribution of the sum of non-central squares in his study of the power function of the analysis of variance test.

In this note we extend the Fisher-Cochran theorem to non-central random variables. If the random variables x_μ are independently distributed with probability densities given by (6), Fisher and Tang have shown that if $\chi'^2 = \frac{1}{\sigma^2}\sum_\nu x_\nu^2$, then the probability density of χ'^2 is given by

$$p(\chi'^2) = \tfrac{1}{2} e^{-\lambda} (\tfrac{1}{2}\chi'^2)^{\frac{1}{2}n-1} e^{-\frac{1}{2}\chi'^2} \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2}\lambda\chi'^2)^\nu}{\nu!\,\Gamma(\frac{1}{2}n + \nu)}, \tag{7}$$

where $\lambda = \frac{1}{2\sigma^2}\sum_\nu a_\nu^2$.

We now give necessary and sufficient conditions that a set of quadratic forms in normally and independently distributed random variables should themselves be independently distributed in χ'^2-distributions.

THEOREM II. *Let $x_1, \cdots, x_n$ be independently distributed random variables, the random variable x_μ having probability density* (6). *Denote $\sum_\nu x_\nu^2$ by q, and denote $\frac{1}{2\sigma^2}\sum_\nu a_\nu^2$ by λ. Let $q_1, \cdots, q_m$, be quadratic forms,*

$$q_\gamma = \sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} x_\mu x_\nu$$

such that $\sum_\gamma q_\gamma = q$, and let the rank of q_γ be denoted by n_γ.

[3] R. A. Fisher, "The General Sampling Distribution of the Multiple Correlation Coefficient," *Proc. Royal Soc. of London,* (A), Vol. 121 (1928), pp. 654–673.

[4] P. C. Tang, "The Power Function of the Analysis of Variance Tests with Tables and Illustrations of their Use,". *Statistical Research Memoirs,* Vol. 2 (1938), pp. 126–149.

A necessary and sufficient condition that the quadratic forms $\chi_\gamma'^2$, $\left(\chi_\gamma'^2 = \frac{q_\gamma}{\sigma^2}\right)$, *be independently distributed with joint probability density*

$$p(\chi_1'^2, \cdots, \chi_m'^2) = \prod_\gamma p(\chi_\gamma'^2), \tag{8}$$

where $p(\chi_\gamma'^2)$ *is given by* (7) *with* n_γ *and* λ_γ *in place of* n *and* λ, *and*

$$\lambda_\gamma = \frac{1}{2\sigma^2} \sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} a_\mu a_\nu \tag{9}$$

is $n' = n$.

PROOF. *Necessity.* Tang[5] has shown that the distribution of χ'^2 is given by (7) and that if the $\chi_\gamma'^2$ have joint distribution (8), then the distribution of $\chi_1'^2 + \cdots + \chi_m'^2$, $(= \chi'^2)$, is (7) with n' in place of n. Upon comparing terms, we see that $n' = n$.

Sufficiency. By Theorem I there exist n orthogonal linear functions (3) such that (2) is true. Then it is easy to see that the random variables $z_1, \cdots, z_n$ are independently distributed with a joint probability density

$$p(z_1, \cdots, z_n) = (2\pi\sigma^2)^{-\frac{1}{2}n} \exp\left[-\tfrac{1}{2} \sum_\nu (z_\nu - a_\nu')^2\right], \tag{10}$$

where

$$\sum_\nu a_\nu'^2 = \sum_\nu a_\nu^2, \quad \text{and} \quad a_\mu' = \sum_\nu c_{\mu\nu} a_\nu .$$

If we set $2\sigma^2\lambda_\gamma = \sum_\alpha a_\alpha'^2$, then we have, from (7) and (10), that the $\chi_\gamma'^2$ are independently distributed with joint probability density (8). It is only necessary to show that $\sum_\alpha a_\alpha'^2 = \sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} a_\mu a_\nu$ in order to complete the proof of the theorem. Now

$$\sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} a_\mu a_\nu = \sum_{i,j} \left(\sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} c_{i\mu} c_{j\nu}\right) a_i' a_j' .$$

On the other hand, by direct substitution for the z's we see that

$$q_\gamma = \sum_\alpha z_\alpha^2 = \sum_{\mu,\nu} \left(\sum_\alpha c_{\mu\alpha} c_{\nu\alpha}\right) x_\mu x_\nu$$

and hence $a_{\mu\nu}^{(\gamma)} = \sum_\alpha c_{\mu\alpha} c_{\nu\alpha}$. Since (1) is an orthogonal transformation,

$$\sum_{\mu,\nu} a_{\mu\nu}^{(\gamma)} c_{i\mu} c_{j\nu} = \sum_{\mu,\nu} \left(\sum_\alpha c_{\mu\alpha} c_{\nu\alpha}\right) c_{i\mu} c_{j\nu} = \sum_\alpha \delta_{\alpha i} \delta_{\alpha j} ,$$

where $\delta_{\alpha i} = 0$, if $\alpha \neq i$ and $= 1$ if $\alpha = i$, which completes the proof.

It is emphasized that the form of λ_γ makes it unnecessary to calculate the matrix of q_γ to determine λ_γ since the values a_ν need only be substituted for the x_ν in the original expression for q_γ to determine λ_γ.

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[5] See 4 p. 140.