THE DISTRIBUTION OF THE RANGE[1]

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1. Summary. The asymptotic distribution of the range w for a large sample taken from an initial unlimited distribution possessing all moments is obtained by the convolution of the asymptotic distribution of the two extremes. Let α and u be the parameters of the distribution of the extremes for a symmetrical variate, and let $R = \alpha(w-2u)$ be the reduced range. Then its asymptotic probability $\Psi(R)$ and its asymptotic distribution $\psi(R)$ may be expressed by the Hankel function of order one and zero. A table is given in the text.

The asymptotic distribution $g(w)$ of the range proper is obtained from $\psi(R)$ by the usual linear transformation. The initial distribution and the sample size influence the position and the shape of the distribution of the range in the same way as they influence the distribution of the largest value. If we take the parameters from the calculated means and standard deviations, the asymptotic distribution of the range gives a good fit to the calculated distributions for normal samples from size 6 onward. Consequently the distribution of the range for normal samples of any size larger than 6 may be obtained from the asymptotic distribution of the reduced range.

The asymptotic probabilities and the asymptotic distributions of the mth range and of the range for asymmetrical distributions are obtained by the same method and lead to integrals which may be evaluated by numerical methods.

2. Introduction. For any initial distribution, and any sample size n, the distribution of the range may easily be written down in the form of an integral. However, for many given initial distributions the integration can be carried out—if at all—only for very small sample sizes, say $n = 2$ or $n = 3$. For larger samples, complicated numerical calculations have to be made, and there is no way of obtaining the distribution for $n + 1$ observations from the distribution for n observations.

Our object is to obtain the *asymptotic* distribution of the range. Nothing is supposed to be known about the initial distribution, except that it is of the exponential type [9] which assures that it is unlimited in both directions, and possesses all moments. It will be shown that this condition is sufficient for the existence of an asymptotic distribution of the range.

With increasing samples sizes the distribution of the range may approach its asymptotic form in a quick, or in a slow way. This behavior depends upon the nature of the initial distribution. Two examples for this approach will be shown.

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3. The exact distribution of the range. Let $\varphi(x)$ be any initial distribution, $\Phi(x)$ the probability of a value equal to, or less than, x. Then, for samples of size n, the joint distribution $\mathfrak{w}_n(x_1, x_n)$ of the smallest value x_1 and the largest value x_n is

$$\mathfrak{w}_n(x_1, x_n) = n(n-1)\varphi(x_1)(\Phi(x_n) - \Phi(x_1))^{n-2}\varphi(x_n). \tag{1}$$

The distribution $g_n(w_n)$ of the range w_n defined by

$$x_n = x_1 + w_n \tag{2}$$

is obtained by integrating over all values $x_1 \leqq x_n$ whence

$$g_n(w_n) = n(n-1) \int_{-\infty}^{+\infty} (\Phi(x + w_n) - \Phi(x))^{n-2} \varphi(x + w_n)\varphi(x)\, dx, \tag{3}$$

where the index 1 has been dropped. The probability $G_n(w_n)$ for the range to be equal to, or less than, w_n is obtained by integration of (3), whence, by reversing the order of integration,

$$G_n(w_n) = n \int_{-\infty}^{+\infty} \int_0^{w_n} (n-1)(\Phi(x + w_n) - \Phi(x))^{n-2}\, d\Phi(x + w_n)\, d\Phi(x),$$

or, after integration,

$$G_n(w_n) = n \int_0^1 (\Phi(x + w_n) - \Phi(x))^{n-1}\, d\Phi,$$

a formula to which Prof. H. Hotelling has drawn my attention. The beauty of this formula is completely marred by the facts that, in general, we cannot express $\Phi(x + w_n)$ by $\Phi(x)$, and that the numerical integration is lengthy and tiresome.

The problem of the range for the normal distribution was first raised twenty five years ago by L. von Bortkiewicz [1, 2]. For $n = 2$ and $n = 3$ the distribution of the normal range may be written down explicitly [12, 13]. For larger normal samples up to $n = 20$, E. S. Pearson [16] and H. O. Hartley [10] have calculated numerical tables of the probability of the range. L. H. C. Tippett [20] has calculated the mean, the standard deviation, and the moment quotients for the range of the normal distribution up to $n = 1000$. He gave formulae for the moments in the form of integrals. Finally "Student" [18] reproduced the distribution of the range for small samples, $n = 2, 3, 4, 5, 6, 10$, by Pearson's type I, and gave a formula for large samples $n = 20, 60$, based on Pearson's type VI, a procedure which is purely empirical and, therefore, unsatisfactory for theoretical purposes. A good resumé of the present knowledge about the range is given in Karl Pearson's Tables [17].

All these studies are confined to the normal distribution and allow no conclusion about the asymptotic distribution of the range. According to Kendall [11] it is not known whether such forms exist and what they are. This question may at once be answered for a special case. If the distribution is limited to the left (or to the right), the asymptotic distribution of the range is equal to the asymptotic distribution of the largest (smallest) value. The asymptotic distribution of the range exists provided that an asymptotic distribution of the largest (smallest) value exists. For the exponential distribution, and for initial distributions of the Pareto type, for example, the asymptotic distribution of the range is equal to the asymptotic distribution of the largest value. The asymptotic distribution of the range for the rectangular distribution has been derived by A. G. Carlton [3].

4. The asymptotic distribution of the reduced range for a symmetrical variate. Instead of the procedures mentioned in the last paragraph, let us consider a large sample. It is generally assumed that the smallest and the largest values are independent in that case. L. H. C. Tippett [20] has shown that the correlation between the extremes is negligible for the normal distribution and for sample sizes $n \geqq 200$. In a previous note [9] it has been shown that independence holds for large samples and for initial distributions of the exponential type unlimited in both directions and possessing all moments. Then the joint distribution (1) splits into the product of the asymptotic distribution $f_1(x_1)$ of the smallest value x_1 and the asymptotic distribution $f_n(x_n)$ of the largest value x_n

$$\mathfrak{w}(x_1, x_n) = f_1(x_1) \cdot f_n(x_n). \tag{4}$$

If, furthermore, the initial distribution is symmetrical about zero, the two asymptotic distributions are

$$f_1(x_1) = \alpha \exp[\alpha(x_1 + u) - e^{\alpha(x_1+u)}]; \; f_n(x_n) = \alpha \exp[-\alpha(x_n - u) - e^{-\alpha(x_n-u)}]. \tag{5}$$

These asymptotic distributions and the corresponding probabilities are traced, in a reduced scale, on Graphs (1) and (2).

Since the two parameters u and α will exist also in the asymptotic distribution of the range, their nature must briefly be explained. The value u is defined as the solution of

$$\Phi(u) = 1 - \frac{1}{n}. \tag{6}$$

Since

$$n(1 - \Phi(u)) = 1, \tag{6'}$$

the largest value u may be called the *expected* largest value. It differs, of course, from the mean of the largest value. It has been shown [6] that u increases as a function of the logarithm of n, the function depending upon the initial distribution.

Criteria for the approach of the distribution of the largest value toward its asymptotic form have been given by R. A. Fisher and L. H. C. Tippett [4].

For our purpose it is sufficient to consider whether n is so large that u is very near to the most probable largest value $\tilde{x}_n$ obtained from

$$\frac{n-1}{\Phi(\tilde{x}_n)}\varphi(\tilde{x}_n) = -\frac{\varphi'(\tilde{x}_n)}{\varphi(\tilde{x}_n)}. \tag{7}$$

If

$$\tilde{x}_n \approx u$$

holds with sufficient approximation, $2u$ may be interpreted as the range of the modes for an initial symmetrical distribution.

The parameter α defined by

$$\alpha = \frac{\varphi(u)}{1-\Phi(u)} \tag{8}$$

also is a function of n. Three cases have to be distinguished: In the first case, α is a constant, or converges with n toward a constant different from zero. In the second (and third) case, α increases with n without limit (decreases with n toward zero). The three cases correspond to three classes of initial distributions of the exponential type. The function α is related to the asymptotic standard error of the largest, and of the smallest value by

$$\alpha^2\sigma_n^2 = \alpha^2\sigma_1^2 = \frac{\pi^2}{6}. \tag{9}$$

If α increases (decreases) with n, or is independent of n, the standard error of the largest value decreases (increases) with the sample size, or is independent of it. This behavior has nothing to do with the fact that the standard error of the mean decreases, of course, with an increasing number of samples.

The determination of the constants u and α from equations (6), (7), (8) is based on the knowledge of the initial distribution and the sample size n from which we take the largest observation. This method cannot be used in many practical applications: 1) It may happen that the initial distribution, or the parameters it contains, are unknown. Therefore the parameters of the largest value cannot be obtained from it. 2) The initial distribution might be known, but the number of observations is insufficient to warrant this procedure, because the most probable largest value $\tilde{x}_n$ differs from the expected value u. In these cases the parameters u and α have to be estimated from the observed distribution of the largest value alone. A similar procedure will be used for the range in paragraph 7.

From (4) and (5) the joint asymptotic distribution $\mathfrak{w}(x, w)$ of the smallest value x_1 and the range w becomes

$$\mathfrak{w}(x_1, w) = \alpha^2\exp[-\alpha(w-2u) - e^{\alpha(x_1+u)} - e^{-\alpha(x_1+w-u)}].$$

The asymptotic distribution $g(w)$ of the range alone is, dropping the index 1,

$$g(w) = \alpha^2 e^{-\alpha(w-2u)}\int_{-\infty}^{+\infty}\exp[-e^{\alpha(x+u)} - e^{-\alpha(x+w-u)}]\,dx. \tag{4'}$$

This distribution contains the two parameters α and u existing in the asymptotic distribution of the largest value. To eliminate the two parameters, a reduced range R is introduced by

$$R = \alpha(w - 2u). \tag{10}$$

The range w is a positive variate unlimited toward the right. The reduced range R is also unlimited toward the right yet limited toward the left by

$$R \geqq -2\alpha u. \tag{10'}$$

The reduced range is not related to one of the averages of the range. It is the range minus the range of the modes divided by a factor which is proportional to the standard error of the extreme value. The distribution $\psi(R)$ of the reduced range R, and the distribution $g(w)$ of the range w are related by

$$\psi(R) = \frac{1}{\alpha} g(w), \tag{11}$$

subject to restriction (10′), whereas the probability $\Psi(R)$ of the reduced range to be equal to, or less than R is equal to the corresponding expression $G(w)$ for the range proper

$$\Psi(R) = G(w). \tag{11'}$$

For the integration in (4′) we put

$$\alpha(x + w - u) = -y$$

whence, from (10),

$$\alpha(x + u) = -y - R.$$

The asymptotic distribution of the reduced range becomes

$$\psi(R) = e^{-R} \int_{-\infty}^{+\infty} \exp[-e^{y} - e^{-y-R}]\, dy \tag{12}$$

and the asymptotic probability $\Psi(R)$ of the range is

$$\Psi(R) = \int_{-\infty}^{+\infty} \exp[y - e^{y} - e^{-y-R}]\, dy \tag{13}$$

an expression which may easily be verified by differentiation.

The asymptotic formulas (12) and (13) hold for any initial symmetrical distribution of the exponential type, for example, for the normal and the logistic distribution (see par. 7). The mean reduced range $\bar{R}$ and the higher moments of the reduced range are easily obtained from the mean $\bar{w}$, the variance σ_w^2, and the invariants λ_ν of order ν of the range proper w given in a previous paper [8]. They are

$$\bar{w} = 2u + \frac{2\gamma}{\alpha}; \qquad \sigma_w^2 = \frac{\pi^2}{3\alpha^2} \tag{14}$$

$$\lambda_\nu = \frac{2(\nu - 1)!}{\alpha^\nu} \sum_{k=1}^{\infty} \frac{1}{k^\nu}; \qquad \nu \geqq 2 \tag{15}$$

where γ stands for Euler's constant.

Consequently the mean $\bar{R}$, the variance σ_R^2 and the invariants λ_ν of the reduced range are

$$\bar{R} = 2\gamma; \quad \sigma_R^2 = \frac{\pi^2}{3}; \quad \lambda_\nu = 2(\nu - 1)! \sum_{k=1}^{\infty} \frac{1}{k^\nu}; \qquad \nu \geqq 2 \tag{16}$$

Equation (14) leads to an interpretation of the reduction (10) which may be written

$$R = \alpha(w - \bar{w}) + 2\gamma$$

or

$$R = \frac{\pi}{\sqrt{3}} \frac{w - \bar{w}}{\sigma_w} + 2\gamma \tag{14'}$$

Thus the transformation (10) is a linear function of the standard transformation $(w - \bar{w})/\sigma_w$ usual in statistics.

5. The probability of the range as a Bessel function. The integrals (12) and (13) may be evaluated by numerical procedures, since tables of the function $\exp(-e^{-y})$ are easily calculated. However, it turned out to be simpler to relate these integrals to the solution of a differential equation. The derivative $\psi'(R)$ of the distribution (12) is

$$\psi'(R) = -\psi(R) + e^{-R} \int_{-\infty}^{+\infty} \exp[-y - R - e^y - e^{-y-R}]\, dy$$

The integral is equal to the probability $\Psi(R)$ since the transformation

$$y + R = -z$$

leads to

$$\int_{-\infty}^{+\infty} \exp[-y - R - e^y - e^{-y-R}]\, dy = \int_{-\infty}^{+\infty} \exp[z - e^{-z-R} - e^z]\, dz$$

Consequently the probability $\Psi(R)$ is subject to the differential equation

$$\Psi'' + \Psi' - e^{-R}\Psi = 0. \tag{17}$$

The mode of the reduced range is a fixed value $\tilde{R}$ such that

(18) $$\psi(\tilde{R}) = e^{-\tilde{R}}\Psi(\tilde{R}).$$

Mr. W. Wasow (Swarthmore College) has drawn my attention to the fact that the probability $\Psi(R)$ of the range can be expressed in terms of a Bessel function.[2] To obtain this simplification of the differential equation we introduce a new positive variable z by

(19) $$z = 2e^{-R/2}$$

and a new function U by

(20) $$\Psi = U \cdot z.$$

The boundary conditions are

(21) $$z = 0, \Psi = 1; \qquad z = \infty; \qquad \Psi = 0.$$

The first derivative becomes, from (19)

$$\frac{d\Psi}{dR} = -\frac{z}{2}\frac{d\Psi}{dz}$$

whence, from (20)

$$\frac{d\Psi}{dR} = -\frac{z}{2}\left(U + z\frac{dU}{dz}\right).$$

The second derivative becomes, by the same procedure

$$\frac{d^2\Psi}{dR^2} = -\frac{z}{2}\frac{d}{dz}\left(-\frac{zU}{2} - \frac{z^2}{2}\frac{dU}{dz}\right).$$

The second member may be written

$$\frac{z}{2}\left(\frac{U}{2} + \frac{3z}{2}\frac{dU}{dz} + \frac{z^2}{2}\frac{d^2U}{dz^2}\right) = \frac{zU}{4} + \frac{3z^2}{4}\frac{dU}{dz} + \frac{z^3}{4}\frac{d^2U}{dz^2}.$$

Thus the differential equation (17) is now

$$\frac{z^3}{4}\frac{d^2U}{dz^2} + \frac{3z^2}{4}\frac{dU}{dz} - \frac{z^2}{2}\frac{dU}{dz} + \frac{zU}{4} - \frac{zU}{2} - \frac{z^3}{4}U = 0.$$

Multiplication by $4z^{-1}$ leads to

(21′) $$z^2\frac{d^2U}{dz^2} + z\frac{dU}{dz} - (z^2 + 1)\,U = 0.$$

This is one of the classical Bessel differential equations of order 1. In the notation used by the British Tables [14] (pp. 264 and 213) one of the solutions is

(22) $$U(z) = K_1(z),$$

[2] I profit of this occasion to thank him for this and other valuable suggestions.

where $K_1(z)$, the modified Bessel function of the second kind (Hankelfunction) is defined by

$$(23)\qquad \mathrm{K}_1(z) = (\gamma - \lg 2 + \lg z) \sum_0^\infty \frac{1}{\nu!(\nu+1)!}\left(\frac{z}{2}\right)^{2\nu+1} + \frac{1}{z} - \sum_1^\infty \frac{1}{(\nu-1)!\,\nu!}\left(\frac{z}{2}\right)^{2\nu-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{\nu} - \frac{1}{2\nu}\right).$$

The relation between the functions $\mathrm{K}_\nu(z)$ and the Hankelfunction $H_\nu^{(1)}(z)$ is

$$(23a)\qquad \mathrm{K}_\nu(z) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz).$$

The asymptotic probability for the range is, from (20) and (22),

$$(24)\qquad \Psi(R) = zK_1(z)$$

or, from (19)

$$(25)\qquad \Psi(R) = 2e^{-R/2}K_1(2e^{-R/2}).$$

This is the only Bessel function satisfying the boundary conditions (21). The asymptotic probability $\Psi(R)$ of the range may be written finally from (25), (23) and (10)

$$(25a)\qquad 1 - \Psi(R) = \sum_1^\infty \frac{\exp(-R\nu)}{\nu!\,(\nu-1)!}\left(R - 2\gamma + 2S_\nu - \frac{1}{\nu}\right)$$

where

$$S_0 = 0; \quad S_\nu = \sum_{\lambda=1}^{\nu} \frac{1}{\lambda}.$$

The distribution

$$\psi(R) = \frac{d\Psi(R)}{dz} \cdot \frac{dz}{dR}$$

of the reduced range R is, from (24) and (19)

$$\psi(R) = -\frac{z}{2}\,(K_1(z) + zK_1'(z)).$$

Now, the derivative $K_1'(z)$ is linked to the modified Bessel function $K_0(z)$ of the second kind and of order zero by

$$zK_1'(z) = -K_1(z) - zK_0(z).$$

Consequently the distribution is

$$(26)\qquad \psi(R) = \frac{z^2}{2}\,K_0(z)$$

or, from (19),

$$\psi(R) = 2e^{-R}K_0(2e^{-R/2}) \tag{27}$$

where the function $K_0(z)$ is defined by

$$K_0(z) = -(\gamma - \lg 2 + \lg z)\sum_0^\infty \left(\frac{z}{2}\right)^{2\nu}\frac{1}{\nu!\,\nu!} + \sum_1^\infty \left(\frac{z}{2}\right)^{2\nu}\frac{1}{\nu!\,\nu!}\left(1 + \frac{1}{2} + \cdots \frac{1}{\nu}\right). \tag{28}$$

Finally the asymptotic distribution $\psi(R)$ of the reduced range may be written from (27) and (28)

$$\psi(R) = \sum_0^\infty \frac{\exp[-(\nu+1)R]}{\nu!\,\nu!}(R - 2\gamma + 2S_\nu) \tag{28a}$$

We first investigate the analytic behavior and the order of magnitude of the probability $\Psi(R)$ and the distribution $\psi(R)$ for large negative, and large positive values of the reduced range, i.e. for large and small values of the positive variable z. If z is so large that

$$z^{-3} = \frac{e^{(3R/2)}}{8} \ll 1 \tag{29}$$

the expressions for $K_1(z)$ and $K_0(z)$ become [14], p. 271,

$$K_1(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1 + \frac{3}{8z} - \frac{15}{128z^2}\right)$$

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1 - \frac{1}{8z} + \frac{9}{128z^2}\right)$$

The probability $\Psi(R)$ becomes, from (24) and (19),

$$\Psi(R) = \sqrt{\pi}\exp\left[-\frac{R}{4} - 2e^{-(R/2)}\right]\left(1 + \frac{3}{16}e^{R/2} - \frac{15}{512}e^{R}\right). \tag{25'}$$

The condition (29) holds, say, for $R = -4$. The numerical calculation leads, for $\Psi(-4)$, to the order of magnitude 10^{-6}.

In the same way, the distribution $\psi(R)$ becomes, from (26) and (19), for large negative reduced ranges

$$\psi(R) = \sqrt{\pi}\exp\left[-\frac{3R}{4} - 2e^{-(R/2)}\right]\left(1 - \frac{e^{R/2}}{16} + \frac{9}{512}e^{R}\right). \tag{27'}$$

This expression cannot be obtained from (25′) since the approximations for $K_0(z)$ and $K_1(z)$ used do not fulfill the relations between the derivatives given above. The order of magnitude of $\psi(-4)$ is 10^{-5}.

Thus the probability $\Psi(R)$ and the distribution $\psi(R)$ may be neglected for $R \leqq -4$. This removes the importance of the lower limit $R \geqq -2\alpha u$ stated in (10′). If $\alpha u \geqq 2$, the distribution of the range may be dealt with as if it were practically unlimited toward the left.

For large positive reduced ranges to which correspond small values of z, say

$$z^3 = 8e^{-(3R/2)} \ll 1 \tag{29′}$$

the Bessel functions $K_1(z)$ and $K_0(z)$ become, from (23) and (28)

$$K_1(z) = \left(\gamma + \lg\frac{z}{2}\right)\left(\frac{z}{2} + \frac{z^3}{16}\right) + \frac{1}{z} - \left(\frac{z}{4} + \frac{5z^3}{64}\right) \tag{23′}$$

$$K_0(z) = -\left(\gamma + \lg\frac{z}{2}\right)\left(1 + \frac{z^2}{4}\right) + \frac{z^2}{4} + \frac{3z^4}{128}\,. \tag{28′}$$

In this case we are interested to know how far the probability $\Psi(R)$ differs from unity. Consequently we calculate $1 - \Psi(R)$ and obtain, from (24) and (23′)

$$1 - \Psi(R) = -\frac{z^2}{2}\left[\left(\gamma + \lg\frac{z}{2}\right)\left(1 + \frac{z^2}{8}\right) - \frac{1}{2} - \frac{5z^2}{32}\right].$$

The right side becomes, from (19)

$$2e^{-R}\left[\left(-\gamma + \frac{R}{2}\right)\left(1 + \frac{e^{-R}}{2}\right) + \frac{1}{2} + \frac{5}{8}e^{-R}\right]$$
$$= e^{-R}\left[(R - 2\gamma)\left(1 + \frac{e^{-R}}{2}\right) + 1 + \frac{5}{4}e^{-R}\right]$$

or

$$e^{-R}\left[R - 2\gamma + 1 + \frac{e^{-R}}{2}\left(R - 2\gamma + \frac{5}{2}\right)\right] = e^{-R}(R - 2\gamma + 1)\left(1 + \frac{e^{-R}}{2}\right) + \frac{3e^{-2R}}{4}\,.$$

If R is so large that

$$e^{-R} \ll 1$$

we simply have

$$1 - \Psi(R) = e^{-R}(R - 2\gamma + 1)\,. \tag{25″}$$

For example, for $R = 10$, the preceding condition is satisfied and $1 - \Psi(R)$ is of the order 5.10^{-4}.

In the same manner we calculate the density of probability $\psi(R)$ for large reduced ranges. From (26), (19) and (28′) we obtain

$$\psi(R) = 2e^{-R}\left[\left(\frac{R}{2} - \gamma\right)(1 + e^{-R}) + e^{-R} + \frac{3}{8}e^{-2R}\right].$$

By neglecting $e^{-2R} \ll R$, the right side becomes

$$e^{-R}[(R - 2\gamma)(1 + e^{-R}) + 2e^{-R}] = e^{-R}[R - 2\gamma + e^{-R}(R - 2\gamma + 2)]$$

whence

$$\psi(R) = e^{-R}(R - 2\gamma)(1 + e^{-R}) + 2e^{-2R}.$$

In first approximation we obtain

$$\psi(R) = e^{-R}(R - 2\gamma) \tag{27''}$$

a formula which may also be derived directly from (25″). The density of probability is of the order 10^{-4} for $R = 10$.

From the formulae (25′) and (27′) valid for large negative values of R, and from the formulae (25″) and (27″) valid for large positive values of R follow the boundary conditions

$$\lim_{R=-\infty} \frac{\psi(R)}{\Psi(R)} = e^{-(R/2)}; \lim_{R=+\infty} \frac{\psi(R)}{1 - \Psi(R)} = \frac{R - 2\gamma}{R - 2\gamma + 1}$$

For the construction of tables of the distribution $\psi(R)$ and the probability $\Psi(R)$ of the reduced range it is sufficient to consider the interval

$$-3 < R < 10.$$

The two functions $K_1(z)$ and $K_0(z)$ have been tabulated [14] and [19]. Hence the probability and the distribution could be calculated from such tables of the Bessel functions. This procedure, however, was only used to obtain boundary values. The tables I and Ia are based on computations made in the Calculation and Ballistics Department at the Naval Proving Ground Dahlgren by stepwise integration of the differential equation (17) using the special Relay Calculator of the International Business Machines Corporation.[3]

Table I gives the probability $\Psi(R)$ (col. 2) and the distribution $\psi(R)$ (col. 4) for the reduced ranges $-3 \leqq R \leqq 10.5$ in intervals $\Delta R = 0.5$. The differences $\Delta\Psi$ given in col. 3 are taken from the original figures.

For different uses it is necessary to know the reduced range as a function of its probability. This relation is shown in Table Ia. The first column gives the probability, the first line gives the last decimal of this probability, and the cells give the reduced range corresponding to the probability obtained from the combination of the first column and the first line. For example: The reduced range $R = -3.20$ corresponds to the probability $\Psi(R) = 0.0002$, and the reduced range $R = 10.44$ corresponds to the probability $\Psi(R) = 0.9997$.

This table may be used for obtaining the percentage points of the reduced range. The mode $\tilde{R}$, the median $\check{R}$ calculated by the Naval Proving Ground and the mean $\bar{R}$ obtained from (14) and (10) are

$$\tilde{R} = 0.506366440; \quad \check{R} = 0.928597642; \quad \bar{R} = 1.154431330. \tag{30}$$

A probability paper for the range may be constructed in the following way: The observed ranges w are plotted on the vertical axis; the reduced ranges R on a horizontal axis. The abscissa shows the probabilities

$$\Psi(R) = G(w)$$

[3] The author wishes to express his sincere appreciation for the permission to use these computations. The original tables give the probability and the distribution to 8 significant decimal places at intervals $\Delta R = 1/100$. Lack of space prevents the reproduction of these tables.

TABLE I

Asymptotic Probability and Asymptotic Distribution of the Reduced Range

1	2	3	4
Reduced Range R	Probability Ψ (R)	Difference $\Delta\Psi$	Distribution ψ (R)
−3.0	.00050		.00212
		.00274	
−2.5	.00324		.01057
		.01032	
−2.0	.01356		.03386
		.02693	
−1.5	.04048		.07705
		.05251	
−1.0	.09299		.13419
		.08141	
− .5	.17440		.18969
		.10533	
.0	.27973		.22779
		.11821	
.5	.39794		.24075
		.11859	
1.0	.51654		.23021
		.10891	
1.5	.62545		.20346
		.09327	
2.0	.71872		.16898
		.07557	
2.5	.79429		.13360
		.05860	
3.0	.85289		.10157
		.04386	
3.5	.89675		.07483
		.03192	
4.0	.92867		.05375
		.02270	
4.5	.95136		.03783
		.01584	
5.0	.96721		.02618
		.01089	
5.5	.97810		.01787
		.00739	
6.0	.98549		.01205
		.00496	

TABLE I—*Concluded*

1	2	3	4
Reduced Range R	Probability Ψ (R)	Difference ΔΨ	Distribution ψ (R)
6.5	.99045		.00805
		.00330	
7.0	.99375		.00534
		.00218	
7.5	.99594		.00351
		.00143	
8.0	.99737		.00230
		.00093	
8.5	.99830		.00150
		.00061	
9.0	.99891		.00097
		.00039	
9.5	.99930		.00062
		.00025	
10.0	.99955		.00040
		.00016	
10.5	.99972		.00026

corresponding to the reduced ranges R. If the observations follow the theory, the observed ranges are scattered around the straight line

$$w = 2u + \frac{R}{\alpha} \tag{10'}$$

If the samples are drawn simultaneously, and if there is a constant interval of time between the drawings, this interval may be used as unit of time for the construction of the return periods $T(R)$ and ${}_1T(R)$ of a range equal to, or larger than (smaller than) R where

$$T(R) = \frac{1}{1 - \Psi(R)} \; ; \; {}_1T(R) = \frac{1}{\Psi(R)} \; .$$

The first (second) notion applies to the range above (below) the median. The return periods are shown in an upper parallel to the abscissa.

A scheme for this paper is given in Fig. 3. Such a paper will allow a graphical test for the fit of the observed ranges to our theory, and avoids any numerical calculations. Obviously this method may only be used if the initial distribution is symmetrical, unlimited, and of the exponential type, and if the sample size is so large that the asymptotic distribution holds.

6. The range, the midrange, and the extremes. The asymptotic distribution (27) of the reduced range was obtained by convolution of the asymptotic distributions (5) of the extremes. The same method leads to the asymptotic distribution of the reduced midrange [8]

$$v = \alpha(x_1 + x_n). \tag{31}$$

TABLE IA

The Reduced Range R as Function of Its Probability Ψ (R)

Ψ (R)	0	1	2	3	4	5	6	7	8	9
.000	—	*	−3.20	−3.12	−3.05	−3.00	−2.96	−2.92	−2.89	−2.86
.00	—	−2.83	−2.64	−2.52	−2.43	−2.36	−2.30	−2.25	−2.20	−2.16
.0	—	−2.12	−1.84	−1.65	−1.51	−1.39	−1.28	−1.19	−1.10	−1.02
.1	−0.95	−0.88	−0.81	−0.75	−0.69	−0.63	−0.58	−0.52	−0.47	−0.42
.2	−0.37	−0.32	−0.27	−0.22	−0.18	−0.13	−0.09	−0.04	0.00	0.04
.3	0.09	0.13	0.17	0.22	0.26	0.30	0.34	0.38	0.43	0.47
.4	0.51	0.55	0.59	0.63	0.68	0.72	0.76	0.80	0.84	0.89
.5	0.93	0.97	1.02	1.06	1.10	1.15	1.19	1.24	1.28	1.33
.6	1.38	1.43	1.47	1.52	1.57	1.62	1.67	1.73	1.78	1.84
.7	1.89	1.95	2.01	2.07	2.13	2.19	2.26	2.33	2.40	2.47
.8	2.54	2.62	2.70	2.79	2.88	2.97	3.07	3.18	3.29	3.41
.9	3.54	3.69	3.85	4.03	4.23	4.46	4.75	5.11	5.61	6.45
.99	6.45	6.57	6.71	6.87	7.05	7.26	7.52	7.85	8.31	9.10
.999	9.10	9.22	9.35	9.50	9.67	9.88	10.12	10.44	*	*

* These values have not been calculated.

On the other hand, the asymptotic distributions of the reduced extremes are obtained by introducing the transformations

$$y_1 = \alpha(x_1 + u); \qquad y_n = \alpha(x_n - u) \tag{32}$$

into formulas (5). It is interesting to compare these four distributions and four probabilities with each other. This is done in Figures 1 and 2. The probability and the distribution of the midrange are practically identical with the probability and distribution of the smallest value, for small values of the midrange, and become practically identical with the probability and distribution of the largest value for large values of the midrange. Fig. 2 shows that the asymptotic distribution of the reduced range is less asymmetrical than the asymptotic distributions of the reduced extremes.

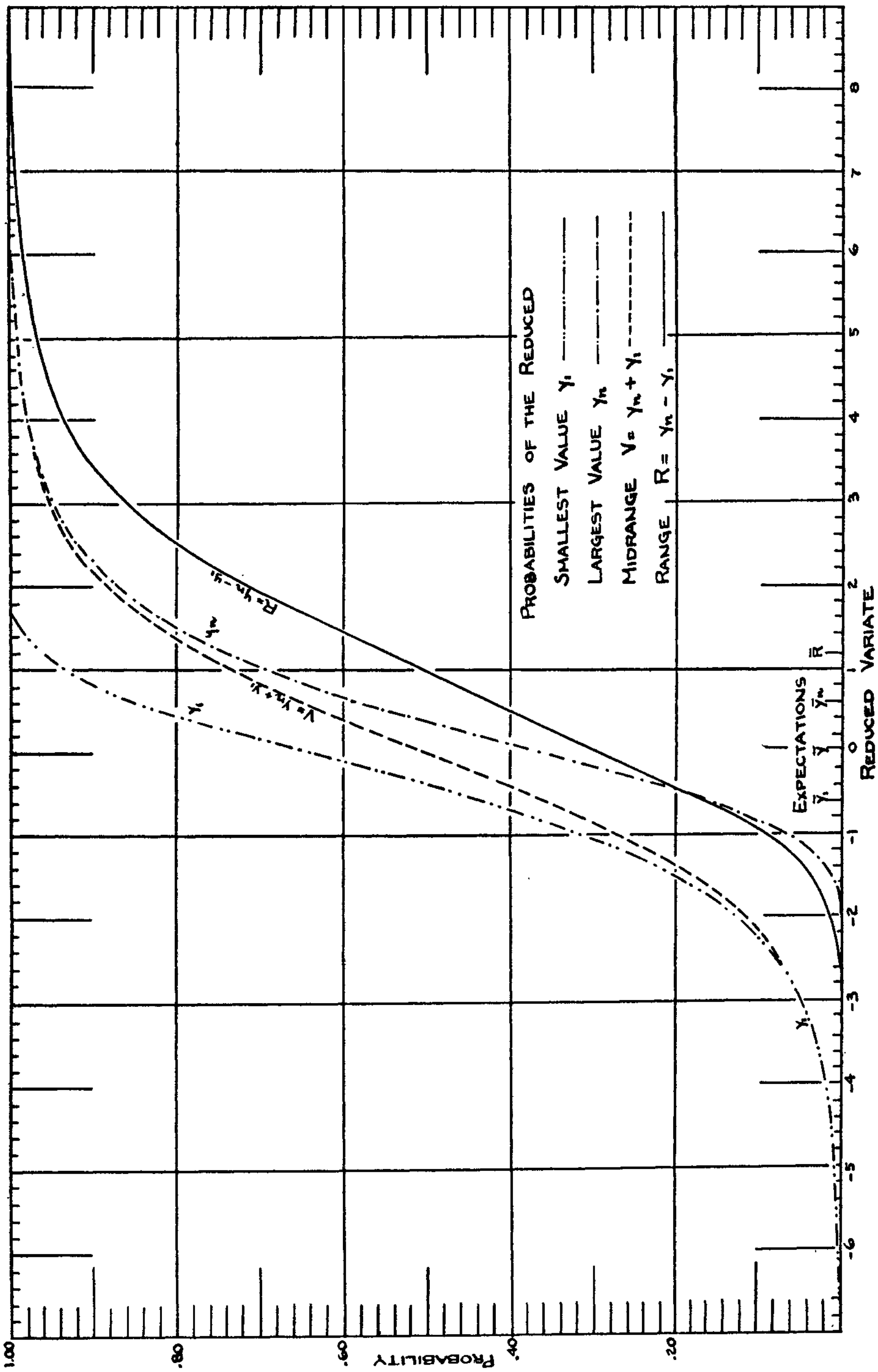
PROBABILITIES OF THE REDUCED
SMALLEST VALUE y1
LARGEST VALUE yn
MIDRANGE V = yn + y1
RANGE R = yn - y1
R = yn - y1
V = yn + y1
EXPECTATIONS
REDUCED VARIATE
PROBABILITY
1.00
.80
.60
.40
.20
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8

FIG. 1

Table II contains some characteristic values for these four asymptotic distributions. The first three columns are obtained from previous publications [6, 8]. The mean range is equal to the range of the means for the extremes. The median of the range is larger than the range from the median of the largest to the median of the smallest value. The mode of the range is slightly smaller than the mean of the largest value. These statements hold, of course, only for the reduced variates.

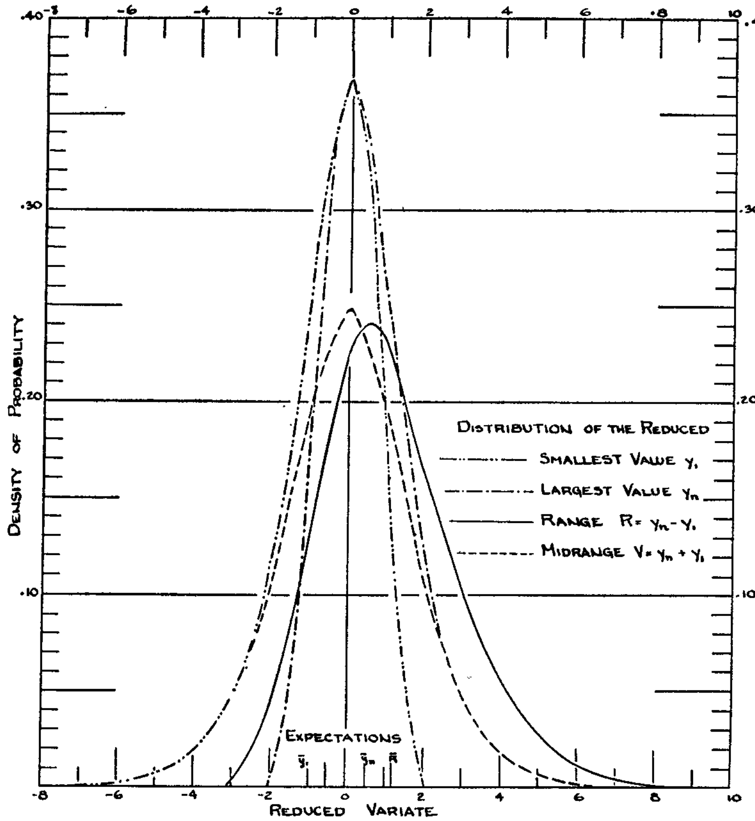

FIG. 2

From the mode $\tilde{R}$ of the reduced range given in equation (30) and the transformation (10), the mode $\tilde{w}$ of the range itself is obtained as

$$\tilde{w} = 2u + \frac{\tilde{R}}{\alpha}$$

whereas the difference of the modes of the largest and of the smallest values is

$$\tilde{x}_n - \tilde{x}_1 = 2u.$$

Consequently

$$\tilde{w} = \tilde{x}_n - \tilde{x}_1 + \frac{\tilde{R}}{\alpha}\,. \tag{33}$$

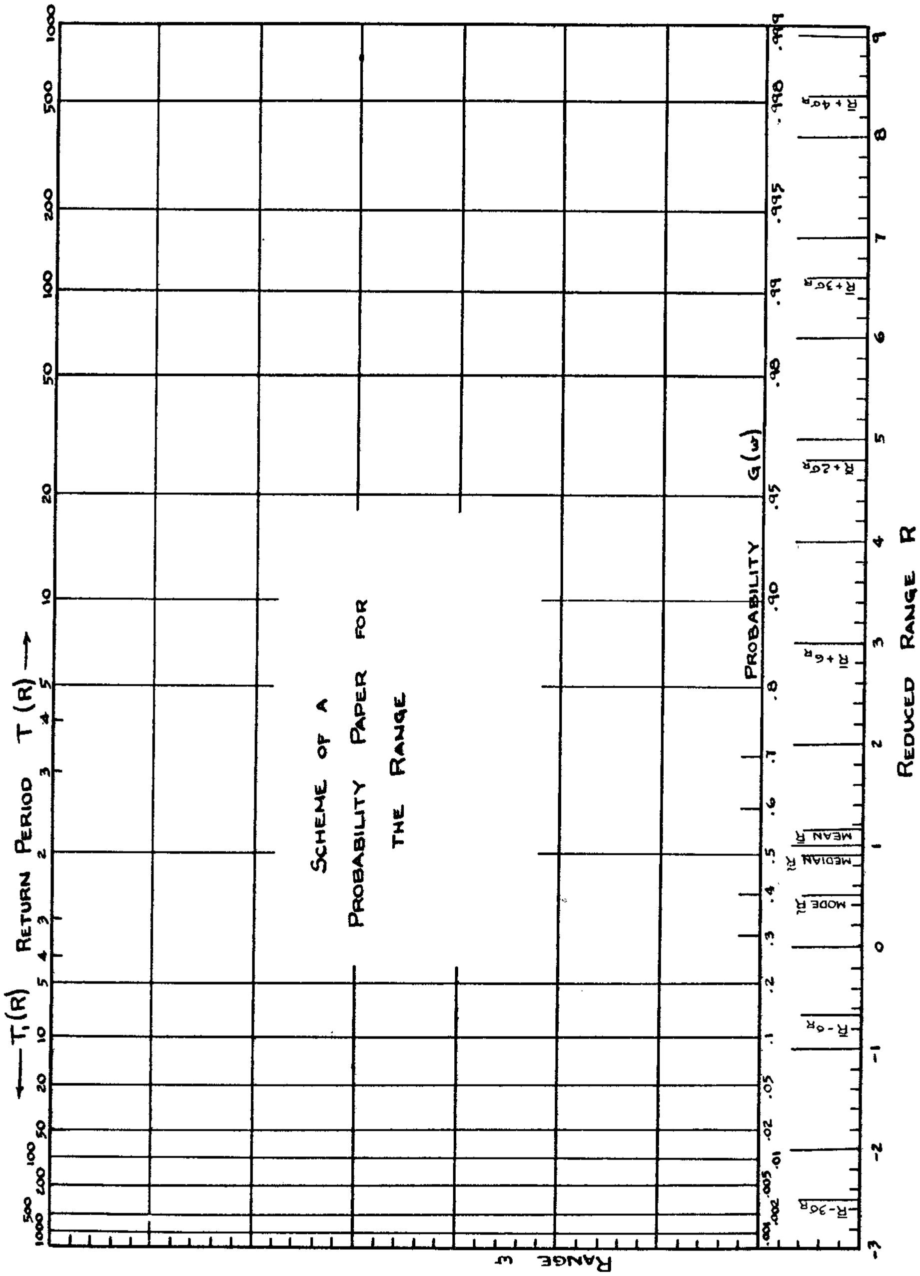
Scheme of a Probability Paper for the Range
Return Period T(R)
T₁(R)
Probability G(ω)
Reduced Range R
Range ω

Fig. 3

For a symmetrical initial distribution of the exponential type the mode of the range converges toward the range of the modes of the smallest and of the largest value, provided that the parameter α increases without limit with the sample size. Thus this convergence does not hold for all symmetrical distributions.

The last two lines in Table II give the four probabilities corresponding to the intervals from the mean μ minus once (twice) the standard deviation σ, up to the mean plus once (twice) the standard deviation. The first probability for

TABLE II

Characteristics for the 4 Asymptotic Reduced Distributions

1 Characteristic	2 Largest Value	3 Smallest Value	4 Midrange	5 Range
Mode	0	0	0	.506
Expectation	$\gamma = .57722$	$= -.57722$	0	$2\gamma = 1.15444$
Median	$-\lg\lg 2 = .36651$	$= -.36651$	0	.929
Seminvariant char. function	$\Gamma(1-t)$	$\Gamma(1+t)$	$\Gamma(1-t)\cdot\Gamma(1+t)$	$\Gamma^2(1-t)$
Variance	$\frac{\pi^2}{6} = 1.64493$	$= 1.64493$	$\frac{\pi^2}{3}$	$= 3.28986$
First + second moment quotient	$\beta_1 = 1.29857$ $\beta_2 = 5.4$	-1.29857 5.4	0 4.2	.64928 4.2
95% Probability	2.97	1.10	2.94	4.46
99% Probability	4.60	1.53	4.60	6.45
$F(\mu+\sigma) - F(\mu-\sigma)$	.72	.72	.72	.71
$F(\mu+2\sigma) - F(\mu-2\sigma)$	.90	.90	.95	.95

the four distributions is about the same as for the normal distribution. The second probability for the range and the midrange is about the same as for the normal one.

7. The asymptotic distribution of the range for a symmetrical variate. The asymptotic distribution of the range R is, of course, independent of the sample size, and parameter-free. Both statements do not hold for the distribution $g(w)$ of the range proper which is, from (11)

$$g(w) = \alpha\psi[\alpha(w - 2u)]. \tag{34}$$

In this formula, the range is expressed in the same units as the initial variate. The parameters α and u are functions of the sample size n, the function depending upon the initial distribution. From equations (6), (8), (14) follows that an increase of the sample size has two influences on the distribution of the range. The increase of the parameter u shifts the distribution toward the right without changing its form, whereas the parameter α influences the shape of the distribution. If α increases (decreases) with n, the distribution of the range shrinks (spreads) with increasing sample size. If α is independent of n, an increase of the sample size does not change the shape of the distribution. Only in the first case may we increase the precision of the range by increasing the sample size. The two parameters thus influence the range in the same way as they influence the extreme values.

To use equation (34) for a given initial distribution and a given sample size, we have to determine the expected largest value u and the parameter α as functions of n. We may use the definitions (6), (7), (8) if the initial distribution is known and of the exponential type, and if the sample size is so large that the most probable largest value is sufficiently near to the solution of (7).

As a first example, consider the so-called logistic distribution. This probability is

$$\Phi(x) = (1 + e^{-x})^{-1}. \tag{35}$$

The initial distribution is

$$\varphi(x) = \Phi(x)(1 - \Phi(x)) \tag{35'}$$

and the derivative is

$$\varphi'(x) = \Phi(x)(1 - \Phi(x))(1 - 2\Phi(x)). \tag{35''}$$

Equation (6) becomes

$$1 + e^{-u} = \frac{n}{n-1}$$

whence the expected largest value

$$u = \lg(n - 1). \tag{36}$$

The most probable largest value $\tilde{x}_n$ for n observations is obtained from (7). This equation becomes, from equation (35)

$$(n - 1)(1 - \Phi(\tilde{x}_n)) = -1 + 2\Phi(\tilde{x}_n)$$

whence

$$\Phi(\tilde{x}_n) = \frac{n}{n+1}$$

Equation (35) leads to the most probable largest value

$$\tilde{x}_n = lg\, n. \tag{36'}$$

Even for n as small as 30 the difference between $\tilde{x}_n$ and u is less than 1%. Consequently the asymptotic form of the distribution of the range may be used even for small samples. The two parameters are

$$u = \lg n; \qquad \alpha = \frac{n}{n+1}. \tag{37}$$

Since α converges toward unity, an increase of the sample size shifts the distribution of the range toward the right without influencing its shape: the precision of any estimate made from the range cannot be increased by increasing the sample size.

The characteristic ranges introduced in paragraph 5 are obtained immediately: the mean $\bar{w}$, the mode $\tilde{w}$, the median range $\check{w}$ and the ranges $w_{.95}$ and $w_{.99}$

$$\bar{w} = \lg n + 1.154; \quad \tilde{w} = \lg n + .506;$$
$$\check{w} = \lg n + .929; \quad w_{.95} = \lg n + 4.46; \quad w_{.99} = \lg n + 6.45$$

are parallel straight lines if traced as functions of the sample size n on semi-logarithmic paper.

For the normal distribution we cannot expect such simple results. Here, u and α can only be calculated as numerical functions of n although limiting forms of these functions are known. The parameter α increases with n, and the standard error of the range decreases without limit although very slowly. The logistic distribution belongs to the first, the normal distribution to the second class of initial distributions of the exponential type.

The probabilities and the distributions of the range for normal samples of size 5, 10, and 20 as calculated by E. S. Pearson and H. O. Hartley [16] are traced in Figures 4 and 5. Our aim is to trace the corresponding asymptotic probabilities and distributions in order to see how far the asymptotic ranges differ from the exact ones. However, we have first to settle the preliminary question how far the most probable largest value $\tilde{x}_n$ differs from the expected largest value u. The most probable largest value $\tilde{x}_n$ is obtained from (7) which becomes, for the normal distribution,

$$\tilde{x}_n \Phi(\tilde{x}_n) = (n - 1)\varphi(\tilde{x}_n). \tag{38}$$

The results $\tilde{x}_n$ as functions of n are shown in Table III cols. 1 and 2. The expected values u obtained from (6) are given in col. 3. For small samples, the two values $\tilde{x}_n$ and u differ widely, as might be expected. We are inclined to conclude that the asymptotic distribution of the range cannot hold for small samples. However, the only legitimate conclusion to be drawn is, that we cannot calculate the two parameters in the way stated before (6) and (8). Instead, we estimate them directly from the observations. The question of the most efficient estimates of these parameters is not yet solved. The simplest way is to use the mean range $\bar{w}_n$ and the standard deviation of the range $\sigma_{w,n}$ as given by Tippett [20] and Pearson [15]. To distinguish these estimates from the asymptotic values, we write the estimates with an index n. From (14) we obtain

$$\frac{1}{\alpha_n} = \frac{\sqrt{3}}{\pi}\sigma_{w,n}; \qquad 2u_n = \bar{w}_n - \frac{2\gamma}{\alpha_n}. \tag{39}$$

Table III gives the calculated means w_n and standard deviations $\sigma_{w,n}$ of the range, and the estimates $1/\alpha_n$ and $2u_n$. Fig. 6 shows how the most probable largest values $\bar{x}_n$ approach the expected largest value u with increasing sample size. The estimate u_n quickly approaches u. Besides we trace the mean range $\bar{w}_n$, the standard error of the range $\sigma_{w,n}$, and $1/\alpha_n$ which is proportional to it.

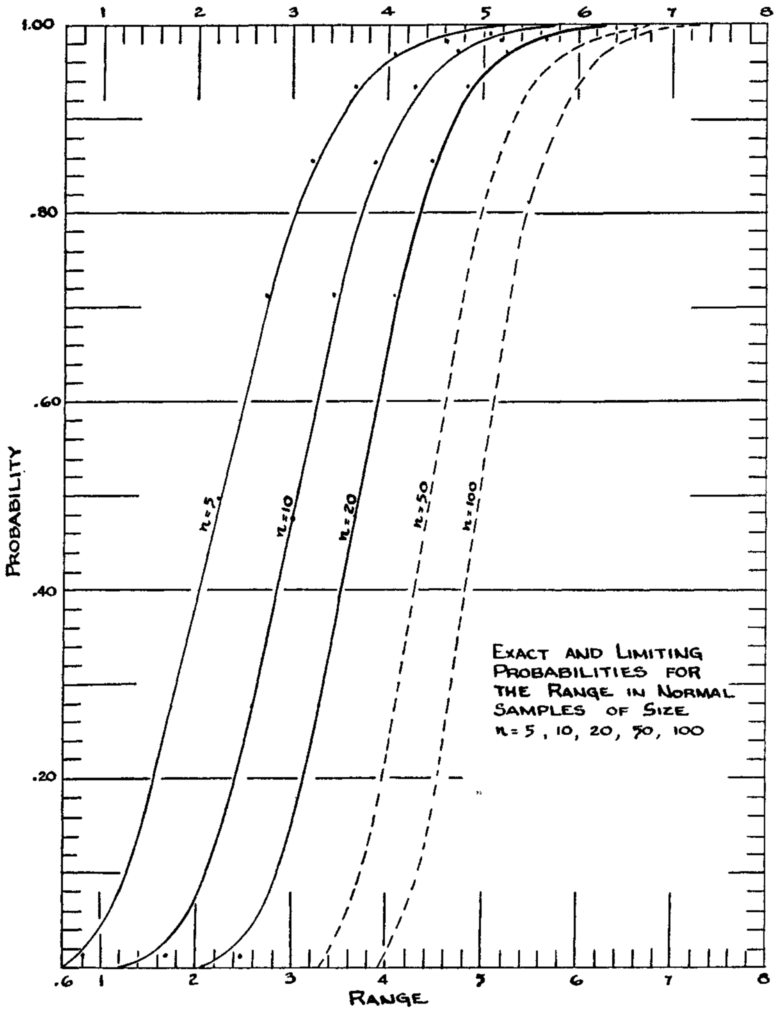

FIG. 4

From col. 8 follows that the condition $\alpha u \geqq 2$ is fulfilled from $n \geqq 6$ onward. The ranges obtained from the transformations

$$w = 2u_n + \frac{R}{\alpha_n} \tag{40}$$

are given in Table IV, cols. 3–7. The asymptotic probabilities of the range as obtained from the combination of columns 3–7, and col. 2 of Table IV are traced in Fig. 4 as separated points. The asymptotic probabilities are situated very near to the exact ones. Therefore the same method was used to calculate the asymptotic probabilities of the range for $n = 50$ and $n = 100$ which have not been calculated by Pearson. They too are traced in Fig. 4.

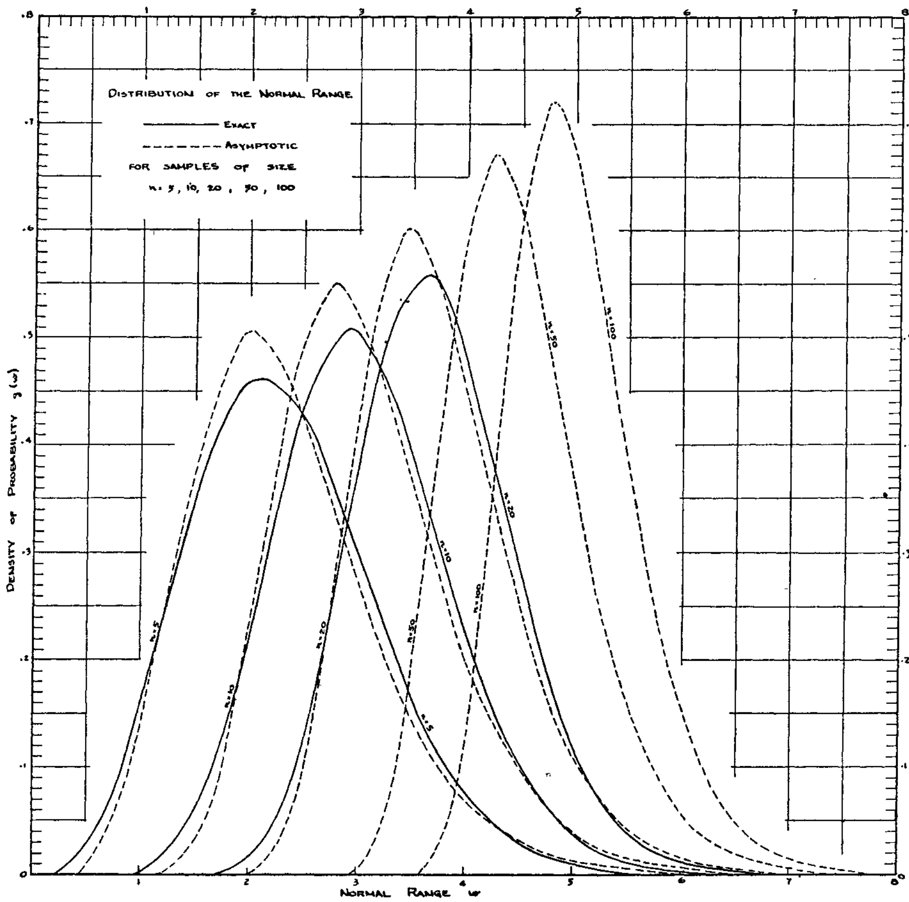

FIG. 5

The asymptotic probabilities of the range hold even for small normal samples. However, the parameters obtained from the exact distribution differ considerably from their asymptotic values. In other words: *The asymptotic probabilities of the range hold even for small normal samples provided that the parameters are taken from the observations.*

To compare the asymptotic distributions of the normal range to the calculated distributions, we attribute the asymptotic differences $\Delta\Psi/\alpha_n$ for a unit interval $\Delta w = 1$ to the middle of the corresponding intervals. The results are traced in Fig. 5 for $n = 5, 10, 20, 50, 100$. On the other hand, we take the differences

$\Delta\Psi$ for unit intervals from Pearson's tables, and trace them in the same graph. The fit of the calculated to the asymptotic values may be considered satisfactory.

TABLE III

Estimate of Parameters from the Calculated Distributions of the Normal Range

1	2	3	4	5	6	7	8
Sample size n	Largest Value		Mean Range $\bar{w}_n$	Standard deviation $\sigma_{w,n}$	Estimated parameters of the range		Lower limit $2\alpha_n u_n$
	Modal $\tilde{x}_n$	Expected u			$1/\alpha_n$	$2u_n$	
3	.765	.431	1.693	.8884	.4898	1.128	2.30
4	.938	.674	2.059	.8798	.4851	1.499	3.09
5	1.061	.842	2.326	.8641	.4764	1.776	3.73
10	1.419	1.282	3.078	.797	.439	2.571	5.86
20	1.740	1.645	3.735	.729	.402	3.271	8.14
50	2.126	2.054	4.498	.653	.360	4.082	11.34
100	2.377	2.326	5.015	.605	.334	4.630	13.86

TABLE IV

Asymptotic Probabilities for Normal Ranges Taken from Small Samples

1	2	3	4	5	6	7
Reduced range R	Probability $G(w) = \Psi(R)$	Normal ranges $w = 2u_n + R/\alpha_n$ for sample sizes				
		n = 5	n = 10	n = 20	n = 50	n = 100
−3	.000	.35	1.25	2.07	3.00	3.62
−2	.014	.82	1.69	2.47	3.36	3.96
−1	.093	1.30	2.13	2.87	3.72	4.30
0	.280	1.78	2.57	3.27	4.08	4.63
1	.517	2.52	3.01	3.67	4.44	4.96
2	.719	2.73	3.45	4.07	4.80	5.30
3	.853	3.21	3.89	4.48	5.16	5.63
4	.929	3.68	4.33	4.88	5.52	5.97
5	.967	4.16	4.77	5.28	5.88	6.30
6	.985	4.63	5.20	5.68	6.24	6.63
7	.994	5.11	5.64	6.09	6.60	6.97

Fig. 5 shows furthermore how the distributions of the range are shifted toward the right and become more concentrated for increasing sample sizes.

As an example for the practical application of the asymptotic distribution of the range, we use an observed distribution of 50 ranges taken from samples of

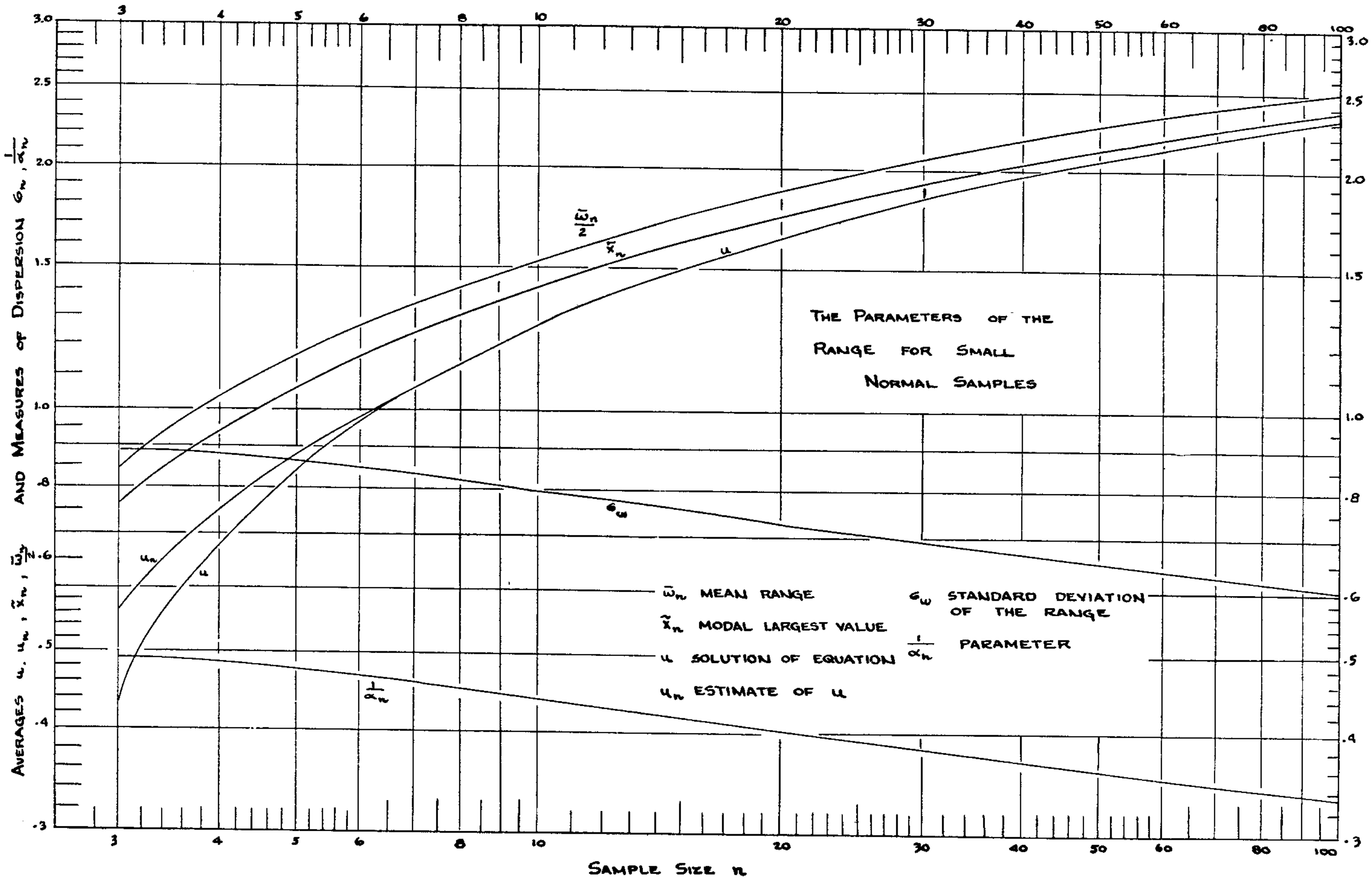
The Parameters of the
Range for Small
Normal Samples
$\bar{\omega}_n$ Mean Range
$\tilde{x}_n$ Modal Largest Value
u Solution of Equation
u_n Estimate of u
σ_w Standard Deviation of the Range
$\frac{1}{\alpha_n}$ Parameter
$\frac{\bar{\omega}_n}{2}$
$\tilde{x}_n$
u
u_n
σ_w
$\frac{1}{\alpha_n}$
Sample Size n
Averages u, u_n, $\tilde{x}_n$, $\frac{\bar{\omega}_n}{2}$ and Measures of Dispersion σ_n, $\frac{1}{\alpha_n}$

Fig. 6

$n = 14$ normal values given in Freeman's book [5] p. 128. The observed step function is traced in Fig. 7. For reasons given in a previous article [7] we attribute the cumulative frequency .5 to the smallest range 3, and the cumulative frequency 49.5 to the largest range 18. To compare this step function with the

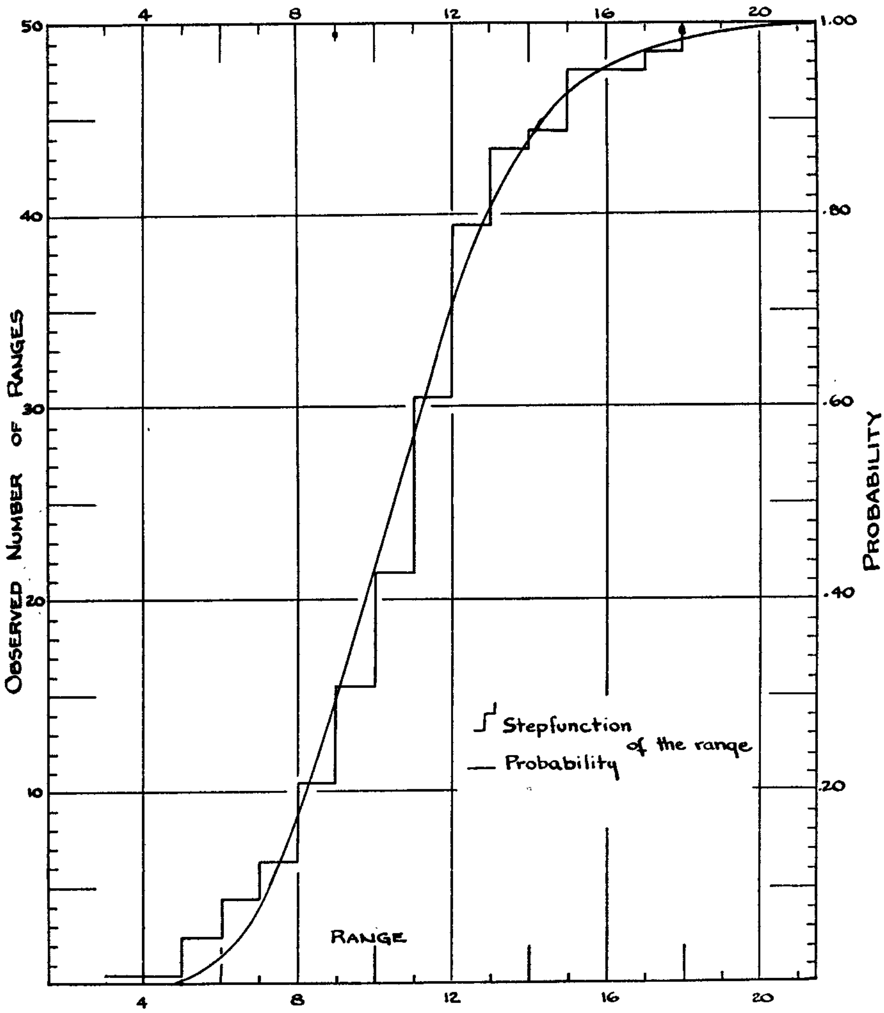

FIG. 7

probability $G(w)$, we estimate the two parameters u_n and α_n from formula (39). The mean range $\bar{w}_n$ and the estimate $s_{w,n}$ of the standard deviation of the ranges are

$$\bar{w} = 10.68; \qquad s_{w,n} = 2.93.$$

Consequently we obtain, from (39)

$$\frac{1}{\alpha_n} = 1.61; \qquad 2u_n = 8.82.$$

The theoretical ranges are thus, from (40),

$$w = 8.82 + 1.61\,R.$$

The corresponding probabilities $G(w)$ taken from Table I are traced in Fig. 7. The fit of the theory to the observations is certainly satisfactory, especially if we take into account that the ranges are given in integer numbers only.

8. The mth range and the asymmetrical case. An obvious generalization of the theory as established in paragraph 4 consists in the construction of the asymptotic distribution of the mth range for an unlimited symmetrical distribution of the exponential type. The mth range is the positive distance from the mth observation from above, x_m, to the mth observation from below, ${}_mx$. We suppose m to be very small compared to the sample size. Under the conditions stated in the beginning, the joint distribution $\mathfrak{w}_n({}_mx,\, x_m)$ of the mth extreme values splits into the product of the asymptotic distribution of the mth extreme value from above, $f_m(x_m)$, by the asymptotic distribution of the mth extreme value from below, ${}_mf({}_mx)$. Here, [6]

$$f_m(x_m) = \alpha_m \exp\,[-m\alpha_m(x_m - u_m) - me^{-\alpha_m(x_m - u_m)}]$$

$${}_mf({}_mx) = \alpha_m \exp\,[m\alpha_m({}_mx + u_m) - me^{\alpha_m({}_mx + u_m)}]$$

The sample size must be so large that the most probable mth extreme value $\tilde{x}_m$ is sufficiently near to u_m which is defined as the solution of

$$\Phi(u_m) = 1 - \frac{m}{n}.$$

The factor α_m defined by

$$\alpha_m = \frac{\varphi(u_m)}{1 - \Phi(u_m)}$$

is related to the asymptotic standard error σ_m of the mth extreme value by

$$\alpha_m\,\sigma_m = \sum_{\nu=m}^{\infty} \frac{1}{\nu^2}.$$

The joint asymptotic distribution $\mathfrak{w}({}_mx,\, x_m)$ of the mth smallest value and the mth range

$$w_m = x_m - {}_mx \tag{41}$$

is

$$\mathfrak{w}({}_mx,\, w_m) = \alpha_m^2 \exp\,[-\, m\alpha_m(w_m - 2u_m) - me^{\alpha_m({}_mx + u_m)} - me^{-\alpha_m({}_mx + w_m - u_m)}].$$

The asymptotic distribution $g(w_m)$ of the mth range is, dropping the index m of the variable ${}_mx$,

$$g(w_m) = \alpha_m^2 e^{-m\alpha_m(w_m - 2u_m)} \int_{-\infty}^{+\infty} \exp\,[-\, me^{\alpha_m(x + u_m)} - me^{-\alpha_m(x + w_m - u_m)}]\, dx.$$

Again we introduce a reduced range R_m defined by

$$\alpha_m(w_m - 2u_m) = R_m \geqq -2\alpha_m u_m \tag{42}$$

and put for the integration

$$\alpha_m (x + u_m) = y.$$

Then the asymptotic distribution $\psi(R_m)$ of the reduced mth range is

$$\psi(R_m) = e^{-mR_m} \int_{-\infty}^{+\infty} \exp[-me^y - me^{-y-R_m}]\, dy. \tag{43}$$

The probability $\Psi(R_m)$ for the mth range

$$\Psi(R_m) = \int_{-2\alpha_m u_m}^{R_m} \psi(z)\, dz$$

cannot be reduced to a single integral. This is due to the fact that the probabilities of the mth extreme values cannot be written down except in the integral form [6]. No differential equation similar to (17) exists. However, the function (43) could be calculated by numerical methods. The mean $\bar{R}_m$, the generating function and the moments of the mth range have been given in a previous paper [8].

For sake of completeness, consider finally an unlimited asymmetrical initial distribution of the exponential type. In this case, the joint distribution of the smallest and of the largest value splits again, for large samples, into the product of the asymptotic distributions $f_1(x_1)$ and $f_n(x_n)$ of the smallest and of the largest values which are now [6]

$$f_1(x_1) = \alpha_1 \exp[\alpha_1(x_1 - u_1) - e^{\alpha_1(x_1-u_1)}];$$

$$f_n(x_n) = \alpha_n \exp[-\alpha_n(x_n - u_n) - e^{-\alpha_n(x_n-u_n)}].$$

Here, α_n and u_n are defined, as previously, by (6) and (8). The sample must be so large that the most probable smallest value $\tilde{x}_1$ is sufficiently near to the solution of

$$\Phi(u_1) = \frac{1}{n}.$$

The factor α_1 defined by

$$\alpha_1 = \frac{\varphi(u_1)}{\Phi(u_1)}$$

is related to the asymptotic standard error of the smallest value by

$$\alpha_1 \sigma_1 = \frac{\pi}{\sqrt{6}}.$$

The joint asymptotic distribution of the smallest value x_1 and the range w

$$\mathfrak{w}(x_1, w) = \alpha_1 \alpha_n \exp[\alpha_1(x_1 - u_1) - \alpha_n(x_1 + w - u_n) - e^{\alpha_1(x_1-u_1)} - e^{-\alpha_n(x_1+w-u_n)}]$$

contains four parameters instead of the two which exist in the symmetrical case. However, the number of parameters may be reduced to one. We introduce a reduced range R defined by

$$R = \alpha_n(w - u_n + u_1) \tag{44}$$

being the range itself minus the range of the modes divided by a factor proportional to the standard error of the largest value. If we put

$$\alpha_1(x_1 - u_1) = y; \qquad \frac{\alpha_n}{\alpha_1} = \beta \tag{45}$$

the distribution $\psi(R)$ of the reduced range becomes, in the asymmetrical case,

$$\psi(R) = e^{-R} \int_{-\infty}^{+\infty} \exp[y(1 - \beta) - e^y - e^{-\beta y - R}]\, dy \tag{46}$$

and the probability $\Psi(R)$ for the reduced range is

$$\Psi(R) = \int_{-\infty}^{+\infty} \exp[y - e^y - e^{-\beta y - R}]\, dy \tag{47}$$

a formula which may immediately be verified by differentiation with respect to R. The mode $\tilde{R}$ of the range is the solution of

$$\psi(\tilde{R}) = e^{-\tilde{R}} \int_{-\infty}^{+\infty} \exp[y(1 - 2\beta) - R - e^y - e^{-\beta y - R}]\, dy.$$

Contrary to the symmetrical case, the latter integral cannot be expressed by the probability, and no simple differential equation similar to (17) exists. The expressions (46) and (47) contain a single constant β measuring the asymmetry of the initial distribution. In the symmetrical case, $\beta = 1$, we obtain, of course, the previous formulas (12) and (13). In the asymmetrical case, the mean, the variance, and the higher moments of the mth range may be derived from the generating function given in a previous paper [8].

The asymptotic distribution of the mth range in the asymmetrical case can easily be obtained by combining the two procedures used in this paragraph.

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ADDITION AT PROOF READING:

G. Elfving's article "The asymptotical distribution of range in samples from a normal population", *Biometrika*, Vol. 35 (1947), appeared when this manuscript was ready for print. Elfving considers a probability transformation of the range whereas we deal with the range itself. His distribution requires the knowledge of the initial distribution and of the sample size, whereas this knowledge is not required in our asymptotic formula.