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DIVISOR CLASS GROUPS AND GRADED CANONICAL MODULES OF MULTISECTION RINGS

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Abstract. We describe the divisor class group and the graded canonical module of the multisection ring $T(X;D_1,\dots,D_s)$ for a normal projective variety X and Weil divisors $D_1,\dots,D_s$ on X under a mild condition. In the proof, we use the theory of Krull domain and the equivariant twisted inverse functor.

§1. Introduction

We will describe the divisor class groups and the graded canonical modules of multisection rings associated with a normal projective variety.

Suppose that $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ are the set of integers, nonnegative integers, and positive integers, respectively.

Let X be a normal projective variety over a field k with the function field $k(X)$. We always assume that $\dim X > 0$. We denote by $C^1(X)$ the set of closed subvarieties of X of codimension 1. For $V \in C^1(X)$ and $a \in k(X)^\times$, we define

$$\operatorname{ord}_V(a) = \ell_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/\alpha\mathcal{O}_{X,V}) - \ell_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/\beta\mathcal{O}_{X,V}),$$
$$\operatorname{div}_X(a) = \sum_{V\in C^1(X)} \operatorname{ord}_V(a)\cdot V \in \operatorname{Div}(X) = \bigoplus_{V\in C^1(X)} \mathbb{Z}\cdot V,$$

where α and β are elements in $\mathcal{O}_{X,V}$ such that $a = \alpha/\beta$, and $\ell_{\mathcal{O}_{X,V}}(\ )$ denotes the length as an $\mathcal{O}_{X,V}$-module.

We call an element in $\operatorname{Div}(X)$ a *Weil divisor* on X. For a Weil divisor $D = \sum n_V V$, we say that D is *effective*, and we write $D \ge 0$ if $n_V \ge 0$ for any $V \in C^1(X)$. For a Weil divisor D on X, we put

$$H^0\bigl(X,\mathcal{O}_X(D)\bigr) = \bigl\{a \in k(X)^\times \bigm| \operatorname{div}_X(a) + D \ge 0\bigr\} \cup \{0\}.$$

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Here we note that $H^0(X, \mathcal{O}_X(D))$ is a k-vector subspace of $k(X)$.

Let $D_1, \ldots, D_s$ be Weil divisors on X. We define the multisection rings $T(X; D_1, \ldots, D_s)$ and $R(X; D_1, \ldots, D_s)$ associated with $D_1, \ldots, D_s$ as follows:

$$\begin{aligned}
&T(X; D_1, \ldots, D_s) \\
&\quad = \bigoplus_{(n_1, \ldots, n_s) \in \mathbb{N}_0^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i\Big)\Big) t_1^{n_1} \cdots t_s^{n_s} \\
&\quad \subset k(X)[t_1, \ldots, t_s] R(X; D_1, \ldots, D_s) \\
&\quad = \bigoplus_{(n_1, \ldots, n_s) \in \mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i\Big)\Big) t_1^{n_1} \cdots t_s^{n_s} \\
&\quad \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}].
\end{aligned} \tag{1.1}$$

We want to describe the divisor class groups and the graded canonical modules of the above rings.

For a Weil divisor F on X, we set

$$M_F = \bigoplus_{(n_1, \ldots, n_s) \in \mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + F\Big)\Big) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}];$$

that is, M_F is a $\mathbb{Z}^s$-graded reflexive $R(X; D_1, \ldots, D_s)$-module with

$$[M_F]_{(n_1, \ldots, n_s)} = H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + F\Big)\Big) t_1^{n_1} \cdots t_s^{n_s}.$$

We denote by $\overline{M_F}$ the isomorphism class of the reflexive module M_F in $\mathrm{Cl}(R(X; D_1, \ldots, D_s))$.

For a normal variety X, we denote by $\mathrm{Cl}(X)$ the class group of X, and for a Weil divisor F on X, we denote by $\overline{F}$ the residue class represented by the Weil divisor F in $\mathrm{Cl}(X)$.

In the case where $\mathrm{Cl}(X)$ is freely generated by $\overline{D_1}, \ldots, \overline{D_s}$, the ring $R(X; D_1, \ldots, D_s)$ is usually called the *Cox ring* of X and is denoted by $\mathrm{Cox}(X)$.

Remark 1.1. Assume that D is an ample divisor on X. In this case, $T(X; D)$ coincides with $R(X; D)$, and it is a Noetherian normal domain by a famous result of Zariski (see [6, Lemma 2.8]). It is well known that $\mathrm{Cl}(T(X; D))$ is isomorphic to $\mathrm{Cl}(X)/\mathbb{Z}\overline{D}$. Mori in [8] constructed a lot of examples of non-Cohen–Macaulay factorial domains using this isomorphism.

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It is well known that the canonical module of $T(X;D)$ is isomorphic to M_{K_X} and that the canonical sheaf ω_X coincides with $\widetilde{M_{K_X}}$. Watanabe proved a more general result in [12, Theorem 2.8].

We want to establish the same type of the above results for multisection rings.

For $R(X;D_1,\ldots,D_s)$, we had already proven the following.

THEOREM 1.2 ([2, Theorem 1.1], [5, Theorem 1.2]). *Let X be a normal projective variety over a field such that $\dim X>0$. Assume that $D_1,\ldots,D_s$ are Weil divisors on X such that $\mathbb{Z}D_1+\cdots+\mathbb{Z}D_s$ contains an ample Cartier divisor. Then, we have the following.*

(1) *The ring $R(X;D_1,\ldots,D_s)$ is a Krull domain.*
(2) *The set $\{P_V \mid V\in C^1(X)\}$ coincides with the set of homogeneous prime ideals of $R(X;D_1,\ldots,D_s)$ of height 1, where $P_V=M_{-V}$.*
(3) *We have an exact sequence*

$$0\longrightarrow \sum_i \mathbb{Z}\overline{D_i}\longrightarrow \mathrm{Cl}(X)\xrightarrow{\ p\ }\mathrm{Cl}\big(R(X;D_1,\ldots,D_s)\big)\longrightarrow 0$$

such that $p(\overline{F})=\overline{M_F}$.
(4) *Assume that $R(X;D_1,\ldots,D_s)$ is Noetherian. Then $\omega_{R(X;D_1,\ldots,D_s)}$ is isomorphic to M_{K_X} as a $\mathbb{Z}^s$-graded module. Therefore, $\omega_{R(X;D_1,\ldots,D_s)}$ is $R(X;D_1,\ldots,D_s)$-free if and only if $\overline{K_X}\in\sum_i\mathbb{Z}\overline{D_i}$ in $\mathrm{Cl}(X)$.*

Suppose that $\mathrm{Cl}(X)$ is a finitely generated free $\mathbb{Z}$-module generated by $\overline{D_1},\ldots,\overline{D_s}$. By the above theorem, the Cox ring $\mathrm{Cox}(X)$ is factorial, and

$$\omega_{\mathrm{Cox}(X)}=M_{K_X}=\mathrm{Cox}(X)(\overline{K_X}),$$

where we regard $\mathrm{Cox}(X)$ as a $\mathrm{Cl}(X)$-graded ring.

The main result of this paper is the following.

THEOREM 1.3. *Let X be a normal projective variety over a field k such that $d=\dim X>0$. Assume that $D_1,\ldots,D_s$ are Weil divisors on X such that $\mathbb{N}D_1+\cdots+\mathbb{N}D_s$ contains an ample Cartier divisor. Put*

$$U=\big\{j \,\big|\, \mathrm{tr.deg}_k\,T(X;D_1,\ldots,D_{j-1},D_{j+1},\ldots,D_s)=d+s-1\big\}.$$

Then, we have the following.

(1) *The ring $T(X;D_1,\ldots,D_s)$ is a Krull domain.*

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(2) *The set*

$$\{Q_V \mid V \in C^1(X)\} \cup \{Q_j \mid j \in U\}$$

coincides with the set of homogeneous prime ideals of $T(X;D_1,\dots,D_s)$ *of height* 1*, where*

$$Q_V = P_V \cap T(X;D_1,\dots,D_s)$$

and

$$Q_j = \bigoplus_{\substack{n_1,\dots,n_s \in \mathbb{N}_0 \\ n_j>0}} T(X;D_1,\dots,D_s)_{(n_1,\dots,n_s)}.$$

(3) *We have an exact sequence*

$$0 \longrightarrow \sum_{j\notin U} \mathbb{Z}\overline{D_j} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}\big(T(X;D_1,\dots,D_s)\big) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{M_F \cap k(X)[t_1,\dots,t_s,\{t_j^{-1} \mid j \notin U\}]}$.

(4) *Assume that* $T(X;D_1,\dots,D_s)$ *is Noetherian. Then* $\omega_{T(X;D_1,\dots,D_s)}$ *is isomorphic to*

$$M_{K_X} \cap t_1 \cdots t_s k(X)\big[t_1,\dots,t_s,\{t_j^{-1} \mid j \notin U\}\big]$$

as a $\mathbb{Z}^s$*-graded module. Further, we have*

$$q\Big(\overline{K_X + \sum_i D_i}\Big) = \overline{\omega_{T(X;D_1,\dots,D_s)}}.$$

Therefore, $\omega_{T(X;D_1,\dots,D_s)}$ *is* $T(X;D_1,\dots,D_s)$*-free if and only if*

$$\overline{K_X + \sum_i D_i} \in \sum_{j\notin U} \mathbb{Z}\overline{D_j}$$

in $\mathrm{Cl}(X)$.

Here, $\mathrm{tr.deg}_k T$ denotes the transcendence degree of the fractional field of T over a field k.

Remark 1.4. With notation as in Theorem 1.3, $\mathrm{ht}(Q_j)=1$ if and only if $j \in U$. This will be proved in Lemma 3.3. Since $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor, $Q_j \neq (0)$ for any j. Therefore, $\mathrm{ht}(Q_j) \geq 2$ if and only if $j \notin U$.

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§2. Examples

EXAMPLE 2.1. Let X be a normal projective variety with $\dim X > 0$. Assume that all D_i are ample Cartier divisors on X. Then, $T(X; D_1, \ldots, D_s)$ is Noetherian by a famous result of Zariski (see [6, Lemma 2.8]).

Assume that $s = 1$. By definition, $U = \emptyset$ since $\dim X > 0$. By Theorem 1.3(3), $\mathrm{Cl}(T(X; D_1))$ is isomorphic to $\mathrm{Cl}(X)/\mathbb{Z}\overline{D_1}$. By Theorem 1.3(4), $\omega_{T(X;D_1)}$ is a $T(X; D_1)$-free module if and only if

$$\overline{K_X} \in \mathbb{Z}\overline{D_1}$$

in $\mathrm{Cl}(X)$ (see Remark 1.1).

Next, assume that $s \geq 2$. In this case, $U = \{1, 2, \ldots, s\}$. By Theorem 1.3(3), $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Cl}(T(X; D_1, \ldots, D_s))$. By Theorem 1.3(4), $\omega_{T(X;D_1,\ldots,D_s)}$ is a $T(X; D_1, \ldots, D_s)$-free module if and only if

$$\overline{K_X} = \overline{-D_1 - \cdots - D_s}$$

in $\mathrm{Cl}(X)$. When this is the case, $-K_X$ is ample; that is, X is a Fano variety.

EXAMPLE 2.2. Set $X = \mathbb{P}^m \times \mathbb{P}^n$. Let p_1 (resp., p_2) be the first (resp., second) projection.

Let H_1 be a hyperplane of $\mathbb{P}^m$, and let H_2 be a hyperplane of $\mathbb{P}^n$. Put $A_i = p_i^{-1}(H_i)$ for $i = 1, 2$. In this case, $\mathrm{Cl}(X) = \mathbb{Z}\overline{A_1} + \mathbb{Z}\overline{A_2} \simeq \mathbb{Z}^2$, and $K_X = -(m+1)A_1 - (n+1)A_2$.

We have

$$\mathrm{Cox}(X) = R(X; A_1, A_2) = k[x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n].$$

$\mathrm{Cox}(X)$ is a $\mathbb{Z}^2$-graded ring such that x_i (resp., y_j) are of degree $(1, 0)$ (resp., $(0, 1)$).

Let a, b, c, d be positive integers such that $ad - bc \neq 0$. Put $D_1 = aA_1 + bA_2$, and put $D_2 = cA_1 + dA_2$. Then, both D_1 and D_2 are ample divisors. Consider the multisection rings

$$R(X; D_1, D_2) = \bigoplus_{p,q \in \mathbb{Z}} \mathrm{Cox}(X)_{p(a,b)+q(c,d)},$$

$$T(X; D_1, D_2) = \bigoplus_{p,q \geq 0} \mathrm{Cox}(X)_{p(a,b)+q(c,d)}.$$

Here, both $R(X; D_1, D_2)$ and $T(X; D_1, D_2)$ are Cohen–Macaulay rings.

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By Theorem 1.2(4), we know that

$$\begin{aligned} R(X;D_1,D_2) \text{ is a Gorenstein ring} &\Longleftrightarrow \overline{K_X} \in \mathbb{Z}\overline{D_1} + \mathbb{Z}\overline{D_2} \text{ in } \mathrm{Cl}(X) \\ &\Longleftrightarrow (m+1, n+1) \in \mathbb{Z}(a,b) + \mathbb{Z}(c,d). \end{aligned}$$

In this case, we have $U = \{1,2\}$ since all of a, b, c, and d are positive. By Theorem 1.3(4), we have

$$\begin{aligned} T(X;D_1,D_2) \text{ is a Gorenstein ring} &\Longleftrightarrow \overline{K_X + D_1 + D_2} = 0 \text{ in } \mathrm{Cl}(X) \\ &\Longleftrightarrow m+1 = a+c \text{ and } n+1 = b+d. \end{aligned}$$

EXAMPLE 2.3. Let a, b, c be pairwise coprime positive integers. Let $\mathfrak{p}$ be the kernel of the k-algebra map $S = k[x,y,z] \to k[T]$ given by $x \mapsto T^a$, $y \mapsto T^b$, $z \mapsto T^c$.

Let $\pi : X \to \mathbb{P} = \mathrm{Proj}(k[x,y,z])$ be the blowup at $V_+(\mathfrak{p})$, where $a = \deg(x)$, $b = \deg(y)$, $c = \deg(z)$. Put $E = \pi^{-1}(V_+(\mathfrak{p}))$. Let A be a Weil divisor on X satisfying $\pi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_X(A)$. In this case, we have $\mathrm{Cl}(X) = \mathbb{Z}\overline{E} + \mathbb{Z}\overline{A} \simeq \mathbb{Z}^2$, and $K_X = E - (a+b+c)A$.

Then, we have

$$\begin{aligned} \mathrm{Cox}(X) = R(X;-E,A) = R'_s(\mathfrak{p}) &:= S[t^{-1}, \mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots] \subset S[t^{\pm 1}], \\ T(X;-E,A) = R_s(\mathfrak{p}) &:= S[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots] \subset S[t]. \end{aligned}$$

By Theorem 1.2(4), we have

$$\omega_{R'_s(\mathfrak{p})} = M_{K_X} = R'_s(\mathfrak{p})(\overline{K_X}) = R'_s(\mathfrak{p})(-1, -a-b-c).$$

In this case, $U = \{1\}$. By Theorem 1.3(4), we have

$$\begin{aligned} \omega_{R_s(\mathfrak{p})} &= M_{K_X} \cap t_1 t_2 k(X)[t_1, t_2^{\pm 1}] \\ &= \omega_{R'_s(\mathfrak{p})} \cap t_1 t_2 k(X)[t_1, t_2^{\pm 1}] \\ &= R'_s(\mathfrak{p})(-1, -a-b-c) \cap t_1 t_2 k(X)[t_1, t_2^{\pm 1}] \\ &= R_s(\mathfrak{p})(-1, -a-b-c). \end{aligned}$$

Therefore, both of $R'_s(\mathfrak{p})$ and $R_s(\mathfrak{p})$ are quasi-Gorenstein rings that were first proved by Simis and Trung [11, Corollary 3.4]. The Cohen–Macaulayness of such rings are deeply studied by Goto, Nishida, and Shimoda [3].

Divisor class groups of ordinary and symbolic Rees rings were studied by, for example, Shimoda [10] and Simis and Trung [11].

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§3. Proof of Theorem 1.3

Throughout this section, we assume that X is a normal projective variety over a field k such that $d = \dim X > 0$, and we assume that $D_1, \ldots, D_s$ are Weil divisors on X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor.

We need the following lemmas. They are well known, but the author could not find a reference.

LEMMA 3.1. *Let G be an integral domain containing a field k. Let P be a prime ideal of G. Assume that both* $\text{tr.deg}_k\, G$ *and* $\text{tr.deg}_k\, G/P$ *are finite.*

Then, the height of P is less than or equal to

$$\text{tr.deg}_k\, G - \text{tr.deg}_k\, G/P.$$

Proof. Assume the contrary. Then there exists a ring G' which satisfies the following five conditions:

- $k \subset G' \subset G$;
- G' is finitely generated (as a ring) over k;
- $\text{tr.deg}_k\, G = \text{tr.deg}_k\, G'$;
- $\text{tr.deg}_k\, G/P = \text{tr.deg}_k\, G'/(G' \cap P)$; and
- $\text{tr.deg}_k\, G - \text{tr.deg}_k\, G/P < \text{ht}(G' \cap P)$.

However, using the dimension formula (e.g., [7, p. 119]), we have

$$\text{ht}(G' \cap P) = \text{tr.deg}_k\, G' - \text{tr.deg}_k\, G'/(G' \cap P) = \text{tr.deg}_k\, G - \text{tr.deg}_k\, G/P.$$

This is a contradiction. □

LEMMA 3.2. *Let r be a positive integer. Let $F_1, \ldots, F_r$ be Weil divisors on X. Let S be the set of all nonzero homogeneous elements of $T(X; F_1, \ldots, F_r)$. Then the following conditions are equivalent.*

(1) *There exist nonnegative integers $q_1, \ldots, q_r$ such that $\sum_{i=1}^{r} q_i F_i$ is linearly equivalent to a sum of an ample Cartier divisor and an effective Weil divisor.*

(2) *There exist positive integers $q_1, \ldots, q_r$ such that $\sum_{i=1}^{r} q_i F_i$ is linearly equivalent to a sum of an ample Cartier divisor and an effective Weil divisor.*

(3) *We have $S^{-1}(T(X; F_1, \ldots, F_r)) = k(X)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$.*

(4) *We have $Q(T(X; F_1, \ldots, F_r)) = k(X)(t_1, \ldots, t_r)$, where $Q(\ )$ denotes the field of fractions.*

(5) *We have* $\text{tr.deg}_k\, T(X; F_1, \ldots, F_r) = \dim X + r$.

Using [1, Theorem 1.5.5], it is easy to see that $T(X; F_1, \ldots, F_r)$ is Noetherian if and only if $T(X; F_1, \ldots, F_r)$ is finitely generated (as a ring) over the field $H^0(X, \mathcal{O}_X)$. Therefore, if $T(X; F_1, \ldots, F_r)$ is Noetherian, then condition (5) is equivalent to stating that the Krull dimension of $T(X; F_1, \ldots, F_r)$ is $\dim X + r$.

Proof. Here (2) ⇒ (1), and (3) ⇒ (4) ⇒ (5) are trivial.

First we will prove that (1) ⇒ (3). Suppose that

$$\sum_{i=1}^{r} q_i F_i \sim D + F,$$

where q_i are nonnegative integers, D is a very ample Cartier divisor, and F is an effective divisor. We put

$$\begin{aligned} C = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{(n_1,\ldots,n_r) \in \mathbb{N}_0^r} & H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i F_i + mD\Big)\Big) t_1^{n_1} \cdots t_r^{n_r} t_{r+1}^m \\ & \subset k(X)[t_1, \ldots, t_r, t_{r+1}^{\pm 1}]. \end{aligned} \tag{3.1}$$

We regard C as a $\mathbb{Z}^{r+1}$-graded ring with

$$C_{(n_1,\ldots,n_r,m)} = H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i F_i + mD\Big)\Big) t_1^{n_1} \cdots t_r^{n_r} t_{r+1}^m.$$

Then, we have

$$T(X; F_1, \ldots, F_r) = \bigoplus_{(n_1,\ldots,n_r) \in \mathbb{N}_0^r} C_{(n_1,\ldots,n_r,0)},$$

so $T(X; F_1, \ldots, F_r)$ is a subring of C. Thus, $S^{-1}C$ is a $\mathbb{Z}^{r+1}$-graded ring such that

$$S^{-1}T(X; F_1, \ldots, F_r) = \bigoplus_{(n_1,\ldots,n_r) \in \mathbb{N}_0^r} (S^{-1}C)_{(n_1,\ldots,n_r,0)}.$$

Since $\sum_{i=1}^{r} q_i F_i - D$ is linearly equivalent to an effective divisor F, there exists a nonzero element a in

$$H^0\Big(X, \mathcal{O}_X\Big(\sum_i q_i F_i - D\Big)\Big).$$

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For any $0 \neq b \in H^0(X, \mathcal{O}_X(D))$,

$$(at_1^{q_1} \cdots t_r^{q_r} t_{r+1}^{-1})(bt_{r+1})$$

is contained in S. Therefore, $S^{-1}C$ contains $(bt_{r+1})^{-1}$. Hence, $k(X)$ is contained in $S^{-1}C$. Since $k(X) = (S^{-1}C)_{(0,\ldots,0)}$, $k(X)$ is contained in $S^{-1}T(X; F_1, \ldots, F_r)$.

By assumption (1), there exists a positive integer ℓ such that

$$(S^{-1}C)_{(\ell q_1, \ldots, \ell q_r, 0)} \neq 0$$

and

$$(S^{-1}C)_{(\ell q_1 + 1, \ell q_2, \ldots, \ell q_r, 0)} \neq 0.$$

Then, it is easy to see that $t_1 \in S^{-1}C$. Therefore, $S^{-1}C$ contains $k(X)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Hence, $S^{-1}T(X; F_1, \ldots, F_r)$ coincides with $k(X)[t_1^{\pm}, \ldots, t_r^{\pm}]$.

Next, we will prove (5) $\Rightarrow$ (2). Let D be a very ample divisor. Consider the ring

$$R(X; F_1, \ldots, F_r, D).$$

First, assume that

$$H^0\Bigl(X, \mathcal{O}_X\Bigl(\sum_i u_i F_i - vD\Bigr)\Bigr) \neq 0$$

for some integers $u_1, \ldots, u_r$, v such that $v > 0$. By assumption (5), there exist positive integers $u'_1, \ldots, u'_r$ such that

$$H^0\Bigl(X, \mathcal{O}_X\Bigl(\sum_i u'_i F_i\Bigr)\Bigr) \neq 0.$$

Therefore, we may assume that there exist positive integers $u_1, \ldots, u_r$ and v such that

$$H^0\Bigl(X, \mathcal{O}_X\Bigl(\sum_i u_i F_i - vD\Bigr)\Bigr) \neq 0.$$

Here, we have

$$\sum_i u_i F_i = vD + \Bigl(\sum_i u_i F_i - vD\Bigr).$$

Therefore, $\sum_i u_i F_i$ is the sum of an ample divisor vD and the divisor $\sum_i u_i F_i - vD$, which is linearly equivalent to an effective divisor.

Next, assume that for any integers $u_1, \dots, u_r$ and v,

$$H^0\Big(X, \mathcal{O}_X\Big(\sum_i u_i F_i - vD\Big)\Big) = 0 \tag{3.2}$$

if $v > 0$. We put

$$P = \bigoplus_{\substack{(n_1,\dots,n_r,m)\in\mathbb{Z}^{r+1}\\ m>0}} R(X; F_1, \dots, F_r, D)_{(n_1,\dots,n_r,m)}.$$

By assumption (5), P is a prime ideal of $R(X; F_1, \dots, F_r, D)$ of height 1 by Lemma 3.1. (Here, since D is an ample divisor, $\operatorname{tr.deg}_k R(X; F_1, \dots, F_r, D) = \dim X + r + 1$. Note that P is an ideal of $R(X; F_1, \dots, F_r, D)$ by (3.2) above. By (5), $\operatorname{tr.deg}_k R(X; F_1, \dots, F_r, D)/P = \dim X + r$.) However, $R(X; F_1, \dots, F_r, D)$ has no homogeneous prime ideal of height 1 that contains

$$H^0\big(X, \mathcal{O}_X(D)\big)t_{r+1}$$

by Theorem 1.2(2). This is a contradiction. ∎

Put $A = k(X)[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$, and put $B = k(X)[t_1, \dots, t_s]$. Recall that $D_1, \dots, D_s$ are Weil divisors on a normal projective variety X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor. We denote $T(X; D_1, \dots, D_s)$ and $R(X; D_1, \dots, D_s)$ simply by T and R, respectively.

Since

$$T = R \cap B,$$

T is a Krull domain. We have proved Theorem 1.3(1).

By Theorem 1.2(2), we have

$$R = \Big(\bigcap_{V\in C^1(X)} R_{P_V}\Big) \cap A,$$

$$A = \bigcap_{P\in \mathrm{NHP}^1(R)} R_P,$$

where $\mathrm{NHP}^1(R)$ is the set of nonhomogeneous prime ideals of R of height 1.

It is easy to see that $R_P = T_{P\cap T}$ for $P \in \mathrm{NHP}^1(R)$. Therefore, we have

$$A = \bigcap_{P\in \mathrm{NHP}^1(R)} T_{P\cap T}.$$

Since $T_{P\cap T}$ is a discrete valuation ring, $P \cap T$ is a nonhomogeneous prime ideal of T of height 1.

For $V \in C^1(X)$, put $Q_V = P_V \cap T$. Then, $R_{P_V} = T_{Q_V}$, since $\sum_i \mathbb{N}D_i$ contains an ample divisor. Therefore, Q_V is a homogeneous prime ideal of T of height 1.

On the other hand, we have $Q_i = T \cap t_i B_{(t_i)}$ and $T_{Q_i} \subset B_{(t_i)}$. Note that

$$B = A \cap \Big(\bigcap_{j=1}^{s} B_{(t_j)} \Big).$$

Then, we have

$$\begin{aligned} T &= R \cap B \\ &= \Big(\bigcap_{V \in C^1(X)} R_{P_V} \Big) \cap A \cap B \qquad (3.3) \\ &= \Big(\bigcap_{V \in C^1(X)} T_{Q_V} \Big) \cap \Big(\bigcap_{P \in \mathrm{NHP}^1(R)} T_{P \cap T} \Big) \cap \Big(\bigcap_{j=1}^{s} B_{(t_j)} \Big). \end{aligned}$$

Put

$$T_j = \bigoplus_{(n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_s) \in \mathbb{N}_0^{s-1}} H^0\Big(X, \mathcal{O}_X\Big(\sum_{i \neq j} n_i D_i\Big)\Big) t_1^{n_1} \cdots t_{j-1}^{n_{j-1}} t_{j+1}^{n_{j+1}} \cdots t_s^{n_s}.$$

We need the following lemma.

Lemma 3.3. *With notation as above, the following conditions are equivalent:*

(1) $T_{Q_j} = B_{(t_j)}$;
(2) *the height of* Q_j *is* 1;
(3) *the height of* Q_j *is less than* 2; *and*
(4) $j \in U$, *that is,* $\mathrm{tr.deg}_k T_j = d + s - 1$.

Proof. By Lemma 3.2, we have $Q(T) = Q(B)$. It is easy to see that $B_{(t_j)}$ is a discrete valuation ring. Since Q_j is a nonzero prime ideal of a Krull domain T, the equivalence of (1), (2), and (3) is easy to see.

Here, we will prove (1) ⇒ (4). Note that $T/Q_j = T_j$. Then, we have

$$Q(T_j) = T_{Q_j}/Q_j T_{Q_j} = B_{(t_i)}/(t_i)B_{(t_i)} = k(X)(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_s).$$

The implication that (4) ⇒ (3) immediately follows from

$$\mathrm{ht}(Q_j) \leq \mathrm{tr.deg}_k T - \mathrm{tr.deg}_k(T_j) = 1.$$

This inequality follows from Lemma 3.1 and from the fact that $T_j = T/Q_j$. ∎

By (3.3), Lemma 3.3, and [7, Theorem 12.3], we know that

$$\{Q_V \mid V \in C^1(X)\} \cup \{Q_j \mid j \in U\}$$

is the set of homogeneous prime ideals of T of height 1, and that

$$\{P \cap T \mid P \in \mathrm{NHP}^1(R)\}$$

is the set of nonhomogeneous prime ideals of T of height 1. Further, we obtain

$$T = \Big(\bigcap_{V \in C^1(X)} T_{Q_V} \Big) \cap \Big(\bigcap_{P \in \mathrm{NHP}^1(R)} T_{P \cap T} \Big) \cap \Big(\bigcap_{j \in U} T_{Q_j} \Big).$$

The proof of Theorem 1.3(2) is completed.

Let

$$\mathrm{Div}(X) = \bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot V$$

be the set of Weil divisors on X. Let

$$\mathrm{HDiv}(T) = \Big(\bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_V) \Big) \oplus \Big(\bigoplus_{j \in U} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_j) \Big)$$

be the set of homogeneous Weil divisors of $\mathrm{Spec}(T)$.

Here, we define

$$\phi : \mathrm{Div}(X) \longrightarrow \mathrm{HDiv}(T)$$

by $\phi(V) = \mathrm{Spec}(T/Q_V)$ for each $V \in C^1(X)$. Then, it satisfies the following.

- For each $a \in k(X)^\times$, we have

$$\phi\big(\mathrm{div}_X(a)\big) = \mathrm{div}_T(a) \in \bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_V) \subset \mathrm{HDiv}(T).$$

- If $j \in U$, then

$$\mathrm{div}_T(t_j) = \mathrm{Spec}(T/Q_j) + \phi(D_j).$$

- If $j \notin U$, then

$$\mathrm{div}_T(t_j) = \phi(D_j).$$

They are proven essentially in the same way as in [2, pp. 631–632]. Then, we have an exact sequence

$$0 \longrightarrow \sum_{j \notin U} \mathbb{Z}\overline{D_j} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}(T) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{\phi(F)}$ in $\mathrm{Cl}(T)$. Here, remember that $\mathrm{Cl}(T)$ coincides with $\mathrm{HDiv}(T)$ divided by the group of homogeneous principal divisors (see, e.g., [9, Proposition 7.1]).

It is easy to see that the class of the Weil divisor $q(\overline{F})$ corresponds to the isomorphism class of the reflexive module

$$\begin{aligned} M_F \cap \Big(\bigcap_{j \in U} T_{Q_j}\Big) &= M_F \cap A \cap \Big(\bigcap_{j \in U} T_{Q_j}\Big) \\ &= M_F \cap k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big]. \end{aligned}$$

The proof of Theorem 1.3(3) is completed.

REMARK 3.4. It is easy to see that

$$t_1^{d_1} \cdots t_s^{d_s} M_{F + \sum_i d_i D_i} = M_F$$

for any integers $d_1, \dots, d_s$. Therefore, we have

$$\begin{aligned} &M_F \cap t_1^{d_1} \cdots t_s^{d_s} k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big] \\ &\quad = t_1^{d_1} \cdots t_s^{d_s} \big(M_{F + \sum_i d_i D_i} \cap k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big]\big). \end{aligned}$$

Hence,

$$M_F \cap t_1^{d_1} \cdots t_s^{d_s} k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big]$$

is isomorphic to

$$M_{F + \sum_i d_i D_i} \cap k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big] \tag{3.4}$$

as a T-module. Note that this is not an isomorphism as a $\mathbb{Z}^s$-graded module. The isomorphism class to which module (3.4) belongs coincides with $q(\overline{F + \sum_i d_i D_i})$.

In the rest, we assume that T is Noetherian. We will prove that ω_T is isomorphic to

$$M_{K_X} \cap t_1 \cdots t_s k(X)\big[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}\big]$$

as a $\mathbb{Z}^s$-graded module. (Suppose that it is true. If we forget the grading, it is isomorphic to

$$M_{K_X+\sum_i D_i} \cap k(X)\big[t_1,\dots,t_s,\{t_j^{-1} \mid j \notin U\}\big]$$

by Remark 3.4, that is, corresponding to $q(\overline{K_X+\sum_i D_i})$ in $\mathrm{Cl}(T)$. Therefore, we know that ω_T is T-free if and only if

$$\overline{K_X+\sum_i D_i} \in \sum_{j\notin U} \mathbb{Z}\overline{D_j}$$

in $\mathrm{Cl}(X)$.)

Put $X' = X \setminus \mathrm{Sing}(X)$. We choose positive integers $a_1,\dots,a_s$ and sections $f_1,\dots,f_t \in H^0(X,\sum_i a_iD_i)$ such that

- $\sum_i a_iD_i$ is an ample Cartier divisor,
- $X' = \bigcup_k D_+(f_k)$, and
- all of the D_i are principal Cartier divisors on $D_+(f_k)$ for $k=1,\dots,t$.

Put $W = \{\underline{n} \in \mathbb{Z}^s \mid n_i \geq 0 \text{ if } i \in U\}$. Put $D'_i = D_i|_{X'}$ for $i=1,\dots,s$. Consider the morphism

$$Y = \mathrm{Spec}_{X'}\Big(\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}\Big(\sum_i n_iD'_i\Big)t_1^{n_1}\cdots t_s^{n_s}\Big) \xrightarrow{\pi} X'.$$

Further, we have the natural map

$$\xi : Y \longrightarrow \mathrm{Spec}(T).$$

The group $\mathbb{G}_m^s$ naturally acts on $\mathrm{Spec}(T)$ and Y and trivially acts on X'. Both π and ξ are equivariant morphisms.

Claim 3.5. *There exists an equivariant open subscheme Z of both Y and* $\mathrm{Spec}(T)$ *such that*

- *the codimension of $Y \setminus Z$ in Y is greater than or equal to 2, and*
- *the codimension of* $\mathrm{Spec}(T)\setminus Z$ *in* $\mathrm{Spec}(T)$ *is greater than or equal to 2.*

Proof. For $u \in U$, there exist integers $c_{1u},\dots,c_{su}$ such that

- $H^0(X,\mathcal{O}_X(\sum_i c_{iu}D_i)) \neq 0$,
- $c_{uu} = -a_u$, and
- $c_{iu} > 0$ if $i \neq u$.

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In fact, if $u \in U$, there exist positive integers $q_1, \ldots, q_{u-1}$, $q_{u+1}, \ldots, q_s$ such that

$$\sum_{i \neq u} q_i D_i$$

is a sum of an ample divisor D and a Weil divisor F, which is linearly equivalent to an effective divisor by Lemma 3.2. Then,

$$H^0\Big(X, \mathcal{O}_X\Big(q\Big(\sum_{i \neq u} q_i D_i\Big) - a_u D_u\Big)\Big) = H^0\big(X, \mathcal{O}_X\big(q(D+F) - a_u D_u\big)\big) \neq 0$$

for $q \gg 0$.

For each $u \in U$, we set

$$(b_{1u}, \ldots, b_{su}) = (c_{1u}, \ldots, c_{su}) + (a_1, \ldots, a_s).$$

Here, note that $b_{uu} = 0$ and $b_{iu} > 0$ if $i \neq u$.

We choose

$$0 \neq g_u \in H^0\Big(X, \mathcal{O}_X\Big(\sum_i c_{iu} D_i\Big)\Big)$$

for each $u \in U$.

Consider the closed set of $\mathrm{Spec}(T)$ defined by the ideal J generated by

$$\{f_k t_1^{a_1} \cdots t_s^{a_s} \mid k = 1, \ldots, t\}$$

and

$$\{g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}} \mid k = 1, \ldots, t; u \in U\}.$$

By Theorem 1.3(2), we know that the height of J is greater than or equal to 2 since there is no prime ideal of T of height 1 which contains J.

We choose $d_{ki} \in k(X)^\times$ satisfying

$$H^0\big(D_+(f_k), \mathcal{O}_X(D_i)\big) = d_{ki} H^0\big(D_+(f_k), \mathcal{O}_X\big)$$

for each k and i. Then

$$(3.5) \qquad Y = \bigcup_{k=1}^{t} \pi^{-1}\big(D_+(f_k)\big) \qquad \text{and} \qquad \pi^{-1}\big(D_+(f_k)\big) = \mathrm{Spec}(C_k),$$

where

$$C_k = H^0\big(D_+(f_k), \mathcal{O}_X\big)\big[d_{k1}t_1, \ldots, d_{ks}t_s, \big\{(d_{kj}t_j)^{-1} \bigm| j \notin U\big\}\big].$$

We put

$$Z = \mathrm{Spec}(T) \setminus V(J).$$

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Then we have

$$
\begin{aligned}
Z = \bigcup_{k=1}^{t} \Big[& \operatorname{Spec}\big(T\big[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}\big]\big) \\
& \cup \Big\{ \bigcup_{u \in U} \operatorname{Spec}\big(T\big[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big]\big) \Big\} \Big].
\end{aligned} \tag{3.6}
$$

Here, we have

$$
\begin{aligned}
T\big[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}\big] &= H^0\big(D_+(f_k), \mathcal{O}_X\big)\big[(d_{k1}t_1)^{\pm 1}, \ldots, (d_{ks}t_s)^{\pm 1}\big] \\
&= C_k\Big[\Big(\prod_{j \in U} (d_{kj}t_j)\Big)^{-1}\Big].
\end{aligned} \tag{3.7}
$$

On the other hand,

$$
\begin{aligned}
& T\big[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big] \\
&\quad = \bigoplus_{(\underline{n}) \in \mathbb{Z}^s} T\big[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big]_{(n_1, \ldots, n_s)} \\
&\quad = \bigoplus_{\substack{(\underline{n}) \in \mathbb{Z}^s \\ n_u \geq 0}} R\big[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big]_{(n_1, \ldots, n_s)} \\
&\quad = \bigoplus_{\substack{(\underline{n}) \in \mathbb{Z}^s \\ n_u \geq 0}} R\big[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big]_{(n_1, \ldots, n_s)} \\
&\quad = C_k\big[\big\{(d_{kj}t_j)^{-1} \bigm| j \neq u\big\}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big].
\end{aligned}
$$

Let β_{ku} be an element in $H^0(D_+(f_k), \mathcal{O}_X)$ such that

$$
g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}} = \beta_{ku} (d_{k1}t_1)^{b_{1u}} \cdots (d_{ks}t_s)^{b_{su}}
$$

for $k = 1, \ldots, t$ and $u \in U$. Then,

$$
\begin{aligned}
& C_k\big[\big\{(d_{kj}t_j)^{-1} \bigm| j \neq u\big\}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}\big] \\
&\quad = C_k\Big[\Big(\beta_{ku} \prod_{\substack{j \in U \\ j \neq u}} (d_{kj}t_j)\Big)^{-1}\Big].
\end{aligned} \tag{3.8}
$$

By (3.5), (3.6), (3.7), and (3.8), we know that Z is an open subscheme of Y. The ideal of C_k generated by

$$\prod_{j\in U}(d_{kj}t_j) \qquad \text{and} \qquad \Big\{\beta_{ku}\prod_{\substack{j\in U\\ j\neq u}}(d_{kj}t_j)\,\Big|\, u\in U\Big\}$$

is the unit ideal or of height 2. (If $U=\emptyset$, then $Z=Y$ by the construction. If $U=\{u\}$ and if β_{ku} is a unit element, then this ideal is the unit. In other cases, this ideal is of height 2.) Therefore, the codimension of $Y\setminus Z$ in Y is greater than or equal to two. ∎

We can define the graded canonical module as in [5, Definition 3.1] using the theory of the equivariant twisted inverse functor (see [4]).

By Claim 3.5 above and [5, Remark 3.2], we have $\omega_T = H^0(Y,\omega_Y)$. On the other hand, we have

$$\begin{aligned}\omega_Y &= \bigwedge^{s}\Omega_{Y/X'}\otimes\pi^*\mathcal{O}_{X'}(K_{X'})\\ &= \pi^*\mathcal{O}_{X'}\Big(\sum_i D'_i\Big)(-1,\ldots,-1)\otimes_{\mathcal{O}_Y}\pi^*\mathcal{O}_{X'}(K_{X'})\\ &= \pi^*\mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)(-1,\ldots,-1),\end{aligned}$$

where $(-1,\ldots,-1)$ denotes the shift of degree (see [4, Theorem 28.11]).

Then, we have

$$H^0(Y,\omega_Y) = H^0\Big(X', \pi_*\pi^*\mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)(-1,\ldots,-1)\Big).$$

By the projection formula (see [4, Lemma 26.4]),

$$\begin{aligned}&\pi_*\pi^*\mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)(-1,\ldots,-1)\\ &= \Big(\mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)\otimes\pi_*\mathcal{O}_Y\Big)(-1,\ldots,-1)\\ &= \Big(\mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)\otimes\Big[\bigoplus_{\underline{n}\in W}\mathcal{O}_{X'}\Big(\sum_i n_i D'_i\Big)\Big]\Big)(-1,\ldots,-1)\end{aligned}$$

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$$= \Big(\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}\Big(\sum_i (n_i+1)D'_i + K_{X'}\Big)\Big)(-1,\dots,-1)$$
$$= \bigoplus_{\underline{n}\in W+(1,\dots,1)} \mathcal{O}_{X'}\Big(\sum_i n_i D'_i + K_{X'}\Big).$$

Therefore, we have

$$\begin{aligned} H^0(Y,\omega_Y) &= H^0\Big(X', \bigoplus_{\underline{n}\in W+(1,\dots,1)} \mathcal{O}_{X'}\Big(\sum_i n_i D'_i + K_{X'}\Big)\Big) \\ &= \bigoplus_{\underline{n}\in W+(1,\dots,1)} H^0\Big(X', \mathcal{O}_{X'}\Big(\sum_i n_i D'_i + K_{X'}\Big)\Big) \\ &= \bigoplus_{\underline{n}\in W+(1,\dots,1)} H^0\Big(X, \mathcal{O}_{X}\Big(\sum_i n_i D_i + K_{X}\Big)\Big) \\ &= M_{K_X} \cap t_1\cdots t_s k(X)\big[t_1,\dots,t_s,\{t_j^{-1} \mid j \notin U\}\big]. \end{aligned}$$

We have completed the proof of Theorem 1.3.

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