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The Distribution of the Maximum Vertex Degree in Random Planar Maps

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We determine the limiting distribution of the maximum vertex degree Δ_n in a random triangulation of an n-gon, and show that it is the same as that of the maximum of n independent identically distributed random variables G_2, where G_2 is the sum of two independent *geometric*(1/2) random variables. This answers affirmatively a question of Devroye, Flajolet, Hurtado, Noy and Steiger, who gave much weaker almost sure bounds on Δ_n. An interesting consequence of this is that the asymptotic probability that a random triangulation has a unique vertex with maximum degree is about 0.72. We also give an analogous result for random planar maps in general.

1. INTRODUCTION

Throughout this paper, a *map* is a connected graph G embedded in the plane with no edge crossings. Loops and multiple edges are allowed in G. A map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. The distinguished vertex and edge are called the root vertex and the root edge of the map. The face on the distinguished side of the root edge is called the root face. A rooted *triangulation of an n-gon* is a topological triangular disection of an n-gon (with no loops or multiple edges) which has a root edge on the n-gon and the root face is the

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external face bounded by the n-gon. Equivalently, this is a rooted map with no loops or multiple edges, with the boundary of the root face a simple n-cycle and with all non-root faces of degree 3. (It would be equivalent for counting purposes to define the n-gon to be convex and the disection to be into triangles with straight sides.) Two rooted maps or triangulations are considered the same if there is a homeomorphism from the plane to itself which transforms one rooted map to the other and preserves the rooting. Throughout this paper, all probability distributions are uniform over a given family of rooted maps. **P**, **E**, and **V** are used to denote the probability, expectation, and variance of a random variable, respectively. We use Δ_n, and $\zeta_{k,n}$ (or ζ_k for short), to denote the maximum vertex degree, and the number of vertices of degree k, respectively, of a random rooted map in a given family.

Devroye *et al.* [6, Theorem 1] point out the connection between Δ_n for random triangulations of an n-gon, and the maximal distance between successive external nodes in a random binary tree. Their main results on random triangulations of an n-gon are that $\mathbf{E}(\Delta_n) \sim \log n/\log 2$, and

$$\mathbf{P}(|\Delta_n - \log n/\log 2| \leqslant (1+\varepsilon)\log\log n/\log 2) \to 1$$

for any constant $\varepsilon > 0$. They asked if Δ_n has the same limiting distribution as the maximum of n independent variables each distributed as the degree of one vertex.

We shall make improvements on these results, answer affirmatively their question on the limiting distribution, and derive analogous results for random planar maps. Our basic method is to study the distribution of the random variables ζ_k for appropriate values of k. To do this we use the method of moments. This seems to represent the most direct application so far of probabilistic methods to the theory of random maps, a theory in which most results rely on the use of generating functions.

Our main results on Δ_n are the following. Although the last statements in Theorems 1 and 2 are sharpened by Corollary 1, we state these results separately because their proof is much simpler.

THEOREM 1. *For rooted triangulations of an n-gon,*

$$\mathbf{E}(\Delta_n) = \frac{\log n + \log\log n}{\log 2} + O(1), \qquad \mathbf{V}(\Delta_n) = O(\log n),$$

and

$$\mathbf{P}(|\Delta_n - (\log n + \log\log n)/\log 2| \leqslant \Omega_n) = 1 - O(1/\log n + (1/2)^{\Omega_n}),$$

for any function $\Omega_n \to \infty$.

THEOREM 2. *For rooted maps with n edges,*

$$\mathbf{E}(\Delta_n) = \frac{\log n - (1/2)\log\log n}{\log(6/5)} + O(1), \qquad \mathbf{V}(\Delta_n) = O(\log n),$$

and

$$\mathbf{P}(|\Delta_n - (\log n - (1/2)\log\log n)/\log(6/5)| \leqslant \Omega_n) = 1 - O(1/\log n + (5/6)^{\Omega_n}),$$

for any function $\Omega_n \to \infty$.

Our more precise results are based on showing that for values of k close to the expected maximum degree the ζ_k are asymptotically independent Poisson random variables. To identify this value, in the context of triangulations of a polygon define $\alpha = \frac{1}{2}$, and $\gamma = \gamma_P(n)$ so that $n\gamma\alpha^\gamma = 1$. Similarly, for rooted maps define $\alpha = \frac{5}{6}$, and $\gamma = \gamma_M(n)$ so that $n(10\pi\gamma)^{-1/2}\alpha^\gamma = 1$.

THEOREM 3. *For both rooted triangulations of a polygon and rooted planar n-edged maps, there exists a function* $\omega = \omega(n) \to \infty$ *(sufficiently slowly as* $n \to \infty$*) so that the following holds. For* $k = \lfloor \gamma - \omega \rfloor$, *the total variation distance between the distribution of* $(\zeta_k, \zeta_{k+1}, ...)$, *and that of* $(Z_k, Z_{k+1}, ...)$ *tends to* 0, *where the* Z_j *are independent Poisson random variables with* $\mathbf{E}Z_j = \alpha^{j-\gamma}$.

COROLLARY 1. *For rooted triangulations of an n-gon,*

$$\mathbf{P}(\Delta_n < x + \gamma_P(n)) \sim \exp(-2^{1-x})$$

uniformly for $|x|$ *bounded with* $x + \gamma_P(n)$ *an integer, and for rooted maps with n edges,*

$$\mathbf{P}(\Delta_n < x + \gamma_M(n)) \sim \exp(-6(6/5)^{1-x})$$

uniformly for $|x|$ *bounded with* $x + \gamma_M(n)$ *an integer.*

Theorem 3 implies the asymptotic joint distribution of the j largest vertex degrees for any fixed j, but we state only the following interesting special case.

COROLLARY 2. *The probability that a random rooted map has a unique vertex with maximum degree is asymptotic to* $\sum_{m=-\infty}^{\infty} 2^{m+y_P(n)} \exp(-2^{m+y_P(n)+1}) \approx 0.7215$ *for triangulations of an n-gon and* $\sum_{m=-\infty}^{\infty} (6/5)^{m+y_M(n)} \exp(-6(6/5)^{m+y_M(n)}) \approx 0.9141$ *for all maps, where* $y_P(n)$ *and* $y_M(n)$ *are the fractional parts of* $\gamma_P(n)$ *and* $\gamma_M(n)$ *respectively.*

By numerical calculation, the probabilities here are functions of y_P and y_M with oscillations of magnitude approximately 10^{-4} and 10^{-22} respectively. They are equal to $\frac{1}{2}S_2(y_P(n)+1)=\frac{1}{2}S_2(y_P(n))$ and $\frac{1}{6}S_{6/5}(y_M(n)+\log_{6/5} 6)$ where $S_c(x)=\sum_{m=-\infty}^{\infty} c^{m+x}\exp(-c^{m+x})$, a function with period 1. B. D. Hughes pointed out to us that $S_c(x)$ can be better understood using the classic Poisson summation formula (see Olver [11]), which gives

$$S_c(x)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} c^{t+x}\exp(-c^{t+x})\exp(2\pi imt)\,dt.$$

Setting $z=c^{t+x}$ gives

$$S_c(x)=\frac{1}{\log c}\sum_{m=-\infty}^{\infty}\exp(-2\pi imx)\int_0^{\infty}\exp(-z+2\pi im\log z/\log c)\,dz,$$

and the integral is just $\Gamma(2\pi im/\log c+1)$. The small fluctuations are explained by noting that the term $m=0$ dominates easily for c close to 1, and indeed as $|y|\to\infty$ for fixed x,

$$|\Gamma(x+iy)|\sim(2\pi)^{1/2}\,|y|^{x-1/2}\exp(-(\pi/2)\,|y|).$$

Related functions occur in the theory of random walks as discussed by Hughes [10].

Corollary 2 contrasts with the result in [9], where it was shown that almost all maps in a certain family have a unique maximum "component", for various families.

In [12] it was shown that almost all maps are asymmetric. Similarly, almost all triangulations of a polygon are asymmetric. (This is much simpler to prove: the numbers with given symmetries can be derived, and each is exponentially smaller than the total number of maps. Since the number of symmetries is at most $2n$ for an n-gon, the result follows.) So an immediate corollary is that the distributions given in Theorem 3, and its corollaries, also apply to random unrooted polygons and maps.

The basic relation between maps (or n-gons) with distinguished vertices of given degrees, and the moments of the variables ζ_k for the numbers of vertices of given degree, is given next. For r distinct positive integers $k_1, k_2, ..., k_r$, let $P_n[k_1^{m_1}, k_2^{m_2}, ..., k_r^{m_r}]$ be the number of rooted triangulations of an n-gon which has root vertex degree k_1, m_1-1 other distinguished vertices of degree k_1 among which there is a specified linear ordering, ..., and m_r distinguished vertices of degree k_r among which there is a specified linear ordering. Similarly define $M_n[k_1^{m_1}, k_2^{m_2}, ..., k_r^{m_r}]$ for all rooted maps. Let P_n (M_n) be the number of rooted triangulations (maps) with n edges. We use the notation $[i]_m$ for the falling factorial $i(i-1)\cdots(i-m+1)$.

LEMMA 1. (i) *For triangulations of an n-gon,*

$$\mathbf{E}([\zeta_{k_1}]_{m_1}[\zeta_{k_2}]_{m_2}\cdots[\zeta_{k_r}]_{m_r})=\frac{nP_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]}{P_n}.$$

(ii) *For all rooted maps,*

$$\mathbf{E}([\zeta_{k_1}]_{m_1}[\zeta_{k_2}]_{m_2}\cdots[\zeta_{k_r}]_{m_r})=\frac{2nM_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]}{k_1M_n}.$$

Proof. Call a rooted map *multiply distinguished* if it has a set of distinguished vertices of which m_1 are of degree k_1, m_2 of degree k_2, ..., and m_r of degree k_r, where the distinguished vertices of the same degree are additionally given a linear ordering. A *doubly rooted* map is a multiply distinguished rooted map with an additional secondary rooting at the first vertex of degree k_1. (Note that the secondary rooting may coincide with the primary rooting.) We will assume both rootings have the same orientation, so that the number of ways of affixing the secondary rooting is exactly k_1. On the other hand, the doubly rooted maps can be obtained from all those rooted maps counted by $M_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]$ by calling the rooting of such a map secondary, and choosing any of $2n$ places for a primary rooting. Hence the number of multiply distinguished maps is

$$\frac{2nM_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]}{k_1}. \tag{1}$$

For multiply distinguished triangulations of an n-gon, the argument is similar except that there is only one way to affix the secondary rooting, and n ways to choose a primary rooting, so their number is

$$nP_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]. \tag{2}$$

For all rooted maps, we have

$$\begin{aligned}
&\mathbf{E}([\zeta_{k_1}]_{m_1}[\zeta_{k_2}]_{m_2}\cdots[\zeta_{k_r}]_{m_r})\\
&\quad=\frac{1}{M_n}\sum_{i_1,i_2,...,i_r}[i_1]_{m_1}[i_2]_{m_2}\cdots[i_r]_{m_r}\\
&\qquad\times(\#\text{rooted maps with } n \text{ edges, } i_j \text{ vertices of degree } k_j,\ j=1,...,r)\\
&\quad=\frac{1}{M_n}\times(\#\text{multiply distinguished maps})\\
&\quad=\frac{2nM_n[k_1^{m_1},k_2^{m_2},...,k_r^{m_r}]}{k_1M_n}
\end{aligned}$$

from (1), which gives (ii). The proof of (i) is essentially the same, but using (2). ▮

In Section 2, we give several analytic lemmas needed to prove our main theorems. Lemmas 2 and 3 are multivariate versions of some "transfer theorems" discussed by Flajolet and Odlyzko [7]; they are interesting in their own right and are useful for other asymptotic enumeration problems. In Sections 3, we calculate information on the first two moments of ζ_k for the two sorts of maps. Then in Section 4 we prove Theorems 1 and 2 using the second moment method. We compute all the joint moments in Section 5. The calculation in this section is quite technical, which is typical in map enumerations, and some familiarity with [2] will be helpful. Readers may skip the proofs in this section if desired. We prove Theorem 3 and its corollaries in Section 6.

2. ANALYTIC LEMMAS

We need to introduce some notation before stating the basic analytic lemmas we will use to obtain asymptotics for coefficients of generating functions. In the following, ε will denote a small positive constant, ϕ is a constant satisfying $0<\phi<\pi/2$, and $\mathbf{y}=(y_1, y_2, ..., y_d)$. Define

$$\Delta_x(\varepsilon, \phi)=\{x: |x|\leqslant 1+\varepsilon,\ x\neq 1,\ |\mathrm{Arg}(x-1)|\geqslant \phi\},$$

$$\Delta_j(\varepsilon, \phi)=\{y_j: |y_j|\leqslant 1+\varepsilon,\ y_j\neq 1,\ |\mathrm{Arg}(y_j-1)|\geqslant \phi\},$$

$$\mathscr{R}(\varepsilon, \phi)=\{(x, \mathbf{y}): |y_j|<1,\ 1\leqslant j\leqslant d,\ x\in \Delta_x(\varepsilon, \phi)\}.$$

Let $\beta_j>0$ for $1\leqslant j\leqslant d$, and α be any real number.

Definition 1. We write

$$f(x, \mathbf{y})=\tilde{O}\left((1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}\right)$$

if there are $\varepsilon>0$ and $0<\phi<\pi/2$ such that in $\mathscr{R}(\varepsilon, \phi)$

(i) $f(x, \mathbf{y})$ is analytic, and

$$f(x, \mathbf{y})=O\left(|1-x|^{-\alpha}\prod_{j=1}^{d}(1-|y_j|)^{-\beta_j}\right)$$

as $(1-x)(1-y_j)^{-p}\to 0$, for $1\leqslant j\leqslant d$, and some $p\geqslant 0$.

(ii)

$$f(x, \mathbf{y}) = O\left(|1-x|^{-\alpha'} \prod_{j=1}^{d} (1-|y_j|)^{-q} \right)$$

for some $q \geqslant 0$ and some real number α'.

DEFINITION 2. We write

$$f(x, \mathbf{y}) \approx c(1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j}$$

if $f(x, \mathbf{y})$ can be expressed as

$$f(x, \mathbf{y}) = c(\mathbf{y})(1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j} + \sum_{j=0}^{d} C_j(x, \mathbf{y}) + E(x, \mathbf{y})$$

such that

(i) $C_0(x, \mathbf{y})$ is a polynomial in x, and for $1 \leqslant j \leqslant d$, $C_j(x, \mathbf{y})$ is a polynomial in y_j.

(ii)

$$E(x, \mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha'} \prod_{j=1}^{d} (1-y_j)^{-\beta'_j} \right),$$

for some $\alpha' < \alpha$ and $\beta'_j \geqslant 0$, $1 \leqslant j \leqslant n$.

(iii) $c(\mathbf{y}) = c + O(\sum_{j=1}^{d} |1-y_j|)$ and is analytic in $\{\mathbf{y}\colon y_j \in \Delta_j(\varepsilon, \phi)\}$, and $c(\mathbf{1}) = c \neq 0$.

Note that the range of k_j in the following lemmas can obviously be relaxed, but only $k_j = O(\log n)$ is needed in our application, and makes a result which is simpler to state.

LEMMA 2. *Suppose*

$$f(x, \mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j} \right).$$

Then

(i) *as* $n \to \infty$ *and* $1 \leqslant k_j = O(\log n)$ $(j = 1, ..., d)$,

$$[x^n \mathbf{y}^{\mathbf{k}}]\, f(x, \mathbf{y}) = O\left(n^{\alpha-1} \prod_{j=1}^{d} k_j^{\beta_j} \right);$$

(ii) *for any* $0<\varepsilon'<1$ *and all* n, k_j,

$$[x^n\mathbf{y}^{\mathbf{k}}]\, f(x,\mathbf{y}) = O\left(n^{\alpha-1}\prod_{j=1}^{d}(1-\varepsilon')^{-k_j}\right).$$

Proof. Let $\Gamma_x = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ be the (positively oriented) contour, with

$$\Gamma_1 = \left\{x\colon |x-1| = \frac{1}{n},\ |\mathrm{Arg}(x-1)| \geqslant \phi\right\}$$

$$\Gamma_2 = \left\{x\colon |x-1| \geqslant \frac{1}{n},\ |x| \leqslant 1+\varepsilon,\ \mathrm{Arg}(x-1) = \phi\right\}$$

$$\Gamma_3 = \{x\colon |x| = 1+\varepsilon,\ |\mathrm{Arg}(x-1)| \geqslant \phi\}$$

$$\Gamma_4 = \left\{x\colon |x-1| \geqslant \frac{1}{n},\ |x| \leqslant 1+\varepsilon,\ \mathrm{Arg}(x-1) = -\phi\right\}.$$

Let $\Gamma_{y_j} = \{y_j\colon |y_j| = 1-1/(k_j+2)\}$ be a positively oriented contour. Using Definition 1(i) and Cauchy's formula, we have

$$\begin{aligned}[x^n\mathbf{y}^{\mathbf{k}}]\, f(x,\mathbf{y}) &= (2\pi i)^{-(d+1)}\int_{\Gamma_x}\int_{\Gamma_{y_1}}\cdots\int_{\Gamma_{y_d}} x^{-(n+1)}\mathbf{y}^{-(\mathbf{k}+\mathbf{1})} f(x,\mathbf{y})\,dx\,d\mathbf{y}\\ &= O\left(\int_{\Gamma_x}\int_{\Gamma_{y_1}}\cdots\int_{\Gamma_{y_d}} |x|^{-(n+1)}\,|f(x,\mathbf{y})|\,|dx|\,|d\mathbf{y}|\right).\end{aligned}$$

For a large positive constant C, define

$$\Gamma'_x = \Gamma_3\cup\{x\colon |x-1|\geqslant C\log n/n,\ |x|\leqslant 1+\varepsilon,\ |\mathrm{Arg}(x-1)| = \phi\},$$

and let Γ''_x be the remaining part of Γ_x. Let I' be the contribution to the above integral from $x\in\Gamma'_x$, and I'' be the contribution from $x\in\Gamma''_x$. Then using Definition 1, we have

$$I' = O\left(\prod_{j=1}^{d} k_j^{q}\right)\int_{\Gamma'_x} |1-x|^{-\alpha'}\,|x|^{-(n+1)}\,|dx|,$$

and

$$I'' = O\left(\prod_{j=1}^{d} k_j^{\beta_j}\right)\int_{\Gamma''_x} |1-x|^{-\alpha}\,|x|^{-(n+1)}\,|dx|.$$

Using

$$\left|1+\frac{t}{n}e^{i\phi}\right| \geqslant 1+\frac{t}{n}\cos\phi$$

for $z=1+(t/n)\,e^{i\phi}$ on Γ_2 and Γ_4, we get quite easily (as in the proof of [7, Theorem 1]) that

$$\int_{\Gamma'_x} |1-x|^{-\alpha'}\,|x|^{-(n+1)}\,|dx| = O(n^{-C/2}),$$

and

$$\int_{\Gamma''_x} |1-x|^{-\alpha}\,|x|^{-(n+1)}\,|dx| = O(n^{\alpha-1}),$$

and hence (i) follows.

The statement in (ii) follows by modifying the argument used above, with $\Gamma_{y_j}=\{y_j\colon |y_j|=1-\varepsilon'\}$ for each j. ∎

Lemma 3. *Let $d\geqslant 1$ and*

$$f(x,\mathbf{y}) \approx c(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j},$$

where α is neither a negative integer nor 0, and $c\neq 0$. Then as $n\to\infty$ and $k_j\to\infty$, $k_j=O(\log n)$ $(j=1,\ldots,d)$,

$$[x^n\mathbf{y}^{\mathbf{k}}]\,f(x,\mathbf{y}) = \frac{c}{\Gamma(\alpha)}\prod_{j=1}^{d}(k_j^{\beta_j-1}/\Gamma(\beta_j))\,n^{\alpha-1}\left(1+O\left(\sum_{j=1}^{d}(1/k_j)\right)\right).$$

Proof. Since n and the k_j all go to ∞, the polynomials $C_j(x,\mathbf{y})$ in Definition 2 contribute nothing. Thus, by the previous lemma, we have, for $k_j=O(\log n)$, that

$$[x^n\mathbf{y}^{\mathbf{k}}]\,f(x,\mathbf{y}) = [x^n](1-x)^{-\alpha}\,[\mathbf{y}^{\mathbf{k}}]\,c(\mathbf{y})\prod_{j=1}^{d}(1-y_j)^{-\beta_j} + O\left(n^{\alpha'-1}\prod_{j=1}^{d}k_j^{\beta'_j}\right),$$

for some $\alpha'<\alpha$, $\beta'_j\geqslant 0$, and $c(\mathbf{y})=c(\mathbf{1})+O(\sum_{j=1}^{d}|1-y_j|)$ analytic in $\{\mathbf{y}\colon y_j\in\Delta_j(\varepsilon,\phi)\}$ with $c(\mathbf{1})=c$. Use the contour Γ_x, as used in the proof of the previous lemma, for each y_j (replacing n by k_j, of course), and compare with evaluation of the same coefficient in $\prod_{j=1}^{d}(1-y_j)^{-\beta_j}$ by the same integral. In this way we obtain

$$[\mathbf{y}^{\mathbf{k}}]\, c(\mathbf{y}) \prod_{j=1}^{d} (1-y_j)^{-\beta_j} = c(\mathbf{1})[\mathbf{y}^{\mathbf{k}}] \prod_{j=1}^{d} (1-y_j)^{-\beta_j} \left(1+O\left(\sum_{j=1}^{d} 1/k_j\right)\right).$$

Therefore

$$[x^n\mathbf{y}^{\mathbf{k}}]\, f(x, \mathbf{y}) = \frac{c}{\Gamma(\alpha)} \prod_{j=1}^{d} (k_j^{\beta_j-1}/\Gamma(\beta_j))\, n^{\alpha-1} \times \left(1+O\left(\sum_{j=1}^{d} 1/k_j + n^{\alpha'-\alpha} \prod_{j=1}^{d} k_j^{\beta_j'-\beta_j+1}\right)\right). \quad \blacksquare$$

The following lemma will be used later to verify that the generating functions in our applications satisfy the conditions of Lemmas 2 and 3.

LEMMA 4. *Let $f(x) = \sum f_n x^n$ be a generating function with non-negative coefficients. Suppose*

(i) *There is a nonempty set J of indices satisfying $f_j > 0$ for $j \in J$ and*

$$\gcd\{j-j' : j, j' \in J\} = 1.$$

(ii) *$f(x)$ is analytic in $\Delta_x(\varepsilon, \phi)$ and $f(1) = 1$.*

(iii) *$f(x) = 1 - c(1-x)^{1/2} + o((1-x)^{1/2})$ for some $c > 0$ and $x \in \Delta_x(\varepsilon, \phi)$.*

Then there exists $0 < \varepsilon' \leqslant \varepsilon$ such that $|f(x)| \leqslant 1$ for $x \in \Delta(\varepsilon', \phi)$.

Proof. It follows easily from (i) and (ii) that $|f(x)| < f(1)$ for all $|x| \leqslant 1$ and $x \neq 1$. (See e.g. [9, Lemma 1] for a proof.) Therefore, there exists a small positive ε' such that

$$|f(x)| \leqslant 1 \qquad \text{for all} \quad x \in \Delta(\varepsilon', \phi) - \{x: |x-1| \leqslant \varepsilon'\}.$$

For $|x-1| \leqslant \varepsilon'$ and $x \in \Delta(\varepsilon, \phi)$, we have

$$(1-x)^{1/2} = re^{i\theta}, \qquad \text{with} \quad 0 < r \leqslant (\varepsilon')^{1/2}, \quad |\theta| \leqslant (\pi-\phi)/2.$$

It follows from assumption (iii) that

$$|f(x)| = |1 - cre^{i\theta}| + o(r) = 1 - cr\cos\theta + o(r) < 1$$

for sufficiently small r. $\blacksquare$

3. THE FIRST TWO MOMENTS

In this section we do the counting required to obtain the first two moments of the number of vertices of given degree. This requires keeping track of the degree of a distinguished vertex, besides the root vertex.

We first consider rooted triangulations of an n-gon. For simplicity since we only need the first two moments here, let $P_{n,k}$ denote the number of these triangulations with n vertices and the root vertex of degree k, and $P_{n,k,l}$ the number of these with additionally another distinguished vertex of degree l.

LEMMA 5. *Let ε' be any constant satisfying $0 < \varepsilon' < 2$. We have*

(i)

$$\mathbf{E}(\zeta_k) = O(n(2-\varepsilon')^{-k})$$

for all k and n;

(ii)

$$\mathbf{E}(\zeta_k) = nk2^{-k}(1 + O(1/k))$$

and

$$\mathbf{E}(\zeta_k(\zeta_k - 1)) = (\mathbf{E}(\zeta_k))^2 \, (1 + O(1/k))$$

for $k = O(\log n)$.

Proof. Define generating functions

$$P(x, y) = \sum P_{n,k} x^n y^k, \qquad \bar{P}(x, y, z) = \sum P_{n,k,l} x^n y^k z^l.$$

Let $\bar{P}_1(x, y, z)$ be the contribution to $\bar{P}(x, y, z)$ from those triangulations in which the distinguished vertex is incident with the root edge, and $\bar{P}_2(x, y, z) = \bar{P}(x, y, z) - \bar{P}_1(x, y, z)$.

Figure 1 shows a decomposition of a rooted triangulation of a polygon into a number of other rooted triangulations of polygons, based on deleting the root vertex. (Root vertices are circled, and the convention is that the root edges is the next edge clockwise from the root vertex around the perimeter of the polygon.) Using this decomposition, we obtain

$$P(x, y) - x^2 y = x^2 \sum_{k \geqslant 1} (P(x, 1)/x)^k \, y^{k+1} = \frac{xy^2 P(x, 1)}{1 - P(x, 1) \, y/x}.$$

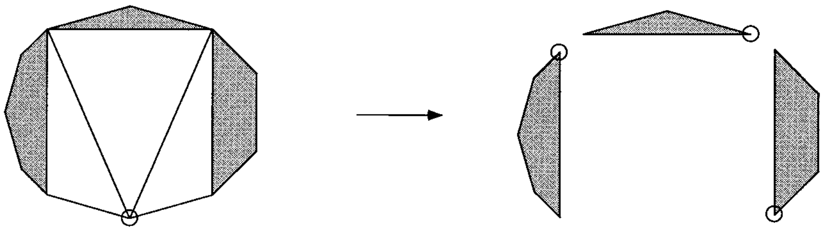

FIG. 1. Decomposition of a rooted triangulation of a polygon into several triangulations.

Setting $y=1$ and solving for $P(x, 1)$, we obtain

$$P(x, 1)=x(1-(1-4x)^{1/2})/2, \tag{3}$$

and hence

$$P_n=\frac{1}{16\sqrt{\pi}}n^{-3/2}4^n(1+O(1/n)). \tag{4}$$

It is convenient to use the parameter

$$t=t(x)=(1-(1-4x)^{1/2})/2, \qquad \text{i.e.} \quad x=t(1-t). \tag{5}$$

We then have

$$P(x, y)=x^2y/(1-yt). \tag{6}$$

Now we analyze $\bar{P}(x, y, z)$. Using the decomposition depicted in Fig. 2, where the root edge is deleted and two polygons result, we obtain

$$\bar{P}_1(x, y, z)=x^2yz+x^{-1}yzP(x, y)\,P(x, z), \tag{7}$$

and

$$\bar{P}_2(x, y, z)=x^{-1}yP(x, z)\,\bar{P}_1(x, , y, z) + x^{-1}yP(x, 1)\,\bar{P}_2(x, y, z)+x^{-1}yP(x, y)\,\bar{P}_2(x, 1, z).$$

Hence

$$\bar{P}_2(x, y, z)=x^{-1}y(1-ty)^{-1}\,(P(x, z)\,\bar{P}_1(x, y, z)+P(x, y)\,\bar{P}_2(x, 1, z)). \tag{8}$$

Setting $y=1$ and solving for $\bar{P}_2(x, 1, z)$, we obtain

$$\bar{P}_2(x, 1, z)=(1-4x)^{-1/2}\,x^{-1}P(x, z)\,\bar{P}_1(x, 1, z). \tag{9}$$

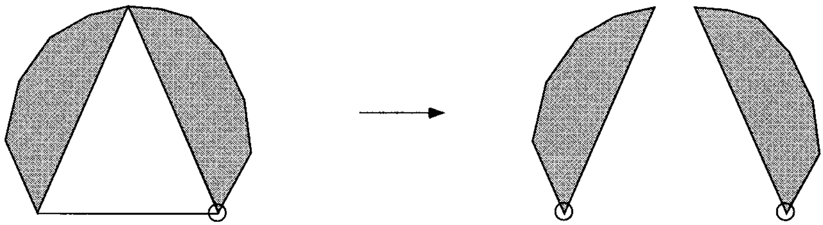

FIG. 2. Decomposition of a rooted triangulation of a polygon into two triangulations.

It is easy to see that $2t(x/4)$ satisfies the conditions of Lemma 4, and hence $|2t(x/4) \leqslant 1$. It follows from (6)–(9) that $P(x/4, 2y)$ and $\bar{P}(x/4, 2y, 2z)$ are analytic in a region $\mathscr{R}(\varepsilon, \phi)$ required in Definition 2 and

$$P(x/4, 2y) = O\left(\frac{1}{1-|y|}\right). \tag{10}$$

We claim that

$$P(x/4, 2y) \approx -\tfrac{1}{8}(1-x)^{1/2}(1-y)^{-2}. \tag{11}$$

For this we let

$$E(x, y) = P(x/4, 2y) - (-y^2(1-x)^{1/2}(1-y)^{-2}/8 + x^2y(1-y)^{-1}/8).$$

It follows from (10) that

$$E(x, y) = O((1-|y|)^{-2}), \qquad \text{for} \quad (x, y) \in \mathscr{R}(\varepsilon, \phi).$$

We also have from (6) that, as $(1-x)(1-y)^{-2} \to 0$ and $(x, y) \in \mathscr{R}(\varepsilon, \phi)$,

$$\begin{aligned} P(x/4, 2y) &= \frac{x^2 y}{8(1-y+y(1-x)^{1/2})} \\ &= \frac{x^2 y}{8(1-y)} \frac{1}{1+y(1-x)^{1/2}(1-y)^{-1}} \\ &= -y^2(1-y)^{-2}(1-x)^{1/2}/8 + x^2 y(1-y)^{-1}/8 \\ &\quad + O((1-x)(1-|y|)^{-3}), \end{aligned}$$

that is

$$E(x, y) = O((1-x)(1-|y|)^{-3}).$$

Hence

$$E(x, y) = \tilde{O}((1-x)(1-y)^{-3}).$$

It is easy to see that $P(x/4, 2y)$ satisfies the conditions (i) and (iii) in Definition 2, and hence (11) follows.

Similarly, we have from (6)–(8), that

$$\bar{P}(x/4, 2y, 2z) \approx \tfrac{1}{16}(1-x)^{-1/2}(1-y)^{-2}(1-z)^{-2}. \tag{12}$$

By Lemmas 2 and 3, we have

$$P_{n,k} = 4^n 2^{-k}[x^n y^k]\, P(x/4, 2y) = \frac{1}{16\sqrt{\pi}} k 2^{-k} n^{-3/2} 4^n (1 + O(1/k)),$$

$$P_{n,k,k} = 4^n 2^{-2k}[x^n y^k z^k]\, \bar{P}(x/4, 2y, 2z) = \frac{1}{16\sqrt{\pi}} k^2 2^{-2k} n^{-1/2} 4^n (1 + O(1/k)),$$

for $k = O(\log n)$, and

$$P_{n,k} = O((2-\varepsilon')^{-k}\, n^{-3/2} 4^n)$$

for all k, n. Now the lemma follows from Lemma 1. ∎

Next, define $M_{n,k}$ and $M_{n,k,l}$ analogous to $P_{n,k}$ and $P_{n,k,l}$ but for all rooted maps. In the following, ζ_k refers to the number of vertices of degree k in a random rooted map.

Lemma 6. *Let ε' be any constant satisfying $0 < \varepsilon' < 6/5$. We have*

(i)

$$\mathbf{E}(\zeta_k) = O(n(6/5-\varepsilon')^{-k}) \qquad \textit{for all } k, n;$$

(ii)

$$\mathbf{E}(\zeta_k) = \frac{1}{\sqrt{10\pi}} k^{-1/2} n (6/5)^{-k}(1 + O(1/k))$$

and

$$\mathbf{E}(\zeta_k(\zeta_k - 1)) = (\mathbf{E}(\zeta_k))^2 (1 + O(1/k))$$

for $k = O(\log n)$.

Proof. Define generating functions

$$M(x, y) = \sum M_{n,k} x^n y^k, \qquad \bar{M}(x, y, z) = \sum M_{n,k,l} x^n y^k z^l.$$

Define $X=(1-12x)^{1/2}$ with the determination being positive for $x<1/12$, and

$$f(x)=6/(X+5), \tag{13}$$

$$\begin{aligned} A(x, y) &= (1-(5+X)\, y/6) \\ &\quad \times (1-y/3-5y^2/12+(y/3+y^2/6)\, X+y^2X^2/4)^{1/2} \\ &= (1-(5+X)\, y/6)(1-yf_1(x))^{1/2}\,(1-yf_2(x))^{1/2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_1(x) &= \frac{(3X+5)}{8(1+X/2)^{1/2}\,(1-X)^{-1/2}-2}, \\ f_2(x) &= \frac{(3X+5)}{8(1+X/2)^{1/2}\,(1-X)^{-1/2}+2}. \end{aligned} \tag{15}$$

It was shown in [3] that (noting there is a typographical error in (4.12): M and $\bar{M}$ are equal to S and D respectively, by duality)

$$M(x, y)=(A(x, y)-1+y-xy^2)/(2xy^2(y-1)), \tag{16}$$

$$\bar{M}(x, 1, z)=\frac{(f-1)\, z}{z-f}\,(zM(x, z)-fM(x, f)), \tag{17}$$

and

$$\bar{M}(x, y, z)=A(x, y)^{-1} \times \left(xy\bar{M}(x, 1, z)-\frac{(y-1)\, xyz}{z-y}\,(zM(x, z)-yM(x, y))\right). \tag{18}$$

To apply Lemmas 2 and 3, we need to show that $|f_1(x/12)|\leqslant 6/5$ and $|f_2(x/12)|\leqslant 6/5$ for $x\in \Delta_x(\varepsilon, \phi)$.

Since

$$\operatorname{Re}((1+X/2)^{1/2}\,(1-X)^{-1/2})\geqslant 0,$$

it is clear that

$$|f_2(x/12)|\leqslant |f_1(x/12)| \tag{19}$$

for $x\in \Delta_x(\varepsilon, \phi)$. We now

Claim. $|f_1(x/12)|\leqslant 6/5$ for $x\in \Delta(\varepsilon, \phi)$.

To see this, we set

$$w=x^{1/2}=t(1-t^2)^{1/2}\,(1+2t^2)^{-1}$$

so that t is positive for small positive w. It follows from (16) that

$$f_1/w = (2+t)(1-t^2)^{-1/2}.$$

By Lagrange's inversion formula, we see that f_1/w has a power series expansion in w with positive coefficients. We also have from (16) that

$$6f_1/5 = 1 - (2/5)\, X + O(X^2)$$

for x near 1/12. Now the claim follows from Lemma 4.

Noting that the factors $z-y$ and $1-(5+X)\, y/6$ in the denominator of (18) do not cause any singularity in $\bar{M}(x, y, z)$, we have, from (13)–(18), that

$$A(x/12, 6y/5) \approx (8/625)(8/5)^{-1/2}\,(1-x)^{3/2}\,(1-y)^{-3/2},$$

$$M(x/12, 6y/5) \approx (4/15)(8/5)^{-1/2}\,(1-x)^{3/2}\,(1-y)^{-3/2},$$

and

$$\bar{M}(x/12, 6y/5, 6z/5) \approx -(1/10)(1-x)^{1/2}\,(1-y)^{-3/2}\,(1-z)^{-1/2},$$

as $(1-x)(1-y)^{-2} \to 0$ and $(1-x)(1-z)^{-2} \to 0$.

Applying Lemmas 2 and 3, we obtain

$$M_{n,k} = 12^n(5/6)^k\,[x^n y^k]\, M(x/12, 6y/5)$$
$$= \frac{(2/5)^{1/2}}{3\Gamma(3/2)\,\Gamma(-3/2)}\, k^{1/2} n^{-5/2} (5/6)^k\, 12^n (1+O(1/k)), \tag{20}$$

$$M_{n,k,k} = 12^n(5/6)^{2k}\,[x^n y^k z^k]\, \bar{M}(x/12, 6y/5, 6z/5)$$
$$= \frac{-1}{10\Gamma(3/2)\,\Gamma(1/2)\,\Gamma(-1/2)}\, n^{-3/2} (5/6)^{2k}\, 12^n (1+O(1/k)), \tag{21}$$

for $k = O(1/\log n)$, and

$$M_{n,k} = O((6/5-\varepsilon')^{-k}\, n^{-5/2} 12^n). \tag{22}$$

Now the lemma follows from (20)–(22), Lemma 1 and the fact

$$M_n = \frac{2}{\sqrt{\pi}}\, n^{-5/2} 12^n (1+O(1/n)) \qquad \text{(see e.g. [3, (5.2)])}.$$

4. PROOF OF THEOREMS 1 AND 2

Theorems 1 and 2 will follow immediately from Lemmas 5, 6 and the following.

LEMMA 7. *Let $\zeta_k = \zeta_k(n)$ be the number of vertices of degree k in a random map with n edges from a given family, and $\mu_k = \mu_k(n) = \mathbf{E}(\zeta_k)$. Let α, β and δ be constants with $\alpha > 0$, $\delta > 0$. Suppose*

(i)

$$\mu_k = O(ne^{-\delta' k})$$

for some positive constant δ', and

(ii)

$$\mu_k = \alpha k^{\beta} n e^{-\delta k}(1 + O(1/k)), \qquad \mathbf{E}(\zeta_k(\zeta_k - 1)) = \mu_k^2(1 + O(1/k))$$

for $k = O(\log n)$.

Then

$$\mathbf{E}(\Delta_n) = \frac{\log n + \beta \log \log n}{\delta} + O(1), \qquad \mathbf{V}(\Delta_n) = O(\log n),$$

and

$$\mathbf{P}(|\Delta_n - (\log n + \beta \log \log n)/\delta| \leqslant \Omega_n) = 1 - O(1/\log n + \exp(-\delta \Omega_n))$$

for any function $\Omega_n \to \infty$.

Proof. it is clear that we only need to consider the case where $\Omega_n = O(\log \log n)$. Put

$$\kappa(n) = \lceil (\log n + \beta \log \log n)/\delta \rceil,$$

and $k = \kappa(n) - \lfloor \Omega_n \rfloor$. We have, from (ii) that, for some positive constant c and for n sufficiently large,

$$\mu_k > c(\log n)^{\beta}\, n \exp(-\log n - \beta \log \log n + \delta \Omega_n) = c \exp(\delta \Omega_n),$$

and

$$\mathbf{V}(\zeta_k) = \mathbf{E}(\zeta_k(\zeta_k - 1)) + \mu_k - \mu_k^2 = O(1/k + 1/\mu_k)\, \mu_k^2.$$

Hence

$$\mathbf{V}(\zeta_k) = O(1/\log n + \exp(-\delta\Omega_n))\, \mu_k^2.$$

By Chebyshev's inequality,

$$\mathbf{P}(\zeta_k = 0) \leqslant \mathbf{V}(\zeta_k)/\mu_k^2 = O(1/\log n + \exp(-\delta\Omega_n)),$$

and hence

$$\begin{aligned}
&\mathbf{P}(\Delta_n > (\log n + \beta \log\log n)/\delta - \Omega_n) \\
&\quad \geqslant \mathbf{P}(\zeta_k \geqslant 1) \\
&\quad \geqslant 1 - O(1/\log n + \exp(-\delta\Omega_n)).
\end{aligned} \tag{23}$$

On the other hand, we have, for $k' = \kappa(n) + \lfloor \Omega_n \rfloor - 1$ and $C > 5/\delta'$, that

$$\begin{aligned}
\sum_{k \geqslant k'} \mu_k &= \sum_{k=k'}^{C \log n} \mu_k + O\left(\sum_{k > C \log n} n \exp(-\delta' k) \right) \qquad \text{(by (i))} \\
&= O\left((\log n)^\beta\, n \sum_{k \geqslant k'} \exp(-\delta k) \right) + O(1/n^4) \qquad \text{(by (ii))} \\
&= O(\log n)^\beta\, n \exp(-\log n - \beta \log\log n - \delta\Omega_n) + O(1/n^4) \\
&= O(\exp(-\delta\Omega_n) + 1/n^4).
\end{aligned}$$

By Markov's inequality,

$$\mathbf{P}\left(\sum_{k \geqslant k'} \zeta_k \geqslant 1 \right) \leqslant \sum_{k \geqslant k'} \mu_k = O(\exp(-\delta\Omega_n) + 1/n^4),$$

and hence

$$\begin{aligned}
&\mathbf{P}(\Delta_n \geqslant (\log n + \beta \log\log n)/\delta + \Omega_n) \\
&\quad \leqslant \mathbf{P}\left(\sum_{k \geqslant k'} \zeta_k \geqslant 1 \right) \\
&\quad = O(\exp(-\delta\Omega_n) + 1/n^4).
\end{aligned} \tag{24}$$

The last part of the lemma follows from (23) and (24). The required lower bound for $\mathbf{E}(\Delta_n)$ can be obtained by noting

$$\mathbf{E}(\Delta_n) = \sum_{k \geqslant 1} \mathbf{P}(\Delta_n \geqslant k) \geqslant k_0 \mathbf{P}(\Delta_n \geqslant k_0) + \sum_{k_0 + 1 \leqslant k \leqslant \kappa(n)} \mathbf{P}(\Delta_n \geqslant k)$$

for $k_0 = \lfloor (2/3\delta) \log n \rfloor$ and using (23). The upper bound comes from

$$\mathbf{E}(\Delta_n) \leqslant \kappa(n) + \sum_{k=\kappa(n)+1}^{2n} \mathbf{P}(\Delta_n \geqslant k)$$

and (24).

To obtain the required bound for $\mathbf{V}(\Delta_n)$, we note

$$\begin{aligned}
\mathbf{V}(\Delta_n) &= \sum_{1 \leqslant k \leqslant 2n} (k - \mathbf{E}(\Delta_n))^2 \, \mathbf{P}(\Delta_n = k) \\
&\leqslant \sum_{1 \leqslant k \leqslant \kappa(n) - \sqrt{\log n}} (k - \mathbf{E}(\Delta_n))^2 \, \mathbf{P}(\Delta_n = k) \\
&\quad + \sum_{\kappa(n) - \sqrt{\log n} \leqslant k \leqslant \kappa(n)} (k - \mathbf{E}(\Delta_n))^2 \, \mathbf{P}(\Delta_n = k) \\
&\quad + \sum_{\kappa(n) \leqslant k \leqslant 2n} (k - \mathbf{E}(\Delta_n))^2 \, \mathbf{P}(\Delta_n = k) \\
&\leqslant O((\log n)^2 \, (1/\log n)) \qquad \text{by (23)} \\
&\quad + O((\log n)^2 \, (1/\log n)) \\
&\quad + O\left(\sum_{0 \leqslant j \leqslant 2n} j^2 (1/n^4 + \exp(-\delta j)) \right) \qquad \text{by (24).}
\end{aligned}$$

This completes the proof of Lemma 7. ∎

5. HIGHER MOMENTS

In this section we derive formulae for the higher joint factorial moments of the variables counting the number of vertices of given degree in a random polygon or map. The lemmas proved here are used in the next section to determine the distribution of the maximum degree in each case.

Consider rooted triangulations of a polygon with a set I of distinguished vertices other than the two ends of the root edge. Let $P(x, y_1, y_2, z_I)$ be the generating function of such triangulations, with x marking the number of vertices, y_1 marking the root vertex degree, y_2 marking the degree of the other end-vertex of the root edge, and z_i marking the degree of the ith distinguished vertex. For convenience, we use $P(x, y_1, y_2)$ to denote $P(x, y_1, y_2, \varnothing)$, and include the term $x^2 y_1 y_2$ to cope with the degenerate triangulation which is just a single edge. We first prove the following results on the behaviour of the generating functions.

LEMMA 8. *For* $|I| = m \geqslant 1$,

$$P(x/4, 2y_1, 2y_2, 2z_I) = \tilde{O}\left((1-x)^{1/2-|I|}(1-y_1)^{-2}(1-y_2)^{-2}\prod_{i\in I}(1-z_i)^{-2}\right),$$

$$P(x/4, 2y, 1, 2z_I) = P(x/4, 1, 2y, 2z_I) \approx a_I(1-x)^{1/2-|I|}(1-y)^{-2}\prod_{i\in I}(1-z_i)^{-2},$$

and

$$P(x/4, 1, 1, 2z_I) \approx a_I(1-x)^{1/2-|I|}\prod_{i\in I}(1-z_i)^{-2},$$

with

$$a_I = a_m = \frac{(2m-2)!}{(m-1)!}4^{-m-1}.$$

Proof. We have from (6) and (7) that

$$P(x, y_1, y_2) = x^2 y_1 y_2 + \frac{x^3 y_1^2 y_2^2}{(1-y_1 t)(1-y_2 t)}, \tag{25}$$

where t is defined in (5). Hence

$$\begin{aligned} P(x/4, 2y, 1) &= P(x/4, 1, 2y) \\ &= \tfrac{1}{8}y(1-y)^{-1} - \tfrac{1}{8}y^2(1-y)^{-2}(1-x)^{1/2} \\ &\quad + \tilde{O}((1-x)(1-y)^{-3}), \end{aligned} \tag{26}$$

and

$$P(x/4, 1, 1) = \tfrac{1}{4} - \tfrac{1}{4}(1-x)^{1/2} - \tfrac{1}{4}(1-x) + \tfrac{1}{4}(1-x)^{3/2}. \tag{27}$$

Let I be a nonempty set and consider triangulations with distinguished vertices indexed by I. We use $J \subset I$ to mean $J \subseteq I$, $J \neq \emptyset, I$. For $J \subseteq I$ and $J_i \subseteq I - \{i\}$, we define $\bar{J} = I - J$ and $\bar{J}_i = I - \{i\} - J_i$. Let v be the vertex adjacent to both ends of the root edge.

Case 1. v is not a distinguished vertex. The contribution from this case is

$$x^{-1} y_1 y_2 \sum_{J \subseteq I} P(x, y_1, 1, z_J)\, P(x, y_2, 1, z_{\bar{J}}).$$

Case 2. v is a distinguished vertex. The contribution from this case is

$$x^{-1}y_1y_2\sum_{i\in I}\sum_{J_i\subseteq I-\{i\}} P(x, y_1, z_i, z_{J_i})\, P(x, y_2, z_i, z_{\bar{J}_i}).$$

Therefore we have, for $I\neq\emptyset$,

$$\begin{aligned}P(x, y_1, y_2, z_I) &= x^{-1}y_1y_2\Bigg(P(x, y_1, 1)\, P(x, y_2, 1, z_I)+P(x, y_2, 1)\, P(x, y_1, 1, z_I)\\ &\quad+\sum_{J\subset I} P(x, y_1, 1, z_J)\, P(x, y_2, 1, z_{\bar{J}})\\ &\quad+\sum_{i\in I}\sum_{J_i\subseteq I-\{i\}} P(x, y_1, z_i, z_{J_i})\, P(x, y_2, z_i, z_{\bar{J}_i})\Bigg).\end{aligned} \tag{28}$$

Setting $y_2=1$ in (28) and solving for $P(x, y_1, 1, z_I)$, we obtain

$$\begin{aligned}P(x, y_1, 1, z_I) = (1-ty_1)^{-1}\, x^{-1}y_1\Bigg(&P(x, y_1, 1)\, P(x, 1, 1, z_I)\\ &+\sum_{J\subset I} P(x, y_1, 1, z_J)\, P(x, 1, 1, z_{\bar{J}})\\ &+x^{-1}y_1\sum_{i\in I}\sum_{J_i\subseteq I-\{i\}} P(x, y_1, z_i, z_{J_i})\, P(x, 1, z_i, z_{\bar{J}_i})\Bigg).\end{aligned} \tag{29}$$

Setting $y_1=y_2=1$ in (28), solving for $P(x, 1, 1, z_I)$, and noting

$$1-2P(x, 1, 1)/x=(1-4x)^{1/2},$$

we obtain

$$\begin{aligned}P(x, 1, 1, z_I) = (1-4x)^{-1/2}\Bigg(&x^{-1}\sum_{J\subset I} P(x, 1, 1, z_J)\, P(x, 1, 1, z_{\bar{J}})\\ &+x^{-1}\sum_{i\in I}\sum_{J_i\subseteq I-\{i\}} P(x, 1, z_i, z_{J_i})\, P(x, 1, z_i, z_{\bar{J}_i})\Bigg).\end{aligned} \tag{30}$$

Now we can prove the lemma by induction on $|I|$ using (28)–(30). It is helpful to note that $P(x, y_1, y_2, z_I)$ is symmetric in y_1 and y_2. When $|I|=1$, we obtain from (26)–(30) that

$$\begin{aligned}P(x/4, 1, 1, 2z) &= (1-x)^{-1/2}\,(x/4)^{-1}\, P^2(x/4, 1, 2z)\\ &\approx \tfrac{1}{2}(1-z)^{-2}\,(1-x)^{-1/2},\end{aligned} \tag{31}$$

$$\begin{aligned}P(x/4, 2y, 1, 2z) &= P(x/4, 1, 2y, 2z)\\ &\approx 8(1-x)^{-1/2}\,(1-y)^{-2}\,(1-z)^{-2},\end{aligned} \tag{32}$$

and

$$P(x/4, 2y_1, 2y_2, 2z) = \tilde{O}((1-x)^{-1/2}(1-y_1)^{-2}(1-y_2)^{-2}(1-z)^{-2}). \tag{33}$$

Hence the lemma holds for $|I|=1$. for $|I|\geqslant 2$, assume that the lemma holds for all proper subsets of I. It is important to note that, in (28) and (29), only the first term is dominant. In (30), only the first summation is dominant. By (30), we have

$$P(x/4, 1, 1, 2z_I) \approx a_I(1-x)^{1/2-|I|}\prod_{i\in I}(1-z_i)^{-2},$$

with

$$a_I = 4\sum_{J\subset I} a_J a_{\bar{J}} = 4\sum_{j=1}^{m-1}\binom{m}{j} a_j a_{m-j}. \tag{34}$$

By (29), we have

$$\begin{aligned} P(x/4, 2y, 1, 2z_I) &= P(x/4, 1, 2y, 2z_I) \\ &\approx a_I(1-x)^{1/2-|I|}(1-y)^{-2}\prod_{i\in I}(1-z_i)^{-2}. \end{aligned}$$

By (28), we have

$$\begin{aligned} &P(x/4, 2y_1, 2y_2, 2z_I) \\ &\quad = \tilde{O}\left((1-x)^{1/2-|I|}(1-y_1)^{-2}(1-y_2)^{-2}\prod_{i\in I}(1-z_i)^{-2}\right). \end{aligned}$$

To derive the desired expression for a_m, we introduce the exponential generating function

$$a(x) = \sum_{m\geqslant 1} a_m u^m/m!.$$

It follows from (34) and the initial condition $a_1 = 1/16$, that

$$a(x) - x/16 = 4a^2(x).$$

Hence

$$a(x) = \frac{1-\sqrt{1-x}}{8},$$

and

$$a_m = m!\,[x^m]\,a(x) = \frac{(2m-2)!}{(m-1)!}\,4^{-m-1}. \quad ∎$$

LEMMA 9. *For random rooted triangulations of an n-gon, defining* $\mu_k = \mathbf{E}(\zeta_k)$ *we have*

$$\mathbf{E}([\zeta_{k_1}]_{m_1}\,[\zeta_{k_2}]_{m_2}\cdots[\zeta_{k_r}]_{m_r}) = \prod_{j=1}^{r} \mu_{k_j}^{m_j}(1+O(1/k_j))$$

provided that $k_j = O(\log n)$, $1 \leqslant j \leqslant r$ *as* $n \to \infty$.

Proof. Let $M_l = \sum_{j=1}^{l} m_j$ and $I = \{1, 2, ..., M_r - 1\}$. We have

$$\begin{aligned} P_n[k_1^{m_1}, ..., k_r^{m_r}] = {} & \left[x^n y_1^{k_1} \prod_{i=1}^{M_1-1} z_i^{k_1} \prod_{i=M_1}^{M_2-1} z_i^{k_2} \cdots \prod_{i=M_{r-1}}^{M_r-1} z_i^{k_r} \right] P(x, y_1, 1, I) \\ & + \left[x^n y_1^{k_1} y_2^{k_1} \prod_{i=2}^{M_1-1} z_i^{k_1} \prod_{i=M_1}^{M_2-1} z_i^{k_2} \cdots \prod_{i=M_{r-1}}^{M_r-1} z_i^{k_r} \right] \\ & P(x, y_1, y_2, I - \{1\}) \\ & + \cdots + \left[x^n y_1^{k_1} y_2^{k_r} \prod_{i=1}^{M_1-1} z_i^{k_1} \prod_{i=M_1}^{M_2-1} z_i^{k_2} \cdots \prod_{i=M_{r-1}}^{M_r-2} z_i^{k_r} \right] \\ & P(x, y_1, I - \{M_r - 1\}). \end{aligned}$$

Applying Lemmas 8, 2 and 3, we have

$$P_n[k_1^{m_1}, ..., k_r^{m_r}] = \frac{a_I}{\Gamma(|I| - 1/2)}\,4^n n^{|I|-3/2} \prod_{j=1}^{r} k_j^{m_j} 2^{-m_j k_j}(1 + O(1/k_j)). \tag{35}$$

Now the lemma follows from Lemma 1 and

$$\frac{a_I}{\Gamma(|I| - 1/2)} = \frac{1}{16\sqrt{\pi}}. \quad ∎$$

Now we turn to all rooted maps. Let I be a finite set. Define generating functions $M(x, y, z_I)$ for rooted maps with distinguished vertices labeled by a set I, with x marking the number of edges, y marking the root vertex degree, z_i marking the degree of the ith distinguished vertex. By duality, $M(x, y, z_I)$ also counts rooted maps by the root face degree and other distinguished face degrees. For convenience, we simply use $M(x, y, I)$ to denote $M(x, y, z_I)$, and use $M(x, y)$ to denote $M(x, y, \varnothing)$.

Let X and A be as defined in Section 3, with A at (14). It follows from [2, (2.2)] that, for $I \neq \emptyset$,

$$\frac{A(x, y)}{xy} M(x, y, I) = y(y-1) \sum_{J \subset I} M(x, y, J)\, M(x, y, \bar{J}) + M(x, 1, I) + \sum_{i \in I} \frac{(1-y)\, z_i}{z_i - y} (z_i M(x, z_i, I-i) - y M(x, y, I-i)). \tag{36}$$

Let

$$M^{(t)}(x, I) = \frac{\partial^t}{(\partial y)^t} M(x, y, I)\bigg|_{y = 6/(X+5)}.$$

We first establish the following.

LEMMA 10. *Let* $|I| = m \geqslant 1$. *Then*

$$M^{(t)}(x/12, 6z_I/5) \approx c_{m,t}(1-x)^{3/4 - t/2 - m} \prod_{i \in I} (1 - z_i)^{-1/2},$$

where $c_{m,t}$ *satisfy the following recursion*

$$\frac{125}{3} \sum_{j=1}^{t} d_j c_{m, t-j} + \sum_{j=0}^{t} \sum_{k=1}^{m-1} \binom{m}{k} \binom{t}{j} c_{k,j} c_{m-k, t-j} = 0, \qquad t \geqslant 1, \quad m \geqslant 2,$$

with the initial values

$$c_{1,t} = \sqrt{5/12}\, [u^t](1 - 25u/18)^{-1/2}.$$

Proof. It follows from [2, Lemma 2] that

$$A^{(1)}(x/12) = \sqrt{2/3}\, (1-x)^{1/2} + O((1-x)^{3/2}),$$

$$A^{(t)}(x/12) \approx d_t (1-x)^{3/4 - t/2},$$

and

$$M^{(t)}(x/12, \emptyset) \approx (125/6)\, d_t (1-x)^{3/4 - t/2},$$

where d_t has the following exponential generating function

$$D(u) = -\sqrt{2/3}\, u \sqrt{1 - 25u/18}. \tag{37}$$

Now we prove the lemma by induction on (m, t). We first deal with the initial case $(m, t) = (1, 0)$. Using recursion (36), with the help of Maple, it is routine to show

$$M(x/12, 6/(X+5), 6z/5) \approx \sqrt{5/12}\,(1-x)^{-1/4}\,(1-z)^{-1/2}.$$

Hence the lemma holds for $(m, t) = (1, 0)$.

For $t \geqslant 1$ and $m = 1$, applying $\partial^{t+1}/(\partial y)^{t+1}$ to both sides of (36), setting $y = 6/(X+5)$, and using the induction hypothesis for smaller values of t, we obtain

$$10(t+1)\,A^{(1)}M^{(t)}(x/12, 6z/5) \approx -10 \sum_{j=2}^{t+1} \binom{t+1}{j} A^{(j)}M^{(t+1-j)}(x/12, 6z/5).$$

Hence

$$M^{(t)}(x/12, 6z/5) \approx c_{1,t}(1-x)^{-1/4-t/2}\,(1-z)^{-1/2},$$

with

$$c_{1,t} = -\sum_{j=2}^{t+1} \binom{t+1}{j} d_j c_{1,t+1-j}. \tag{38}$$

Define the exponential generating function

$$B(u) = \sum_{t \geqslant 1} c_{1,t} u^t/t!.$$

We obtain from (38)

$$D(u)\,B(u) + (\sqrt{10}/6)\,u = 0.$$

Hence

$$B(u) = \sqrt{5/12}\,(1 - 25u/18)^{-1/2}, \tag{39}$$

which establishes the lemma for $m = 1$.

For $|I| = m \geqslant 2$, we apply $\partial^{t+1}/(\partial y)^{t+1}$ to both sides of (36) and set $y = 6/(X+5)$. Noting that the second summation in (36) is negligible, we obtain

$$\frac{-125}{3} A^{(1)} M^{(t)}(x/12, 6z_I/5)$$

$$\approx \sum_{j=2}^{t+1} \binom{t+1}{j} A^{(j)} M^{(t+1-j)}(x/12, 6z_I/5)$$

$$+ \sum_{j=0}^{t+1} \sum_{k=1}^{m-1} \binom{m}{k} \binom{t+1}{j} M^{(j)}(x/12, 6z_J/5)\, M^{(t+1-j)}(x/12, 6z_{\bar{J}}/5).$$

By the induction hypothesis, we obtain

$$M^{(t)}(x/12, 6z_I/5) \approx c_{m,t}(1-x)^{3/4-t/2-m} \prod_{i \in I} (1-z_i)^{-1/2},$$

with $c_{m,t}$ satisfying the desired recursion. ∎

LEMMA 11. *For random rooted maps, defining* $\mu_k = \mathbf{E}(\zeta_k)$ *we have*

$$\mathbf{E}([\zeta_{k_1}]_{m_1} [\zeta_{k_2}]_{m_2} \cdots [\zeta_{k_r}]_{m_r}) = \prod_{j=1}^{r} \mu_{k_j}^{m_j} (1 + O(1/k_j))$$

provided that $k_j = O(\log n)$, $1 \leqslant j \leqslant r$ *as* $n \to \infty$.

Proof. Setting $y = 6/(X+5)$ in (36), applying Lemma 10, and noting that the second summation is negligible, we obtain

$$M(x/12, 1, 6z_I/5) \approx (6/25) \sum_{j=1}^{m-1} \binom{m}{j} c_{j,0} c_{m-j,0} (1-x)^{3/2-m} \prod_{i \in I} (1-z_i)^{-1/2}.$$

To obtain the desired expression for the coefficient, we introduce the exponential generating function

$$C(u, v) = \sum_{m \geqslant 1} \sum_{t \geqslant 0} c_{m,t} \frac{u^t}{t!} \frac{v^m}{m!}.$$

We then have

$$M(x/12, 1, 6z_I/5) \approx (6/25)\, m!\, [v^m]\, C^2(0, v)(1-x)^{3/2-m} \prod_{i \in I} (1-z_i)^{-1/2},$$

and from (36) that

$$M(x/12, 6y/5, 6z_I/5) \approx \tfrac{1}{10} \sqrt{\tfrac{5}{8}}\, (6/25)\, m!\, [v^m]\, C^2(0, v)$$
$$\times (1-x)^{3/2-m} (1-y)^{-3/2} \prod_{i \in I} (1-z_i)^{-1/2}. \quad (40)$$

Using the recursion stated in Lemma 10, we obtain

$$\tfrac{125}{3}\, D(u)(C(u,v) - vB(u)) + C^2(u,v) - C^2(0,v) = 0.$$

We can use the "quadratic method" to solve for $C^2(0, v)$. Rewrite the above equation as

$$\left(C(u,v) + \frac{125D(u)}{6}\right)^2 = -\frac{125}{9}\sqrt{\frac{5}{2}}\,uv + C^2(0,v) + \left(\frac{125D(u)}{6}\right)^2. \tag{41}$$

Let $u = \lambda(v)$ satisfy $C(\lambda(v), v) + 125D(\lambda(v))/6 = 0$. Then we obtain from (41) that

$$-\frac{125}{9}\sqrt{\frac{5}{2}}\,\lambda v + C^2(0,v) + \left(\frac{125D(\lambda)}{6}\right)^2 = 0,$$

$$-\frac{125}{9}\sqrt{\frac{5}{2}}\,v + \frac{d}{d\lambda}\left(\frac{125D(\lambda)}{6}\right)^2 = 0.$$

Treating λ as a parameter, we obtain from (37) and the above equations that

$$v = \frac{125}{6}\sqrt{\frac{8}{5}}\,\lambda\left(1 - \frac{25\lambda}{12}\right),$$

$$C^2(0,v) = \left(\frac{125}{6}\right)^2 \frac{2}{3}\,\lambda^2\left(1 - \frac{25\lambda}{9}\right).$$

Using Lagrange's inversion formula, we obtain

$$\begin{aligned}[v^m]\, C^2(0,v) &= \frac{1}{m}\,[\lambda^{m-1}]\left(\left(\frac{6}{125}\sqrt{\frac{5}{8}}\right)^m (1 - 25\lambda/12)^{-m}\right.\\ &\qquad \left.\times \left(\frac{125}{6}\right)^2 \frac{2}{3}\,(2\lambda - 25\lambda^2/3)\right)\\ &= \frac{4}{3}\,\frac{(2m-4)!}{(m-2)!\,m!}\left(\frac{5}{8}\right)^{m/2}\left(\frac{1}{10}\right)^{m-2}.\end{aligned} \tag{42}$$

Substituting (42) into (40), we obtain

$$\begin{aligned}&M(x/12,\, 6y/5,\, 6z_I/5)\\ &\quad \approx 32\left(\frac{1}{10}\sqrt{\frac{5}{8}}\right)^{m+1} \frac{(2m-4)!}{(m-2)!}\,(1-x)^{3/2-m}\,(1-y)^{-3/2}\prod_{i\in I}(1-z_i)^{-1/2}.\end{aligned}$$

Applying Lemma 3, and using

$$\Gamma(m-3/2)=4^{2-m}\frac{(2m-4)!}{(m-2)!}\sqrt{\pi},$$

we obtain

$$\begin{aligned} M_n[k_1^{m_1},...,k_r^{m_r}] &= \left[x^n y^{k_1}\prod_{i=1}^{M_1-1} z_i^{k_i}\cdots\prod_{i=M_{r-1}}^{M_r-1} z_i^{k_r}\right] M(x,y,z_I)\\ &= \frac{1}{\sqrt{\pi}}k_1 n^{-7/2}12^n\prod_{j=1}^{r}((10\pi k_j)^{-1/2}(5/6)^{k_j}n)^{m_j}(1+O(1/k_j)), \end{aligned}$$

for $k_j=O(\log n)$, $1\leqslant j\leqslant r$, and $n\to\infty$. Now the lemma follows from Lemma 1. ∎

6. PROOF OF THEOREM 3 AND COROLLARIES

The following lemma can be used for both families of maps.

LEMMA 12. *Suppose that $X_1,...,X_n=X_1(n),...,X_n(n)$ is a set of non-negative integer variables on a probability space Λ_n, $n=1,2,...$, and there is a sequence of positive reals $\gamma(n)$ and constants $0<\alpha$, $c<1$ such that*

(i) *$\gamma(n)\to\infty$ and $n-\gamma(n)\to\infty$;*

(ii) *for any fixed r and sequences $k_i(n)$ with $|k_i(n)-\gamma(n)|=O(1)$, for $1\leqslant i\leqslant r$ we have*

$$\mathbf{E}([X_{k_1(n)}]_{m_1}[X_{k_2(n)}]_{m_2}\cdots[X_{k_r(n)}]_{m_r})\sim\prod_{j=1}^{r}\alpha^{(k_j(n)-\gamma(n))m_j},$$

(iii) *$\mathbf{P}(X_{k(n)}>0)=O(c^{k(n)-\gamma(n)})$ uniformly for all $k(n)>\gamma(n)$.*

Then there exists a function $\omega=\omega(n)\to\infty$ (sufficiently slowly as $n\to\infty$) so that the following holds. For $k=\lfloor\gamma-\omega\rfloor$, the total variation distance between the distribution of $(\zeta_k,\zeta_{k+1},...)$, and that of $(Z_k,Z_{k+1},...)$ tends to 0, where the $Z_j=Z_j(n)$ are independent Poisson random variables with $\mathbf{E}Z_j=\alpha^{j-\gamma(n)}$.

Proof. Condition (i) is included merely to ensure the $X_{k_i(n)}$ are defined. From (ii) and [4, p. 23, Theorem 21] it follows that the $X_{k_i(n)}$ for $|k_i(n)-\gamma(n)|=O(1)$ are asymptotically independent Poisson random variables with expectations asymptotic to $\alpha^{k_i(n)-\gamma(n)}$. This in turn implies that the same holds provided $|k_i(n) - \gamma(n)| < \omega(n)$ for some sufficiently slowly growing function $\omega(n) \to \infty$. Since (iii) implies $\mathbf{P}(\bigwedge_{j \geqslant \gamma + \omega} \{X_j(n) = 0\}) \to 0$, the lemma follows. (The total variation distance tends to 0 if $\max_A \{\mathbf{P}((\zeta_k, \zeta_{k+1}, ...) \in A) - \mathbf{P}((Z_k, Z_{k+1}, ...) \in A)\} \to 0$; see Barbour *et al.* [1] for much more on Poisson approximation.) ∎

Theorem 3 follows from Lemmas 5, 9, 6, 11 and 12.

To obtain Corollary 1, note that with Z_j as in Lemma 12 and k an integer with $|k - \gamma(n)|$ bounded,

$$\mathbf{P}\left(\bigwedge_{j \geqslant k} \{Z_j(n) = 0\}\right) \sim \prod_{j=k}^{\infty} \exp(-\alpha^{j-\gamma(n)}) = \exp(-\alpha^{k-\gamma(n)}/(1-\alpha))$$

from which the corollary follows.

Corollary 2 is obtained similarly, noting that $\mathbf{P}(\Delta_n < \gamma - \omega) \to 0$ from Corollary 1, and so the probability there is a unique vertex with maximum degree is asymptotic to

$$\begin{aligned}\sum_{k \geqslant \gamma - \omega} \mathbf{P}(\Delta_n = k \wedge \zeta_k = 1) &\sim \sum_{k \geqslant \gamma - \omega} \mathbf{P}(\{\max\{i : Z_i > 0\} = k\} \wedge \{Z_k = 1\}) \\ &= \sum_{k=-\infty}^{\infty} \alpha^{k-\gamma} \exp(-\alpha^{k-\gamma}/(1-\alpha)).\end{aligned}$$

REFERENCES

1. A. D. Barbour, L. Holst, and S. Janson, "Poisson Approximation," Clarendon, Oxford, 1992.
2. E. A. Bender and E. R. Canfield, The asymptotic number of rooted maps on a surface, *J. Combin. Theory, Ser. A* **43** (1986), 244–257.
3. E. A. Bender, E. R. Canfield, and R. W. Robinson, The enumeration of maps on the torus and the projective plane, *Canad. Math. Bull.* **31** (1988), 257–271.
4. B. Bollobás, "Random Graphs," Academic Press, London, 1985.
5. W. G. Brown, Enumeration of triangulations of the disk, *Proc. London Math. Soc.* **14** (1964), 746–768.
6. L. Devroye, P. Flajolet, F. Hurtado, M. Noy, and W. Steiger, Properties of Random Triangulations and Trees, *Discrete and Computational Geometry* (to appear).
7. P. Flajolet and A. Odlyzko, Singularity Analysis of generating functions, *SIAM J. Disc. Math.* **3** (1990), 216–240.
8. Z. C. Gao and L. B. Richmond, Root vertex valency distributions of rooted maps and rooted triangulations, *Europ. J. Combin.* **15** (1994), 483–490.

9. Z. C. Gao and N. C. Wormald, The size of the largest components in random planar maps, *SIAM J. Disc. Math.* **12** (1999), 217–228.
10. B. D. Hughes, "Random Walks and Random Environments," Clarendon, Oxford, 1995.
11. F. W. J. Olver, "Asymptotics and Special Functions," Academic, New York, 1974.
12. L. B. Richmond and N. C. Wormald, Almost all maps are asymmetric, *J. Combin. Theory Ser. B* **63** (1995), 1–7.