

m	A	B	C	p	$\left(\frac{A+B\sqrt{m}}{p}\right)$	$(-1)^{(p-1)/8}$, if $p \equiv 1(8)$ $(-1)^{(p+m-1)/8}$, if $p \equiv 7(8)$	$\left(\frac{p}{m/2}\right)_4$
10	50	9	13	31	-1	-1	1
10	50	13	9	31	-1	-1	1
26	130	11	23	17	-1	1	-1
26	130	17	19	23	-1	1	-1
34	34	3	5	47	1	1	1
34	170	3	29	47	1	1	1
58	58	3	7	23	1	1	1
58	58	7	3	23	1	1	1
74	74	5	7	41	1	-1	-1
74	74	7	5	41	1	-1	-1
82	82	1	9	23	-1	-1	1
82	82	9	1	23	-1	-1	1

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The divisor problem for arithmetic progressions

by

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1. Introduction. Let $n \geq 1$ and $r \geq 2$ be integers and let $d_r(n)$ denote the number of ordered r -tuples (n_1, \dots, n_r) of positive integers for which

$$\prod_{1 \leq j \leq r} n_j = n.$$

For $(a, q) = 1$ define

$$D_r(X, q, a) = \sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} d_r(n).$$

We are interested here in finding real numbers θ_r , as large as possible, such that the following statement holds.

(S) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$D_r(X, q, a) - \frac{X}{\varphi(q)} P_r(\log X) \ll_\varepsilon \frac{X^{1-\delta}}{\varphi(q)},$$

provided that $q < X^{\theta_r - \varepsilon}$.

Here $P_r(\log X)$ is the residue at $s = 1$ of $s^{-1} L(s, \chi_0) X^{s-1}$, where χ_0 is the principal character of modulus q .

It was discovered independently by Selberg and by Hooley that Weil's estimate for the Kloosterman sum yielded the above statement with $\theta_2 = 2/3$. The authors [2] recently proved that one may take $\theta_3 = 1/2 + 1/230$. The result with $\theta_4 = 1/2$ seems harder to trace but was known to Linnik. In this paper we are able to improve the results $\theta_r = 8/(3r+4)$ for $r \geq 5$ which are due to Lavrik [5].

THEOREM. *The statement (S) holds in the following cases:*

(I) $\theta_5 = 9/20$,

(II) $r \geq 6$ and $\theta_r = \min\{8/3r, 5/12\}$,

(III) q is restricted to cube-free integers, $r \geq 7$, and $\theta_r = \min\{4/r, 5/12\}$.

Although the proof of this result involves some fairly deep arguments, these are for the most part already recorded in the literature and we shall quote liberally therefrom.

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The main ingredients in the proof are the Burgess estimates [1] for character sums and the recent work by several authors [3] on the difference between consecutive primes. Indeed, it is no coincidence that $1 - \frac{5}{12}$ and $1 - \frac{9}{20}$ have been numbers of significance in this latter area.

2. Lemmata.

LEMMA 1 (Burgess [1]). For q a positive integer, let

$$\gamma = \gamma(q) = \begin{cases} \frac{1}{4} & \text{if } q \text{ is cube-free,} \\ \frac{3}{8} & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$. There exists $\delta(\varepsilon) > 0$ such that, for all non-principal characters $\chi \pmod q$ and all $L \geq q^{\gamma+\varepsilon}$, we have

$$\sum_{l \leq L} \chi(l) \ll_{\varepsilon} Lq^{-\delta}.$$

LEMMA 2. For $T \geq 2$ we have

$$\sum_{\chi \pmod q} \int_{-T}^T |L(\frac{1}{2} + it, \chi)|^4 dt \ll qT \log^4 qT.$$

Lemma 2 follows at once from Theorem 10.1 of [6].

LEMMA 3.

$$\sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \chi(n) \right|^4 \ll N^2 q \log^6 q.$$

Proof. We may assume that N is half an odd integer and is less than q , whence

$$\sum_{n \leq N} \chi(n) = \frac{1}{2\pi i} \int_{U-iT}^{U+iT} L(s, \chi) N^s \frac{ds}{s} + O(NT^{-1} \log N),$$

where $U = 1 + \log^{-1} N$, $T = (qN)^{1/2}$.

Since

$$|L(\sigma \pm iT)| \ll_{\varepsilon} (qT)^{(1-\sigma)/2+\varepsilon} \quad \text{for } \frac{1}{2} \leq \sigma \leq U,$$

we have

$$S = \left| \sum_{n \leq N} \chi(n) \right| \ll N^{1/2} \int_{1/2-iT}^{1/2+iT} |L(s, \chi)| \left| \frac{ds}{s} \right| + O(N^{1/2}).$$

Hölder's inequality gives

$$S^4 \ll N^2 \log q \int_{1/2-iT}^{1/2+iT} |L(s, \chi)|^4 \left| \frac{ds}{s} \right| + O(N^2).$$

We sum over χ , then apply Lemma 2 and partial summation. This completes the proof.

LEMMA 4. Let $L, M, N \geq 1$, $LMN = X$, $(a, q) = 1$, $\varepsilon > 0$, and $\gamma = \gamma(q)$ as in Lemma 1. Assume

- (1) $q < X^{9/20-\varepsilon}$,
- (2) $X^{\varepsilon} q^{\gamma} < L$,
- (3) $M, N < X^{1-\varepsilon} q^{-6/5}$.

Let $\{\alpha_m | m \leq M\}$, $\{\beta_n | n \leq N\}$ be sequences of complex numbers with $|\alpha_m| < X^{\varepsilon}$, $|\beta_n| < X^{\varepsilon}$, for all m, n . Then, there exists $\delta(\varepsilon) > 0$ such that

$$\sum_{\chi \neq \chi_0} \left| \sum_{l \leq L} \chi(l) \right| \left| \sum_{m \leq M} \alpha_m \chi(m) \right| \left| \sum_{n \leq N} \beta_n \chi(n) \right| \ll_{\varepsilon} X^{1-\delta}.$$

We omit the proof of Lemma 4 as it is an almost *verbatim* duplicate of the argument on pages 102–104 of [4], the only difference being that there one dealt with Dirichlet polynomials on $\sigma = 1/2$ rather than $\sigma = 0$ as here. It should be mentioned however that this proof is the heart of the matter. In addition to using Lemmata 1 and 3 it appeals to the large sieve and the Halász–Montgomery–Huxley method. It is perhaps best described as an analogue for arithmetic progressions of the lemma of Heath-Brown used in [3] to estimate the difference of consecutive primes.

3. Proof of the theorem. Let N_1, \dots, N_r satisfy

$$(4) \quad N_1 \geq N_2 \geq \dots \geq N_r \geq 1, \quad \prod_{j=1}^r N_j = X.$$

Let $\Delta = X^{-\eta}$ where $\eta > 0$ is fixed. Let

$$E(N_1, \dots, N_r) = \sum_{\substack{(1-\Delta)N_j < n_j \leq N_j \\ n_1 \dots n_r \equiv a \pmod q}} \dots \sum_{\substack{(1-\Delta)N_j < n_j \leq N_j \\ (n_1 \dots n_r, q) = 1}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(1-\Delta)N_j < n_j \leq N_j \\ (n_1 \dots n_r, q) = 1}} 1.$$

By an elementary argument (see, for example, the first part of the proof of Theorem 5 of [2]), the proof may be reduced to the demonstration that, for an arbitrary box N_1, \dots, N_r satisfying (4), we have

$$E(N_1, \dots, N_r) \ll \frac{1}{\varphi(q)} X^{1-\delta}.$$

Now,

$$\begin{aligned} E(N_1, \dots, N_r) &= \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) \sum_{(1-\Delta)N_j < n_j \leq N_j} \chi(n_1 \dots n_r) \\ &\ll \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \sum_{n_1} \chi(n_1) \right| \left| \sum_{m \leq M} \alpha_m \chi(m) \right| \left| \sum_{n \leq N} \beta_n \chi(n) \right|. \end{aligned}$$

Here we have fixed a partition \mathcal{M}, \mathcal{N} of the set $\{2, \dots, r\}$, let

$M = \prod_{j \in \mathcal{M}} N_j$, $N = \prod_{j \in \mathcal{N}} N_j$, with α_m being the number of representations of m in the form

$$\prod_{\substack{(1-\delta)N_j < n_j \leq N_j \\ j \in \mathcal{M}}} n_j = m,$$

and β_n defined similarly. Note that

$$\alpha_m \leq d_{r-1}(m), \quad \beta_n \leq d_{r-1}(n).$$

By Lemma 4, it remains to show that, given any N_1, \dots, N_r satisfying (4), we have $N_1 > q^\nu x^\varepsilon$ and there exists a partition \mathcal{M}, \mathcal{N} for which (3) holds. Since $N_1 \geq X^{1/r}$ by (4), the first requirement follows easily. Moreover, by choosing say $\mathcal{M} = \{2, 3, \dots, k\}$ where $M = \prod_{j=2}^k N_j \leq X^{1/2}$ with k maximal, we have $M \geq X^{1/2} N_1^{-1}$ so $N \leq X^{1/2}$. Combining this with (3) we get the theorem for $r \geq 6$ (as well as the statement with $\theta_5 = 5/12$). The remaining estimate with $\theta_5 = 9/20$ requires a little more effort.

Case I: If $N_2 \leq q^{1/2+\delta}$, then $\mathcal{M} = \{2, 3\}$, $\mathcal{N} = \{4, 5\}$ gives a decomposition with $M, N \leq N_2^2 \leq q^{1+2\delta}$. Since $q < Xq^{-6/5}$ if $q < X^{5/11} < X^{9/20-\varepsilon}$, (3) holds and Lemma 4 gives the result in this case if $\delta(\varepsilon)$ is sufficiently small.

Case II: If $N_2 > q^{1/2+\delta}$ we abandon Lemma 4 and write

$$E(N_1, \dots, N_5) \leq \frac{1}{\varphi(q)} \sum_{x \neq x_0} \left| \sum_{n_1} \chi(n_1) \right| \left| \sum_{n_2} \chi(n_2) \right| \sum_{m \leq M} \alpha_m \chi(m)$$

where $m = n_3 n_4 n_5$ and α_m is defined by the same prescription as before. By Hölder's inequality,

$$E(N_1, \dots, N_5) \leq \frac{1}{\varphi(q)} \left(\sum_{x \neq x_0} \left| \sum_{n_1} \chi(n_1) \right|^4 \right)^{1/4} \left(\sum_{x \neq x_0} \left| \sum_{n_2} \chi(n_2) \right|^4 \right)^{1/4} \left(\sum_x \left| \sum_m \alpha_m \chi(m) \right|^2 \right)^{1/2},$$

and, for any $\varepsilon > 0$, this is

$$\ll_\varepsilon \frac{1}{\varphi(q)} q^{1/2} (M+q)^{1/2} X^{1/2+\varepsilon}$$

by Lemma 3 and the large sieve inequality. Since $qM = qXN_1^{-1}N_2^{-1} < Xq^{-2\delta} < X^{1-\delta_1}$ (we may assume $q > X^{1/3}$), and since $q < X^{1/2-\delta_2}$, the result follows.

Remark. The estimate $\theta_r = 5/12$ for $r \geq 6$ cannot be improved by this method as can be seen by considering the case $N_1 = \dots = N_6 = X^{1/6}$, $q = X^{5/12}$. Here, for any partition, either $M \geq X^{1/2}$ or $N \geq X^{1/2}$ so (3) fails to hold. A similar phenomenon has already been observed by several people in connection with the problem of consecutive primes.

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