Ark. Mat., **32** (1994), 449–454


# The distribution of square-full integers

H.-Q. Liu

## 1. Introduction

A positive integer $n$ is called to be square-full, if $p|n$ implies that $p^2|n$, here $p$ denotes prime numbers. Let $Q(x)$ be the number of square-full numbers not exceeding $x$, and

$$\Delta(x) := Q(x) - \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} - \frac{\zeta(2/3)}{\zeta(2)} x^{1/3}.$$

The best unconditional upper bound estimate is given in [6], that is,

$$\Delta(x) = O(x^{1/6} \exp(-A(\log x)^{3/5} (\log\log x)^{-1/5}),$$

where $A$ is a positive number. The above estimate cannot be improved unconditional due to our current knowledge concerning the zero-free region of the zeta-function. Assuming the Riemann hypothesis, richer information for $\Delta(x)$ has been given in [6], in which it was shown that

$$(*) \qquad \Delta(x) = O(x^{(1-\varphi)/(7-12\varphi)} \exp(A(\log x)(\log\log x)^{-1})),$$

here $\varphi$ is a number such that

$$(\#) \qquad \sum_{a^2 b^3 \le x} 1 = \zeta\left(\tfrac{3}{2}\right) x^{1/2} + \zeta\left(\tfrac{2}{3}\right) x^{1/3} + O(x^{\varphi}).$$

What is the most optimal value of $\varphi$ one can expect? It is known that (cf. (8) of Schmidt [5], or cf. [6])

$$(1) \qquad \begin{aligned} \sum_{x \ge a^2 b^3} 1 = \zeta\left(\tfrac{3}{2}\right) x^{1/2} + \zeta\left(\tfrac{2}{3}\right) x^{1/3} &- \sum_{n \le x^{1/5}} \psi\left(\left(\frac{x}{n^3}\right)^{1/2}\right) \\ &- \sum_{n \le x^{1/5}} \psi\left(\left(\frac{x}{n^2}\right)^{1/3}\right) + O(1), \end{aligned}$$

where $\psi(t)=t-[t]-\frac{1}{2}$ for a real number $t$. Thus from §4 of [3], we see that (#) holds for $\varphi=14/107+\varepsilon$, where $\varepsilon$ is a sufficiently small positive number. Here $14/107=0.1308...$, and, in view of recent work of Huxley [2], it can be reduced quite satisfactory. But, unless the so-called exponent pair conjecture is true, namely, $\left(\varepsilon, \frac{1}{2}+\varepsilon\right)$ is an exponent pair for all sufficiently small number $\varepsilon\geq 0$, in which case we can take $\varphi=0.1+\varepsilon$ in (#), we can not prove that $\Delta(x)=O(x^{9/58+\varepsilon})$ in (*). But in this paper we can really prove the following.

**Theorem.** *Assuming the Riemann hypothesis, then*

$$\Delta(x)=O\left(x^{9/58+\varepsilon}\right)$$

*for any* $\varepsilon>0$.

## 2. Reduction

Taking an idea from Montgomery–Vaughan [4], we first give a reduction of our problem. Throughout the arguments we assume the Riemann hypothesis for the zeta-function.

**Lemma 1.** *Let* $Y=$ *integer* $+\frac{1}{2}$, $x^{2/15+\varepsilon}\leq Y\leq x^{1/6-\varepsilon}$, *then*

$$\Delta(x)=S_1+S_2+O\left(x^{1/2+\varepsilon}Y^{-5/2}+Y\right),$$

*where*

$$-S_1=\sum_{m^6n^5\leq x,\, m\leq Y}\mu(m)\psi\left(\left(\frac{x}{m^6n^3}\right)^{1/2}\right),$$

$$-S_2=\sum_{m^6n^5\leq x,\, m\leq Y}\mu(m)\psi\left(\left(\frac{x}{m^6n^2}\right)^{1/3}\right),$$

$\mu(\cdot)$ *is the Möbius function.*

*Proof.* Let $Y_1=xY^{-6}$, it is obvious that

$$Q(x)=\sum_{a^2b^3m^6\leq x}\mu(m)=\sum_1+\sum_2-\sum_3,$$

$$\sum_1=\sum_{n\leq Y_1}\tau(n)\sum_{m\leq (xn^{-1})^{1/6}}\mu(m),\quad \tau(n)=\sum_{a^2b^3=n}1,$$

$$\sum_2=\sum_{m\leq Y}\mu(m)\sum_{n\leq xm^{-6}}\tau(n),$$

$$\sum_3=\left(\sum_{n\leq Y_1}\tau(n)\right)\left(\sum_{m\leq Y}\mu(m)\right).$$

From (1) we have

$$\sum_2 = \zeta(\tfrac{3}{2})x^{1/2}\Big(\sum_{m\le Y}\frac{\mu(m)}{m^3}\Big)+\zeta(\tfrac{2}{3})x^{1/3}\Big(\sum_{m\le Y}\frac{\mu(m)}{m^2}\Big)+S_1+S_2+O(Y),$$

which, in conjunction with the facts

$$\sum_{m\le Y}\frac{\mu(m)}{m^3}=\frac{1}{\zeta(3)}+O(Y^{-5/2+\varepsilon})\quad\text{and}\quad\sum_{m\le Y}\frac{\mu(m)}{m^2}=\frac{1}{\zeta(2)}+O(Y^{-3/2+\varepsilon})$$

(both follow from partial summations and the estimate $\sum_{m\le Z}\mu(m)\ll Z^{1/2+\varepsilon}$ for $Z>0$—a consequence of the Riemann hypothesis), gives

$$\sum_2 = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2}+\frac{\zeta(2/3)}{\zeta(2)}x^{1/3}+S_1+S_2+O\big(x^{1/2+\varepsilon}Y^{-5/2}+Y\big).$$

Similarly we get $\sum_1, \sum_3=O(x^{1/2+\varepsilon}Y^{-5/2})$. Lemma 1 then follows.

## 3. Proof of our Theorem

Now we choose $\theta=9/58$, $Y=x^{(1-2^\theta)/5}$ in Lemma 1. It suffices to estimate $S_1$ and $S_2$. We consider subsums of the form $S_1(M)$ and $S_2(M)$, where, for $M\le Y$,

$$-S_1(M)=\sum_{m^6n^5\le x,\, m\sim M,\, m\le Y}\mu(m)\psi\bigg(\Big(\frac{x}{m^6n^3}\Big)^{1/2}\bigg)$$

and

$$-S_2(M)=\sum_{m^6n^5\le x,\, m\sim M,\, m\le Y}\mu(m)\psi\bigg(\Big(\frac{x}{m^6n^3}\Big)^{1/3}\bigg)$$

and $m\sim M$ means $M<m\le 2M$. Denote $X=xm^{-6}$, then from §4 of [3] we get the estimates

$$\sum_{n\le x^{1/5}}\psi\bigg(\Big(\frac{x}{n^3}\Big)^{1/2}\bigg)\ll x^{14/107+\varepsilon},$$

$$\sum_{n\le x^{1/5}}\psi\bigg(\Big(\frac{x}{n^2}\Big)^{1/3}\bigg)\ll x^{7/55+\varepsilon},$$

thus we have

**Lemma 2.**

$$S_1(M)=O((x^{14}M^{23})^{1/107}x^{\varepsilon}),\quad S_2(M)=O((x^7M^{13})^{1/55}x^{\varepsilon}).$$

From Lemma 2 we deduce that, for $M\le x^{0.098}$, $S_1(M)$, $S_2(M)=O(x^{9/58})$. We thus can assume $M>x^{0.098}$, and we give a further splitting of the summation range by considering

$$-S_1(M,N)=\sum_{m^6n^5\le x,\, m\sim M,\, n\sim N,\, m\le Y}\mu(m)\psi\left(\left(\frac{x}{m^6n^3}\right)^{1/2}\right),$$

$$-S_2(M,N)=\sum_{m^6n^5\le x,\, m\sim M,\, n\sim N,\, m\le Y}\mu(m)\psi\left(\left(\frac{x}{m^6n^2}\right)^{1/3}\right),$$

here $N$ is such that $M^6N^5\le x$ and $MN>x^{\theta}$. By an argument using the Fourier expansion of the function $\psi(\cdot)$ (cf. [4]), we get, with $H=MNx^{-\theta}$, the following estimate

$$S_i(M,N)\ll\sum_{h=1}^{\infty}\min\left(\frac{1}{h},\frac{H}{h^2}\right)|L_i(M,N)|+x^{\theta}\log x,\quad i=1,2,$$

$$L_1(M,N)=\sum_{(m,n)\in D}\mu(m)e(hx^{1/2}m^{-3}n^{-3/2}),$$

$$L_2(M,N)=\sum_{(m,n)\in D}\mu(m)e(hx^{1/3}m^{-2}n^{-2/3}),$$

$$D=\{(m,n)\,|\,m\sim M, n\sim N, m^6n^5\le x, m\le Y\}.$$

To estimate $L_i(M,N)$ we appeal to the next lemmas.

**Lemma 3.** *Let $M\le N<N_1\le M_1$, $a_n$ be complex numbers. Then*

$$\left|\sum_{N<n\le N_1}a_n\right|\le\int_{-\infty}^{\infty}K(t)\left|\sum_{M<m\le M_1}a_me(tm)\right|dt,$$

*with* $K(t)=\min(M_1-M+1,(\pi|t|)^{-1},(\pi t)^{-2})$ *and*

$$\int_{-\infty}^{\infty}K(t)\,dt\ll 3\log(2+M_1-M).$$

**Lemma 4.** *Let*

$$\omega_{\phi\psi}(X,Y)=\sum_r\sum_s \phi_r\psi_s e(x_r y_s),$$

*where* $X=(x_r)$, $Y=(y_s)$ *are finite sequences of real numbers with*

$$|x_r|\le P, \quad |y_s|\le Q$$

*and* $\phi_r$, $\psi_s$ *are complex numbers. Then*

$$|\omega_{\phi\psi}(X,Y)|^2\le 20(1+PQ)\omega_\phi(X,Q)\omega_\psi(Y,P),$$

*with*

$$\omega_\phi(X,Q)=\sum_{|x_r-x_{r'}|\le Q^{-1}}|\phi_r\phi_{r'}|,$$

*and* $\omega_\psi(Y,P)$ *being defined similarly.*

Lemmas 3 and 4 are Lemma 2.2 and Lemma 2.4 (with $k=1$) respectively, of [1]. Using Lemma 3 to separate variables, we get

$$(\log x)^{-1}L_1(M,N)\ll\sum_{m\sim M}\left|\sum_{n\sim N}e(tn)e(hx^{1/2}m^{-3}n^{-3/2})\right|, \tag{2}$$

where $t$ is a real number (independent of $m$ and $n$). By Lemma 4 we derive that

$$(\log x)^{-2}L_1^2(M,N)\ll hx^{1/2}M^{-3}N^{-3/2}A_1A_2, \tag{3}$$

here $A_1$ is the number of lattice points $(m,m_1)$ such that

$$|m^{-3}-m_1^{-3}|\ll N^{3/2}(hx^{1/2})^{-1},$$

hence $A_1\ll M(1+M^4N^{3/2}(hx^{1/2})^{-1})$; and $A_2$ is the number of lattice points $(n,n_1)$ such that

$$|n^{-3/2}-n_1^{-3/2}|\ll M^3(hx^{1/2})^{-1},$$

hence $A_2\ll N(1+N^{5/2}M^3(hx^{1/2})^{-1})\ll N$. Thus from (3) we obtain the estimate

$$S_1(M,N)\ll\left(\sqrt[4]{x^{1-2\theta}M^{-2}N}+MN^{1/2}+x^\theta\right)x^{\varepsilon/2}. \tag{4}$$

By an argument analogous to that of (2) and (3), we can get

$$S_2(M,N)\ll\left(\sqrt[6]{x^{1-2\theta}N^4}+MN^{1/2}+x^\theta\right)x^{\varepsilon/2}. \tag{5}$$

Now we have

$$MN^{1/2}\le M^{4/10}(M^6N^5)^{1/10}\ll Y^{4/10}x^{1/10}=x^\theta,$$

$$\sqrt[4]{x^{1-2\theta}M^{-2}N}\ll\sqrt[4]{x^{1-2\theta}M^{-1}}\ll x^\theta,$$

$$\sqrt[6]{x^{1-3\theta}N^4}\ll(x^{1-3\theta}x^{4/11})^{1/6}\ll x^\theta,$$

in view of the facts that $M>x^{0.098}$ and $M^6N^5\le x$ (which also imply that $M\ge N$ and $N\le x^{1/11}$). Our theorem follows from the above estimates in view of (4) and (5).

*Remark* 1. Clearly the limit value of the exponent can be expected from Lemma 1 to be $1/7+\varepsilon=0.14285...+\varepsilon$, while $9/58=0.15517...$.

*Remark* 2. It is of interest whether our result can be improved by invoking the decomposition of the Möbius function, as was carried out in [4].

## References

1. BOMBIERI, E. and IWANIEC, H., On the order of $\zeta(\frac{1}{2}+it)$, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **13** (1986), 449–472.
2. HUXLEY, M. N., Exponential sums and lattice points II, *Proc. London Math. Soc.* **66** (1993), 279–301.
3. LIU, H.-Q., The number of square-full numbers in an interval, *Acta Arith.* **64** (1993), 129–149.
4. MONTGOMERY, H. L. and VAUGHAN, R. C., The distribution of squarefree numbers, in *Recent Progress in Analytic Number Theory, vol. 1* (Halberstam, H. and Hooley, C., eds.), pp. 247–256, Academic Press, London, 1981.
5. SCHMIDT, P. G., Zur Anzahl Abelscher Gruppen gegebener Ordnung, *J. Reine Angew. Math.* **229** (1968), 34–43.
6. SURYANARAYANA, D. and SITARAMACHANDRA, R., The distribution of square-full integers, *Ark. Mat.* **11** (1973), 195–201.

*Received September 6, 1993*

H.-Q. Liu
206-10, Bao Guo St.
Harbin, 150066
P.R. China