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THE DOMAIN RANK OF OPEN SURFACES OF INFINITE GENUS

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ABSTRACT. In a recent paper it was shown that an open surface, i.e. a connected 2-manifold without boundary, has finite domain rank if and only if it has finite genus. In the present paper, it is shown that the domain rank of any open surface of infinite genus is countably infinite.

1. Introduction. Throughout the presentation, the notation and definitions found in [4] will be used. The principal tool used in this paper is a modification of Theorem 3 of [3]. As in [3], if M is an open surface, let (X, Y, Z) denote the triple of compact totally disconnected spaces determining the ideal boundary of M, where X is the entire boundary, Y is the set of nonplanar boundary points, and Z is the set of nonorientable boundary points. In the proof of Theorem 3 of [3], one is mainly concerned with how to identify $C^+(k)$ and $C^-(k)$ to produce an orientable or nonorientable compact surface. The same result can be obtained by using only $C^-(k)$ and properly identifying points of $C^-(k)$ to produce a compact surface which is either a handle or möbius band. This alteration has no effect on the validity of the remainder of the proof and the following modification of Theorem 3 of [3] is obtained.

THEOREM 1.1. *Let* $S^2=\{x\in R^3 \mid \|x\|=1\}$, $E^1=\{x\in S^2 \mid x_3\geqq 0\}$, $E^2=\{x\in S^2 \mid x_3\leqq 0\}$, *and* $S^1=E^1\cap E^2$. *Every open surface M is homeomorphic to the surface formed from S^2 by first removing a closed totally disconnected set $X\subset S^1$ from S^2 and then removing the interiors of a finite or infinite sequence $C_1, C_2, \cdots$ of pairwise disjoint 2-cells in E^2-S^1 and properly identifying the boundaries of these 2-cells to produce handles or möbius bands. The sequence $C_1, C_2, \cdots$ "approaches X" in the sense that if U is an open set of S^2, $X\subset U$, then all but a finite number of the C_i are contained in U.*

If in forming M via the above theorem, bd C_j is identified to produce a handle, C_j will be called *toral*; otherwise, C_j will be called *projective*.

If M is an open surface of infinite genus, then M belongs to one of the following four classes of surfaces: class 1—orientable surfaces;

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class 2—infinitely nonorientable surfaces; class 3—nonorientable surfaces of odd parity; class 4—nonorientable surfaces of even parity. Thus if an open surface M is formed from S^2 via Theorem 1.1, it may and henceforth will be assumed that if (i) $M \in$ class 1, each C_j is toral; (ii) $M \in$ class 2, C_1, C_2, and an infinite number of C_j are projective; (iii) $M \in$ class 3, C_1 is projective and all other C_j are toral; (iv) $M \in$ class 4, C_1, C_2 are projective and all other C_j are toral.

2. **Proof of the theorem.** As in [4], the proof of the theorem depends upon showing that there is an open surface D which is a generator of every open surface M belonging to a certain class of open surfaces. Let X denote the subset of the real line defined by $X = \{0\} \cup \{1/n \mid n \text{ is a positive integer}\}$ and let $A \subset X$, $A = \{0\}$. For each i, $1 \leqq i \leqq 4$. Let D_i denote the open surface of infinite genus belonging to class i such that $(X, A, \varnothing)$, (X, A, A), $(X, A, \varnothing)$, and $(X, A, \varnothing)$ are the ideal boundaries of D_1, D_2, D_3, and D_4 respectively.

THEOREM 2.1. *Let M be an open surface of infinite genus, $M \in$ class q. Then D_q is a generator of M.*

PROOF. Let M be formed from S^2 via Theorem 1.1 and let $\Gamma = \{C_j\}_{j=1}^{\infty}$ be the sequence of 2-cells which "approaches X." Since the sequence "approaches X," there exists a subsequence $\Omega = \{C_j'\}_{j=1}^{\infty}$ of Γ such that $C_j' = C_j$, $j = 1, 2$; C_j' is projective for all j if $q = 2$; and Ω converges to a point $c \in X$. Therefore, there exists a 2-cell $B \subset E^2$ such that $B \cap S^1 = c$; $C_j' \subset \text{int } B$, $j \geqq 1$; and $C_j \cap B \neq \varnothing \Leftrightarrow C_j \in \Omega$. Since (1) $W = S^2 - (\text{int } B \cup X)$ is homeomorphic to a domain of $H_2 = \{x \in R^2 \mid x_2 \geqq 0\}$ which contains bd H_2, (2) $C_j \subset \text{int } W$ for all $C_j \in \Gamma - \Omega$, and (3) Γ "approaches X," there exists a sequence $\{L_k\}_{k=0}^{\infty}$ of compact surfaces of genus 0 contained in S^2 such that

(i) $B = L_0$;

(ii) $L_k = B_k - (\bigcup_{p=1}^{n(k)} \text{int } E_p^k)$ where B_k is a 2-cell in S^2 such that $c \in \text{bd } B_k$ and $\{E_p^k\}_{p=1}^{n(k)}$ is a finite disjoint collection of 2-cells contained in int B_k, $k \geqq 1$;

(iii) $L_k \cap X = c = \text{bd } L_k \cap X$, $k \geqq 0$;

(iv) $(L_k - c) \subset \text{int } L_{k+1}$, $k \geqq 0$;

(v) for each k, there are at most a finite number of 2-cells $C_j \in \Gamma - \Omega$ such that $C_j \cap L_k \neq \varnothing$, in which case $C_j \subset \text{int } L_k$;

(vi) $S^2 - X = \bigcup_{k=1}^{\infty} \text{int } L_k$.

For each $k \geqq 1$, let A_k be a sequence of points in S^2 which converges to c such that $A_k \subset \text{int } L_k - (L_{k-1} \cup (\bigcup_{j=1}^{\infty} C_j))$. Let $Q_k = \text{int } L_k - A_k$. Then for all $k \geqq 1$, Q_k is a domain of $S^2 - X$, $Q_k \subset Q_{k+1}$, and furthermore $S^2 - X = \bigcup_{k=1}^{\infty} Q_k$. Let $p: S^2 - (\bigcup_{j=1}^{\infty} \text{int } C_j) \to M$ be the identification map used in forming M via Theorem 1.1. For each $k \geqq 1$, let $G_k = Q_k - (\bigcup_{j=1}^{\infty} \text{int } C_j)$. Since $C_j \cap Q_k \neq \emptyset \Leftrightarrow C_j \subset Q_k$; the set $M_k = p(G_k)$ is a domain of M. In view of the properties of $\{Q_k\}_{k=1}^{\infty}$, it follows that $M = \bigcup_{k=1}^{\infty} M_k$ and $M_k \subset M_{k+1}$ for $k \geqq 1$. It follows from (i), (iv), and (v) that for all $k \geqq 1$, $M_k \in$ class q. But by (ii), $c \in \text{bd } B_k$ and the sequence of 2-cells Ω and the sequence of points A_k both converge to c. Therefore in view of (v) the ideal boundary (X_k, Y_k, Z_k) of M_k is homeomorphic to the ideal boundary of D_q. Since $M_k \in$ class q, it follows from Theorem 1 of [3] that $M_k \equiv D_q$ for all $k \geqq 1$, and therefore D_q is a generator of M.

As an immediate consequence we have the following:

THEOREM 2.2. *Let M be an open surface. If M has infinite genus, then $DR(M)$ is countably infinite; otherwise, $DR(M)$ is finite.*

PROOF. Let M be an open surface of infinite genus. If D is a proper domain of M, then D is an open surface of infinite or finite genus. If D has infinite genus, then it follows from the previous theorem that D_k is a generator of D for some k, $1 \leqq k \leqq 4$. If D has finite genus, then it follows from Lemma 2.7 of [4] that $Q(\infty)$ is a generator of D, Q a closed surface. Since there are only a countable number of topologically distinct closed surfaces, $DR(M)$ is at most countably infinite. The result now follows from Corollary 2.9 of [4].

Although any open surface has at most countably infinite domain rank, there do exist open 3-manifolds which have noncountable domain rank. Such a 3-manifold M can be constructed from a countable collection $\{M_j\}_{j=1}^{\infty}$ of closed irreducible, orientable 3-manifolds such that $M_q \equiv M_r \Leftrightarrow q = r$ (see for example [2]), by taking M to be the infinite connected sum of this collection. Then using the results of [1] it can be shown that $DR(M)$ is not countably infinite.

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