The Double Points of Mathieu's Differential Equation

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Abstract. Mathieu's differential equation, $y'' + (a - 2q \cos 2x)y = 0$, admits of solutions of period π or 2π for four countable sets of characteristic values, $a(q)$, which can be ordered as $a_r(q)$, $r = 0, 1, \cdots$. The power series expansions for the $a_r(q)$ converge up to the first double point for that order in the complex plane. [At a double point, $a_r(q) = a_{r+2}(q)$.] The present work furnishes the double points for orders r up to and including 15. These double points are singular points, and the usual methods of determining the characteristic values break down at a singular point. However, it was possible to determine two smooth functions in which one could interpolate for both q and $a_r(q)$ at the singular point. The method is quite general and can be used in other problems as well. ■

1. Introduction. Mathieu's differential equation

$$y'' + (a - 2q \cos 2x)y = 0 \tag{1.0}$$

admits of four countable sets of *characteristic* values, $a_r(q)$, corresponding to which the solutions $y(x)$ are periodic, and of period π or 2π. These four sets are associated with solutions defined below.

$$y(q, a_{2m}, x) = \sum_{k=0}^{\infty} A_{2k} \cos 2kx \,, \quad a = a_{2m}(q), m = 0, 1, \cdots , \tag{1.10}$$

$$y(q, a_{2m+1}, x) = \sum_{k=0}^{\infty} A_{2k+1} \cos (2k+1)x \,, \quad a = a_{2m+1}(q), m = 0, 1, \cdots \tag{1.11}$$

$$u(q, b_{2m}, x) = \sum_{k=1}^{\infty} B_{2k} \sin 2kx \,, \quad a = b_{2m}, m = 1, 2, \cdots , \tag{1.12}$$

$$u(q, b_{2m+1}, x) = \sum_{k=0}^{\infty} B_{2k+1} \sin (2k+1)x \,, \quad a = b_{2m+1}, m = 0, 1, \cdots . \tag{1.13}$$

When $q = 0$, $a_r(q) = b_r(q) = r^2$, $r = 0, 1, \cdots$, and the corresponding solutions are $y(q, r^2, x) = \cos rx$, $u(q, r^2, x) = \sin rx$.* If $q \neq 0$, the four sets of eigenvalues are distinct, and there is only one periodic solution corresponding to a characteristic value. The second, independent solution of (1.0), associated with the eigenvalue, is not periodic.

If q is real and different from zero, it is known that the eigenvalues are all real and simple. They can be ordered as follows:

$$a_0 < b_1 < a_1 < b_2, < \cdots , \quad q > 0$$
$$a_0 < a_1 < b_1 < b_2, < \cdots , \quad q < 0 .$$

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* If $r = 0$, $\sin rx$ is a trivial solution; the odd solutions begin with $r = 1$.

If q is real, the sets $\{a_r(q)\}$ and $\{b_r(q)\}$ are characterized by (1.14) and (1.15) below:

(1.14) The solutions $y(q, a_r, x)$, $u(q, b_r, x)$ have r zeros in the interval $0 \leqq x < \pi$.

(1.15) As $q \to 0$, $a_r(q) \to r^2$, $b_r(q) \to r^2$—valid in the complex q-plane.

Another important property is given in (1.2); it holds in the complex q-plane.

$$a_{2m}(-q) = a_{2m}(q); \quad b_{2m}(-q) = b_{2m}(q); \quad a_{2m+1}(-q) = b_{2m+1}(q). \tag{1.2}$$

Power-series expansions for the characteristic values, as functions of q, were first developed by Mathieu [4]. An algorithm, suitable for computers, by means of which one may obtain the successive coefficients of the power series has been given in [6]. The radii of convergence of these power series, however, remained largely unknown since these depend on a knowledge of the double points (singular points) in the complex plane. The present work supplies these singular points for orders $r \leqq 15$.

Mulholland and Goldstein [5] published the first multiple eigenvalue. They found that for imaginary q, namely $q = is$, there is a singular point at $s = 1.468 \cdots$ where $a_0(q)$ and $a_2(q)$ have a common value. [It can be shown that, aside from the origin, double points can arise only between members of the same set; there can be no double points connecting orders of different sets.] The value of $a(q)$ at the singular point, however, was obtained in [5] only in the order of magnitude. These authors noted that a_0 and a_2 are real up to the singular point, and become complex conjugates of each other after the singular point. They conjectured that if q is purely imaginary, similar situations will hold for a_{4m} and a_{4m+2}, for all m, and for b_{2m+2} and b_{2m+4}. [It should be noted that the eigenvalues of odd order have no singular points on the 90°-ray.]

Bouwkamp [2] verified and improved the first singular point, giving $s = 1.468769$, but he gave the value of $a(q)$ to only 3 decimals, namely $a = 2.088$. The value of $a(q)$ at a singular point is indeed difficult to obtain by the methods employed by the authors cited. In the method to be explained below this difficulty disappears. Moreover, the procedure is general and is applicable to other problems as well.

From (1.0) and (1.2), it is sufficient to determine $a(q)$ and the singular points for values of q in the first quadrant of the complex plane. For, if $a_r(q)$ and $b_r(q)$ are known for $q = \rho \exp(i\phi)$, then $a_r(-q)$ is known from (1.2). Moreover, an examination of (1.0) shows that $\bar{a}_r(q)$, [or $\bar{b}_r(q)$], when associated with $\bar{y}(q, a_r, x)$, [or $\bar{u}(q, b_r, x)$] satisfies (1.0) when q is replaced by $\bar{q}$. Hence, in what follows, define

$$q = \rho e^{i\phi}, 0 < \varphi \leqq 90^\circ; \qquad a_r(q) = a_r(\rho, \phi). \tag{1.3}$$

[Values on the real axis will not be discussed, since they are amply tabulated, and there are no multiple eigenvalues, except when $q = 0$.]

2. Auxiliary Functions, Useful near a Singular Point. The continued-fraction method formed the basis for the present calculations. A full discussion of the method is given in [1]. In addition, a comprehensive code now exists [3] for obtaining all solutions of Mathieu's equation, including the eigenvalues, for $q > 0$. A part of this code was modified to operate with complex arithmetic. Certain other modifications were necessary, since one could no longer assume that all eigenvalues are simple ones. For the sake of conciseness, the derivation of the particular continued fraction forms will not be repeated here. The availability of [1] will be assumed and only the necessary modifications will be explained below.

In essence, there is a complex-valued function, say $T(a, q)$, such that, a necessary and sufficient condition for $a(q)$ to be an eigenvalue is that $T(a, q) = 0$. The order, r, is not determined. It is obtained from continuity beginning with the eigenvalue for $\rho = 0$, where the order is known and continuing at an interval, $\Delta\rho$, (for a fixed ϕ) which is sufficiently small for adequate extrapolation of a first approximation. In the discussion to follow, the symbols a_r and $a_r(q)$ will be used to imply members of any one of the four sets, since the discussion applies equally well to those eigenvalues giving rise to odd solutions as to even solutions. In the few cases where a distinction between the two is made, the fact will be stated. For brevity let

$$(2.01)\qquad \begin{aligned} &T_0 = T(a, q)\,, \qquad T_1 = T_1(a, q) = \partial T(a, q)/\partial a\,, \\ &T_2 = T_2(a, q) = \partial^2 T(a, q)/\partial a^2\,. \end{aligned}$$

Assume that in the neighborhood of an eigenvalue, T_0, T_1, and T_2 are continuous functions of a. [No assumptions are made about $|da/d\rho|$ or $|da/dq|$; these do become infinite at a multiple eigenvalue.] It is shown in [1] that for real, positive values of q, $|T_1|$ is bounded away from zero—indeed if q is positive, $|T_1| \geqq 1/q$. In the complex plane, however, this is no longer true, since a double point is characterized precisely by $T_0 = T_1 = 0$. However, if one is not too close to a singular point, then it is possible to use Newton's method, the same as in the real case. Thus, let a_r^{k-1} be a sufficiently close approximation to $a_r(q)$. Define

$$(2.02)\qquad \Delta a_r^{k-1} = -T_0(a_r^{k-1}, q)/T_1(a_r^{k-1}, q)$$

$$(2.03)\qquad a_r^k = a_r^{k-1} + \Delta a_r^{k-1}\,.$$

In practice, convergence to within a preset tolerance was obtained after four iterations or less in the great majority of cases; rarely were more than 9 iterations required. Suitable precautionary tests have to be included to insure that the new approximation, a_r^k, is within a reasonable distance from $a_r(\rho - h, \phi)$, so as to insure that the approximation approaches the rth eigenvalue and no other.

Consider the Taylor series for $T(a, q)$, namely

$$(2.04)\qquad T(a + \Delta a, q) = T_0 + \Delta a T_1 + \tfrac{1}{2}(\Delta a)^2 T_2 + O(\Delta a)^3\,.$$

Dropping terms in $(\Delta a)^3$, and solving for a zero of $T(a + \Delta a, q)$, one obtains

$$(2.05)\qquad \Delta a = -(T_1/T_2) + \sigma((T_1/T_2)^2 - (2T_0/T_2))^{1/2}\,, \quad \sigma = \pm 1\,.$$

The approximation (2.05) is more suitable near a singular point than (2.03). Since the terms in (2.04) are in general complex numbers, the sign of σ is more difficult to determine than in the real case. Let

$$(2.06)\qquad w_1 = ((T_1/T_2)^2 - (2T_0/T_2))^{1/2}\,,$$

assuming that one of the two values of the radical has been taken. Define

$$(2.07)\qquad \Delta_1 a = -(T_1/T_2) + w_1\,, \qquad \Delta_2 a = -(T_1/T_2) - w_1\,.$$

If the iterative process is to converge, then eventually $|\Delta a|$ should approach zero.

It is therefore reasonable to choose that value of Δa which is smaller in magnitude. The ambiguous case, when both values of Δa are equal in magnitude, occurs only in exceptional cases near a singular point. The method of dealing with it will be further discussed in Section 3.

Consider (2.05) when

$$|T_1/T_2|^2 \gg |2T_0/T_2| . \tag{2.10}$$

Let us factor $(T_1/T_2)^2$ from the radical; in view of the assumption (2.10), the radical can be expanded by the binomial theorem and is in fact determined—again because we choose the smaller of the two possible values of $|\Delta a|$. In this case (2.05) reduces to

$$\begin{aligned} \Delta a &= -(T_1/T_2) + (T_1/T_2)(1 - (2T_0T_2/T_1^2))^{1/2} \\ &= -(T_0/T_1)(1 + (\tfrac{1}{2}T_0T_2/T_1^2) + \cdots) . \end{aligned} \tag{2.11}$$

It is clear that Δa of (2.11) differs little in nature from (2.02). This situation will be true in regions where $|T_1|$ is sufficiently large.

On the other hand, consider a region where

$$|2T_0/T_2| \gg |(T_1/T_2)^2| . \tag{2.12}$$

Again factoring the numerically dominant term of the radical, one obtains

$$\Delta a = -(T_1/T_2) + \sigma(-2T_0/T_2)^{1/2}\cdot(1 - (T_1^2/T_0T_2))^{1/2} . \tag{2.13}$$

In (2.13) the behavior of Δa is radically different from that in (2.02). Whether or not the eigenvalue $a(q)$ is a simple one, $T(a^k, q)$ must approach zero as a^k approaches $a(q)$. If $a(q)$ is not a simple eigenvalue, T_1 will also approach zero, in such a way that $(1 - (T_1^2/T_0T_2))^{1/2}$ remains finite. The radical $(-2T_0/T_2)^{1/2}$ in (2.13) gives an insight into the behavior of $a(q)$ near a singular point. Suppose $a_r(q) = a_{r+2}(q)$. As the branches $a_r(q)$ and $a_{r+2}(q)$ are generated, the values $T(a^k, q)$ will tend to be the same, when the a^k of the two branches approach each other—as they must. Let

$$w = (-2T_0/T_2)^{1/2}\cdot(1 - (T_1^2/T_0T_2))^{1/2} \tag{2.14}$$

assuming either choice of the radical. It is to be expected that if $\Delta_1 a = (-T_1/T_2) + w$ is a suitable increment for $a_r^k(q)$, then $\Delta_2 a = (-T_1/T_2) - w$ will be the corresponding increment for $a_{r+2}^k(q)$.

It is important to observe the following:

Near a singular point, the radical in (2.13) *is eliminated in the functions* (2.15) *and* (2.16) *defined below.*

$$FA(q) = \tfrac{1}{2}(a_r(q) + a_{r+2}(q)) = FA_1 + iFA_2, \textit{ say} . \tag{2.15}$$

$$FB(q) = (a_{r+2}(q) - a_r(q))^2 = FB_1 + iFB_2, \textit{ say} . \tag{2.16}$$

The functions $FA(q)$ and $FB(q)$ are smooth in the neighborhood of the singular point, when T_2 is smooth. They may have singularities elsewhere. For example, if $a_r(q_1)$ also has a double point with $a_{r-2}(q_1)$, but not with $a_{r+2}(q_1)$, then in the neighborhood of q_1, FA and FB will mirror the singularities at this point, and they will not be smooth functions. However, the fact that both $FA(q)$ and $FB(q)$ are smooth near the singularity is of great importance in computation. For it permits us to bypass a region close to the singular point, and to obtain the value of q at which $a(q)$ is singular by interpolation in a smooth function. In this way the double eigenvalue can be obtained to any preassigned accuracy.

From a knowledge of $FA(q)$ and $FB(q)$, both $a_r(q)$ and $a_{r+2}(q)$ are determined, up to an ambiguity of the subscripts. Thus let

$$a_r(q) = c_1 + id_1 , \qquad a_{r+2}(q) = c_2 + id_2 . \tag{2.20}$$

Then

$$FA(q) = \tfrac{1}{2}(c_1 + c_2) + i\tfrac{1}{2}(d_1 + d_2) = FA_1 + iFA_2 , \tag{2.21}$$

$$FB(q) = (c_2 - c_1)^2 - (d_2 - d_1)^2 + i2(c_2 - c_1)(d_2 - d_1) = FB_1 + iFB_2 . \tag{2.22}$$

Three cases arise:

Case 1. $FB_2 \neq 0$. Then $(c_2 - c_1)$ and $(d_2 - d_1)$ are different from zero. Define

$$d_2 - d_1 = \lambda(c_2 - c_1) . \tag{2.23}$$

Substituting (2.23) into (2.22) one obtains

$$FB_1 = (c_2 - c_1)^2(1 - \lambda^2) , \qquad FB_2 = 2\lambda(c_2 - c_1)^2 . \tag{2.24}$$

Observe that λ must have the sign of FB_2. From (2.24) λ is known; namely

$$\lambda = -(FB_1/FB_2) + p(1 + (FB_1/FB_2)^2)^{1/2} , \quad p = \pm 1 . \tag{2.25}$$

Since the radical in (2.25) is always greater than $|FB_1/FB_2|$, the sign of λ is the same as the sign of p. However, it has already been noted that λ must have the sign of FB_2. It follows that p is uniquely determined by the sign of FB_2, and so is λ. With λ known, (2.22) yields

$$c_2 - c_1 = \tau g , \tag{2.26}$$

$$d_2 - d_1 = \tau\lambda g , \tag{2.27}$$

$$g = (FB_2/2\lambda)^{1/2} , \qquad \tau = \pm 1 . \tag{2.28}$$

From (2.26)–(2.28) and (2.21), one now obtains

$$c_1 = FA_1 - \tfrac{1}{2}\tau g , \qquad d_1 = FA_2 - \tfrac{1}{2}\tau\lambda g , \tag{2.30}$$

$$c_2 = FA_1 + \tfrac{1}{2}\tau g , \qquad d_2 = FA_2 + \tfrac{1}{2}\tau\lambda g . \tag{2.31}$$

It is clear from (2.30) and (2.31) that changing the sign of τ merely interchanges $a_r(q)$ and $a_{r+2}(q)$.

Case 2. $FB_2(q) = 0$. Either $(c_2 - c_1) = 0$ or else $(d_2 - d_1) = 0$. Suppose $FB_1 \neq 0$. If $FB_1 < 0$, the first equation of (2.22) shows that in this case $(c_2 - c_1) = 0$. Similarly, if $FB_1 > 0$, then $(d_2 - d_1) = 0$. Thus

If $FB_1 < 0$, $c_2 - c_1 = 0$; $d_2 - d_1 = \tau(-FB_1)^{1/2}$.

If $FB_1 > 0$, $d_2 - d_1 = 0$; $c_2 - c_1 = \tau(FB_1)^{1/2}$.

One may again solve for c_k, d_k, $k = 1, 2$, as in (2.30)–(2.31).

Case 3. $FB_1(q) = FB_2(q) = 0$. This is a necessary and sufficient condition for $a(q)$ to be a multiple eigenvalue. In this case $a_r(q) = a_{r+2}(q) = FA(q)$.

3. Method of Computation. *Phase* 1. This involved tabulation of $a_r(q)$ for

$\phi = 90°(-5°)5°$, $\rho \leqq 100$, $r = 0(1)15$. The interval, $\Delta\rho$, ranged between 0.1 and 0.5, with the smaller intervals for low orders r. Along with $a_r(q)$ and $a_{r+2}(q)$, which were computed simultaneously, the functions $FA(q)$ and $FB(q)$, defined in (2.15)–(2.16), were also generated. This phase of the computations was performed with 8-significant digit arithmetic, using an IBM 7094 computer. Since the power-series expansion for $a_r(q)$ converges for sufficiently small values of $|q|$, the code [3] was adequate in a region where $|q| \leqq 4h$, $h = \Delta\rho$. Thereafter, for a fixed ϕ, the extrapolation routine of [3] was used. From this point on modifications had to be introduced, as outlined below.

Given an approximation $a_r^k(q)$, $k = 0, 1, \cdots$, one obtained T_0, T_1, T_2, as defined in (2.01). The next approximation depended on the magnitude, $|T_1|$, as follows:

Case (a). If $|T_1| \geqq 0.1$, the method of (2.02)–(2.03) was adequate.

Case (b). If $|T_1| < 0.1$, formula (2.05) was used. It remains to be explained how σ was chosen. For even orders on the 90°-ray, $a_r(q)$ is real up to the singular point connecting $a_r(q)$ and $a_{r+2}(q)$, and thereafter the two become complex conjugates of each another. The sign of the imaginary component was taken so that the values on the 90°-ray would be continuous with those obtained on a neighboring ray—taken here as $\phi = 89.99°$. [Actual computation of $a_r(q)$ on this ray was made within the computer, in the neighborhood of the point where an imaginary component began to enter.] It turned out that in all cases, the imaginary component of $a_{4r}(q)$ was negative, and that of a_{4r+2} positive, in the immediate neighborhood of the singularity. In the case of the eigenvalues associated with odd solutions of (1.0), b_{4r+2} had a negative imaginary component and b_{4r+4} had the positive component. [In [5], the authors also assigned the same signs to the imaginary component in the few cases they treated, from considerations of the asymptotic behavior of the functions—namely the fact that on the real axis, $a_r \to b_{r+1}$. However, the asymptotic behavior beyond the singular point is not the same on the imaginary axis as it is on the real axis, and there is as yet no proof that the property in question holds on the imaginary axis.]

On other rays, that value of Δa was chosen which gave the smaller magnitude of $|\Delta a|$.* Ambiguity, when both values of $|\Delta a|$ were the same up to a pre-assigned tolerance, could occur only in the very close vicinity of a singular point. Since this first tabulation was a coarse grid in the complex plane and the singular points form only a countable set, the probability of ambiguity was small. An indication of any ambiguity was read out for further examination and one additional test was performed. Of the two possible choices of $a_r^k(q)$, that one was taken which made $|a_r(\rho - h, \phi) - a_r^k(\rho, \phi)|$ least. In all cases, the ambiguity was resolved within the computer. [Part of this coarse tabulation will be published in book form at a future date.] For the higher orders, it was necessary to carry the calculations considerably beyond $\rho = 100$, in order to explore regions containing singularities.

A necessary and sufficient condition for a singular point is that both the real and imaginary components of $FB(q)$ equal zero. It was therefore only necessary to inspect the tabulations for changes in sign of FB_1, and to note whether FB_2 also changed sign within the same region. This inspection did not require a computer.

Phase 2. This consisted of a more elaborate routine, carried out with double-precision arithmetic around the region in the $(\rho - \phi)$ plane where a double point was expected. It will be easiest to give an example.

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* In the case of odd orders, this choice was also made on the 90°-ray.

Example. Even periodic solutions, $r = 4$. An examination of the coarse tabulation showed that there is a double point in the range $17.6 < \rho < 18.8$, and $35° > \phi > 25°$. The "critical" region read into the computer was processed, and in a second attempt, the ϕ-region was reduced to

$\phi_0 = 30.5°$ (initial value of ϕ), $\phi_1 = 29°$ (final value of ϕ).

$\rho_0 = 17.6$, initial value of ρ, $\rho_1 = 18.8$, final value of ρ.

$h = \Delta\rho = .05$, $\Delta\phi = -.05°$.

The computation began with the first ray, $\phi = 30.5°$. On that ray, $a_4(q)$ and $a_6(q)$ were generated simultaneously, beginning with $\rho = 0$, by the method explained in Section 2. [In this region, no singularity connecting these two orders exists.] Beginning with ρ_0 a new method was used for extrapolating an approximation to $a_4(q)$ and $a_6(q)$, since these functions are not smooth near the expected singularity. In this range the extrapolation was on the functions $FA(q)$ and $FB(q)$; not on $a_r(q)$ and $a_{r+2}(q)$. From the extrapolated values of FA and FB, a_r^0 and a_{r+2}^0 (the first approximation) was obtained through (2.30) and (2.31). Let

$$U = |a_r(\rho - h, \phi) - a_r^0|^2 + |a_{r+2}(\rho - h, \phi) - a_{r+2}^0|^2;$$

the sign of τ in (2.30) was chosen so that U was the lesser of the two values of U. If both values of U were the same, the first τ tested was assigned. Since convergence of the successive iterations guaranteed that the final value obtained was an eigenvalue, to within an assigned tolerance, the possible ambiguity of the initial approximation could only mean that the value might have converged to $a_{r+2}(q)$ rather than to $a_r(q)$. Such a situation would not affect the eventual determination of the double point. With this initial approximation, either (2.02)–(2.03) or (2.05) was used, depending on the magnitude of $|T_1|$. In practice the initial approximation started with the computation of $a_{r+2}(q)$. Once this value was obtained to within the required accuracy, the extrapolated value of $FA(q)$ and the known value of $a_{r+2}(q)$ determined the initial approximation for a_r^0. At the interval chosen, the extrapolated value of $FA(q)$ was good to at least 4 decimal places—in many cases it was good to 8 decimals. This assured that the initial approximation would converge to the companion-eigenvalue, a_r. A test was made after $a_r(q)$ was obtained. If $FA(q)$, as computed from the generated values of $a_{r+2}(q)$ and $a_r(q)$, differed by more than a preassigned, close tolerance from the extrapolated value of $FA(q)$, this value of a_r was discarded, and the value obtained from extrapolation was entered. A warning was read out, for a posteriori examination. [It turned out that in practice, no such warnings were read out in the computations leading to the published eigenvalues.] Another test was made upon the set $a_{r+2}(q)$, $a_r(q)$. Such a test was necessary, since close to a singular point, an initial approximation to $a_{r+2}(q)$ might indeed have converged to $a_r(q)$. This test consisted of the following.

Let

$$U_1 = |a_{r+2}(\rho - h, \phi) - a_{r+2}(\rho, \phi)|^2 + |a_r(\rho - h, \phi) - a_r(\rho, \phi)|^2,$$

$$U_2 = |a_{r+2}(\rho - h, \phi) - a_r(\rho, \phi)|^2 + |a_r(\rho - h, \phi) - a_{r+2}(\rho, \phi)|^2.$$

Whenever $U_1 \leqq U_2$, the values $a_{r+2}(q)$ and $a_r(q)$ were accepted. Whenever this was not true, the subscripts were interchanged, and a warning to this effect was read out. In practice, there were several such interchanges. Examination of the final results indicated that the interchange was indeed necessary.

Once a set $a_{r+2}(q)$ and $a_r(q)$ was computed, the associated values of $FA(q)$ and $FB(q)$ were obtained and stored. For a fixed ϕ, the stored values of FB_1 were tested for a change of sign, as successive values of ρ were entered in the tabulation. Once a change of sign was noted, tabulation continued until there were at least 9 values in storage, with at least 4 values beyond the sign change. When that was available, Aitken's method was used to compute ρ_s where $FB_1(\rho_s) = 0$. Corresponding to this value of ρ, values of $a_r(\rho_s, \phi)$ and $a_{r+2}(\rho_s, \phi)$ were generated from first principles, and corresponding value of FB_2 was obtained. This ended the computations for that particular value of ϕ. The interpolations were made with both 8-point and 7-point formulas, and both sets of results were stored. The computations then proceeded to the next ϕ of the grid. When at least 4 values of ϕ had been stored, the values of $FB_2(\rho_s, \phi_j)$ were tested for a change in sign. Once a change in sign was noted, only 4 additional values of ϕ were processed. The value of ϕ_d for which $FB_2(\rho_s, \phi) = 0$ was again obtained by Aitken's method. Once ϕ_d was obtained, the corresponding value of ρ_d at the double point was again obtained by Aitken's method, from interpolation in the tabulated values of ρ_s. In a similar manner, $FA(\rho_d, \phi_d)$ was obtained by interpolation. The value of $a_r(q) = a_{r+2}(q) = FA(\rho_d, \phi_d)$ was read out, along with corresponding values of T_0, T_1, T_2. Table 1, which follows, shows the behavior of the functions $\rho_s(\phi)$ and of $FB_2(\rho_s, \phi)$ for the present example, along with the interpolated values of ϕ_d, ρ_d, and $a_r(q)$ at the double point. In all cases, acceptably small values of $|T_1|$ were noted.

Two further checks were performed. Whenever the interpolations by the 8-point and 7-point formulas differed before the 9th decimal place, they were discarded, and a finer grid in ρ, ϕ or both was processed. In addition, the following functions were differenced, by ordinary or divided differences:

Argument	*Dependent function*	*Type of differences*
$FB_2(\rho_s, \phi)$	ϕ	Divided differences
ϕ	$\rho_s(\phi)$	Ordinary differences
ϕ	FA_1 and FA_2	Ordinary differences

The numerically largest differences, of orders 2, 4, 6, 7, 8 were read out of the computer for a posteriori examination. Whenever the 8th difference would have affected the 8th decimal place of the final result, the computations were discarded, and a finer grid was processed.

TABLE 1. *Computations relating to* $a_4(q) = a_6(q)$

ϕ (in degrees)	$\rho_s(\phi)$ [At (ρ_s, ϕ), $FB_1(\rho_s, \phi) = 0$.]	$FB_2(\rho_s, \phi)$
30.50	17.82825 50422	−4.74878 03757
30.45	17.85118 76733	−3.37330 03697
30.40	17.87436 99065	−1.98690 50855
30.35	17.89780 70715	−0.58930 72418
30.30	17.92150 46855	+0.81979 12146
30.25	17.94546 84630	+2.24069 96596
30.20	17.96970 43251	+3.67373 93790
30.15	17.99421 84104	+5.11924 41945

Interpolated values:

At double point

$$\phi_d = 30.32903\,89079°; \rho_d = 17.90770\,95980 ,$$

$$a_4(q) = a_6(q) = 33.54015\,64324 + i\,6.36251\,87840 ,$$

$$T(a, q) = .6(10^{-15}) + i\,.35(10^{-14}); T_1(a, q) = .51(10^{-15}) + i\,.32(10^{-14}) ,$$

$$T_2(a, q) = -.00186 - i\,.0178 .$$

Note. Within the computer, all values were listed to 15 significant figures. The above table lists only ten decimals, and only the order of magnitude of $T_k(a, q)$, $k = 0, 1, 2$.

TABLE 2

Double points of Mathieu's equation, associated with even periodic solutions.

r	ϕ (degrees)	ρ	$a_r(q)$ Real Part	$a_r(q)$ Imag. Part	r+2
0	90.	1.46876861	2.08869890	0.0	2
1	59.18208061	3.76995749	6.17647404	1.23177966	3
2	44.60975039	7.26814689	12.79971624	2.76304492	4
3	36.02304851	11.97821151	21.92533616	4.49002890	5
4	90.	16.47116589	27.31912767	0.0	6
4	30.32903891	17.90770960	33.54015643	6.36251878	6
5	77.74433895	22.85524712	38.40883857	2.53293279	7
5	26.26120049	25.06087566	47.63741382	8.35068598	7
6	68.63569460	30.42738210	52.02534500	5.55189444	8
6	23.20168627	33.44030379	64.21313050	10.43474552	8
7	61.57215455	39.19378450	68.15680853	8.96150250	9
7	20.81211404	43.04769498	83.26475268	12.60061661	9
8	90.	47.80596570	80.65826424	0.0	10
8	55.91955555	49.16014417	86.79479850	12.69861754	10
8	18.89115596	53.88422425	104.79053631	14.83777144	10
9	82.35333500	58.27413845	98.76912388	3.83025506	11
9	51.28456166	60.33123310	107.93306428	16.71813422	11
9	17.31131065	65.95073725	128.78923395	17.13804526	11
10	76.00421757	69.92930518	119.40038738	8.20296334	12
10	47.40927141	72.71097078	131.56682190	20.98611513	12
10	15.98778925	79.24786295	155.25992075	19.49492409	12
11	70.63818332	82.77468530	142.54619965	13.04302555	13
11	44.11709801	86.30257222	157.69231520	25.47604566	13
11	14.86194679	93.77608193	184.20189088	21.90309228	13

TABLE 2—*Continued*

r	ϕ (degrees)	ρ	$a_r(q)$ Real Part	$a_r(q)$ Imag. Part	r+2
12	90.	95.47527271	162.10702112	0.0	14
12	66.03683674	96.81379444	168.20157306	18.29431821	14
12	41.28283447	101.10868908	186.30653256	30.16660867	14
12	13.89188815	109.53576981	215.61459283	24.35813133	14
13	84.44343693	110.02736921	187.24248763	5.12750451	15
13	62.04316195	112.05003644	196.36226473	23.91319567	15
13	38.81510667	117.13152570	217.40701681	35.04027512	15
13	13.04686266	126.52722577	249.49758698	26.85631162	15
14	79.59090305	125.76627897	214.89467225	10.82481143	16
14	58.54107283	128.48655463	227.02465063	29.86467710	16
14	36.64559325	134.37293031	250.99173315	40.08236608	16
14	12.30377417	144.75069208	285.85051698	29.39444380	16
15	75.31192241	142.69395383	245.06010153	17.03092757	17
15	55.44272850	146.12619098	260.18561672	36.12005618	17
15	34.72213986	152.83446572	287.05897499	45.28040307	17
15	11.64492867	164.20636770	324.67308978	31.96977006	17

TABLE 3

Double points of Mathieu's equation, associated with odd periodic solutions.

r	ϕ (degrees)	ρ	$b_r(q)$ Real Part	$b_r(q)$ Imag. Part	r+2
2	90.	6.92895476	11.19047360	0.0	4
3	72.46057467	11.27098527	18.77370055	1.88381571	5
4	60.97874908	16.80308983	28.88860879	4.19467426	6
5	52.82618856	23.53467876	41.51634588	6.82630952	7
6	90.	30.09677284	50.47501616	0.0	8
6	46.71423788	31.47295165	56.64571353	9.71571559	8
7	80.58233121	38.52292501	65.07456904	3.18163148	9
7	41.94897328	40.62318483	74.26939582	12.82090012	9
8	73.08912353	48.13638186	82.19724671	6.88343235	10
8	38.12170543	50.98928567	94.38230111	16.11176782	10
9	66.96914596	58.94150633	101.83496931	11.02097811	11

TABLE 3—*Continued*

r	ϕ (degrees)	ρ	$b_r(q)$		r+2
			Real Part	Imag. Part	
9	34.97532055	62.57420650	116.98071992	19.56564754	11
10	90.	69.59879328	117.86892416	0.0	12
10	61.86698774	70.94273869	123.98133068	15.53425785	12
10	32.33961544	75.38022473	142.06185385	23.16482626	12
11	83.56378920	82.10894361	139.49186015	4.47887410	13
11	57.54201185	84.14413219	148.63118156	20.37826431	13
11	30.09725025	89.40913113	169.62353277	26.89507274	13
12	78.06133695	95.80595671	163.63313127	9.51589661	14
12	53.82495450	98.54925096	175.78032210	25.51790587	14
12	28.16459857	104.66235807	199.66403556	30.74469753	14
13	73.29652000	110.69230161	190.28830309	15.04368354	15
13	50.59302351	114.16118710	205.42527964	30.92500454	15
13	26.48038795	121.14106880	232.18197149	34.70392490	15
14	90.	125.43541131	213.37256864	0.0	16
14	69.12577961	126.77081443	219.45339815	21.00996902	16
14	47.75482811	130.98261358	237.56314715	36.57640523	16
14	24.99865911	138.84622074	267.17619891	38.76445692	16
15	85.11157324	142.02943128	242.02085606	5.77614871	17
15	65.44128256	144.04436333	251.12488713	27.37294857	17
15	45.24085886	149.01584316	272.19146415	42.45272306	17
15	23.68423768	157.77861135	304.64576791	42.91916094	17

The entries in Table 3 show that $b_1(q)$ has no double points when q is in the first quadrant of the complex plane. However, since $b_1(-q) = a_1(q)$, there is a double point of $b_1(q)$ in the third quadrant—and also its conjugate in the second quadrant. From the present tabulation, it is now known for the first time that the power series expansions for $a_1(q)$ and $a_3(q)$ converge up to $\rho = 3.7699 \cdots$. Similarly, one may obtain the limit of convergence of the power series for orders up to 15 from the present tabulation.

If one rearranges the values in Tables 2 and 3, listing the first double point, the second point, etc., it seems plausible that all the double points have been obtained for orders less than or equal to 15. However, there is as yet no mathematical proof of this conjecture. It is hoped the present tabulation will aid in obtaining more accurate asymptotic approximations in the various regions of the complex plane. With the aid of these, it may be possible to describe more completely the behavior of the eigenvalues for large values of $|q|$.

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