Transactions on Engineering Sciences vol 6, © 1994 WIT Press, www.witpress.com, ISSN 1743-3533

The dual boundary element method for dynamic analysis of cracked pin-loaded lugs

P. Fedelinski,[a*] M.H. Aliabadi,[a] D.P. Rooke[b]
[a] *Wessex Institute of Technology, University of Portsmouth, Ashurst Lodge, Ashurst, Southampton SO40 7AA, UK*
[b] *Defence Research Agency, Farnborough, Hampshire GU14 6TD, UK*
* *On leave from Silesian Technical University of Gliwice, Poland*

Abstract - The dual boundary element method (DBEM) and the time domain method are applied for the determination of dynamic stress intensity factors (DSIF) for a general mixed-mode crack problem. The DBEM generates a distinct set of boundary integral equations by applying the displacement equation to one of the crack surfaces and the traction equation to the other. The present method does not require any domain discretisation. The boundary of the body is divided into quadratic elements and quarter-point elements (QPE) are used near the crack tips. The temporal variations of the boundary quantities are either piecewise constant or piecewise linear. The DSIF are calculated using the crack opening displacements of the QPE and a least-square error minimization. The method is applied to two problems. The solution for the first example is compared with the solutions obtained by using other methods and shows good agreement. The second example, a dynamic analysis of a pin-loaded cracked lug is an application of the method to a new mixed-mode problem.

Introduction

The determination of dynamic stress intensity factors (DSIF) plays an important role in fracture mechanics. Peak values of the DSIF are usually much higher than the static values, and accurate methods are needed to calculate them. The application of the boundary element method and the time domain approach to dynamic fracture mechanics has been presented in many works [1 - 6]. Usually symmetric problems are analyzed, where only a half or a quarter of the structure and one of the crack surfaces need to be discretized. For mixed-mode crack problems the structure is divided into subdomains along the crack surfaces and the subdomains are assembled using equilibrium and compatibility conditions. The application of the dual boundary element method in static fracture mechanics was presented by Portela, Aliabadi and Rooke [7]. This method uses two different equations: the displacement and the traction boundary integral equation for coincident points on the crack surfaces. The DBEM can solve a general mixed-mode crack problem by discretizing the boundary of the structure only. The DBEM and the dual reciprocity approach in dynamic fracture mechanics was presented by Fedelinski, Aliabadi and Rooke [8]. In the present paper the DBEM and the time domain app~oach are presented. The displacement and the traction boundary integral equation are formulated. Information about space and time interpolation and integration are given. As the result of discretization and integration the matrix equation of motion is obtained. The solution of this system of equations gives the unknown displacements and tractions. The DSIF are calculated by minimizing the difference between analytical and numerical displacements of the QPE. Finally the method is applied to two problems.

Dual boundary integral equations and time-domain approach

Consider a linear elastic homogeneous and isotropic body enclosed by the boundary Γ. For a body which is not subjected to body forces and which has zero initial displacements and velocities, the displacement of a point $\mathbf{x}'$ can be represented by the following boundary integral equation:

$$c_{ij}(\mathbf{x}')u_j(\mathbf{x}',t) = \int_0^t \Big[\int_\Gamma U_{ij}(\mathbf{x}',t;\mathbf{x},\tau)t_j(\mathbf{x},\tau)d\Gamma(\mathbf{x})\Big]d\tau - \int_0^t \Big[\int_\Gamma T_{ij}(\mathbf{x}',t;\mathbf{x},\tau)u_j(\mathbf{x},\tau)d\Gamma(\mathbf{x})\Big]d\tau; \tag{1}$$

where $U_{ij}(\mathbf{x}',t;\mathbf{x},\tau)$, $T_{ij}(\mathbf{x}',t;\mathbf{x},\tau)$ are fundamental solutions of elastodynamics; $u_j(\mathbf{x},\tau)$, $t_j(\mathbf{x},\tau)$ are displacements and tractions respectively, at the boundary; $c_{ij}(\mathbf{x}')$ is a constant which depends on the position of the collocation point $\mathbf{x}'$; $\mathbf{x}$ is the boundary point; t is the observation time.

The traction equation is obtained by differentiating the above equation, followed by the application of Hooke's law and multiplication by the normal at the collocation point. For a point which belongs to the smooth boundary the traction equation is given by

$$\frac{1}{2}t_j(\mathbf{x}',t) = n_i(\mathbf{x}')\Big\{\int_0^t \Big[\int_\Gamma U_{kij}(\mathbf{x}',t;\mathbf{x},\tau)t_k(\mathbf{x},\tau)d\Gamma(\mathbf{x})\Big]d\tau - \int_0^t \Big[\int_\Gamma T_{kij}(\mathbf{x}',t;\mathbf{x},\tau)u_k(\mathbf{x},\tau)d\Gamma(\mathbf{x})\Big]d\tau\Big\}, \tag{2}$$

where $n_i(\mathbf{x}')$ are components of the outward normal at the collocation point $\mathbf{x}'$ and $U_{kij}(\mathbf{x}',t;\mathbf{x},\tau)$, $T_{kij}(\mathbf{x}',t;\mathbf{x},\tau)$ are other fundamental solutions of elastodynamics.

Numerical formulation

The numerical solution of a general mixed-mode crack problem is obtained by discretizing space and time. The boundary Γ of the body is divided into M boundary elements with P nodes per element. Similarly, the observation time t is divided into N time steps. The temporal variation of boundary quantities is specified by Q values within the time step. Displacements and tractions are approximated within each element using interpolation functions $N^p(\xi)$ and within each time step using interpolation functions $M^q(\tau)$. After the approximation, the displacement and the traction equation are:

$$c_{ij}^l u_j^{lN} = \sum_{m=1}^{M}\sum_{p=1}^{P}\sum_{n=1}^{N}\sum_{q=1}^{Q}\Big\{t_j^{mnpq}\int_{-1}^{1}\Big[\int_{t_{n-1}}^{t_n} U_{ij}^{lN}(\xi,\tau)M^q(\tau)d\tau\Big]N^p(\xi)J^m(\xi)d\xi$$

$$-\,u_j^{mnpq}\int_{-1}^{1}\Big[\int_{t_{n-1}}^{t_n} T_{ij}^{lN}(\xi,\tau)M^q(\tau)d\tau\Big]N^p(\xi)J^m(\xi)d\xi\Big\}, \qquad (3)$$

$$\frac{1}{2}t_j^{lN} = n_i^l \sum_{m=1}^{M}\sum_{p=1}^{P}\sum_{n=1}^{N}\sum_{q=1}^{Q}\Big\{t_k^{mnpq}\int_{-1}^{1}\Big[\int_{t_{n-1}}^{t_n} U_{kij}^{lN}(\xi,\tau)M^q(\tau)d\tau\Big]N^p(\xi)J^m(\xi)d\xi$$

$$-\,u_k^{mnpq}\int_{-1}^{1}\Big[\int_{t_{n-1}}^{t_n} T_{kij}^{lN}(\xi,\tau)M^q(\tau)d\tau\Big]N^p(\xi)J^m(\xi)d\xi\Big\}, \qquad (4)$$

where $l = 1, 2, ..., L$; L is the total number of nodes; J^m is the Jacobian and ξ is the local coordinate.

In order to solve a general mixed-mode crack problem the following modelling strategy is employed. The displacement equation (3) is applied for the collocation points along the external boundary Γ_a and along one of the crack faces Γ_b, and the traction equation (4) for the opposite surface of the crack Γ_c (see Fig. 1).

The boundary is divided into quadratic boundary elements. The displacements and tractions are interpolated using: continuous elements for the external boundary Γ_a, semi-discontinuous elements at junctions with the cracks, straight discontinuous elements on the crack faces Γ_b and Γ_c. The geometry is approximated by using continuous elements. At the crack tips the quarter-point elements are used. For the displacement equation (3) the displacements are approximated within each time step by using linear interpolating functions and the tractions are piecewise constant (see [6]). For the traction equation (4), both, the displacements and the tractions are assumed to be piecewise linear.

The fundamental solutions of elastodynamics have the same order of spatial singularity as the elastostatic solutions. The coefficients c_{ij}^l are calculated analytically [6]. The boundary integrals are integrated semi-analytically or analytically when the collocation point belongs to the element and numerically for other elements using Gaussian integration. The time integration along the time interval is performed analytically as shown for the displacement equation in [6] and for the stress equation in [4].

After the discretization and integration the following matrix equation is obtained

$$\mathbf{H}^{NN}\mathbf{u}^N = \mathbf{G}^{NN}\mathbf{t}^N + \sum_{n=1}^{N-1}(\mathbf{G}^{Nn}\mathbf{t}^n - \mathbf{H}^{Nn}\mathbf{u}^n), \qquad (5)$$

where $\mathbf{u}^n$, $\mathbf{t}^n$ contain nodal values of displacements and tractions at the time step n, $\mathbf{H}^{Nn}$, $\mathbf{G}^{Nn}$ depend on fundamental solutions and interpolating functions. The columns of matrices $\mathbf{H}^{NN}$, $\mathbf{G}^{NN}$ are reordered according to the boundary conditions. Finally the following system of equations is obtained

$$\mathbf{A}\mathbf{X}^N = \mathbf{F}^N, \tag{6}$$

where $\mathbf{A}$ contains the reordered matrices, $\mathbf{X}^N$ is a vector of unknown displacements and tractions, $\mathbf{F}^N$ depends on the reordered matrices multiplied by the known boundary conditions and the last term of equation (5), which corresponds to the known displacements and tractions at previous steps. In each time step only the matrices, which correspond to the maximum difference $N - n$ are computed. The matrix $\mathbf{A}$ is the same at each time step. The rest of the matrices is known from the previous steps. The system of equations (6) is solved step-by-step giving the unknown displacements and tractions at each time step.

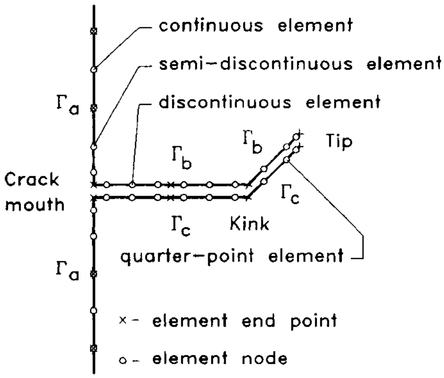

Figure 1: Modelling of the boundary

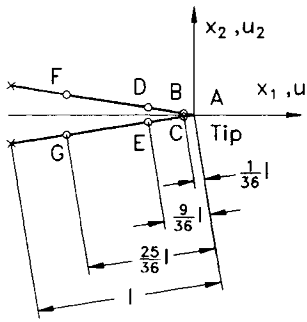

Figure 2: Modelling of the crack using the quarter-point elements

More information about the numerical implementation of the method will be presented in [9].

Dynamic stress intensity factors

The dynamic stress intensity factors (DSIF) are determined by using the crack opening displacements (COD). In order to improve the accuracy of displacements near the crack tip the quarter-point elements (QPE) have been implemented. For the QPE the local coordinate ξ $(-1 \leq \xi \leq 1)$ is a square-root function of the distance r from the crack tip

$$\xi = 1 - 2\sqrt{\frac{r}{l}}, \tag{7}$$

where l is the length of the element and the crack tip is at $\xi = 1$. When the mid-node is placed at a quarter of the length of the straight element the square-

Transactions on Engineering Sciences vol 6, © 1994 WIT Press, www.witpress.com, ISSN 1743-3533

root behaviour of displacements near the crack tip is better represented. The position of the nodes of the QPE is shown in Fig. 2. The DSIF are calculated by minimizing the sum of squared differences between the analytical and numerical values of crack opening displacements for two pair of nodes near the crack tip. The least-square minimization gives the following expressions for the DSIF

$$K_I = \frac{6\mu}{5(\kappa+1)}\sqrt{\frac{\pi}{2l}}\left[(u_2^B - u_2^C) + 3(u_2^D - u_2^E)\right], \tag{8}$$

$$K_{II} = \frac{6\mu}{5(\kappa+1)}\sqrt{\frac{\pi}{2l}}\left[(u_1^B - u_1^C) + 3(u_1^D - u_1^E)\right], \tag{9}$$

where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, and u_1 and u_2 are, respectively, the displacement along and perpendicular to the crack; B, C, D and E are nodes on the crack faces, defined in Fig. 2.

Numerical examples

The method presented in the previous sections is used to solve two problems. The solution of the first example is compared with the available solutions. The second example is a new application of the method.

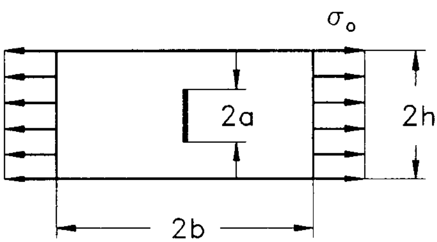

Figure 3: Rectangular plate with an internal crack

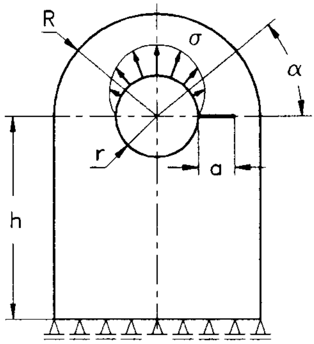

Figure 4: Pin-loaded lug with an edge crack

Rectangular plate with a central crack

A rectangular plate with a central crack, shown in Fig. 3, is instantaneously loaded by a uniform tensile stress σ_o at time $t = 0$. The plate has the following dimensions: the length $2b = 40\ mm$; the width $2h = 20\ mm$; and the half crack length $a = 2.4\ mm$. The plate has the following material properties: the shear modulus $\mu = 76.92 \cdot 10^9\ Pa$; Poisson's ratio $\nu = 0.3$; and the density $\rho = 5000\ kgm^{-3}$. A state of plane strain is assumed. The boundary is divided into 32 boundary elements and the time step $\Delta t = 0.4\ \mu s$. The DSIF is normalized with respect to

$$K_o = \sigma_o\sqrt{\pi a}. \tag{10}$$

The results are presented in Fig. 5 and compared with those of Chen [10], who used the finite difference method, and Dominguez and Gallego [5], who used a subregion technique in the BEM; good general agreement is obtained.

Pin-loaded lug with a single edge crack

The straight lug with a cracked hole shown in Fig. 4 is considered. The hole is of radius r ($r/R = 0.4$) and is concentric with the circular end. The width of the lug is $2R$ and the distance of the centre of the hole from the lug base is h ($h/R = 2$). The radial crack of length a is perpendicular to the lug symmetry axis. The radius of the circular end is $R = 20\ mm$. The upper half of the hole is loaded by a suddenly applied, at $t = 0$, normal pressure σ with a sine distribution $\sigma = \sigma_o sin\alpha$. The lug base is constrained with rollers. The material properties of the lug are: Young's modulus $E = 0.2 \cdot 10^{12}\ Pa$; the mass density $\rho = 8000\ kgm^{-3}$; Poisson's ratio $\nu = 0.3$ and plane strain is assumed. The boundary of the lug is divided into 41 boundary elements (for a long crack 43 elements are used) and the time step is taken to be $\Delta t = 0.8\ \mu s$. The DSIF are calculated for three different lengths of the crack ($a/r = 0.2,\ 0.5,\ 0.8$).

The results are normalized with respect to the static SIF for the crack of length $2(r+a)$ in an infinite sheet with forces $P = \sigma_o \pi r/2$ acting on the opposite faces at the centre of the crack

$$K_o = \frac{\sigma_o \pi r}{2\sqrt{\pi(r+a)}}. \tag{11}$$

The dynamic and the static SIF are shown in Fig. 6. The values of the static SIF are obtained by using the present method and assuming a very small mass density of the material. They agree very well with those given by Rooke and Aliabadi [11]. The peak values of DSIF are about twice the static ones. The time dependence is similar for different configurations. For longer cracks the peaks occur at later times. The mode II static and dynamic SIF are much smaller than the mode I SIF.

Conclusions

The dual boundary element method and the time domain approach are applied to determine the dynamic stress intensity factors. They are calculated using the crack opening displacements of the quarter-point elements. In this formulation, only the boundary of the structure needs to be discretized. Two problems have been solved showing that the method can be used efficiently. The static and dynamic SIFs were calculated for the pin-loaded lug with an edge crack. The static SIFs were obtained by using the present method and by assuming small mass density of the material. It has been shown that the peak values of dynamic SIFs for this particular lug are twice static ones.

Transactions on Engineering Sciences vol 6, © 1994 WIT Press, www.witpress.com, ISSN 1743-3533

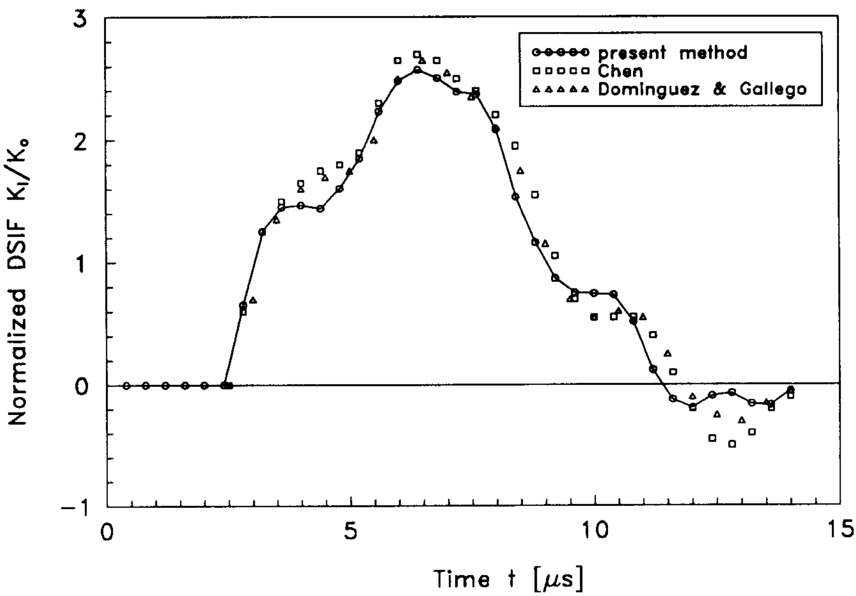

Figure 5: Normalized dynamic stress intensity factor K_I/K_o for the rectangular plate with an internal crack

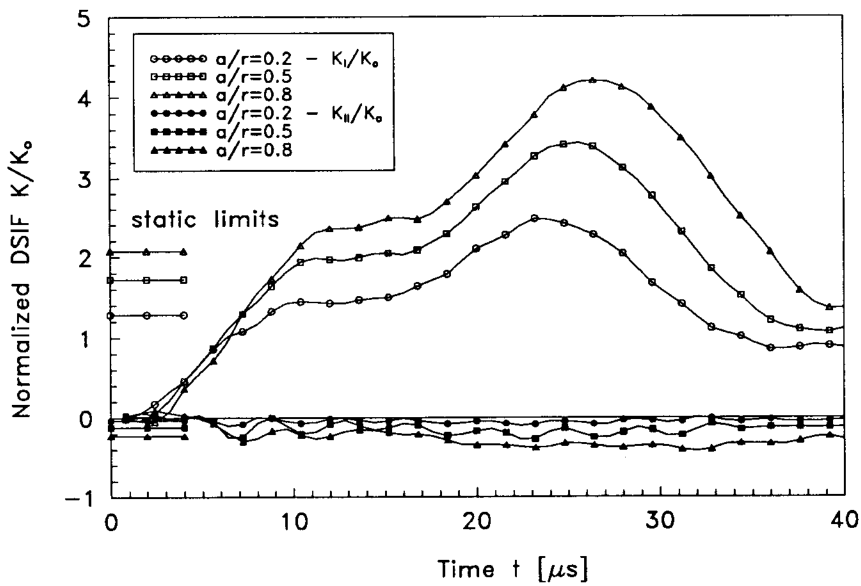

Figure 6: Normalized dynamic stress intensity factors K/K_o for the pin-loaded lug with a single edge crack

Acknowledgement

This work was conducted with the support of the Defence Research Agency, Farnborough, UK, under the Ministry of Defence Strategic Research Programme.

References

1. J.W.Nicholson and S.R.Mettu, 'Computation of dynamic stress intensity factors by the time domain boundary integral equation method - I. Analysis', *Engng Fracture Mech.*, **31**, No. 5, 759-767 (1988).

2. S.R.Mettu and J.W.Nicholson, 'Computation of dynamic stress intensity factors by the time domain boundary integral equation method - II. Examples', *Engng Fracture Mech.*, **31**, No. 5, 769-782 (1988).

3. A.S.M.Israil and G.F.Dargush, 'Dynamic fracture mechanics studies by time-domain BEM', *Engng Fracture Mech.*, **39**, No. 2, 315-328 (1991).

4. A.S.M.Israil and P.K.Banerjee, 'Interior stress calculations in 2-D time-domain transient BEM analysis', *Int. J. Solids Structures*,**27**, No.7, 915-927 (1991).

5. J.Dominguez and R.Gallego, 'Time domain boundary element method for dynamic stress intensity factor computations', *Int. J. Num. Meth. Engng*, **33**, 635-647 (1992).

6. J.Dominguez, *Boundary Elements in Dynamics.* Computational Mechanics Publications, Southampton, Elsevier Applied Science, London 1993.

7. A.Portela, M.H.Aliabadi and D.P.Rooke, 'The dual boundary element method: effective implementation for crack problems', *Int. J. Num. Meth. Engng*, **33**, 1269-1287 (1992).

8. P.Fedelinski, M.H.Aliabadi and D.P.Rooke, 'The dual boundary element method in dynamic fracture mechanics' *Engng Anal. Bound. Elem.*, **12**, No.3, 203-210 (1993).

9. P.Fedelinski, M.H.Aliabadi and D.P.Rooke, 'A new boundary element approach to dynamic fracture mechanics' (in preparation).

10. Y.M.Chen, 'Numerical computation of dynamic stress intensity factors by a Lagrangian finite-difference method (the HEMP code), *Engng Fracture Mech.* ,**7**, 653-660 (1975).

11. D.P.Rooke and M.H.Aliabadi, 'The use of fundamental fields to obtain weight functions for mixed-mode cracks', *Int. J. Engng Sci.*, **32**, No. 1, 155-166 (1994).