The Distribution of Shadows

H. CHERNOFF & J. F. DALY

Communicated by M. ROSENBLATT

1. Summary. This paper deals with the problem of characterizing the probability distribution of lengths of intervals of shade cast on a line by disks randomly distributed in a plane with a point source of light at P. The methods developed for this problem involve the use of certain Markoff processes and a form of the wave equation and are extendable to a generalized version of the problem. This generalized version is relevant to certain type II counter problems and traffic problems. In particular, some recent results of TAKÁCS [1] for type II counters can be obtained easily for a special version of this generalized problem. We solve a traffic problem dealing with the time required to wait to cross an intersection while the traffic flow is decreasing.

2. Introduction. Suppose that a source of light is at a point P and a worm is crawling in a given direction along a line L which does not go through P. Suppose also that there are many circular disks distributed randomly throughout the plane containing P and L. These disks (which are permitted to overlap) cast shadows upon the line. If the worm can travel only in the shade, what can we say about the distribution of the distance that the worm can travel from a given starting point?

A similar question may be raised for the worm that travels only in light. To specify the question more precisely we must describe the random distribution of disks. For the time being we shall take the case where the disks are all open disks with the same radius r and the number of disk centers on a set of measure A has a Poisson distribution with mean λA. Furthermore, we assume that the number of disk centers on two non-overlapping sets is independently distributed. This situation is a limiting case of the one where a large number of disks are independently given positions on a large area according to a uniform distribution on this area. Note that if a disk covers P, the whole line is in shadow. We shall later treat the case where the radii may vary and the distribution of disk centers

This work was sponsored by the Army, Navy, and Air Force through the Joint Services Advisory Committee for Research Groups in Applied Mathematics and Statistics by Contract No. N6onr-25140 (NR342-022).

Journal of Mathematics and Mechanics, Vol. 6, No. 4 (1957).

may be modified. Let d_0 be the perpendicular distance from P to L and let us designate points on L by their distances from the foot of the perpendicular in the direction traveled by the worm.

The starting point t_0 will not be shaded if and only if the set of points which are a distance less than r from the line segment t_0P contains no disk centers. Let S represent that subset of L which is in shade and $\tilde{S}$ its complement. Then

$$P\{t_0 \in \tilde{S}\} = \exp\{-\lambda[2rd_{t_0} + \pi r^2]\} \tag{1}$$

where

$$d_{t_0} = \sqrt{d_0^2 + t_0^2} \tag{2}$$

is the length of the line segment t_0P.

Similarly, we compute the probability that the interval $[t_0, a]$ is unshaded.

$$\begin{aligned} P\{[t_0, a] \subset \tilde{S}\} = \exp\{-\lambda\{(a - t_0)\, d_0 \\ + r[d_{t_0} + d_a + (a - t_0)] + \pi r^2\}\} = e^{-g(t_0, a)}. \end{aligned} \tag{3}$$

Here $g(t_0, a)$ is λ times the area of $G(t_0, a)$, the set of points within a distance r of the triangle (P, t_0, a). If X represents the first point of shade that the light-traveling worm encounters, its c.d.f. is given by

$$P\{X \leqq a\} = F(a) = 1 - P\{[t_0, a] \subset \tilde{S}\} = 1 - e^{-g(t_0, a)}. \tag{4}$$

The problem of the shade-traveling worm is not so trivial. To treat this problem we shall find it convenient to introduce the following definitions.

3. Definitions.

Definition 3.1. *Let $U(t)$ be the first unshaded point on L greater than or equal to t.*

We are interested in the distribution of $U(t_0)$.

Definition 3.2. *If t is unshaded, let $T(t) = t$. Otherwise, let*

$$T(t) = \sup\{t' : \text{there is a disk which shades } t \text{ and } t'\}. \tag{5}$$

Note that

$$U(t) = \lim_{n\to\infty} T^{(n)}(t) \tag{6}$$

where

$$T^{(n+1)}(t) = T[T^{(n)}(t)]. \tag{7}$$

In fact, if

$$T^{(n_0+1)}(t) = T^{(n_0)}(t), \tag{8}$$

then

$$U(t) = T^{(n_0)}(t). \tag{9}$$

Definition 3.3. *Let* $\delta\{x, E\}$ *be the distance from* x *to the set* E.

Definition 3.4. *Let* $A\{E\}$ *be the area of the set* E.

Definition 3.5. *Let*

$$H(t, a) = \{x: \delta(x, tP) < r, \delta(x, aP) < r\} \quad \text{for} \quad t \leqq a. \tag{10}$$

Definition 3.6. *Let*

$$h(t, a) = \lambda A\{H(t, a)\}. \tag{11}$$

4. Basic Results.

Theorem 4.1.

(12)

$$(a) \quad P\{T(t) \leqq a\} = e^{-h(t,a)}$$

$$(b) \quad For \quad t_2 \geqq t_1$$

$$P\{T(t_2) \leqq a \mid T(t_1) = \tau_1\} = e^{-[h(t_2,a)-h(t_1,a)]} \quad for \quad a \geqq \max(t_2, \tau_1)$$

$$P\{T(t_2) \leqq a \mid T(t_1) = \tau_1\} = 0 \qquad otherwise.$$

$$(c) \quad For\ t_2 \geqq t_1$$

$$P\{T(t_2) \leqq a \mid T(t) = \tau(t), t \leqq t_1\} = P\{T(t_2) \leqq a \mid T(t_1) = \tau(t_1)\}$$

i.e., $T(t)$ *is a Markoff process.*

Proof: $T(t) \leqq a$ if and only if there are no disk centers in $H(t, a)$. Part (a) follows immediately.

The following relations will prove useful:

$$H(t_1, t_3) \subset \{x: \delta(x, t_2P) < r\} \quad \text{if} \quad t_1 \leqq t_2 \leqq t_3 \tag{15}$$

and therefore

$$H(t_1, t_4) \subset H(t_2, t_3) \quad \text{if} \quad t_1 \leqq t_2 \leqq t_3 \leqq t_4, \tag{16}$$

$$H(t_1, s_2) = H(t_1, s_1) \cap H(t_2, s_2) \quad \text{if} \quad t_1 \leqq s_1 \leqq s_2 \quad \text{and} \quad t_1 \leqq t_2 \leqq s_2, \tag{17}$$

and

$$\{x: \delta(x, t_1P) < r, \delta(x, s_1P) = r\} \cap H(t_2, s_2) = \emptyset \tag{18}$$

$$\text{if } t_1 \leqq s_1 \leqq s_2 \quad \text{and} \quad t_1 \leqq t_2 \leqq s_2.$$

Now let us suppose that $a \geqq \max(\tau_1, t_2)$ since the other case is trivial. The event $T(t_1) = \tau_1$ is that there are no disk centers in $H(t_1, \tau_1)$ and, if $\tau_1 > t_1$, there is at least one on $\{x:\delta(x, t_1P) < r, \delta(x, \tau_1P) = r\}$. The first of these sets intersects $H(t_2, a)$ in $H(t_1, a)$ while the second does not intersect $H(t_2, a)$. Part (b) follows from the assumption of independence of the number of disk centers in non-overlapping sets.

To prove part (c) let us assume $a \geqq \max(\tau(t_1), t_2)$. It now suffices to note that

(19) $$H(t_2, a) \bigcap \{\bigcup_{t \leqq t_1} H(t, \tau(t))\} = \bigcup_{t \leqq t_1} [H(t_2, a) \bigcap H(t, \tau(t))] = \bigcup_{t \leqq t_1} H(t, a) = H(t_1, a).$$

Equation 6 and Theorem 4.1 furnish an iterative method for computing the distribution of $U(t_0)$. In fact, if $n \geqq 2$,

(20) $$\begin{aligned} P\{T^{(n)}(t) \leqq a \mid T^{(n-1)}(t) = \tau_{n-1}, T^{(n-2)}(t) = \tau_{n-2}\} &= e^{-[h(\tau_{n-1},a)-h(\tau_{n-2},a)]} \quad \text{for } a \geqq \tau_{n-1} \geqq \tau_{n-2}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If we regard (20) as giving the joint distribution of $T^{(n)}$ and $T^{(n-1)}$ for given $T^{(n-1)}$ and $T^{(n-2)}$ and define

(21) $$\Phi_n(a, b) = P\{T^{(n-1)}(t) \leqq a, T^{(n)}(t) \leqq b\},$$

we have

Theorem 4.2.

(22) $$\Phi_1(a, b) = e^{-h(t,b)} \qquad \textit{for} \quad t \leqq a \leqq b$$

(23) $$\Phi_n(a, b) = \int_{\{t \leqq x \leqq y \leqq a\}} e^{-[h(y,b)-h(x,b)]} \, d\Phi_{n-1}(x, y).$$

Using this theorem, we may compute

(24) $$P\{T^{(n)}(t) \leqq b\} = \Phi_n(b, b)$$

and

(25) $$P\{U(t) \leqq b\} = \lim_{n\to\infty} \Phi_n(b, b)$$

Another approach to the problem of characterizing the distribution of $U(t)$ involves the wave equation. Select a number b which will remain fixed throughout the discussion. For $t \leqq \tau$ and $t \leqq b$

(26) $$I(t, \tau) = P\{U(t) \leqq b \mid T(t) = \tau\}.$$

We shall devote our attention to $I(t, \tau)$ because of its relation to the distribution of $U(t)$. That is given by

(27) $$\begin{aligned} P\{U(t) \leqq b\} &= \int_{\tau \geqq t} I(t, \tau) \, dP\{T(t) \leqq \tau\} \\ &= \int_{\tau > t} I(t, \tau) \, d[e^{-h(t,\tau)}] + I(t, t)e^{-h(t,t)}. \end{aligned}$$

Theorem 4.3. *For $t \leqq \tau \leqq b$ the function $I(t, \tau)$ satisfies the following relations:*

(28) $$I_\tau(t, \tau) = -K(\tau)e^{h(t,\tau)} \quad \textit{where} \quad K(\tau) \geqq 0$$

and

$$I_{t\tau}(t, \tau) = h_t(t, \tau) I_\tau(t, \tau). \tag{29}$$

The function $K(x)$ is determined by

$$\int_t^b K(x) e^{h(t,x)} dx = 1 - e^{-[h(b,b) - h(t,b)]}, \tag{30}$$

and I is determined by K and the fact that $I(t, t) = 1$. That is to say,

$$I(t, \tau) = 1 - \int_t^\tau K(y) e^{h(t,y)} \, dy. \tag{31}$$

Proof: Since T is a Markoff process, we have for $t < t_1 \leqq \tau \leqq b$

$$I(t, \tau) = \int_{y \geqq \tau} I(t_1, y) \, dP\{T(t_1) \leqq y \mid T(t) = \tau\}. \tag{32}$$

In the appendix rigorous proofs are given for a generalization of this problem to show that $I(t, \tau)$ is increasing in t, decreasing in τ, left continuous in both, and that a generalized form of Theorem 4.3 applies (see Lemmas 8.5 and 8.6 of the appendix). We prefer to give a rather formal proof here, assuming continuity and differentiability where needed. Equation (32) is equivalent to

$$I(t, \tau) = \int_{y > \tau} I(t_1, y) \, d[e^{-[h(t_1,y) - h(t,y)]}] + I(t_1, \tau) e^{-[h(t_1,\tau) - h(t,\tau)]}. \tag{33}$$

Differentiating formally,

$$I_\tau(t, \tau) = I_\tau(t_1, \tau) e^{-[h(t_1,\tau) - h(t,\tau)]} \tag{34}$$

and equations (28) and (29) follow. It is interesting that equation (29) is a form of the wave equation. (The fact that K is positive follows from the monotonicity proved in the appendix.) The boundary conditions which determine the solution of the wave equation are the values of I along $t = \tau$ and along $\tau = b$. These are

$$I(t, t) = 1 \quad \text{for} \quad t \leqq b \tag{35}$$

and

$$I(t, b) = P\{T(b) \leqq b \mid T(t) = b\} = e^{-[h(b,b) - h(t,b)]}. \tag{36}$$

Equation (31) follows from (28) and (35). The boundary condition (36) now yields equation (30) which determines K.

Theorem 4.4 furnishes an expression for $P\{U(t) \leqq b\}$ in terms of $K(x)$.

Theorem 4.4.

$$P\{U(t) \leqq b\} = e^{-h(b,b)} + \int_t^b K(\tau) \, d\tau. \tag{37}$$

Proof: Formally we may integrate one of the integrals of (27) by parts and apply (36). (See Lemma 8.7 of the appendix.)

One may be interested in the distribution of shade given the starting point of an interval of shade. Here we are interested in

$$P^* = \lim_{\epsilon \to 0+} P\{U(t+\epsilon) \leqq b \mid T(t) = t, T(t+\epsilon) > t+\epsilon\},$$

$$P^* = \lim_{\epsilon \to 0+} \int_{t+\epsilon}^{b} I(t+\epsilon, y)\, dP\{T(t+\epsilon) \leqq y \mid T(t) = t, T(t+\epsilon) > t+\epsilon\},$$

$$P\{T(t+\epsilon) \leqq y \mid T(t) = t, T(t+\epsilon) > t+\epsilon\} = \frac{e^{-[h(t+\epsilon, y) - h(t, y)]} - e^{-[h(t+\epsilon, t+\epsilon) - h(t, t+\epsilon)]}}{1 - e^{-[h(t+\epsilon, t+\epsilon) - h(t, t+\epsilon)]}}, \tag{38}$$

$$P\{T(t+\epsilon) \leqq y \mid T(t) = t, T(t+\epsilon) > t+\epsilon\} \to 1 - \frac{h_t(t, y)}{h_t(t, t)},$$

$$P^* = \int_t^b I(t, y)\, d\left[1 - \frac{h_t(t, b)}{h_t(t, t)}\right]$$

$$= I(t, b)\left[1 - \frac{h_t(t, b)}{h_t(t, t)}\right] + \int_t^b \left[1 - \frac{h_t(t, y)}{h_t(t, t)}\right] e^{h(t,y)} K(y)\, dy,$$

and we have

Theorem 4.5.

$$P^* = e^{-[h(b,b) - h(t,b)]}\left[1 - \frac{h_t(t, b)}{h_t(t, t)}\right] + \int_t^b \left[1 - \frac{h_t(t, y)}{h_t(t, t)}\right] e^{h(t,y)} K(y)\, dy. \tag{39}$$

A generalization of this result is given in Lemma 8.8 of the appendix.

Because of the relatively complicated form of $h(t, \tau)$ no effort will be made here for a more specific evaluation in this problem where the shade is cast by disks.

5. Extensions of the Problem and Applications. The characterization of the distribution of $U(t)$ in Section 4 depends on the applicability of Theorem 4.1. In turn, Theorem 4.1 depends on the assumption that given two non-overlapping sets, the events that there are no disk centers in these sets are independent events.

Suppose now that the disk radii are not constant but have a distribution given by the density $f(r)$. Disk radii are assumed independent of the location of the center. Intuitively, we know that two disks of equal radii cast different size shadows depending on the location of their centers. It would be surprising if the assumption of fixed radius were basic to our approach. In fact, let us represent a disk by a point in three-dimensional space, where the first two coordinates of the point are the location x of the center of the disk and the third is the radius r. Then the event that t is unshaded is the event that no disk centers are in

$$R = \{(x, r): \delta(x, tP) < r\}.$$

The numbers of disk centers in non-overlapping three-dimensional sets R_1 and R_2 are independent Poisson variables with means given by

$$\lambda \int_{R_i} f(r)\, da\, dr, \qquad i = 1,2.$$

Hence

$$P\{t \epsilon \tilde{S}\} = \exp\{-\lambda \int_R f(r)\, da\, dr\}.$$

Similarly,

$$P\{[t, a] \subset \tilde{S}\} = e^{-g(t,a)} \tag{40}$$

where

$$g(t, a) = \lambda \int_{G(t,a)} f(r)\, da\, dr \tag{41}$$

and

$$G(t, a) = \{(x, r): \delta(x, \Delta tPa) < r\}. \tag{42}$$

We may also extend Definitions 3.5 and 3.6 to

Definition 5.1.

$$H(t, a) = \{(x, r): \delta(x, tP) < r, \delta(x, aP) < r\} \quad \textit{for} \quad t \leqq a \tag{43}$$

and

Definition 5.2.

$$h(t, a) = \lambda \int_{H(t,a)} f(r)\, dx\, dr. \tag{44}$$

In fact, our methods and results would still be valid if we applied

Definition 5.3.

$$h(t, a) = \lambda \int_{H(t,a)} d\mu(x, r) \tag{45}$$

where μ represents a measure on (x, r)-space which yields the expected number of disk centers corresponding to sets in the (x, r)-space.

This permits us to treat, among others, the case where the distribution of disk centers may be confined to a region between L and a parallel line L'. This latter case is of special interest if we wish to let P go to ∞. In fact, if P goes to ∞ and disks are randomly distributed between L and L', the U and T processes become stationary.

Another case of interest is that where the disks are replaced by line segments of length $2r$ parallel to L. While there is positive probability that P is covered by a disk (in the original problem) and the line L is completely shaded, this is not the case for the problem with disks replaced by line segments. In fact, this latter problem with the point P at ∞ is equivalent to those which arise in certain counter and traffic problems. In this problem the definition of δ must be modified so that $\delta(x, E)$ represents the horizontal distance from x to E.

To be more specific, let us state several problems and elaborate on them somewhat.

Problem I: (Type II Counter)

Several types of counters are used to count the rate at which particles are emitted by a specimen of radioactive material. A type II counter is one where each particle gives rise to a dead time which begins at the emission and during which no other particles will be counted. However, a particle will give rise to the dead time whether it is or is not counted. We assume that the lengths of dead times are random variables distributed independently of time of emission or of the state of the counter (dead or otherwise) at the time of emission. We are interested in the distribution of the number of counts in a given period of time.

This problem may be phrased as a shadow problem. Let P go to ∞. Let shadows be cast by line segments of length c (dead time) whose centers (and therefore whose left hand end points) are randomly distributed along a line L' parallel to L. That is, the number of centers in an interval of length a along L' is a Poisson variate with mean λa. Then the dead time for a particle corresponds to the shade cast by an interval. The number of counts between times t_1 and t_2 would be the number of shadows between t_1 and t_2 (unless the point t_1 is shaded, in which case we subtract one). The standard tool in finding the distribution of this number was developed by FELLER [2] and applied by others (*e.g.*, HAMMERSLEY [3], TAKÁCS [1]) and makes basic use of the distribution of the length of shadows. TAKÁCS obtained the characteristic function of this distribution and by a limiting argument the distribution of $U(t)$ also. His approach permits him to treat type I counters also and seems to be substantially different from ours. We shall later indicate how his results for type II counters can be obtained also by our methods.

Problem II: (Circular Counter Problem)

HAMMERSLEY felt it was convenint to replace the random intervals of shade on the line L by random intervals on a circle C. Here again the approach we have used would be applicable. Replace L by the circle C and P by the center of the circle. For the counter problem the intervals which cast shade may be assumed to lie on a concentric circle C'. The same methods apply as before.

Problem III: (Traffic Problem)

A man wishes to cross a busy highway. If there is only one lane of traffic and it takes him c seconds to cross, he must start c seconds before a car reaches the crossing point. If the traffic flow is free, the arrival times are randomly distributed. A car arriving at time t produces a shadow from $t - c$ to t during which the man may not start to cross. $U(t)$ represents the time he must wait if he arrives at time t. This problem is equivalent to the type II counter problem with fixed dead time. However, it may be modified somewhat. First, we may introduce several lanes of traffic. This case is still equivalent to the type II counter problem with fixed dead time. Second, we may introduce a non-stationarity by assuming that the rate of flow changes with time. Thus the number of cars arriving between t_1 and t_2 is a Poisson variable with mean $\int_{t_1}^{t_2} f(x)dx$. This problem does not seem to be solvable by the method of TAKÁCS. Finally, we may modify the distribution of times at which cars reach the crossing point just so long as the probability of no car arriving in either of two non-overlapping time intervals is the product of the probabilities for each of the time intervals. Thus, our method is applicable to cases where certain arrival times are known.

Problem IV: (Shadows Cast by Intervals in the Plane)

Here we return to the problem where P is a finite point and intervals of length $2r$ parallel to L are randomly distributed throughout the plane, casting shade on L. Suppose that the number of centers in a set R of points (x, r) is given by

$$\lambda \int_R f(r)\, da\, dr$$

where $f(r)$ is the density distribution of the radii of the intervals. It is interesting that in this problem $T(t)$ and $U(t)$ have stationary distributions. The basic "cause" of this result is the fact that the length of shade cast by an interval depends only on (1) its length and (2) the vertical distances from the interval to L and P, and does not depend on the horizontal distance of the interval to P.

In fact, if we assume the vertical distance from P to L is one, half the length of shade cast by a random interval has distribution

$$f^*(r) = \frac{1}{r^2} \int_0^r x f(x)\, dx \tag{46}$$

and thus Problem IV is equivalent to a type II counter problem.

6. The Type II Counter Problem. Here we shall briefly indicate how our method yields TAKÁCS' results for type II counters. Basic to FELLER'S approach are the probability distributions of $U(t)$, of $\lim_{\epsilon \to 0+} U(t + \epsilon)$ given that the counter was "alive" when a particle was emitted at time t, and of their Laplace transforms.

As indicated in the appendix, the basic expression required is $h(t, \tau)$ which represents the expected number of particles emitted at a time before t with accompanying dead time extending past τ. If the length c of dead time has density $f(c)$ and particles are emitted at a rate λ, then

$$h(t, \tau) = \lambda \int_{\tau-t}^{\infty} f(c)[c - (\tau - t)]\, dc = h^*(\tau - t) \tag{47}$$

$$h(t, t) = \lambda \int_0^{\infty} cf(c)\, dc = \lambda\mu \tag{48}$$

which we assume finite.

$$h_t(t, \tau) = \lambda \int_{\tau-t}^{\infty} f(c)\, dc = \lambda P\{c > \tau - t\} = \lambda[1 - Q(\tau - t)] \tag{49}$$

using TAKÁCS' definition of Q(c.d.f. of c)

$$h(t, \tau) = h^*(\tau - t) = \lambda\mu + \lambda \int_0^{\tau-t} [1 - Q(x)]\, dx. \tag{50}$$

Since $U(t)$ is a stationary process,

$$P\{U(t) \leqq t + x\} = P\{U(0) \leqq x\} = P\{U(b - x) \leqq b\}.$$

Applying Theorem 4.4, it follows that the distribution of $U(0)$ consists of the discrete part which assumes the value 0 with probability $e^{-\lambda\mu}$ and the continuous part which has density

$$f_1(x) = K(b - x). \tag{51}$$

Let

$$f_2(x) = e^{h^*(x)} \tag{52}$$

and

$$f_3(x) = 1 - e^{-\lambda\mu + h^*(x)} \tag{53}$$

and let their Laplace transforms be φ_1, φ_2, and φ_3, respectively. Equation (30) states that f_3 is the convolution of f_1 and f_2. Hence

$$\varphi_1(s) = \frac{\varphi_3(s)}{\varphi_2(s)}. \tag{54}$$

But the Laplace transform of $dP\{U(0) \leqq x\}$ is given by

$$\varphi(s) = \int_0^{\infty} e^{-sx}\, dP\{U(0) \leqq x\} = e^{-\lambda\mu} + \varphi_1(s).$$

$$\varphi(s) = e^{-\lambda\mu} + \frac{\frac{1}{s} - \psi(s)}{e^{\lambda\mu}\psi(s)} = \frac{e^{-\lambda\mu}}{s\psi(s)} \tag{55}$$

where

$$\psi(s) = \int_0^\infty e^{-sx} e^{-\lambda\mu + h^*(x)}\, dx. \tag{56}$$

This result coincides with that of Takács. Of course, the distribution of U can be obtained by inverting φ.

To evaluate the distribution of lengths of shadows we note that

$$\begin{aligned} P^*(x) &= \lim_{\epsilon\to 0+} P\{U(b - x + \epsilon) \leqq b \mid T(b - x) = b - x, \\ &\qquad\qquad T(b - x + \epsilon) > b - x + \epsilon\} \\ &= e^{-\lambda\mu + h^*(x)}[Q(x)] + \int_0^x Q(x - u) e^{h^*(x-u)} f_1(u)\, du. \end{aligned} \tag{57}$$

Let

$$f_4(x) = Q(x) e^{h^*(x)} \tag{58}$$

and $\varphi_4(s)$ be its Laplace transform. Then the Laplace transform of $P^*(x)$ is

$$\begin{aligned} \varphi^*(s) &= \varphi_4(s) e^{-\lambda\mu} + \varphi_4(s)\varphi_1(s) \\ &= \varphi_4(s)\left[e^{-\lambda\mu} + \frac{\frac{1}{s} - \psi(s)}{e^{\lambda\mu}\psi(s)}\right] = \frac{\varphi_4(s) e^{-\mu\lambda}}{s\psi(s)}. \end{aligned}$$

But

$$f_4(x) = e^{\lambda\mu}\{e^{-\lambda\mu + h^*(x)} + \frac{1}{\lambda}\frac{d}{dx}[e^{-\lambda\mu + h^*(x)}]\},$$

$$\varphi_4(s) = e^{\lambda\mu}\left\{\psi(s) + \frac{s}{\lambda}\psi(s) - \frac{1}{\lambda}\right\},$$

and hence

$$\varphi^*(s) = \frac{(\lambda + s)\psi(s) - 1}{\lambda s \psi(s)} \tag{59}$$

which result is the same as that of Takács.

7. Traffic Problem with Decreasing Flow of Traffic. Representing each car by an interval of shade length c, let us assume that the number of cars which arrive at a point between times t_1 and t_2 has a Poisson distribution with mean

$$\lambda \int_{t_1}^{t_2} \frac{1}{x}\, dx = \lambda \log\left(\frac{t_2}{t_1}\right) \quad \text{for} \quad 0 < t_1 < t_2 .$$

Then

$$h(t, \tau) = \lambda \int_\tau^{t+c} \frac{1}{x}\, dx = \lambda \log \left(\frac{t+c}{\tau}\right) \quad \text{for} \quad t \leqq \tau \leqq t+c, \tau > 0. \tag{60}$$

Applying Theorem 4.3, we have

$$\begin{aligned} &\int_t^b K(x) \left(\frac{t+c}{x}\right)^\lambda dx = 1 - \left(\frac{t+c}{b+c}\right)^\lambda \quad \text{for} \quad b-c < t \leqq b \\ &\int_t^{t+c} K(x) \left(\frac{t+c}{x}\right)^\lambda dx = 1 - \left(\frac{b}{b+c}\right)^\lambda \quad \text{for} \quad -c < t \leqq b-c. \end{aligned} \tag{61}$$

Differentiating $K(t) = \lambda t^\lambda/(t+c)^{\lambda+1}$ for $b - c < t \leqq b$,

$$\frac{K(t)}{t^\lambda} = \frac{K(t+c)}{(t+c)^\lambda} + \frac{\lambda}{(t+c)^{\lambda+1}}\left[1 - \left(\frac{b}{b+c}\right)^\lambda\right] \quad \text{for} \quad -c < t \leqq b-c. \tag{62}$$

In general

$$K(t) = \lambda t^\lambda \left\{\left[1 - \left(\frac{b}{b+c}\right)^\lambda\right] \sum_{j=1}^{i-1} \frac{1}{(t+jc)^{\lambda+1}} + \frac{1}{(t+ic)^{\lambda+1}}\right\} \tag{63}$$

$$\text{for} \quad b < t + ic \leqq b + c, \qquad t > -c$$

and we can evaluate $P\{U(t) \leqq b\}$ by integrating. That is,

$$P\{U(t) \leqq b\} = \left(\frac{b}{b+c}\right)^\lambda + \int_t^b K(x)\, dx. \tag{64}$$

8. Appendix. We shall give rigorous proofs of the results which were heuristically established in section 4. At the same time these proofs will apply to a very general scheme of shadows. These shadows need not be cast by disks. The distribution of shadows need not be so strongly tied in with the Poisson distribution. In fact, in this general scheme, it is possible to have certain shadows occur with certainty. While the arguments do not apply to shadows cast on a circle, they can be modified to do so without much difficulty. It should be noted that the shadows are open intervals and thus two abutting shadows must be considered as separate and non-overlapping shadows. Furthermore, we shall not assume that the shadows are bounded nor shall we assume that it is always possible for a shadow to cover any two points t and b.

There is a non-negative measure μ defined on a Borel field of subsets of a space Z of elements z. There is a function $(u(z), v(z))$ defined on Z to the set of points $\{(u, v): u < v \leqq \infty\}$. The point (u, v) respresents the open shadow from u to v. We say that the points of the interval $(u(z), v(z))$ are shaded when the point z is "occupied".

Definition 8.1. Let

$$H(t, \tau) = \{z: u(z) < t \leqq \tau < v(z)\} \quad \textit{for} \quad t \leqq \tau. \tag{65}$$

Lemma 8.1.
(a) $H(t, \tau)$ *is increasing*[1] *in* t *and decreasing in* τ,
(b) $H(t, \tau)$ *is left continuous in* t *and right continuous in* τ.

The proof is trivial and is omitted.

Assumption 1. *We assume that the sets* $H(t, \tau)$ *are measurable. (This is equivalent to the statement that the function* $(u(z), v(z))$ *is a Borel measurable function on* Z.) *We also assume that the probability that there are no occupied points in a measurable set* A *is given by* $e^{-\mu(A)}$.

Definition 8.2. *Let*

$$h(t, \tau) = \mu[H(t, \tau)]. \tag{66}$$

Lemma 8.2.
(a) $h(t, \tau)$ *is increasing in* t *and decreasing in* τ.
(b) $h(t, \tau)$ *is left continuous in* t *and right continuous in* τ.

The proof is trivial and will be omitted.

It is evident that Theorem 4.1 applies.[2] In fact, the argument is relatively easy to visualize by referring to the space $\{(u, v): u < v\}$. To permit us to treat the case where $T(t)$ may be bounded, we introduce

Definition 8.3. *Let*

$$\rho(t) = \inf \{x: P\{T(t) \leqq x\} = 1\}. \tag{67}$$

Then $I(t, \tau)$ is defined for $t \leqq \tau < \rho(t)$ and possibly for $\tau = \rho(t)$. Since $I(t, \tau) = 0$ for $b < \tau$, we shall investigate $I(t, \tau)$ for

$$t \leqq \tau \leqq \min(\rho(t), b).$$

Lemma 8.3. $I(t, \tau)$ *is increasing in* t *and decreasing in* τ.

Proof: The proof will consist of showing that if $T(t)$ is increased, the conditional distribution of $T^{(2)}$ is shifted to the right. Similarly, those of $T^{(3)}$, $T^{(4)}, \cdots$ are shifted to the right and finally $I(t, \tau) \geqq I(t, \tau + \delta)$. If $t^* < t$ and if in two realizations of the stochastic process we have $T(t^*) = \tau$ and $T(t) = \tau$, respectively, then the conditional distribution of $T^{(2)}(t^*) = T(\tau)$ in the first is to the right of $T^{(2)}(t) = T(\tau)$ in the second. The same argument as before now gives $I(t^*, \tau) \leqq I(t, \tau)$. More specifically,

$$P\{T(t) \leqq x | T(t) = \tau\}$$

---

[1] We use the term increasing to mean non-decreasing as distingushed from strictly increasing.

[2] In this theorem the term $h(t_2, a) - h(t_1, a)$ may be undefined since both $h(t_1, a)$ and $h(t_2, a)$ may be ∞. In this case we should interpret the difference as $\mu[H(t_2, a) - H(t_1, a)]$. Here $T(t)$ is defined essentially as in section 3. That is, $T(t) = t$ if t is unshaded. Otherwise $T(t) = \sup \{v(z): u(z) < t < v(z), z \varepsilon Z\}$.

and

$$P\{T^{(2)}(t) \leqq x|T(t) = \tau\} = P\{T(\tau) \leqq x|T(t) = \tau\} = e^{-[h(\tau,x)-h(t,x)]}$$

are decreasing in τ. Suppose $P\{T^{(i)}(t) \leqq x \,|T(t) = \tau\}$ is descreasing in τ for $i = n - 1$ and n. Then

$$P\{T^{(n+1)}(t) \leqq x|T(t) = \tau\} = \int_{y \geqq \tau} P\{T^{(n+1)}(t) \leqq x|T^{(n)}(t) = y, T(t) = \tau\} \cdot dP\{T^{(n)}(t) \leqq y|T(t) = \tau\}$$

is decreasing in τ if $P\{T^{(n+1)}(t) \leqq x|T^{(n)}(t) = y, T(t) = \tau\}$ is decreasing in y and τ. But this latter expression is equal to

$$\int_{y^* \geqq \tau} P\{T^{(n+1)}(t) \leqq x|T^{(n)}(t) = y, T^{(n-1)}(t) = y^*\} \, dP\{T^{(n-1)}(t) \leqq y^*|T(t) = \tau\}$$
$$= \int_{y^* \geqq \tau} e^{-[h(y,x)-h(y^*,x)]} \, dP\{T^{(n-1)}(t) \leqq y^*|T(t) = \tau\},$$

which is decreasing in y and τ. Hence

$$I(t, \tau) = \lim_{n \to \infty} P\{T^{(n)}(t) \leqq x|T(t) = \tau\} \tag{68}$$

is decreasing in τ. Now

$$P\{T^{(2)}(t) \leqq x \,|T(t) = \tau\} = P\{T(\tau) \leqq x|T(t) = \tau\} = e^{-[h(\tau,x)-h(t,x)]}$$

is increasing in t. Suppose $P\{T^{(n)}(t) \leqq x|T(t) = \tau\}$ is increasing in t. Then

$$P\{T^{(n+1)}(t) \leqq x|T(t) = \tau\}$$
$$= \int_{y \geqq \tau} P\{T^{(n)}(\tau) \leqq x|T(\tau) = y\} \, dP\{T(\tau) \leqq y|T(t) = \tau\}$$

is increasing in t and hence so is $I(t, \tau)$.

Lemma 8.4. *$I(t, \tau)$ is left continuous in t and τ.*

Proof: For $t \leqq \tau$

$$\begin{aligned} I(t, \tau) &= \int_{y \geqq \tau} I(\tau, y) \, dP\{T(\tau) \leqq y|T(t) = \tau\} \\ &= \int_{y \geqq \tau} I(\tau, y) \, d[e^{-[h(\tau,y)-h(t,y)]}]. \end{aligned} \tag{69}$$

Since $h(t, y)$ is left continuous in t and $I(\tau, y)$ is monotone in y and therefore has for discontinuities a denumerable set of jumps, it follows that I is left continuous in t.

For $t < \tau - \delta < \tau$

$$0 \leqq I(t, \tau - \delta) - I(t, \tau) = A_1 + A_2 + A_3$$

where

$$A_1 = \int_{\tau-\delta<y\leqq\tau} I(\tau - \delta, y)\, d[e^{-[h(\tau-\delta,y)-h(t,y)]}]$$

$$A_2 = e^{-[h(\tau-\delta,\tau-\delta)-h(t,\tau-\delta)]} - e^{-[h(\tau,\tau)-h(t,\tau)]}$$

$$A_3 = \int_{y>\tau} I(\tau - \delta, y)\, d[e^{-[h(\tau-\delta,y)-h(t,y)]}] - \int_{y>\tau} I(\tau, y)\, d[e^{-[h(\tau,y)-h(t,y)]}].$$

But

$$A_1 \leqq e^{-[h(\tau-\delta,\tau)-h(t,\tau)]} - e^{-[h(\tau-\delta,\tau-\delta)-h(t,\tau-\delta)]}$$

$$\lim_{\delta\to 0+} \sup\ (A_1 + A_2) \leqq 0$$

$$A_3 \leqq \int_{y>\tau} I(\tau, y)\, d[e^{-[h(\tau-\delta,y)-h(t,y)]} - e^{-[h(\tau,y)-h(t,y)]}] \to 0$$

and the desired result follows.

If $I(t, \rho(t))$ is not defined, Lemma 8.4 permits us to extend the definition of $I(t, \tau)$ by $I(t, \rho(t)) = \lim_{\tau\to\rho(t)-} I(t, \tau)$.

Lemma 8.5. *Taking differentials with respect to* τ *there is a decreasing left continuous function* $A(\tau)$ *such that*

$$dI(t, \tau) = e^{h(t,\tau-)}\, dA(\tau).^3 \tag{70}$$

Proof: Let $t < t_1 \leqq \tau - \delta < \tau$.

(71)

$$I(t, \tau) = \int_{y\geqq\tau} I(t_1, y)\, dP\{T(t_1) \leqq y | T(t) = \tau\},$$

$$I(t, \tau) = \int_{y>\tau} I(t_1, y)\, d[e^{-[h(t_1,y)-h(t,y)]}] + I(t_1, \tau)e^{-[h(t_1,\tau)-h(t,\tau)]}.$$

It follows that

$$[I(t, \tau - \delta) - I(t, \tau)] = \int_{\tau-\delta<y<\tau} [I(t_1, y) - I(t_1, \tau)]\, d[e^{-[h(t_1,y)-h(t,y)]}] + [I(t_1, \tau - \delta) - I(t_1, \tau)]e^{-[h(t_1,\tau-\delta)-h(t,\tau-\delta)]},$$

$$\frac{1}{\delta}[I(t, \tau - \delta) - I(t, \tau)] = \frac{1}{\delta}[I(t_1, \tau - \delta) - I(t_1, \tau)] \cdot [e^{-[h(t_1,\tau-)-h(t,\tau-)]} + o(1)].$$

---

[3] $h(t, \tau -)$ represents $\lim_{y\to\tau-} h(t, y)$.
$h(t, t -)$ represents $\lim_{t^*\to t-} h(t^*, t-)$.

Hence

$$e^{-h(t,\tau-)}\, dI(t, \tau) = e^{-h(t_1,\tau-)}\, dI(t_1, \tau)$$

and the result follows for $t < \tau$. For $t = \tau$ we need only apply the monotonicity properties of I and the left continuity of $I(t, \tau)$ in t.

Definition 8.3a. *Let*

$$\rho^{(n)}(t) = \rho[\rho^{(n-1)}(t)] \tag{72}$$

$$\rho^*(t) = \min\ (\rho^{(2)}(t), b) \tag{73}$$

$$t^* = \inf \{x: U(x) \quad \text{can exceed} \quad b\}. \tag{74}$$

Since $T(t^*) = t^*$, we are not interested in $A(y)$ for $y \leqq t^*$.

Lemma 8.6. *For* $t^* < t \leqq b$

$$-\int_{t \leqq y < b} e^{a(t,y)}\, dA(y) = 1 - e^{-[h(b,b)-h(t,b)]} \tag{75}$$

where

$$\begin{gathered} a(t, y) = h(t, y-) \qquad \textit{for} \quad y < \rho(t) \\ a(t, y) = h(\rho^{(r)}(t), y-) - h(\rho^{(r)}(t), y) \quad \textit{for} \quad \rho^{(r)}(t) \leqq y < \rho^{(r+1)}(t). \end{gathered} \tag{76}$$

Proof: We shall use the facts that if $\tau \geqq \rho(t)$, $h(t, \tau) = 0$ and that if $U(t)$ can exceed b, $\rho^{(n)}(t) > b$ for some finite n. Now

$$I(t, b) = P\{T(b) = b \mid T(t) = b\} = e^{-[h(b,b)-h(t,b)]} \quad \text{if} \quad \rho(t) \geqq b. \tag{77}$$

But

$$I(t, b) - 1 = I(t, b) - I(t, t) = \int_{t \leqq y < b} e^{a(t,y)}\, dA(y)$$

which is the desired result for $\rho(t) \geqq b$. If $\rho(t) < b$ but $U(t)$ can exceed b,

$$I(t, \rho(t)) = \int_{\rho(t) \leqq y \leqq \rho^*(t)} I(\rho(t), y)\, dP\{T(\rho(t)) \leqq y | T(t) = \rho(t)\}$$

$$\begin{gathered} I(t, \rho(t)) = I(\rho(t), \rho^*(t))e^{-h(\rho(t),\rho^*(t))} - \int_{\rho(t) \leqq y < \rho^*(t)} e^{-[h(\rho(t),y)]}\, dI(\rho(t), y) \\ I(t, \rho(t)) = I(\rho(t), \rho^*(t))e^{-h(\rho(t),\rho^*(t))} - \int_{\rho(t) \leqq y < \rho^*(t)} e^{a(t,y)}\, dA(y). \end{gathered} \tag{78}$$

Suppose $\rho^{(n-1)}(t) < b \leqq \rho^{(n)}(t)$. Then

$$\begin{gathered} I(t, \rho(t)) = I(\rho^{(n-1)}(t), b)e^{-h(\rho^{(n-1)}(t),b)} - \int_{\rho(t) \leqq y < b} e^{a(t,y)}\, dA(y) \\ I(t, \rho(t)) = e^{-[h(b,b)-h(t,b)]} - \int_{\rho(t) \leqq y < b} e^{a(t,y)}\, dA(y). \end{gathered} \tag{79}$$

But

$$I(t, \rho(t)) - 1 = I(t, \rho(t)) - I(t, t) = \int_{t \leqq y < \rho(t)} e^{a(t,y)}\, dA(y),$$

and the desired result follows. Now suppose $t^* = \inf \{x: U(x) \text{ can exceed } b\}$. Then $T(t^*) = t^*$, $I(t, \tau) = 1$ for all $t \leqq t^*$, and

Lemma 8.7.

$$P\{U(t) \leqq b\} = e^{-h(b,b)} - \int_{t \leqq y < b} e^{c(t,y)}\, dA(y) \tag{80}$$

where

$$\begin{aligned} c(t, y) &= h(t, y-) - h(t, y) \quad \textit{for} \quad t \leqq y < \rho(t) \\ &= a(t, y) \qquad\qquad\qquad\quad \textit{for} \quad \rho(t) \leqq y \end{aligned} \tag{81}$$

Proof: Suppose $\rho(t) > b$. Integrating

$$P\{U(t) \leqq b\} = \int_{y \geqq t} I(t, y)\, dP\{T(t) \leqq y\} \tag{82}$$

by parts, we have the desired result. Now suppose $\rho(t) \leqq b$. Our integration yields

$$P\{U(t) \leqq b\} = I(t, \rho(t)) - \int_{t \leqq y < \rho(t)} e^{c(t,y)}\, dA(y).$$

Applying (79) and remembering that $h(t, b) = 0$, we have our result.

Lemma 8.8. *The distribution of the length of a shade interval, given that t is the initial point of the interval, is given by*

$$P^* = e^{-[h(b,b) - h(t+,b)]}\, D(t, b) - \lim_{\epsilon \to 0+} \int_{t+\epsilon \leqq y < b} d_\epsilon(t, y)\, dA(y) \tag{83}$$

where

$$D(t, y) = \lim_{\epsilon \to 0+} D_\epsilon(t, y)$$

$$D_\epsilon(t, y) = \frac{e^{-[h(t+\epsilon,y) - h(t,y)]} - e^{-[h(t+\epsilon,t+\epsilon) - h(t,t+\epsilon)]}}{1 - e^{-[h(t+\epsilon,t+\epsilon) - h(t,t+\epsilon)]}} \tag{84}$$

$$\begin{aligned} d_\epsilon(t, y) &= e^{a(t+\epsilon,y)} && \textit{for} \quad y > \rho(t+\epsilon) \\ d_\epsilon(t, y) &= D_\epsilon(t, y) e^{h(t+\epsilon,y-)} && \textit{for} \quad t + \epsilon < y \leqq \rho(t+\epsilon). \end{aligned} \tag{85}$$

The proof of this lemma is similar to that of Lemma 8.7.

Bibliography

[1] Takács, L., On a Probability Problem Arising in the Theory of Counters, *Proc. Camb. Phil. Soc.* **52** (1956) pp. 488–498.
[2] Feller, W., On Probability Problems in the Theory of Counters, *Courant anniversary volume* (1948) pp. 105–115.
[3] Hammersley, J. M., On Counters with Random Dead Time I., *Proc. Camb. Phil. Soc.* **49** (1953) pp. 623–637.

Stanford University and
Bureau of the Census