PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 2, June 1990

THE DISTRIBUTION OF RADEMACHER SUMS

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(Communicated by William D. Sudderth)

ABSTRACT. We find upper and lower bounds for $\Pr(\sum \pm x_n \geq t)$, where x_1, $x_2, \ldots$ are real numbers. We express the answer in terms of the K-interpolation norm from the theory of interpolation of Banach spaces.

INTRODUCTION

Throughout this paper, we let $\varepsilon_1, \varepsilon_2, \ldots$ be independent Bernoulli random variables (that is, $\Pr(\varepsilon_n = 1) = \Pr(\varepsilon_n = -1) = \frac{1}{2}$). We are going to look for upper and lower bounds for $\Pr(\sum \varepsilon_n x_n > t)$, where $x_1, x_2, \ldots$ is a sequence of real numbers such that $x = (x_n)_{n=1}^{\infty} \in l_2$.

Our first upper bound is well known (see, for example, Chapter II, §59 of [5]):

$$\Pr\left(\sum \varepsilon_n x_n > t\,\|x\|_2\right) \leq e^{-t^2/2}. \tag{1}$$

However, if $\|x\|_1 < \infty$, this cannot also provide a good lower bound, because then we have another upper bound:

$$\Pr\left(\sum \varepsilon_n x_n > \|x\|_1\right) = 0. \tag{2}$$

To look for lower bounds, we might first consider using some version of the central limit theorem. For example, using Theorem 7.1.4 of [2], it can be shown that for some constant c we have

$$\left|\Pr\left(\sum \varepsilon_n x_n > t\,\|x\|_2\right) - \frac{1}{\sqrt{2}\pi}\int_t^{\infty} e^{s^2/2}\,ds\right| \leq c\left(\frac{\|x\|_3}{\|x\|_2}\right)^3.$$

Thus, for some constant c we have that if $r \leq c^{-1}(\log \|x\|_3/\|x\|_2)^{1/2}$, then

$$\Pr\left(\sum \varepsilon_n x_n > t\,\|x\|_2\right) \geq c^{-1}\int_t^{\infty} e^{-s^2/2}\,ds \geq \frac{c^{-2}e^{-t^2/2}}{t}.$$

Received by the editors December 22, 1988 and, in revised form, August 30, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60C05; Secondary 60G50.
Key words and phrases. Rademacher sum, Holmstedt's formula.

However, we should hope for far more. From (1) and (2), we could conjecture something like

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} \inf\{\|x\|_1 , t\|x\|_2\}\right) \geq c^{-1} e^{-ct^2} .$$

Actually such a conjecture is unreasonable—one should not take infimums of norms, but instead one should consider the following quantity:

$$\begin{aligned} K(x, t; l_1, l_2) &= K_{1,2}(x, t) \\ &= \inf\left\{ \left\|x'\right\|_1 + t\left\|x''\right\|_2 : x', x'' \in l_2, x' + x'' = x \right\} . \end{aligned}$$

This norm is well known to the theory of interpolation of Banach spaces (see, for example [1] or [3]). For small t, this norm looks a lot like $t\|x\|_2$, but as t gets much larger, it starts to look more like $\|x\|_1$. In fact, there is a rather nice approximate formula due to T. Holmstedt (Theorem 4.1 of [3]): if we write $(x_n^*)_{n=1}^\infty$ for the sequence $(|x_n|)_{n=1}^\infty$ rearranged into decreasing order, then

$$c^{-1} K_{1,2}(x, t) \leq \sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* + t \left(\sum_{n=\lfloor t^2 \rfloor + 1}^{\infty} \left(x_n^*\right)^2 \right)^{\frac{1}{2}} \leq K_{1,2}(x, t) ,$$

where c is a universal constant.

In this paper, we will prove the following result.

Theorem. *There is a constant c such that for all $x \in l_2$ and $t > 0$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > K_{1,2}(x, t)\right) \leq e^{-t^2/2}$$

and

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x, t)\right) \geq c^{-1} e^{-ct^2} .$$

An interesting example is $x = (n^{-1})_{n=1}^\infty$. Then $c^{-1} \log t \leq K_{1,2}(x, t) \leq c \log t$, and hence

$$c^{-1} \exp(-\exp(ct)) \leq \Pr\left(\sum \varepsilon_n n^{-1} > t\right) \leq c \exp\left(-\exp\left(c^{-1} t\right)\right) .$$

This is quite different behavior than that which we might have expected from the central limit theorem.

We might also consider $x = (n^{-1/p})_{n=1}^\infty$, where $1 < p < 2$. This example leads us to deduce Proposition 2.1 of [7]. More involved methods allow us to rederive the results of [8] (which include the above-mentioned result from [7]). We do not go into details.

We also deduce from the following corollary.

Corollary. *There is a constant c such that for all $x \in l_2$ and $0 < t \leq \|x\|_2/\|x\|_\infty$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} t \|x\|_2\right) \geq c^{-1} e^{-ct^2} .$$

Proof. It is sufficient to show that there is a constant c such that if $0 < t \leq \|x\|_2/\|x\|_\infty$, then

$$K_{1,2}(x, t) \leq t\,\|x\|_2 \leq cK_{1,2}(x, t) .$$

The left-hand inequality follows straight away from the definition of $K_{1,2}(x, t)$. The right-hand side follows easily from Holmstedt's formula; obviously if $t < 1$, and otherwise because

$$\sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* \geq \lfloor t^2 \rfloor \frac{\|x\|_2}{t} \geq \frac{t}{2}\,\|x\|_2 . \quad \square$$

Proof of theorem

In order to prove the theorem, we will need some new norms on l_2, and a few lemmas.

Definition. For $x \in l_2$ and $t > 0$, define the norm

$$J\left(x, t; l_\infty, l_2\right) = J_{\infty,2}(x, t) = \max\left\{\|x\|_\infty , t\,\|x\|_2\right\} .$$

Lemma 1. *For $t > 0$, the spaces $(l_2, K_{1,2}, (\cdot, t))$ and $(l_2, J_{\infty,2}(\cdot, t^{-1}))$ are dual to one another, that is, for $x \in l_2$ we have*

$$K_{1,2}(x, t) = \sup\left\{\sum x_n y_n : y \in l_2, J_{\infty,2}\left(y, t^{-1}\right) \leq 1\right\} .$$

Proof. This is elementary (see, for example Chapter 3, Exercise 1–6 of [1]). $\square$

Definition. For $x \in l_2$ and $t \in \mathbf{N}$, define the norm

$$\|x\|_{P(t)} = \sup\left\{\sum_{m=1}^{t}\left(\sum_{n\in B_m}|x_n|^2\right)^{1/2}\right\} ,$$

where the supremum is taken over all disjoint subsets, $B_1, B_2, \dots, B_t \subseteq \mathbf{N}$.

Lemma 2. *If $x \in l_2$ and $t^2 \in \mathbf{N}$, then*

$$\|x\|_{P(t^2)} \leq K_{1,2}(x, t) \leq \sqrt{2}\,\|x\|_{P(t^2)} .$$

Proof. To show the first inequality, note that we have

$$\|x\|_{P(t^2)} \leq \|x\|_1 \quad \text{and} \quad \|x\|_{P(t^2)} \leq t\,\|x\|_2 .$$

Hence

$$\begin{aligned} K_{1,2}(x, t) &= \inf\left\{\left\|x'\right\|_1 + t\left\|x''\right\|_2 : x' + x'' = x\right\} \\ &\geq \inf\left\{\left\|x'\right\|_{P(t^2)} + \left\|x''\right\|_{P(t^2)} : x' + x'' = x\right\} \\ &\geq \|x\|_{P(t^2)} , \end{aligned}$$

where the last step follows by the triangle inequality.

For the second inequality, we start by using Lemma 1. For any $\delta > 0$, let $y \in l_2$ be such that

$$(1-\delta) K_{1,2}(x, t) \leq \sum x_n y_n \quad \text{and} \quad J_{\infty,2}\left(y, t^{-1}\right) = 1.$$

Now pick numbers $n_0, n_1, n_2, \ldots, n_{t^2} \in \{0, 1, 2, \ldots, \infty\}$ by induction as follows: given $0 = n_0 < n_1 < \cdots < n_m$, let

$$n_{m+1} = 1 + \sup\left\{\nu : \sum_{n=n_m+1}^{\nu} |y_n|^2 \leq 1\right\}.$$

Since $\|y\|_\infty \leq 1$, we have that $\sum_{n=n_m+1}^{n_{m+1}} |y_n|^2 \leq 2$. Also, as $\|y\|_2 \leq t$, it follows that $n_{t^2} = \infty$. Therefore

$$\begin{aligned}(1-\delta) K_{1,2}(x, t) &\leq \sum x_n y_n \\ &\leq \sum_{m=1}^{t^2} \left(\sum_{n=n_{m-1}+1}^{n_m} |y_n|^2\right)^{1/2} \left(\sum_{n=n_{m-1}+1}^{n_m} |x_n|^2\right)^{1/2} \\ &\leq \sqrt{2}\, \|x\|_{P(t^2)}.\end{aligned}$$

Since this is true for all $\delta > 0$, the result follows. □

The following lemma is due to Paley and Zygmund.

Lemma 3. *If $x \in l_2$, then given $0 < \lambda < 1$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > \lambda \|x\|_2\right) \geq \frac{1}{3}\left(1-\lambda^2\right)^2.$$

Proof. See Chapter 3, Theorem 3 of [4]. □

N. ow we proceed with the proof of the theorem. First we will show that

$$\Pr\left(\sum \varepsilon_n x_n > K_{1,2}(x, t)\right) \leq e^{-t^2/2}.$$

Given $\delta > 0$, let x', $x'' \in l_2$ be such that $x' + x'' = x$, and

$$(1+\delta) K_{1,2}(x, t) > \left\|x'\right\|_1 + t\left\|x''\right\|_2.$$

Then

$$\begin{aligned}\Pr\left(\sum \varepsilon_n x_n > (1+\delta) K_{1,2}(x, t)\right) &\leq \Pr\left(\sum \varepsilon_n x'_n > \left\|x'\right\|_1\right) \\ &\quad + \Pr\left(\sum \varepsilon_n x''_n > t\left\|x''\right\|_2\right) \\ &\leq 0 + e^{-t^2/2},\end{aligned}$$

where the last inequality follows from equations (1) and (2) above. Letting $\delta \to 0$, the result follows.

Now we show that for some constant c we have

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x, t)\right) \geq c^{-1} e^{-ct^2}.$$

First let us assume that $t^2 \in \mathbf{N}$. Given $\delta > 0$, let $B_1, B_2, \ldots, B_{t^2} \subseteq \mathbf{N}$ be disjoint subsets such that $\bigcup_{m=1}^{t^2} B_m = \mathbf{N}$ and

$$\|x\|_{P(t^2)} \le (1+\delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2 \right)^{1/2} .$$

Then

$$\begin{aligned}
\Pr\left(\sum \varepsilon_n x_n > \frac{1}{2} K_{1,2}(x, t)\right) &\ge \Pr\left(\sum \varepsilon_n x_n > \frac{1}{\sqrt{2}} \|x\|_{P(t^2)}\right) \\
&\ge \Pr\left(\sum_{m=1}^{t^2} \sum_{n \in B_m} \varepsilon_n x_n \ge \frac{1}{\sqrt{2}} (1+\delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2\right)^{1/2}\right) \\
&\ge \prod_{m=1}^{t^2} \Pr\left(\sum_{n \in B_m} \varepsilon_n x_n \ge \frac{1}{\sqrt{2}} (1+\delta) \left(\sum_{n \in B_m} |x_n|^2\right)^{1/2}\right) \\
&\ge \left(\tfrac{1}{3}\left(1 - \tfrac{1}{2}(1+\delta)^2\right)^2\right)^{t^2} ,
\end{aligned}$$

where the last step is from Lemma 3. If we let $\delta \to 0$, then we see that

$$\Pr\left(\sum \varepsilon_n x_n > \frac{1}{2} K_{1,2}(x, t)\right) \ge \exp\left(-(\log 12)\, t^2\right) .$$

This proves the result for $t^2 \in \mathbf{N}$. For $t \ge 1$, note that

$$K_{1,2}(x, t) \le K_{1,2}(x \lceil t \rceil) \quad \text{and} \quad \lceil t \rceil^2 \le 4t^2 ,$$

and hence the result follows (with $c = 4 \log 12$). For $t < 1$, the result may be deduced straightaway from Holmstedt's formula and Lemma 3. □

Acknowledgments

All the work in this paper appears in my Ph.D. thesis [6], taken at Cambridge University under the supervision of Dr. D. J. H. Garling, to whom I would like to express my thanks. I would also like to express my gratitude to the Science and Engineering Research Council, who financed my studies in Cambridge.

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