

Research Article

Hongxing Wang* and Ju Gao

The dual index and dual core generalized inverse

<https://doi.org/10.1515/math-2022-0592>

received January 4, 2023; accepted May 19, 2023

Abstract: In this article, we introduce the dual index and dual core generalized inverse (DCGI). By applying rank equation, generalized inverse, and matrix decomposition, we give several characterizations of the dual index when it is equal to 1. We realize that if DCGI exists, then it is unique. We derive a compact formula for DCGI and a series of equivalent characterizations of the existence of the inverse. It is worth noting that the dual index of \hat{A} is equal to 1 if and only if its DCGI exists. When the dual index of \hat{A} is equal to 1, we study dual Moore-Penrose generalized inverse (DMPGI) and dual group generalized inverse (DGGI) and consider the relationships among DCGI, DMPGI, DGGI, Moore-Penrose dual generalized inverse, and other dual generalized inverses. In addition, we consider symmetric dual matrix and its dual generalized inverses. Finally, two examples are given to illustrate the application of DCGI in linear dual equations.

Keywords: dual core generalized inverse, dual index, dual Moore-Penrose generalized inverse, dual group generalized inverse, Moore-Penrose dual generalized inverse, dual analog of least-squares solutions

MSC 2020: 15A09, 15A66, 65F05

1 Introduction

The concept of dual number was introduced by Clifford in 1873 [1], and the name of the number is given by Study in 1903 [2]. The dual number consists of a real unit 1 and a Clifford operator ε . The dual number contains two real elements, i.e., $\hat{a} = a + \varepsilon a'$, where the real elements a and a' are called the real part and dual part of \hat{a} , respectively. The rule is $\varepsilon \neq 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$, and $\varepsilon^2 = 0$. When we discuss the geometry of directed lines in space, we can take the angle “ θ ” as the real part and the vertical distance “ s ” as the dual part to form the dual angle, $\hat{\theta} = \theta + \varepsilon s$. As an extension of the concept of dual number, dual vector is to replace the real elements of a real vector with dual numbers and is often used as a mathematical expression for helices. A matrix with dual numbers as elements is called a dual matrix. Denote an $m \times n$ dual matrix as \hat{A} , which is represented as follows:

$$\hat{A} = A + \varepsilon B, \quad (1.1)$$

in which $A \in \mathbb{R}_{m,n}$ and $B \in \mathbb{R}_{m,n}$. The matrix A (B) is called the real(dual) part of the dual matrix \hat{A} . The symbol $\mathbb{D}_{m,n}$ denotes the set of all $m \times n$ dual matrices; I_m is an m -order identity matrix; $\mathcal{R}(\hat{A})$ represents the range of \hat{A} . Furthermore, denote $\hat{A}^T = A^T + \varepsilon B^T$. When $\hat{A}^T = \hat{A}$, we say that \hat{A} is symmetric.

Dual matrices are used in many fields today. In Kinematics, e.g., with the aid of the principle of transference [3], many problems can be initially stated under the condition of spherical motion and then extended

* **Corresponding author: Hongxing Wang**, College of Mathematics and Physics, Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi Minzu University, Nanning 530006, China, e-mail: 3160369785@qq.com

Ju Gao: College of Mathematics and Physics, Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi Minzu University, Nanning 530006, China, e-mail: 3160369785@qq.com

to spiral motion after the dualization of the equation, which makes the dual matrix widely used in space agency kinematics analysis and synthesis [4–8] and robotics [9–13]. Their presence is also felt in other areas of science and engineering, which has raised interest in various aspects of linear algebra and computational methods associated with their use [14–18]. Keler [19], Beyer [20], and others have done pioneering work in the engineering applications of dual algebra.

Pennestrì and Valentini [13] introduced the *Moore-Penrose dual generalized inverse* (MPDGI): let $\hat{A} = A + \varepsilon B$, then the MPDGI of \hat{A} is denoted by \hat{A}^P and is displayed in the form $\hat{A}^P = A^+ - \varepsilon A^+ B A^+$. For a given dual matrix \hat{A} , if there exists a dual matrix \hat{X} satisfying

$$(1) \quad \hat{A} \hat{X} \hat{A} = \hat{A}, \quad (2) \quad \hat{X} \hat{A} \hat{X} = \hat{X}, \quad (3) \quad (\hat{A} \hat{X})^T = \hat{A} \hat{X}, \quad (4) \quad (\hat{X} \hat{A})^T = \hat{X} \hat{A},$$

then we call \hat{X} the *dual Moore-Penrose generalized inverse* (DMPGI) of \hat{A} , and denote it as \hat{A}^+ [17]. It is worth noting that for any dual matrix, its MPDGI always exists, while its DMPGI may not exist. Furthermore, if \hat{X} satisfies $\hat{A} \hat{X} \hat{A} = \hat{A}$, we call \hat{X} a {1}-dual generalized inverse of \hat{A} and denote it as $\hat{A}^{(1)}$; if \hat{X} satisfies $\hat{A} \hat{X} \hat{A} = \hat{A}$ and $(\hat{A} \hat{X})^T = \hat{A} \hat{X}$, we call \hat{X} a {1, 3}-dual generalized inverse of \hat{A} and denote it as $\hat{A}^{(1,3)}$. Recently, Wang has given some necessary and sufficient conditions for a dual matrix to have the DMPGI, and some equivalent relations between the DMPGI and the MPDGI in [21].

Lemma 1.1. [21] *Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{m,n}$, then the following conditions are equivalent:*

- (i) *The DMPGI \hat{A}^+ of \hat{A} exists;*
- (ii) *$(I_m - A A^+) B (I_n - A^+ A) = 0$;*
- (iii) *$\text{rank} \begin{pmatrix} B & A \\ A & 0 \end{pmatrix} = 2 \text{rank}(A)$.*

Furthermore, when the DMPGI \hat{A}^+ of \hat{A} exists,

$$\hat{A}^+ = A^+ - \varepsilon (A^+ B A^+ - (A^T A)^+ B^T (I_m - A A^+) - (I_n - A^+ A) B^T (A A^T)^+). \quad (1.2)$$

Lemma 1.2. [21] *Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{m,n}$, then the DMPGI \hat{A}^+ of \hat{A} exists, and $\hat{A}^+ = \hat{A}^P$ if and only if $(I_m - A A^+) B = 0$ and $B (I_n - A^+ A) = 0$.*

MPDGI and DMPGI are used in many aspects. For example, Pennestrì and Valentini, in their study [13], applied MPDGI to various motions such as rigid body translation and dual angular velocity acquisition. In a study by Pennestrì et al. [17], DMPGI is applied to kinematic synthesis of spatial mechanisms, and a series of numerical examples about calculation and application of DMPGI to kinematic synthesis of linkage mechanisms is given. In addition, MPDGI are also used in many inverse problems of kinematics and analysis of machines and mechanisms in a study by de Falco et al. [4].

Next, Zhong and Zhang [22] introduced the dual group generalized inverse (DGGI): let \hat{A} be an n -square dual matrix. If there exists an n -square dual matrix \hat{G} satisfying

$$(1) \quad \hat{A} \hat{G} \hat{A} = \hat{A}, \quad (2) \quad \hat{G} \hat{A} \hat{G} = \hat{G}, \quad (5) \quad \hat{A} \hat{G} = \hat{G} \hat{A},$$

then \hat{A} is called a dual group generalized invertible matrix, and \hat{G} is the DGGI of \hat{A} , which is recorded as $\hat{A}^\#$. Zhong and Zhang [22] gave some necessary and sufficient conditions for a dual matrix to have DGGI and apply DGGI to study linear dual equations.

Lemma 1.3. [22] *Let $\hat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}_{n,n}$ and $\text{Ind}(A) = 1$, then the DGGI of \hat{A} exists if and only if $(I_n - A A^\#) B (I_n - A A^\#) = 0$.*

Furthermore, if the dual group inverse of \hat{A} exists, then

$$\hat{A}^\# = A^\# + \varepsilon R, \quad (1.3)$$

where

$$R = -A^\#BA^\# + (A^\#)^2B(I_n - AA^\#) + (I_n - AA^\#)B(A^\#)^2. \quad (1.4)$$

Most applications of dual algebra in Kinematics require numerical solutions to linear dual equations. Udwardia [23] introduced the norm of dual vector and used some properties of dual generalized inverses in solving linear dual equations. Various dual generalized inverses are useful for solving consistent linear dual equations or inconsistent linear dual equations. In particular, Qi et al., in their study [24], proposed both a total order and an absolute value function for dual numbers. Then, they gave the definition of the magnitude of a dual quaternion as a dual number. Furthermore, 1-norm, ∞ -norm, and 2-norm are extended to dual quaternion vectors in their article. Furthermore, Qi et al. began to study a series of basic problems in [24–28], e.g., singular value decomposition of dual complex matrices, low rank approximation of dual complex matrices, dual quaternion vectors, generalized inverses of dual quaternion matrices, and others.

It is known that Moore-Penrose inverse and group inverse belong to generalized inverses in complex fields. Other well-known generalized inverses are Drazin inverse, core inverse, and so on. The core inverse means that when the index of $A \in \mathbb{R}_{n,n}$ is 1, there is a unique matrix $X \in \mathbb{R}_{n,n}$, which satisfies $AXA = A$, $AX^2 = X$ and $(AX)^T = AX$. We call it the core inverse of matrix A , which is expressed as A^\oplus . Baksalary and Trenkler, in their study [29], obtained $A^\oplus = A^\#AA^\dagger$. The core inverse has good properties. It can be used to solve many problems, especially in constrained least squares problem.

Although DMPGI, MPDGI, and DGGI are discussed in [21,22], the dual core generalized inverse (DCGI) and the dual index have not been studied yet. On the basis of the above studies, the concepts of the dual index and DCGI are introduced in this article. Furthermore, it is proved that when the dual index of dual matrix is 1, there must be the DCGI of the dual matrix. The sufficient and necessary condition, namely the index is 1, is used to identify the existence of DCGI, which makes the problem more concise and clear. At the same time, we also give other equivalent conditions for the existence of DCGI and the compact formula for DCGI, as well as the relations among DCGI, DGGI, DMPGI, and MPDGI of the dual matrix, and discuss some special dual matrices. Finally, we solve two linear dual equations by applying DCGI.

2 Dual index 1

In complex (real) field, the index is necessary for studying generalized inverse and its related problems. For example, it is known that the group (core) inverse of a matrix exists if and only if the index of the matrix is equal to 1. Moreover, Wei et al. [30–32] considered singular linear structured system with index 1. In this section, we introduce the dual index of a dual matrix. We provide some necessary and sufficient conditions for a dual matrix with the dual index of 1. Furthermore, by using the dual index, we study the dual group generalized invertible matrix.

Definition 2.1. Let \widehat{A} be an n -square dual matrix. If $\mathcal{R}(\widehat{A}^2) = \mathcal{R}(\widehat{A})$, then the dual index of \widehat{A} is equal to 1, and it is recorded as $\text{Ind}(\widehat{A}) = 1$.

Next, we discuss the equivalent characterization of $\mathcal{R}(\widehat{A}^2) = \mathcal{R}(\widehat{A})$, where $\widehat{A} = A + \varepsilon B$. $\mathcal{R}(\widehat{A}^2) \subseteq \mathcal{R}(\widehat{A})$ is constant, which means that $\mathcal{R}(\widehat{A}^2) = \mathcal{R}(\widehat{A})$ if and only if $\mathcal{R}(\widehat{A}) \subseteq \mathcal{R}(\widehat{A}^2)$, i.e., there exists $\widehat{X} = X_1 + \varepsilon X_2$, which makes

$$\widehat{A}^2\widehat{X} = \widehat{A}.$$

Put $\widehat{A} = A + \varepsilon B$ and $\widehat{X} = X_1 + \varepsilon X_2$ into the above equation to obtain

$$A + \varepsilon B = (A + \varepsilon B)^2(X_1 + \varepsilon X_2) = A^2X_1 + \varepsilon(A^2X_2 + (AB + BA)X_1),$$

i.e.,

$$\begin{cases} A^2X_1 = A, & (2.1a) \\ A^2X_2 + (AB + BA)X_1 = B. & (2.1b) \end{cases}$$

From equation (2.1a), we see that $A^2X_1 = A$ is consistent if and only if

$$\text{rank}(A) = \text{rank}(A^2), \quad (2.2)$$

i.e., the index of real matrix A is equal to 1. It is obvious that $(A^2)^\# = (A^\#)^2$ and $A^2(A^\#)^2 = AA^\#$. Therefore, we can obtain the general solution to equation (2.1a) as follows:

$$X_1 = (A^2)^\#A + (I_n - A^2(A^\#)^2)Y = A^\# + (I_n - A^\#A)Y, \quad (2.3)$$

where Y is arbitrary.

By substituting equation (2.3) into equation (2.1b), we obtain $B = A^2X_2 + (AB + BA)(A^\# + (I_n - A^\#A)Y)$. From $A(A^\# + (I_n - A^\#A)Y) = AA^\#$, it follows that $B = A^2X_2 + AB(A^\# + (I_n - A^\#A)Y) + BAA^\# = A^2X_2 + ABA^\# + AB(I_n - A^\#A)Y + BAA^\#$. Therefore,

$$[A^2 \quad AB(I_n - A^\#A)] \begin{bmatrix} X_2 \\ Y \end{bmatrix} = B - ABA^\# - BAA^\#. \quad (2.4)$$

By applying (2.2), we obtain (2.4) if and only if

$$\text{rank}([A^2 \quad AB(I_n - A^\#A) \quad B - ABA^\# - BAA^\#]) = \text{rank}([A^2 \quad AB(I_n - A^\#A)]). \quad (2.5)$$

Since

$$[A^2 \quad AB(I_n - A^\#A)] \begin{bmatrix} I_n & -A^\#B(I_n - A^\#A) \\ 0 & I_n \end{bmatrix} = [A^2 \quad 0],$$

by applying equation (2.2), we obtain

$$\begin{aligned} \text{rank}(A^2) &= \text{rank}(A) = \text{rank}([A^2 \quad AB(I_n - A^\#A) \quad B - (AB + BA)A^\#]) \\ &= \text{rank}([A \quad B - (AB + BA)A^\#]) \\ &= \text{rank}([A \quad B - ABA^\# - BAA^\#]) \\ &= \text{rank}([A \quad B - BAA^\#]) \\ &= \text{rank}([A \quad B(I_n - AA^\#)]). \end{aligned}$$

Then, the consistency of equation (2.4) is equivalent to

$$\text{rank}(A) = \text{rank}([A \quad B(I_n - AA^\#)]). \quad (2.6)$$

According to equations (2.2) and (2.6), the dual index of \widehat{A} is equal to 1, which is equivalent to

$$\text{rank}(A^2) = \text{rank}(A) = \text{rank}([A \quad B(I_n - AA^\#)]). \quad (2.7)$$

Therefore, we have the following theorem.

Theorem 2.1. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ and $\text{rank}(A) = r$, then the dual index of \widehat{A} is equal to 1, which is equivalent to $\text{Ind}(A) = 1$, and*

$$\text{rank}(A) = \text{rank}([A \quad B(I_n - AA^\#)]). \quad (2.8)$$

Next, we present a well-known matrix decomposition [33, Corollary 6] and several corresponding decompositions of generalized inverses, which will be used in the following part of this article. Let $A \in \mathbb{R}_{n,n}$ with $\text{rank}(A) = r$, then

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T, \quad (2.9)$$

where $U \in \mathbb{R}_{n,n}$ is unitary, nonsingular $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$ is the diagonal matrix of singular values of A , $\sigma_1 \geq \dots \geq \sigma_r > 0$, and $K \in \mathbb{R}_{r,r}$ and $L \in \mathbb{R}_{r,n-r}$ satisfy

$$KK^T + LL^T = I_r. \quad (2.10)$$

By applying the decomposition, Baksalary and Trenkler [29] obtain

$$A^+ = U \begin{bmatrix} K^T \Sigma^{-1} & 0 \\ L^T \Sigma^{-1} & 0 \end{bmatrix} U^T. \quad (2.11)$$

Especially, when the index of A is 1, the necessary and sufficient condition for the existence of A^{\oplus} is that K is nonsingular. In [29], by applying equation (2.9) Baksalary and Trenkler also give characterizations of core inverse and group inverse:

$$A^{\oplus} = A^{\#} A A^+ \quad (2.12)$$

$$= U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T, \quad (2.13)$$

$$A^{\#} = U \begin{bmatrix} K^{-1} \Sigma^{-1} & K^{-1} \Sigma^{-1} K^{-1} L \\ 0 & 0 \end{bmatrix} U^T. \quad (2.14)$$

Based on the premise that the index of $A \in \mathbb{R}_{n,n}$ is 1, we analyze equation (2.8). Let the decomposition of A be of the form in equation (2.9). Then, we write

$$B = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T, \quad (2.15)$$

where B_1 is an r -square matrix and $r = \text{rank}(A)$. By substituting equations (2.9), (2.14), and (2.15) into $B(I_n - AA^{\#})$, we obtain

$$B(I_n - AA^{\#}) = U \begin{bmatrix} 0 & -B_1 K^{-1} L + B_2 \\ 0 & -B_3 K^{-1} L + B_4 \end{bmatrix} U^T.$$

It follows from equations (2.8) and (2.9) that

$$\text{rank} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} \Sigma K & \Sigma L & 0 & -B_1 K^{-1} L + B_2 \\ 0 & 0 & 0 & -B_3 K^{-1} L + B_4 \end{bmatrix},$$

which implies that $B_4 = B_3 K^{-1} L$.

In summary, we obtain that $\text{rank}(A) = \text{rank}([A \ B(I_n - AA^{\#})])$ if and only if $B_4 = B_3 K^{-1} L$. Therefore, we have the following Theorem 2.2.

Theorem 2.2. *Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, $\text{rank}(A) = r$, and A and B have the forms as in equations (2.9) and (2.15), respectively. Then, the dual index of \hat{A} is equal to 1, which is equivalent to $\text{Ind}(A) = 1$ and $B_4 = B_3 K^{-1} L$.*

Furthermore, since $\text{Ind}(A) = 1$, applying equations (2.9) and (2.15), it is easy to check that

$$\text{rank} \begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} B_1 & B_2 & \Sigma K & \Sigma L \\ B_3 & B_4 & 0 & 0 \\ \Sigma K & \Sigma L & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2\text{rank}(A) + \text{rank}(B_4 - B_3 K^{-1} L).$$

By applying Theorem 2.2, we have the following Theorem 2.3.

Theorem 2.3. *Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ and $\text{rank}(A) = r$, then the dual index of \hat{A} is equal to 1, which is equivalent to $\text{Ind}(A) = 1$, and*

$$\text{rank} \begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 2\text{rank}(A).$$

In the following theorems, we give some equivalent characterizations with dual index 1.

Theorem 2.4. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, where $\text{rank}(A) = r$, then the dual index of \widehat{A} is 1 if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^+)B(I_n - A^+A) = 0$.

Proof. By applying equations (2.9) and (2.11), we obtain

$$I_n - AA^+ = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^T, \quad (2.16)$$

$$I_n - A^+A = U \begin{bmatrix} I_r - K^TK & -K^TL \\ -L^TK & I_{n-r} - L^TL \end{bmatrix} U^T. \quad (2.17)$$

Then, by equation (2.15)

$$(I_n - AA^+)B(I_n - A^+A) = U \begin{bmatrix} 0 & 0 \\ B_3 - B_3K^TK - B_4L^TK & -B_3K^TL + B_4 - B_4L^TL \end{bmatrix} U^T. \quad (2.18)$$

Let the dual index of \widehat{A} be equal to 1. According to Theorem 2.2, the index of A is 1 and $B_4 = B_3K^{-1}L$. Substituting $B_4 = B_3K^{-1}L$ into $B_3 - B_3K^TK - B_4L^TK$ and $-B_3K^TL + B_4 - B_4L^TL$ and applying equation (2.10), we obtain

$$\begin{aligned} B_3 - B_3K^TK - B_4L^TK &= B_3 - B_3K^TK - B_3K^{-1}LL^TK \\ &= B_3 - B_3K^TK - B_3K^{-1}K + B_3K^{-1}KK^TK \\ &= B_3 - B_3K^TK - B_3 + B_3K^TK = 0 \end{aligned}$$

and

$$\begin{aligned} -B_3K^TL + B_4 - B_4L^TL &= -B_3K^TL + B_3K^{-1}L - B_3K^{-1}LL^TL \\ &= -B_3K^TL + B_3K^{-1}L - B_3K^{-1}L + B_3K^{-1}KK^TL \\ &= -B_3K^TL + B_3K^{-1}L - B_3K^{-1}L + B_3K^TL = 0. \end{aligned}$$

Therefore, from equation (2.18), it follows that $(I_n - AA^+)B(I_n - A^+A) = 0$.

Conversely, let the index of A is 1 and $(I_n - AA^+)B(I_n - A^+A) = 0$. Applying equation (2.18) gives

$$\begin{cases} B_3 - B_3K^TK - B_4L^TK = 0, \\ -B_3K^TL + B_4 - B_4L^TL = 0, \end{cases}$$

i.e.,

$$\begin{cases} B_4L^TK = B_3 - B_3K^TK, & (2.20a) \\ B_3K^TL = B_4 - B_4L^TL. & (2.20b) \end{cases}$$

Since the index of A is 1, it is known that K is a nonsingular matrix. Post-multiplying both sides of equation (2.20a) by $K^{-1}L$, we obtain

$$B_3K^{-1}L - B_3K^TL = B_4L^TL. \quad (2.21)$$

By substituting equation (2.20b) into (2.21), we obtain $B_3K^{-1}L - B_4 + B_4L^TL = B_4L^TL$, i.e., $B_4 = B_3K^{-1}L$. In summary, the index of A is 1 and $B_4 = B_3K^{-1}L$. It follows from Theorem 2.2 that the dual index of \widehat{A} is 1. \square

Theorem 2.5. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, where $\text{rank}(A) = r$, then the dual index of \widehat{A} is 1 if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^\#)B(I_n - AA^\#) = 0$.

Proof. Let $\text{Ind}(A) = 1$. By applying equations (2.9), (2.14), and (2.15), we can obtain

$$\begin{aligned} I_n - AA^\# &= U \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T, \\ (I_n - AA^\#)B(I_n - AA^\#) &= U \begin{bmatrix} 0 & K^{-1}LB_3K^{-1}L - K^{-1}LB_4 \\ 0 & -B_3K^{-1}L + B_4 \end{bmatrix} U^T. \end{aligned} \quad (2.22)$$

If the dual index of \widehat{A} is 1, from Theorem 2.2, we can obtain the index of A is 1 and $B_4 = B_3K^{-1}L$. Therefore, $K^{-1}LB_3K^{-1}L - K^{-1}LB_4 = 0$ and $-B_3K^{-1}L + B_4 = 0$. It follows from equation (2.22) that $(I_n - AA^\#)B(I_n - AA^\#) = 0$.

Conversely, let the index of A is 1 and $(I_n - AA^\#)B(I_n - AA^\#) = 0$. Applying equation (2.22) gives $B_4 = B_3K^{-1}L$. To sum up, the index of A is 1 and $B_4 = B_3K^{-1}L$. Furthermore, according to Theorem 2.2, we obtain that the dual index of \widehat{A} is 1. \square

By applying Lemma 1.3 and Theorem 2.5, we obtain the following Theorem 2.6 that discusses the relationship between DGGI and dual index 1.

Theorem 2.6. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then the dual index of \widehat{A} is 1 if and only if $\widehat{A}^\#$ exists.*

Proof. From Lemma 1.3, we see that the DGGI of \widehat{A} exists if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^\#)B(I_n - AA^\#) = 0$. From Theorem 2.5, we see that the dual index of \widehat{A} is 1 if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^\#)B(I_n - AA^\#) = 0$. Therefore, we obtain that the dual index of \widehat{A} is 1 if and only if $\widehat{A}^\#$ exists. \square

By applying Theorem 2.4, we see that the dual index of \widehat{A} is 1 if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^\#)B(I_n - A^+A) = 0$. By applying Lemma 1.1, we see that the DMPGI \widehat{A}^+ of \widehat{A} exists if and only if $(I_n - AA^\#)B(I_n - A^+A) = 0$. Therefore, we obtain the relationship between dual index 1 and DMPGI in the following Theorem 2.7.

Theorem 2.7. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then the dual index of \widehat{A} is 1 if and only if $\text{Ind}(A) = 1$ and \widehat{A}^+ exists.*

Theorem 2.8. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ and $\text{Ind}(A) = 1$, then \widehat{A}^+ exists if and only if $\text{rank}(A) = \text{rank}([A \ B(I_n - AA^\#)])$.*

Proof. “ \Rightarrow ” If \widehat{A}^+ exists, then $\text{rank}\begin{pmatrix} B & A \\ A & 0 \end{pmatrix} = 2\text{rank}(A)$ is known by the Lemma 1.1. Therefore, when the index of A is 1, the dual index of \widehat{A} is equal to 1 from Theorem 2.3. It is also known from Theorem 2.1 that $\text{rank}(A) = \text{rank}([A \ B(I_n - AA^\#)])$.

“ \Leftarrow ” Let $\text{rank}(A) = \text{rank}([A \ B(I_n - AA^\#)])$. When the index of A is 1, the dual index of \widehat{A} is equal to 1 from Theorem 2.1. So $\text{rank}\begin{pmatrix} B & A \\ A & 0 \end{pmatrix} = 2\text{rank}(A)$ from Theorem 2.3. By conditions (i) and (iii) of Lemma 1.1, we know that \widehat{A}^+ exists. \square

3 DCGI

It is well known that a matrix is group invertible if and only if its index is 1 and is core invertible in $\mathbb{R}_{n,n}$. In Section 2, we obtain that the dual index of $\widehat{A} = A + \varepsilon B$ is 1 if and only if $\widehat{A}^\#$ exists. In this section, we introduce DCGI, give some properties and characterizations of the inverse, and consider relationships among DCGI, DGGI, and dual index 1. Meanwhile, we also give characterizations of some other interesting dual generalized inverses.

3.1 Definition and uniqueness of DCGI

Definition 3.1. Let \widehat{A} be an n -square dual matrix. If there exists an n -square dual matrix \widehat{G} satisfying

$$(1) \quad \widehat{A}\widehat{G}\widehat{A} = \widehat{A}, \quad (2) \quad \widehat{A}\widehat{G}^2 = \widehat{G}, \quad (3) \quad (\widehat{A}\widehat{G})^T = \widehat{A}\widehat{G}, \quad (3.1)$$

then \widehat{A} is called a dual core generalized invertible matrix, and \widehat{G} is the DCGI of \widehat{A} , which is recorded as \widehat{A}^\oplus .

Theorem 3.1. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then the existence of the DCGI of \widehat{A} is equivalent to the existence of G and R , which meet the following requirements: $G = A^{\oplus}$ and

$$\begin{cases} BGA + ARA + AGB = B, & (3.2a) \\ AGR + ARG + BG^2 = R, & (3.2b) \\ (AR + BG)^T = AR + BG. & (3.2c) \end{cases}$$

Furthermore, $\widehat{G} = G + \varepsilon R$ is the DCGI of \widehat{A} .

Proof. From $\widehat{A} = A + \varepsilon B$, $\widehat{G} = G + \varepsilon R$, and

$$\begin{cases} \widehat{A}\widehat{G}\widehat{A} = (A + \varepsilon B)(G + \varepsilon R)(A + \varepsilon B) = AGA + \varepsilon(BGA + ARA + AGB), \\ \widehat{A}\widehat{G}^2 = (A + \varepsilon B)(G + \varepsilon R)^2 = AG^2 + \varepsilon(AGR + ARG + BG^2), \\ (\widehat{A}\widehat{G})^T = ((A + \varepsilon B)(G + \varepsilon R))^T = (AG)^T + \varepsilon(AR + BG)^T, \end{cases}$$

we obtain that $\widehat{A}\widehat{G}\widehat{A} = \widehat{A}$, $\widehat{A}\widehat{G}^2 = \widehat{G}$, and $(\widehat{A}\widehat{G})^T = \widehat{A}\widehat{G}$ are, respectively, equivalent to

$$\begin{cases} AGA = A, & BGA + ARA + AGB = B, \\ AG^2 = G, & AGR + ARG + BG^2 = R, \\ (AG)^T = AG, & (AR + BG)^T = AR + BG. \end{cases}$$

Since $AGA = A$, $AG^2 = G$, and $(AG)^T = AG$, we have $G = A^{\oplus}$. Therefore, if the DCGI of \widehat{A} exists and $\widehat{G} = G + \varepsilon R$ is the DCGI of \widehat{A} , then $G = A^{\oplus}$ and equations (3.2a), (3.2b) and (3.2c) are established.

Conversely, let $\widehat{G} = G + \varepsilon R$ satisfy equations (3.2a), (3.2b), (3.2c) and $G = A^{\oplus}$. By applying Definition 3.1, it is easy to check that \widehat{G} is the DCGI of \widehat{A} . So, the DCGI of \widehat{A} exists. \square

According to Theorem 3.1, we can see that the existence of the core inverse of A is only a necessary condition for the dual core generalized invertibility of \widehat{A} , that is to say, even though the real part of a dual matrix is core invertible, it may be also a dual matrix without DCGI.

Example 3.1. Let

$$\widehat{A} = A + \varepsilon B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad (3.3)$$

where a , b , and b_{33} are not 0 and b_{ij} ($i = 1, 2, 3, j = 1, 2$), b_{13} , and b_{23} are arbitrary real numbers. It is obvious that the real part A is core invertible and

$$A^{\oplus} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (3.4)$$

Let $\widehat{G} = A^{\oplus} + \varepsilon R$, where

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}. \quad (3.5)$$

According to Theorem 3.1, if \widehat{G} is the DCGI of \widehat{A} , then R is a suitable matrix of order n and satisfies equations (3.2a), (3.2b), and (3.2c). Equation (3.2a) requires

$$\Delta = \underbrace{BA^{\oplus}A}_{S_1} + \underbrace{ARA}_{S_2} + \underbrace{AA^{\oplus}B}_{S_3} - B = 0. \quad (3.6)$$

Now we will prove that for any \hat{A} constructed by A and B in equation (3.3), equation (3.6) does not satisfy any three-order matrix R . Therefore, we need to prove $S_{123} - B \neq 0$. As shown below, from equations (3.3)–(3.5), we have

$$\begin{aligned} S_1 = BA^{\oplus}A &= \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix}, \\ S_2 = AA^{\oplus}B &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix}, \\ S_3 = ARA &= \begin{bmatrix} a^2r_1 & abr_2 & 0 \\ abr_4 & b^2r_5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ S_{123} = S_1 + S_2 + S_3 &= \begin{bmatrix} a^2r_1 + 2b_{11} & 2b_{12} + abr_2 & b_{13} \\ 2b_{21} + abr_4 & 2b_{22} + b^2r_5 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}, \\ \Delta = S_{123} - B &= \begin{bmatrix} a^2r_1 + b_{11} & b_{12} + abr_2 & 0 \\ b_{21} + abr_4 & b_{22} + b^2r_5 & 0 \\ 0 & 0 & -b_{33} \end{bmatrix}. \end{aligned}$$

If b_{33} is not 0, then Δ is not 0 no matter what the matrix R is. Therefore, the core inverse condition (3.2a) is not satisfied, and \hat{A} of the set has no DCGI.

In the following Theorem 3.2, we consider the uniqueness of DCGI.

Theorem 3.2. *The DCGI of any dual matrix is unique if it exists.*

Proof. Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, $\text{rank}(A) = r$, and $\hat{A}^{\oplus} = A^{\oplus} + \varepsilon R$. Suppose that \hat{T} is any DCGI of \hat{A} , from Theorem 3.1 and the uniqueness of the core inverse of a real matrix, we can denote \hat{T} as $\hat{T} = A^{\oplus} + \varepsilon \tilde{R}$. Furthermore, we write

$$X = R - \tilde{R}.$$

Next, we prove $X = 0$.

From equation (3.2a), it can be seen that

$$\begin{cases} B = BA^{\oplus}A + ARA + AA^{\oplus}B, \\ B = BA^{\oplus}A + A\tilde{R}A + AA^{\oplus}B. \end{cases} \quad (3.7)$$

Through the first equation minus the second equation in (3.7), we obtain

$$0 = A(R - \tilde{R})A = AXA. \quad (3.8)$$

From equation (3.2b), we have

$$\begin{cases} R = AA^{\oplus}R + ARA^{\oplus} + B(A^{\oplus})^2, \\ \tilde{R} = AA^{\oplus}\tilde{R} + A\tilde{R}A^{\oplus} + B(A^{\oplus})^2. \end{cases} \quad (3.9)$$

Through the first equation minus the second equation in (3.9), we have

$$X = R - \tilde{R} = AA^{\oplus}X + AXA^{\oplus}. \quad (3.10)$$

Similarly, from equation (3.2c), it can be seen that R and \tilde{R} satisfy

$$\begin{cases} (AR + BA^{\oplus})^T = AR + BA^{\oplus}, \\ (A\tilde{R} + BA^{\oplus})^T = A\tilde{R} + BA^{\oplus}. \end{cases} \quad (3.11)$$

Through the first equation minus the second equation in (3.11), we have $(A(R - \tilde{R}))^T = A(R - \tilde{R})$, i.e.,

$$(AX)^T = AX. \quad (3.12)$$

By post-multiplying both sides of equation (3.8) by A^{\oplus} , and by applying equation (3.12), we obtain

$$0 = AXA = AXAA^{\oplus} = (AX)^T(AA^{\oplus})^T = X^T A^T (A^{\oplus})^T A^T = X^T (AA^{\oplus}A)^T = X^T A^T = (AX)^T = AX,$$

i.e., $AX = 0$. Thus, the equation (3.10) is simplified as follows:

$$X = R - \tilde{R} = AA^{\oplus}X. \quad (3.13)$$

Let the decomposition of A be as in equation (2.9). We write

$$X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^T, \quad (3.14)$$

where $X_i \in \mathbb{R}_{r,r}$. Substituting equations (2.9), (2.13), and (3.14) into equation (3.13), we obtain

$$\begin{aligned} AA^{\oplus}X &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^T \\ &= U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^T \\ &= U \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^T = X. \end{aligned}$$

Therefore, $X_3 = X_4 = 0$, i.e.,

$$X = U \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} U^T. \quad (3.15)$$

To make $X = 0$ hold, we only need to prove $X_1 = 0$ and $X_2 = 0$.

By substituting equations (2.9) and (3.15) into equation (3.8), we obtain

$$\begin{aligned} AXA &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T \\ &= U \begin{bmatrix} \Sigma K X_1 & \Sigma K X_2 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T \\ &= U \begin{bmatrix} \Sigma K X_1 \Sigma K & \Sigma K X_1 \Sigma L \\ 0 & 0 \end{bmatrix} U^T = 0. \end{aligned}$$

Therefore,

$$\begin{cases} \Sigma K X_1 \Sigma K = 0, & (3.16a) \\ \Sigma K X_1 \Sigma L = 0. & (3.16b) \end{cases}$$

Since A is core invertible, from Theorem 3.1, we see that K is nonsingular. Thus, from equations (3.15) and (3.16a), we obtain $X_1 = 0$. So

$$X = U \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix} U^T. \quad (3.17)$$

Similarly, by substituting equations (2.9) and (3.17) into equation (3.12), we obtain

$$\left(U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix} U^T \right)^T = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix} U^T,$$

i.e.,

$$\left(U \begin{bmatrix} 0 & \Sigma K X_2 \\ 0 & 0 \end{bmatrix} U^T \right)^T = U \begin{bmatrix} 0 & \Sigma K X_2 \\ 0 & 0 \end{bmatrix} U^T.$$

Continue to simplify the above equation and obtain

$$U \begin{bmatrix} 0 & 0 \\ (\Sigma K X_2)^T & 0 \end{bmatrix} U^T = U \begin{bmatrix} 0 & \Sigma K X_2 \\ 0 & 0 \end{bmatrix} U^T.$$

Thus, $\Sigma K X_2 = 0$. Considering that both Σ and K are nonsingular matrices, we obtain $X_2 = 0$.

To sum up, we obtain $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, and $X_4 = 0$. From equation (3.14), we obtain $X = 0$, which can also be understood as

$$R = \tilde{R}.$$

Therefore, R satisfying equations (3.2a), (3.2b), and (3.2c) is unique, i.e., if the DCGI of \hat{A} exists, then the inverse is unique. \square

3.2 Characterizations and properties of DCGI

Theorem 3.3. *Let $\hat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then its DCGI exists if and only if its dual index is 1.*

Proof. Suppose that A is core invertible. Let the decomposition of A be as in equation (2.9), and the form of B be as in equation (2.15). We write

$$R = U \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} U^T, \quad (3.18)$$

where R_1 is an r -square matrix and $r = \text{rank}(A)$.

“ \Rightarrow ” Assuming that the dual core inverse $\hat{A}^{\oplus} = G + \varepsilon R$ of the dual matrix $\hat{A} = A + \varepsilon B$ exists, it can be seen from Theorem 3.1 that the real part matrix A is core invertible and $G = A^{\oplus}$, so the index of A is 1.

Since the DCGI exists, then equation (3.2a) holds. Substituting equations (2.9), (2.13), (2.15), and (3.18) into $BGA + ARA + AGB = B$, we obtain

$$U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T = U \begin{bmatrix} B_1 + (\Sigma K R_1 + \Sigma L R_3) \Sigma K + B_1 & B_1 K^{-1} L + (\Sigma K R_1 + \Sigma L R_3) \Sigma L + B_2 \\ B_3 & B_3 K^{-1} L \end{bmatrix} U^T.$$

So we have $B_4 = B_3 K^{-1} L$.

To sum up, from Theorem 2.2, when the index of A is 1 and $B_4 = B_3 K^{-1} L$, the dual index of \hat{A} is 1.

“ \Leftarrow ” Let the dual index of \hat{A} be 1. According to Theorem 2.2, the index of A is 1 and $B_4 = B_3 K^{-1} L$. It follows from equation (2.15) that

$$B = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_3 K^{-1} L \end{bmatrix} U^T. \quad (3.19)$$

From equations (3.19) and (2.9), we obtain

$$\hat{A} = A + \varepsilon B = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_3 K^{-1} L \end{bmatrix} U^T.$$

Denote

$$\widehat{G} = G + \varepsilon R = U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \begin{bmatrix} -K^{-1}LB_3(\Sigma K)^{-2} - (\Sigma K)^{-1}B_1(\Sigma K)^{-1} & K^{-1}\Sigma^{-2}(B_3K^{-1})^T \\ B_3(\Sigma K)^{-2} & 0 \end{bmatrix} U^T. \quad (3.20)$$

Then,

$$\begin{aligned} \widehat{A} \widehat{G} \widehat{A} &= AGA + \varepsilon(BGA + ARA + AGB) \\ &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T + \varepsilon \left(U \begin{bmatrix} B_1 & B_1(\Sigma K)^{-1}\Sigma L \\ B_3 & B_3(\Sigma K)^{-1}\Sigma L \end{bmatrix} U^T + U \begin{bmatrix} -B_1 & -B_1(\Sigma K)^{-1}\Sigma L \\ 0 & 0 \end{bmatrix} U^T + U \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} U^T \right) \\ &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_3K^{-1}L \end{bmatrix} U^T \\ &= A + \varepsilon B = \widehat{A}, \\ \widehat{A} \widehat{G}^2 &= AG^2 + \varepsilon(AGR + ARG + BG^2) \\ &= U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \left(\begin{bmatrix} -K^{-1}LB_3(\Sigma K)^{-2} - (\Sigma K)^{-1}B_1(\Sigma K)^{-1} & K^{-1}\Sigma^{-2}(B_3K^{-1})^T \\ 0 & 0 \end{bmatrix} U^T \right. \\ &\quad \left. + \varepsilon U \left(\begin{bmatrix} -B_1(\Sigma K)^{-2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1(\Sigma K)^{-2} & 0 \\ B_3(\Sigma K)^{-2} & 0 \end{bmatrix} U^T \right) \right) \\ &= U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \begin{bmatrix} -K^{-1}LB_3(\Sigma K)^{-2} - (\Sigma K)^{-1}B_1(\Sigma K)^{-1} & K^{-1}\Sigma^{-2}(B_3K^{-1})^T \\ B_3(\Sigma K)^{-2} & 0 \end{bmatrix} U^T \\ &= G + \varepsilon R = \widehat{G} \end{aligned}$$

and

$$\begin{aligned} (\widehat{A} \widehat{G})^T &= (AG + \varepsilon(AR + BG))^T \\ &= \left(U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^T \right)^T + \varepsilon \left(U \begin{bmatrix} 0 & \Sigma^{-1}(B_3K^{-1})^T \\ B_3(\Sigma K)^{-1} & 0 \end{bmatrix} U^T \right)^T \\ &= AG + \varepsilon(AR + BG) = \widehat{A} \widehat{G}. \end{aligned}$$

Therefore, \widehat{G} is the DCGI of \widehat{A} , i.e., $\widehat{G} = \widehat{A}^{\oplus}$ by Definition 3.1. \square

Theorem 3.4. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$. Then, the DCGI \widehat{A}^{\oplus} of \widehat{A} exists if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^+)B(I_n - AA^{\#}) = 0$.*

Proof. Suppose that A is core invertible. Let the decomposition of A be as in (2.9), and the form of B be as in (2.15).

“ \Rightarrow ” Let the DCGI \widehat{A}^{\oplus} of \widehat{A} exists, then we have the index of A as 1, and $B_4 = B_3K^{-1}L$ from Theorem 3.3. Thus, we obtain equation (3.19). Substituting equations (2.9), (2.11), (2.14), and (3.19) into $(I_n - AA^+)B(I_n - AA^{\#})$, we obtain

$$\begin{aligned} (I_n - AA^+)B(I_n - AA^{\#}) &= U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_3K^{-1}L \end{bmatrix} \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T \\ &= U \begin{bmatrix} 0 & 0 \\ B_3 & B_3K^{-1}L \end{bmatrix} \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T \\ &= U \begin{bmatrix} 0 & 0 \\ 0 & -B_3K^{-1}L + B_3K^{-1}L \end{bmatrix} U^T = 0. \end{aligned} \quad (3.21)$$

“ \Leftarrow ” Let $(I_n - AA^+)B(I_n - AA^{\#}) = 0$. Since $A^{\#}$ exists, the index of A is 1. Substituting equations (2.9), (2.11), (2.14), and (2.15) into $(I_n - AA^+)B(I_n - AA^{\#})$, we obtain

$$\begin{aligned}
(I_n - AA^+)B(I_n - AA^\#) &= U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T \\
&= U \begin{bmatrix} 0 & 0 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T \\
&= U \begin{bmatrix} 0 & 0 \\ 0 & -B_3K^{-1}L + B_4 \end{bmatrix} U^T.
\end{aligned} \tag{3.22}$$

It follows from $(I_n - AA^+)B(I_n - AA^\#) = 0$ that $B_4 = B_3K^{-1}L$. Then, there is equation (3.20) from Theorem 3.3. According to Definition 3.1, DCGI exists. \square

Theorem 3.5. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then the DCGI \widehat{A}^{\oplus} of \widehat{A} exists if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^{\oplus})B(I_n - A^{\oplus}A) = 0$.*

Proof. In the real field, it is known that $A^\#A = AA^\# = A^{\oplus}A$ and $AA^+ = AA^{\oplus}$. According to Theorem 3.4, the DCGI \widehat{A}^{\oplus} of \widehat{A} exists if and only if $\text{Ind}(A) = 1$ and $(I_n - AA^{\oplus})B(I_n - A^{\oplus}A) = 0$. \square

Next, we further discuss the characterizations of the existence of DCGI.

Theorem 3.6. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then the following conditions are equivalent:*

- (1) *The DCGI of \widehat{A} exists;*
- (2) *The index of A is equal to 1, and $BA^\#A + AXA + AA^\#B = B$ is consistent;*
- (3) *The index of A is equal to 1, and $BA^+A + AXA + AA^+B = B$ is consistent;*
- (4) *The index of A is equal to 1, and $BA^{\oplus}A + AXA + AA^{\oplus}B = B$ is consistent.*

Proof. From Theorems 2.6, 2.7, and 3.3, we know that the existence of DCGI is equivalent to the existence of DGGI; the existence of DGGI is equivalent to the existence of DMPGI with $\text{Ind}(A) = 1$. Therefore, condition (1) indicates that DCGI exists or DGGI exists or DMPGI exists with $\text{Ind}(A) = 1$.

“(1) \Rightarrow (2)” Let DCGI exists, then $\widehat{A}^\# = A^\# + \varepsilon P$ of $\widehat{A} = A + \varepsilon B$ exists, so the index of A is 1 and $\widehat{A}\widehat{A}^\# \widehat{A} = \widehat{A}$, i.e., $BA^\#A + APA + AA^\#B = B$. Therefore, $BA^\#A + AXA + AA^\#B = B$ is consistent.

“(1) \Leftarrow (2)” Let the index of A be 1 and $BA^\#A + AXA + AA^\#B = B$ be consistent. By applying equations (2.9), (2.10), (2.14), and (2.15) into $BA^\#A + AXA + AA^\#B = B$, we obtain

$$\begin{aligned}
&U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T + U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^T \\
&+ U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T,
\end{aligned} \tag{3.23}$$

where $X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^T$, B_1 is an r -square matrix, and $r = \text{rank}(A)$.

By simplifying equation (3.23), we obtain

$$\begin{aligned}
U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T &= U \begin{bmatrix} B_1 + \Sigma K X_1 \Sigma K + \Sigma L X_3 \Sigma K + B_1 + K^{-1} L B_3 \\ B_3 \end{bmatrix} \\
&\quad \begin{bmatrix} B_1 K^{-1} L + \Sigma K X_1 \Sigma L + \Sigma L X_3 \Sigma L + B_2 + K^{-1} L B_4 \\ B_3 K^{-1} L \end{bmatrix} U^T.
\end{aligned}$$

Therefore, $B_4 = B_3K^{-1}L$.

To sum up, if $\text{Ind}(A) = 1$ and $B_4 = B_3K^{-1}L$, then the dual index of \widehat{A} is 1 from Theorem 2.2. By Theorem 3.3, we obtain that the DCGI of \widehat{A} exists.

Similarly, conditions (1) and (3) are equivalent, and conditions (1) and (4) are equivalent. \square

3.3 Compact formula for DCGI

The following is a compact formula for DCGI.

Theorem 3.7. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ and the DCGI of \widehat{A} exists, then*

$$\widehat{A}^{\oplus} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+ \quad (3.24)$$

$$= A^{\oplus} + \varepsilon(-A^{\oplus} B A^+ + A^{\#} B A^+ - A^{\#} B A^{\oplus} + A^{\oplus} (B A^+)^T (I_n - A A^+) + (I_n - A A^{\#}) B A^{\#} A^{\oplus}). \quad (3.25)$$

Proof. According to Theorems 2.6, 2.7, and 3.3, if \widehat{A}^{\oplus} exists, then $\widehat{A}^{\#}$ and \widehat{A}^+ exist. Write $\widehat{X} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+$. It is easy to check that

$$\widehat{A} \widehat{X} \widehat{A} = \widehat{A} \widehat{A}^{\#} \widehat{A} \widehat{A}^+ \widehat{A} = \widehat{A} \widehat{A}^+ \widehat{A} = \widehat{A}, \quad (3.26)$$

$$\widehat{A} \widehat{X}^2 = \widehat{A} \widehat{A}^{\#} \widehat{A} \widehat{A}^+ \widehat{A}^{\#} \widehat{A} \widehat{A}^+ = \widehat{A} \widehat{A}^+ \widehat{A} \widehat{A}^{\#} \widehat{A}^+ = \widehat{A} \widehat{A}^{\#} \widehat{A}^+ = \widehat{A}^{\#} \widehat{A} \widehat{A}^+ = \widehat{X}. \quad (3.27)$$

Since $(\widehat{A} \widehat{X})^T = (\widehat{A} \widehat{A}^{\#} \widehat{A} \widehat{A}^+)^T = (\widehat{A} \widehat{A}^+)^T = \widehat{A} \widehat{A}^+$ and $\widehat{A} \widehat{X} = \widehat{A} \widehat{A}^{\#} \widehat{A} \widehat{A}^+ = \widehat{A} \widehat{A}^+$, we obtain

$$(\widehat{A} \widehat{X})^T = \widehat{A} \widehat{X}. \quad (3.28)$$

To sum up, we obtain $\widehat{A}^{\oplus} = \widehat{X} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+$.

Substituting equations (1.2) and (1.3) into equation (3.24) gives (3.25). \square

In addition, substituting equations (2.9), (2.10), (2.11), (2.13), (2.14), and (3.19) into equation (3.25), we obtain the following Theorem 3.8.

Theorem 3.8. *Let the DCGI of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists, A and B be as forms in equations (2.9) and (2.15), respectively. Then,*

$$\widehat{A}^{\oplus} = U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T + \varepsilon U \begin{bmatrix} -K^{-1} L B_3 (\Sigma K)^{-2} - (\Sigma K)^{-1} B_1 (\Sigma K)^{-1} & K^{-1} \Sigma^{-2} (B_3 K^{-1})^T \\ B_3 (\Sigma K)^{-2} & 0 \end{bmatrix} U^T. \quad (3.29)$$

3.4 Relationships among some dual generalized inverses

The MPDGI is a very interesting inverse, which is useful for solving different kinematic problems [4]. Obviously, when the real part of a dual matrix is a nonsingular matrix, its MPDGI is equal to its DMPGI. Similarly, we consider the laws and properties of DCGI in the form of $\widehat{A}^{\oplus} = A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus}$ and DGGI in the form of $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#} B A^{\#}$, as well as relationships among those dual generalized inverses.

Theorem 3.9. *Let the DCGI \widehat{A}^{\oplus} of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists, where $\text{rank}(A) = r$. Then $\widehat{A}^{\oplus} = A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus}$ if and only if $\text{Ind}(A) = 1$ and*

$$(I_n - A A^+) B = 0. \quad (3.30)$$

Proof. Suppose that A is core invertible. Let the decomposition of A be as in equation (2.9), and the form of B be as in equation (2.15). Then,

$$-A^{\oplus} B A^{\oplus} = U \begin{bmatrix} -(\Sigma K)^{-1} B_1 (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T, \quad (3.31)$$

$$-A^{\oplus} B A^+ + A^{\#} B A^+ = U \begin{bmatrix} (\Sigma K)^{-1} K^{-1} L (B_3 K^T + B_4 L^T) \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T, \quad (3.32)$$

$$-A^\#BA^\oplus = U \begin{bmatrix} -(\Sigma K)^{-1}B_1(\Sigma K)^{-1} - (\Sigma K)^{-1}K^{-1}LB_3(\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T, \quad (3.33)$$

$$A^\oplus(BA^+)^T(I_n - AA^+) = U \begin{bmatrix} 0 & (\Sigma K)^{-1}(B_3K^T\Sigma^{-1} + B_4L^T\Sigma^{-1})^T \\ 0 & 0 \end{bmatrix} U^T, \quad (3.34)$$

$$(I_n - AA^\#)BA^\#A^\oplus = U \begin{bmatrix} -K^{-1}LB_3(\Sigma K)^{-2} & 0 \\ B_3(\Sigma K)^{-2} & 0 \end{bmatrix} U^T, \quad (3.35)$$

$$(I_n - AA^+)B = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T = U \begin{bmatrix} 0 & 0 \\ B_3 & B_4 \end{bmatrix} U^T. \quad (3.36)$$

“ \Rightarrow ” Let the DCGI \widehat{A}^\oplus of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$. According to Theorem 3.3, we obtain $\text{Ind}(A) = 1$ and $B_4 = B_3K^{-1}L$. To make $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$, from the compact formula for DCGI (3.25), we obtain

$$A^\oplus B A^\oplus = -A^\oplus B A^+ + A^\# B A^+ - A^\# B A^\oplus + A^\oplus (B A^+)^T (I_n - AA^+) + (I_n - AA^\#) B A^\# A^\oplus.$$

It follows from (3.31)–(3.35) that $B_3(\Sigma K)^{-2} = 0$, so $B_3 = 0$. Since $B_4 = B_3K^{-1}L$, then $B_4 = 0$. By applying equation (3.36), $B_3 = 0$ and $B_4 = 0$, we derive $(I_n - AA^+)B = 0$. Thus, $\text{Ind}(A) = 1$ and $(I_n - AA^+)B = 0$ are proved.

“ \Leftarrow ” Let $\text{Ind}(A) = 1$ and $(I_n - AA^+)B = 0$, then $(I_n - AA^+)B(I_n - AA^\#) = 0$. According to Theorem 2.4, the DCGI of \widehat{A} exists. We know that the real part of the DCGI is A^\oplus , i.e., $\widehat{A}^\oplus = A^\oplus + \varepsilon R$. Put equations (2.9), (2.11), and (2.15) into equation (3.36) to obtain $(I_n - AA^+)B = 0$. From $(I_n - AA^+)B = 0$, we have $B_3 = 0$ and $B_4 = 0$. Put $B_3 = 0$ and $B_4 = 0$ into equation (3.25) to obtain

$$\begin{aligned} R &= -A^\oplus B A^+ + A^\# B A^+ - A^\# B A^\oplus + A^\oplus (B A^+)^T (I_n - AA^+) + (I_n - AA^\#) B A^\# A^\oplus \\ &= U \begin{bmatrix} -(\Sigma K)^{-1}B_1(\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T = -A^\oplus B A^\oplus. \end{aligned} \quad (3.37)$$

Therefore, $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$. □

Since it is well known that $(I_n - AA^+)B = 0$ if and only if $\text{rank}([A \ B]) = \text{rank}(A)$, we obtain the following Theorem 3.10.

Theorem 3.10. *Let the DCGI \widehat{A}^\oplus of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists, where $\text{rank}(A) = r$. Then, $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$ is equivalent to $\text{Ind}(A) = 1$ and $\text{rank}([A \ B]) = \text{rank}(A)$.*

Next, we continue to analyze DGGI in the form of $\widehat{A}^\# = A^\# - \varepsilon A^\# B A^\#$.

Theorem 3.11. *Let the DGGI $\widehat{A}^\#$ of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists, where $\text{rank}(A) = r$. Then, $\widehat{A}^\# = A^\# - \varepsilon A^\# B A^\#$ is equivalent to*

$$B(I_n - AA^\#) = 0 \quad \text{and} \quad (I_n - AA^\#)B = 0. \quad (3.38)$$

Proof. Suppose that A is core invertible. Let the decomposition of A be as in equation (2.9), the decomposition of $A^\#$ be as in equation (2.14), and the form of B be as in equation (2.15). Then, we have

$$(A^\#)^2 B (I_n - AA^\#) = U \begin{bmatrix} 0 & (\Sigma K)^{-2}(-B_1K^{-1}L + B_2 - K^{-1}LB_3K^{-1}L + K^{-1}LB_4) \\ 0 & 0 \end{bmatrix} U^T$$

and

$$(I_n - AA^\#)B(A^\#)^2 = U \begin{bmatrix} -K^{-1}LB_3(\Sigma K)^{-2} & -K^{-1}LB_3(\Sigma K)^{-2}K^{-1}L \\ B_3(\Sigma K)^{-2} & B_3(\Sigma K)^{-2}K^{-1}L \end{bmatrix} U^T. \quad (3.39)$$

“ \Rightarrow ” Let the DGGI $\widehat{A}^\#$ of $\widehat{A} \in \mathbb{D}_{n,n}$ exists, then $\text{Ind}(\widehat{A}) = 1$ from Theorem 3.3. Therefore, $\text{Ind}(A) = 1$ and $B_4 = B_3K^{-1}L$ from Theorem 2.2. Thus,

$$(A^\#)^2B(I_n - AA^\#) = U \begin{bmatrix} 0 & -(\Sigma K)^{-2}B_1K^{-1}L + (\Sigma K)^{-2}B_2 \\ 0 & 0 \end{bmatrix} U^T. \quad (3.40)$$

If $\widehat{A}^\# = A^\# - \varepsilon A^\#BA^\#$, by equations (1.3), (3.39), and (3.40), we have $(A^\#)^2B(I_n - AA^\#) = 0$ and $(I_n - AA^\#)B(A^\#)^2 = 0$, so $B_2 = B_1K^{-1}L$ and $B_3 = 0$. It follows that

$$B = U \begin{bmatrix} B_1 & B_1K^{-1}L \\ 0 & 0 \end{bmatrix} U^T. \quad (3.41)$$

Therefore, we have

$$\begin{aligned} B(I_n - AA^\#) &= U \begin{bmatrix} B_1 & B_1K^{-1}L \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T = 0, \\ (I_n - AA^\#)B &= U \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T U \begin{bmatrix} B_1 & B_1K^{-1}L \\ 0 & 0 \end{bmatrix} U^T = 0, \end{aligned}$$

i.e., (3.38) is established.

“ \Leftarrow ” Assuming $B(I_n - AA^\#) = 0$ and $(I_n - AA^\#)B = 0$, it is easy to check that $(I_n - AA^\#)B(I_n - A^\#A) = 0$, $(A^\#)^2B(I_n - AA^\#) = 0$, and $(I_n - AA^\#)B(A^\#)^2 = 0$. According to Lemma 1.3, the DGGI of \widehat{A} exists. Therefore, by using (1.3), we obtain $\widehat{A}^\# = A^\# - \varepsilon A^\#BA^\#$. \square

In Theorems 3.12, 3.13, and 3.14, we consider the relationships among $\widehat{A}^\# = A^\# - \varepsilon A^\#BA^\#$, $\widehat{A}^+ = A^+ - \varepsilon A^+BA^+$, and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus BA^\oplus$.

Theorem 3.12. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, then DGGI exists and $\widehat{A}^\# = A^\# - \varepsilon A^\#BA^\#$ if and only if $\text{Ind}(A) = 1$, DMPGI exists, and $\widehat{A}^+ = A^+ - \varepsilon A^+BA^+$.*

Proof. Suppose that A is group invertible. Let the decomposition of A be as in equation (2.9) and the form of B be as in equation (2.15).

“ \Rightarrow ” Let DGGI exists and $\widehat{A}^\# = A^\# - \varepsilon A^\#BA^\#$, then $B(I_n - AA^\#) = 0$ and $(I_n - AA^\#)B = 0$ from Theorem 3.11. Substituting equations (2.9), (2.14), and (2.15) into $(I_n - AA^\#)B = 0$, we obtain

$$(I_n - AA^\#)B = U \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T = U \begin{bmatrix} -K^{-1}LB_3 & -K^{-1}LB_4 \\ B_3 & B_4 \end{bmatrix} U^T = 0.$$

Therefore, $B_3 = 0$ and $B_4 = 0$. Substituting equations (2.9) and (2.14) into $B(I_n - AA^\#) = 0$, it follows that

$$B(I_n - AA^\#) = U \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -K^{-1}L \\ 0 & I_{n-r} \end{bmatrix} U^T = U \begin{bmatrix} 0 & -B_1K^{-1}L + B_2 \\ 0 & 0 \end{bmatrix} U^T = 0,$$

i.e., $B_2 = B_1K^{-1}L$.

Since $B_3 = 0$, $B_4 = 0$, and $B_2 = B_1K^{-1}L$, we have

$$B = U \begin{bmatrix} B_1 & B_1K^{-1}L \\ 0 & 0 \end{bmatrix} U^T. \quad (3.42)$$

Since $KK^T + LL^T = I_r$, applying equations (2.16), (2.17), and (3.42), we have

$$\begin{aligned} B(I_n - A^+A) &= U \begin{bmatrix} B_1 & B_1K^{-1}L \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} I_r - K^TK & -K^TL \\ -L^TK & I_{n-r} - L^TL \end{bmatrix} U^T \\ &= U \begin{bmatrix} B_1 - B_1K^TK - B_1K^{-1}(I_r - KK^T)K & -B_1K^TL + B_1K^{-1}L - B_1K^{-1}(I_r - KK^T)L \\ 0 & 0 \end{bmatrix} U^T = 0 \end{aligned}$$

and

$$(I_n - AA^+)B = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^T U \begin{bmatrix} B_1 & B_1 K^{-1} L \\ 0 & 0 \end{bmatrix} U^T = 0.$$

By applying Lemma 1.2, we can conclude that DMPGI exists and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$.

Because the DGGI exists, $\text{Ind}(A) = 1$. To sum up, $\text{Ind}(A) = 1$, DMPGI exists, and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$.

“ \Leftarrow ” Let $\text{Ind}(A) = 1$, DMPGI exists, and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$. By applying Lemma 1.2, we know $(I_n - AA^+)B = 0$ and $B(I_n - A^+A) = 0$. Substituting equations (2.9), (2.11), and (2.15) into $(I_n - AA^+)B = 0$, we obtain

$$(I_n - AA^+)B = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^T = U \begin{bmatrix} 0 & 0 \\ B_3 & B_4 \end{bmatrix} U^T = 0,$$

i.e., $B_3 = 0$ and $B_4 = 0$.

Substituting equations (2.9), (2.11), and (2.15) into $B(I_n - A^+A) = 0$ and by applying $B_3 = 0$ and $B_4 = 0$, we obtain

$$\begin{aligned} B(I_n - A^+A) &= U \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r - K^T K & -K^T L \\ -L^T K & I_{n-r} - L^T L \end{bmatrix} U^T \\ &= U \begin{bmatrix} B_1 - B_1 K^T K - B_2 L^T K & -B_1 K^T L + B_2 - B_2 L^T L \\ 0 & 0 \end{bmatrix} U^T = 0. \end{aligned}$$

Therefore,

$$\begin{cases} B_1 = B_1 K^T K + B_2 L^T K = (B_1 K^T + B_2 L^T) K, & (3.43a) \\ B_2 = B_2 L^T L + B_1 K^T L = (B_2 L^T + B_1 K^T) L. & (3.43b) \end{cases}$$

Since $\text{Ind}(A) = 1$, K is nonsingular. Applying (3.43a) gives

$$B_1 K^T + B_2 L^T = B_1 K^{-1}. \quad (3.44)$$

Substituting equation (3.44) into equation (3.43b), we have $B_2 = B_1 K^{-1} L$.

Since $B_3 = 0$, $B_4 = 0$, and $B_2 = B_1 K^{-1} L$ and by substituting equations (2.9) and (2.11) into $B(I_n - AA^\#)$ and $(I_n - AA^\#)B$, we have $B(I_n - AA^\#) = 0$ and $(I_n - AA^\#)B = 0$. According to Theorem 3.11, DGGI exists and $\widehat{A}^\# = A^\# - \varepsilon A^\# B A^\#$. \square

Theorem 3.13. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$. If $\text{Ind}(A) = 1$, DMPGI exists and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$, then DCGI exists and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$.

Proof. Since $\text{Ind}(A) = 1$, DMPGI exists and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$, then $(I_n - AA^+)B = 0$ by applying Lemma 1.2.

According to Theorem 3.9, if $\text{Ind}(A) = 1$ and $(I_n - AA^+)B = 0$, then the DCGI \widehat{A}^\oplus of \widehat{A} exists, and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$. \square

Theorem 3.14. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$. If $\widehat{A}^\#$ exists, and $\widehat{A}^\# = A^\# - \varepsilon A^\# B A^\#$, then DCGI exists and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$.

Proof. If $\widehat{A}^\#$ exists, and $\widehat{A}^\# = A^\# - \varepsilon A^\# B A^\#$, then $\text{Ind}(A) = 1$, DMPGI exists and $\widehat{A}^+ = A^+ - \varepsilon A^+ B A^+$ from Theorem 3.12. Then, DCGI exists and $\widehat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$ from Theorem 3.13. \square

Example 3.2. Let $\widehat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. By applying equations (1.2), (1.3), and (3.25), we have

$$\widehat{A}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \widehat{A}^\# = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}^\oplus = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \widehat{A}^{\text{P}} = A^+ - \varepsilon A^+ B A^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A^{\#} - \varepsilon A^{\#} B A^{\#} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that $\widehat{A}^{\oplus} = A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus}$, $\widehat{A}^+ \neq \widehat{A}^{\text{P}}$, and $\widehat{A}^{\#} \neq A^{\#} - \varepsilon A^{\#} B A^{\#}$. This means that when DCGI exists and $\widehat{A}^{\oplus} = A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus}$, there is not necessarily $\widehat{A}^+ = \widehat{A}^{\text{P}} = A^+ - \varepsilon A^+ B A^+$ or $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#} B A^{\#}$.

3.5 Symmetric dual matrix

We know that in real field, the index of a symmetric matrix must be equal to 1, and its core inverse is equal to its Moore-Penrose inverse and its group inverse. But it is not true for some symmetric dual matrices. Even some symmetric dual matrices do not have DCGIs and DGGIs. For example,

Example 3.3. $\widehat{A} = A + \varepsilon B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b , and c are not 0. Because $\text{rank}(A) = \text{rank}(A^2) = 2$,

$\text{rank}([A \ B(I_3 - AA^{\#})]) = 3$, and $2 \neq 3$, we can obtain that the dual index of \widehat{A} is not 1, \widehat{A} does not satisfy Theorems 2.6 and 3.3. Therefore, \widehat{A} has no DGGI and DCGI, as well as \widehat{A} no DMPGI from Theorem 2.8.

Theorem 3.15. *If the dual matrix \widehat{A} is a symmetric dual matrix, and the dual index of \widehat{A} is 1, then*

$$\widehat{A}^{\#} = \widehat{A}^+ = \widehat{A}^{\oplus}. \quad (3.45)$$

Proof. Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$, $\widehat{A}^{\text{T}} = \widehat{A}$, $\text{rank}(A) = r$, and the dual index of \widehat{A} be 1. According to Theorems 2.6, 2.7, and 3.3, the DGGI, DMPGI, and DCGI of \widehat{A} exist simultaneously. Since $\widehat{A} = \widehat{A}^{\text{T}}$, we obtain that A is symmetrical, B is symmetrical, A^2 is symmetrical, and $(A^2)^{\#} = (A^2)^+ = (AA^{\text{T}})^+ = (A^{\text{T}}A)^+$. Therefore, we have

$$\begin{aligned} A^+ &= A^{\#}, \quad A^+ B A^+ = A^{\#} B A^{\#} \\ (A^{\text{T}}A)^+ B^{\text{T}}(I_n - AA^+) &= (A^{\#})^2 B(I_n - A^{\#}A), \\ (I_n - A^+A) B^{\text{T}}(AA^{\text{T}})^+ &= (I_n - AA^{\#}) B(A^{\#})^2. \end{aligned}$$

Then, by applying equations (1.2) and (1.3), we derive that $\widehat{A}^{\#} = \widehat{A}^+$.

It follows from $\widehat{A}^{\#} = \widehat{A}^+$ and $\widehat{A}^{\oplus} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+$ in equation (3.24) that $\widehat{A}^{\oplus} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+ = \widehat{A}^{\#} \widehat{A} \widehat{A}^{\#} = \widehat{A}^{\#} = \widehat{A}^+$. Therefore, $\widehat{A}^{\#} = \widehat{A}^+ = \widehat{A}^{\oplus}$. \square

Theorem 3.16. *If the DCGI $\widehat{A}^{\oplus} = A^{\oplus} + \varepsilon R$ of $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ exists, and \widehat{A} is a symmetric dual matrix, then*

$$\widehat{A}^{\oplus} = (\widehat{A}^{\oplus})^{\text{T}} = A^{\oplus} + \varepsilon(-A^{\oplus} B A^{\oplus} + (A^{\oplus})^2 B(I_n - AA^{\oplus}) + (I_n - AA^{\oplus}) B(A^{\oplus})^2). \quad (3.46)$$

Proof. According to Theorems 2.6, 2.7, and 3.3, if the DCGI of \widehat{A} exists, then DGGI and DMPGI exist.

Then $\widehat{A}^{\oplus} = \widehat{A}^{\#} \widehat{A} \widehat{A}^+ = \widehat{A}^+ \widehat{A} \widehat{A}^+ = \widehat{A}^+$ by equation (3.45). From $(\widehat{A}^+)^{\text{T}} = \widehat{A}^+$ (see [23]), we have $(\widehat{A}^{\oplus})^{\text{T}} = \widehat{A}^{\oplus}$. Since \widehat{A} is a symmetric dual matrix, we have $A^{\text{T}} = A$ and $B^{\text{T}} = B$. Moreover, $A^{\oplus} = A^{\#} = A^+ = (A^{\oplus})^{\text{T}} = (A^{\#})^{\text{T}} = (A^+)^{\text{T}}$.

According to equation (3.25), we can obtain

$$\begin{aligned} \widehat{A}^{\oplus} &= A^{\oplus} + \varepsilon(-A^{\oplus} B A^+ + A^{\#} B A^+ - A^{\#} B A^{\oplus} + A^{\oplus} (B A^+)^{\text{T}}(I_n - AA^+) + (I_n - AA^{\#}) B A^{\#} A^{\oplus}) \\ &= A^{\oplus} + \varepsilon(-A^{\oplus} B A^{\oplus} + A^{\oplus} (B A^+)^{\text{T}}(I_n - AA^+) + (I_n - AA^{\#}) B A^{\#} A^{\oplus}) \\ &= A^{\oplus} + \varepsilon(-A^{\oplus} B A^{\oplus} + A^{\oplus} (B A^{\oplus})^{\text{T}}(I_n - AA^{\oplus}) + (I_n - AA^{\oplus}) B A^{\oplus} A^{\oplus}) \\ &= A^{\oplus} + \varepsilon(-A^{\oplus} B A^{\oplus} + A^{\oplus} (A^{\oplus})^{\text{T}} B^{\text{T}}(I_n - AA^{\oplus}) + (I_n - AA^{\oplus}) B A^{\oplus} A^{\oplus}) \\ &= A^{\oplus} + \varepsilon(-A^{\oplus} B A^{\oplus} + (A^{\oplus})^2 B(I_n - AA^{\oplus}) + (I_n - AA^{\oplus}) B(A^{\oplus})^2), \end{aligned}$$

i.e., (3.46). \square

Next, we consider some properties of DCGI, DGGI, and DMPGI in special forms when the research object is a symmetric dual matrix.

Lemma 3.17. [34] *Let $A \in \mathbb{R}_{n,n}$ and $B \in \mathbb{R}_{n,n}$. Then, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $\text{rank}([A \ B]) = \text{rank}(A)$.*

Theorem 3.18. *Let $\widehat{A} = A + \varepsilon B$ be an n -order symmetric dual matrix. Then, the following conditions are equivalent:*

- (1) *The DCGI \widehat{A}^{\oplus} of \widehat{A} exists, and $\widehat{A}^{\oplus} = \widehat{A}^{\#} = \widehat{A}^+ = \widehat{A}^P = A^+ - \varepsilon A^+ B A^+$;*
- (2) *$(I_n - A A^+)B = 0$ or $(I_n - A A^{\#})B = 0$ or $(I_n - A A^{\oplus})B = 0$;*
- (3) *$\text{rank}([A \ B]) = \text{rank}(A)$;*
- (4) *$\mathcal{R}(B) \subseteq \mathcal{R}(A)$;*
- (5) *$B(I_n - A A^+) = 0$ or $B(I_n - A A^{\#}) = 0$ or $B(I_n - A A^{\oplus}) = 0$.*

Proof. Let $\widehat{A} \in \mathbb{D}_{n,n}$, $\text{rank}(A) = r$, and $\widehat{A}^T = \widehat{A}$. Since \widehat{A} is a symmetric dual matrix, then the real part A is symmetric, and $A^{\oplus} = A^+ = A^{\#}$. Therefore, condition (1) is $\widehat{A}^{\oplus} = \widehat{A}^{\#} = \widehat{A}^+ = \widehat{A}^P = A^+ - \varepsilon A^+ B A^+ = A^{\#} - \varepsilon A^{\#} B A^{\#} = A^{\oplus} - \varepsilon A^{\oplus} B A^{\oplus}$, and conditions (2) and (5) are equivalent.

“(1) \Rightarrow (2)” Let $\widehat{A}^{\oplus} = \widehat{A}^{\#} = \widehat{A}^+ = \widehat{A}^P = A^{\#} - \varepsilon A^{\#} B A^{\#}$. From Theorem 3.11, we obtain $B(I_n - A A^{\#}) = 0$ and $(I_n - A A^{\#})B = 0$. Therefore, we obtain $(I_n - A A^{\#})B = 0$. Since $A^{\oplus} = A^+ = A^{\#}$, condition (2) is established.

“(1) \Leftarrow (2)” Let $(I_n - A A^{\#})B = 0$. Since \widehat{A} is symmetrical, $B(I_n - A A^{\#}) = 0$. Applying Theorem 3.11 gives that DGGI $\widehat{A}^{\#}$ of $\widehat{A} = A + \varepsilon B$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#} B A^{\#}$.

According to Theorems 2.6, 2.7, and 3.3, when $\widehat{A}^{\#}$ exists, \widehat{A}^{\oplus} and \widehat{A}^+ exist, and the dual index of \widehat{A} is 1. According to Theorem 3.15, $\widehat{A}^{\oplus} = \widehat{A}^+ = \widehat{A}^{\#}$. Then $\widehat{A}^{\oplus} = \widehat{A}^+ = \widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#} B A^{\#}$. Considering that \widehat{A} is a symmetric dual matrix, then $A^{\oplus} = A^+ = A^{\#}$. Therefore, equation (1) is established.

According to Theorem 3.10, conditions (1) and (3) are equivalent. According to Lemma 3.17, conditions (3) and (4) are equivalent. \square

According to Theorems 3.15 and 3.16, it is easy to obtain the following Corollary 3.19.

Corollary 3.19. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}_{n,n}$ be a symmetric dual matrix. If the DCGI, DGGI, and DMPGI of \widehat{A} exist, then they are symmetrical and equal.*

4 Applications of DCGI in linear dual equations

In this section, we use two examples to illustrate some applications of DCGI in solving linear dual equations.

First, we consider solving a consistent linear dual equation by DCGI in Example 4.1. We give a general solution to the consistent dual equation.

Example 4.1. Let $\widehat{A} \widehat{x} = \widehat{b}$ be a consistent equation, where

$$\widehat{A} = A + \varepsilon B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon & \varepsilon \\ \varepsilon & 0 \end{bmatrix},$$

$$\widehat{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon \\ \varepsilon \end{bmatrix}, \text{ and } \widehat{x} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}, \widehat{x}_i = x_i + \varepsilon x'_i, \quad i = 1, 2.$$

It is easy to check that $\text{rank}(A^2) = \text{rank}(A) = \text{rank}([A \ B(I_n - A A^{\#})]) = 1$. By applying Theorem 2.1, we obtain that the dual index of \widehat{A} is 1, i.e., the DCGI \widehat{A}^{\oplus} exists. Applying (3.25) gives

$$\widehat{A}^{\oplus} = G + \varepsilon R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 0 \end{bmatrix}. \quad (4.1)$$

Thus,

$$\widehat{A}^{\oplus} \widehat{b} = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \begin{bmatrix} 1 + \varepsilon \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}.$$

Furthermore, let

$$\widehat{x} = \widehat{A}^{\oplus} \widehat{b} + (I_2 - \widehat{A}^{\oplus} \widehat{A}) \widehat{w} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} + \begin{bmatrix} 0 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix} \widehat{w}, \quad (4.2)$$

where \widehat{w} is an arbitrary n -by-1 dual column vector.

By substituting equation (4.2) into $\widehat{A} \widehat{x} = \widehat{b}$, we can obtain

$$\begin{aligned} \widehat{A} \widehat{x} - \widehat{b} &= \widehat{A} (\widehat{A}^{\oplus} \widehat{b} + (I_2 - \widehat{A}^{\oplus} \widehat{A}) \widehat{w}) - \widehat{b} = \widehat{A} \widehat{A}^{\oplus} \widehat{b} - \widehat{b} + (\widehat{A} - \widehat{A} \widehat{A}^{\oplus} \widehat{A}) \widehat{w} \\ &= \begin{bmatrix} 1 + \varepsilon & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} - \widehat{b} = \begin{bmatrix} 1 + \varepsilon \\ \varepsilon \end{bmatrix} - \begin{bmatrix} 1 + \varepsilon \\ \varepsilon \end{bmatrix} = 0, \end{aligned}$$

i.e., (4.2) is the solution to $\widehat{A} \widehat{x} = \widehat{b}$.

Meanwhile, let \widehat{x} be any solution to $\widehat{A} \widehat{x} = \widehat{b}$. Pre-multiplying $\widehat{A} \widehat{x} = \widehat{b}$ by \widehat{A}^{\oplus} , we obtain $\widehat{A}^{\oplus} \widehat{A} \widehat{x} = \widehat{A}^{\oplus} \widehat{b}$, and then

$$\widehat{x} = \widehat{A}^{\oplus} \widehat{b} + \widehat{x} - \widehat{A}^{\oplus} \widehat{A} \widehat{x} = \widehat{A}^{\oplus} \widehat{b} + (I_2 - \widehat{A}^{\oplus} \widehat{A}) \widehat{x} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} + \begin{bmatrix} 0 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix} \widehat{x}.$$

Therefore, each solution to $\widehat{A} \widehat{x} = \widehat{b}$ can be written as equation (4.2) in which $\widehat{w} = \widehat{x}$.

To sum up, equation (4.2) is the general solution to $\widehat{A} \widehat{x} = \widehat{b}$.

In order to solve inconsistent dual linear equation, Udwardia [23] introduced the norm of the dual vector. Consider the m -by-1 dual vector $\widehat{u}_i = p_i + \varepsilon q_i$. We write

$$\|\widehat{u}_i\|^2 = (p_i + \varepsilon q_i)^T (p_i + \varepsilon q_i) = \|p_i\|^2 + 2\varepsilon p_i^T q_i, \quad (4.3)$$

where $p_i \neq 0$ and $\|p_i\|^2 = p_i^T p_i$. By using the right-most expression in equation (4.3), one norm of the dual vector \widehat{u}_i is given as follows:

$$\langle \widehat{u}_i \rangle = \|p_i\| + \|q_i\|. \quad (4.4)$$

In equation (4.3), the dual norm is used to determine the magnitude of the error. Udwardia [23] introduced the analog of the least squares solution of any inconsistent dual equation $\widehat{A} \widehat{x} = \widehat{b}$ and gives the corresponding solution – analog of the least squares solution $\widehat{x} = \widehat{A}^{(1,3)} \widehat{b} + (I_n - \widehat{A}^{(1,3)} \widehat{A}) \widehat{h}$, where \widehat{h} is an arbitrary dual column vector and $\widehat{A}^{(1,3)}$ exists. It can be found that the real part x of the analog of the least squares solution is the least squares solution to equation $Ax = b$, where A , b , and x are matched with the real parts of \widehat{A} , \widehat{b} , and \widehat{x} , respectively.

Example 4.2. Let the inconsistent equation be $\widehat{A} \widehat{x} = \widehat{b}$, where $\widehat{A} = A + \varepsilon B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 10 & 10 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 4 + 10\varepsilon & 2 + 10\varepsilon \\ 2 + 9\varepsilon & 1 + 7\varepsilon \end{bmatrix}$, $\widehat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 1 \end{bmatrix}$, and $\widehat{x} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}$, $\widehat{x}_i = x_i + \varepsilon x'_i$, $i = 1, 2$, i.e.,

$$\begin{bmatrix} 4 + 10\varepsilon & 2 + 10\varepsilon \\ 2 + 9\varepsilon & 1 + 7\varepsilon \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 1 \end{bmatrix}. \quad (4.5)$$

Then,

$$\widehat{A}^+ = \begin{bmatrix} 0.1600 - 0.6880\varepsilon & 0.0800 - 0.1840\varepsilon \\ 0.0800 - 0.1440\varepsilon & 0.0400 + 0.0080\varepsilon \end{bmatrix}.$$

By applying the Result 12 of Udawadia in [23], the analog of the least squares solution of the inconsistent equation is

$$\hat{x} = \hat{A}^+ \hat{b} + (I_2 - \hat{A}^+ \hat{A}) \hat{h} = \begin{bmatrix} 0.0800 - 0.0240\varepsilon \\ 0.0400 + 0.0880\varepsilon \end{bmatrix} + \begin{bmatrix} 0.2000 + 0.8000\varepsilon & -0.4000 - 0.6000\varepsilon \\ -0.4000 - 0.8000\varepsilon & 0.8000 - 0.8000\varepsilon \end{bmatrix} \hat{h},$$

where \hat{h} is an arbitrary n -by-1 dual vector. The norm of the error

$$\langle \hat{e}^* \rangle = \langle \hat{A} \hat{x} - \hat{b} \rangle = \langle \hat{A} \hat{A}^+ \hat{b} - \hat{b} \rangle = 1.9715. \quad (4.6)$$

According to Theorem 2.1, we have $\text{rank}(A^2) = \text{rank}(A) = \text{rank}([A \ B(I_n - AA^\#)]) = 1$, i.e., the dual index of \hat{A} is 1. From Theorem 3.3, it can be seen that \hat{A}^\oplus exists. Then,

$$\hat{A}^\oplus = G + \varepsilon R = \begin{bmatrix} 0.1600 - 0.6720\varepsilon & 0.0800 - 0.1760\varepsilon \\ 0.0800 - 0.1760\varepsilon & 0.0400 - 0.0080\varepsilon \end{bmatrix}$$

by the compact formula (3.25). Denote

$$\hat{x} = \hat{A}^\oplus \hat{b} + (I - \hat{A}^\oplus \hat{A}) \hat{w} = \begin{bmatrix} 0.0800 - 0.0160\varepsilon \\ 0.0400 + 0.0720\varepsilon \end{bmatrix} + \begin{bmatrix} 0.2000 + 0.7200\varepsilon & -0.4000 - 0.6400\varepsilon \\ -0.4000 - 0.4400\varepsilon & 0.8000 - 0.7200\varepsilon \end{bmatrix} \hat{w}, \quad (4.7)$$

where \hat{w} is an arbitrary n -by-1 dual vector. The norm of the error is

$$\langle \hat{e} \rangle = \langle \hat{A} \hat{x} - \hat{b} \rangle = \langle \hat{A} \hat{A}^\oplus \hat{b} - \hat{b} \rangle = \langle \hat{u}_2 \rangle = \|m_2\| + \|n_2\| = \left\| \begin{bmatrix} 0.4000 \\ -0.8000 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0.2800 \\ 1.0400 \end{bmatrix} \right\| = 1.9715. \quad (4.8)$$

Therefore, from equations (4.6) and (4.8), we see that $\langle \hat{e}^* \rangle = \langle \hat{e} \rangle = 1.9715$, i.e., equation (4.7) is also the analog of the least-squares solution of equation (4.5).

The two examples in this section calculate the DCGIs of the two dual matrices through the compact formula (3.25). When the dual index of any dual matrix is 1, its DCGI exists. On this basis, we can obtain DCGI directly through the compact formula (3.25). However, in order to reduce the amount of calculation, we can first consider equation (3.30) in Theorem 3.9. If the dual matrix satisfies equation (3.30), then $\hat{A}^\oplus = A^\oplus - \varepsilon A^\oplus B A^\oplus$. Otherwise, we have to use the compact formula (3.25).

5 Conclusion

The first part of this article provides some new findings about dual index 1 of dual matrices, including the characterizations of the dual index 1. We obtain that DGGI exists if and only if the dual index is 1. Furthermore, when the dual index is 1, DMPGI exists and the real part index of the dual Moore-Penrose generalized invertible matrix is 1, and *vice versa*. The second part of this article explores DCGI systematically. Some results from the second part of the article are as follows:

- (1) If a DCGI of a dual matrix exists, it is unique.
- (2) If DCGI exists, a compact formula for DCGI is given.
- (3) We provide a series of equivalent characterizations of the existence of DCGI, e.g., the dual index is 1 if and only if DCGI exists.
- (4) Relations among MPDGI, DMPGI, DCGI, and DGGI are proved.

In the third part, DCGI is applied to linear dual equations through a consistent dual equation and an inconsistent dual equation.

Acknowledgement: The authors wish to extend their sincere gratitude to the referees for their precious comments and suggestions.

Funding information: This work was supported partially by the Research Fund Project of Guangxi University for Nationalities [grant number 2019KJQD03], the National Natural Science Foundation of China [grant number 12061015], the Thousands of Young and Middle-aged Key Teachers Training Programme in Guangxi Colleges and Universities [grant number GUIJIAOSHIFAN2019-81HAO], and the Innovation Project of Guangxi Graduate Education [grant number YCSW2022243].

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

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