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THE DYADIC STRUCTURE AND ATOMIC DECOMPOSITION OF Q SPACES IN SEVERAL REAL VARIABLES

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Abstract. This paper contains several results relating Q spaces in several real variables with their dyadic counterparts, which are analogues of theorems for BMO and for Q spaces on the circle. In addition, it gives an atomic (or quasi-orthogonal) decomposition for these Q spaces in terms of the same type of atoms used to decompose BMO.

1. Introduction. In recent years there has been much interest in a new family of function spaces, called Q spaces. These spaces were originally defined by Aulaskari, Xiao and Zhao in [AXZ] as spaces of holomorphic functions on the unit disk. Following the work of Essén and Xiao [EX] on the boundary values of these functions on the unit circle, the definition was extended to the n-dimensional Euclidean space by Essén, Janson, Peng and Xiao in [EJPX].

Fix $\alpha \in (-\infty, \infty)$. For a cube I in $\boldsymbol{R}^n$ with sidelength $\ell(I)$, consider the mean quotient of symmetric differences of a function $f \in L^2(I)$ as follows:

$$O_{f,\alpha}(I) := (\ell(I))^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}}\, dx dy\,. \tag{1.1}$$

We say that $f \in Q_\alpha(\boldsymbol{R}^n)$ if $O_{f,\alpha}(I)$ is uniformly bounded, namely

$$\|f\|_{Q_\alpha(\boldsymbol{R}^n)} := \sup_I (O_{f,\alpha}(I))^{1/2} < \infty\,,$$

where the supremum ranges over all cubes I in $\boldsymbol{R}^n$ with sides parallel to the coordinate axes. Modulo constants, this defines a norm under which $Q_\alpha(\boldsymbol{R}^n)$ becomes a Banach space.

From this definition it is not difficult to see that the spaces $Q_\alpha(\boldsymbol{R}^n)$ bear a close connection to the space BMO($\boldsymbol{R}^n$) of functions of *bounded mean oscillation*, introduced by John and Nirenberg [JN]. Recall that a locally integrable function f belongs to BMO($\boldsymbol{R}^n$) if

$$\|f\|_* := \sup_{\text{cubes } I} \frac{1}{|I|} \int_I |f(x) - f(I)| dx < \infty\,,$$

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where $f(I)$ denotes the mean of f over the cube I, i.e. $f(I) = |I|^{-1} \int_I f(x)dx$. Equivalently (cf. [JN]), one has

$$\|f\|_* \approx \sup_I (\Phi_f(I))^{1/2}$$

with

$$\Phi_f(I) := \frac{1}{|I|} \int_I |f(x) - f(I)|^2 \, dx \,. \tag{1.2}$$

Rewriting the integral on the right as a double integral gives

$$\Phi_f(I) = \frac{1}{2|I|^2} \int_I \int_I |f(x) - f(y)|^2 dx dy$$

and reveals the relationship with $O_{f,\alpha}(I)$ in (1.1).

In fact, the paper [EJPX] showed that

$$Q_\alpha(\boldsymbol{R}^n) = \mathrm{BMO}(\boldsymbol{R}^n) \quad \text{if} \quad \alpha < 0\,,$$

while

$$Q_\alpha(\boldsymbol{R}^n) \subsetneq \mathrm{BMO}(\boldsymbol{R}^n) \quad \text{if} \quad \alpha \geq 0\,.$$

Furthermore, when $\alpha \geq 1$ (for $n \geq 2$) or when $\alpha > 1/2$ (for $n = 1$), $Q_\alpha(\boldsymbol{R}^n)$ contains only constants. Thus the cases of interest are when α is between 0 and $\min(1, n/2)$.

It is also important to note that like BMO($\boldsymbol{R}^n$), $Q_\alpha(\boldsymbol{R}^n)$ is homogeneous of degree zero, namely:

$$\|f \circ \phi\|_{Q_\alpha(\boldsymbol{R}^n)} = \|f\|_{Q_\alpha(\boldsymbol{R}^n)}$$

for any dilation $\phi(x) = \delta x$ of $\boldsymbol{R}^n$, $\delta > 0$. This is in contrast to the case of the homogeneous Sobolev or Besov spaces, whose homogeneity depends on α (see [EJPX] for the relationship between $Q_\alpha(\boldsymbol{R}^n)$ and Besov spaces).

The aim of this paper is the further study of $Q_\alpha(\boldsymbol{R}^n)$ and its dyadic structure, in particular the analogues for $Q_\alpha(\boldsymbol{R}^n)$ of certain well-known results for BMO($\boldsymbol{R}^n$).

In Section 2 we first review some background information on $Q_\alpha(\boldsymbol{R}^n)$ from [EJPX]. We then present higher dimensional analogues of some of Janson's results in [Ja] and give a $Q_\alpha(\boldsymbol{R}^n)$-version of the main result (concerning the relation between BMO($\boldsymbol{R}^n$) and its dyadic counterpart) of Garnett and Jones [GJ].

In Section 3 we obtain a decomposition of functions in $Q_\alpha(\boldsymbol{R}^n)$ into sums of "atoms" of the type used by Uchiyama [U] (following the work of Chang and Fefferman [CF]) to represent BMO($\boldsymbol{R}^n$)-functions. (See also Rochberg and Semmes [RS] for a different decomposition of BMO($\boldsymbol{R}^n$), and Wu and Xie [WX] for decomposition theorems for Q_p spaces in the unit disk.) The key ingredients in the proof are a quasi-orthogonality lemma and the characterization of $Q_\alpha(\boldsymbol{R}^n)$ in terms of fractional Carleson measures, as well as the duality theorem from [DX], identifying $Q_\alpha(\boldsymbol{R}^n)$ with the dual of a certain space of distributions, $HH^1_{-\alpha}(\boldsymbol{R}^n)$. Thus we may view this decomposition for $Q_\alpha(\boldsymbol{R}^n)$ as a kind of dual form of the atomic decomposition of $HH^1_{-\alpha}(\boldsymbol{R}^n)$ which was proved in [DX].

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2. The dyadic structure of $Q_\alpha(\boldsymbol{R}^n)$. We first review some notation and then some facts about the dyadic structure of $Q_\alpha(\boldsymbol{R}^n)$, which are analogues of similar results for BMO($\boldsymbol{R}^n$).

In the following a cube will always mean a cube in $\boldsymbol{R}^n$ with sides parallel to the coordinate axes. We will use the notation $\ell(I)$ for the sidelength of the cube I, $|I|$ for its volume (Lebesgue measure), diam(I) for its diameter, and x_I for its center. For $\delta > 0$, we will denote by δI the dilated cube, whose center is x_I and whose sidelength is $\delta\ell(I)$. Similarly, for $x \in \boldsymbol{R}^n$, $I + x$ will denote the translated cube, namely the cube with center $x_I + x$ and sidelength $\ell(I)$.

By $\mathcal{D}_0 = \mathcal{D}_0(\boldsymbol{R}^n)$ we denote the collection of unit cubes whose vertices have integer coordinates, and we set $\mathcal{D}_k = \mathcal{D}_k(\boldsymbol{R}^n)$, $k \in \boldsymbol{Z}$, to be the collection of all dyadic cubes of sidelength 2^{-k}, namely all cubes of the form $J = \{2^{-k}x; x \in I\}$ for some $I \in \mathcal{D}_0$. The collection of all dyadic cubes is then $\mathcal{D} = \bigcup_{-\infty}^{\infty} \mathcal{D}_k$. Starting with an arbitrary (not necessarily dyadic) cube I, for every $k \geq 0$ we can partition it into 2^{kn} subcubes of sidelength $2^{-k}\ell(I)$, forming the collection $\mathcal{D}_k(I)$. We write $\mathcal{D}(I) = \bigcup_0^{\infty} \mathcal{D}_k(I)$.

We use the notation $U \approx V$ to denote the comparability of the quantities U and V, i.e., the existence of two positive constants C_1 and C_2 satisfying $C_1 V \leq U \leq C_2 V$. For convenience, we will always use the letter C to denote a positive constant, which may change from one equation to the next. The constants usually depend on the dimension n, and may also depend on α and other fixed parameters.

Now, for $\alpha \in (-\infty, \infty)$ and any cube I, let

$$\Psi_{f,\alpha}(I) := \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \Phi_f(J)\,, \tag{2.1}$$

where Φ_f is as in (1.2), and $\mathcal{D}_k(I)$ are the dyadic partitions of I defined above. As shown in [EJPX],

$$\Phi_f(I) \leq \Psi_{f,\alpha}(I) \leq \sum_{k=0}^{\infty} 2^{2\alpha k} \Phi_f(I)\,,$$

and hence $\Psi_{f,\alpha}(I) \approx \Phi_f(I)$ for all $\alpha \in (-\infty, 0)$. Moreover, for α in the positive range we have the following two lemmas (Lemmas 5.8 and 7.7 in [EJPX])

LEMMA 2.1 (EJPX).
(i) *Let $\alpha \in (-\infty, 1/2)$. Then for any cube I and $f \in L^2(I)$,*

$$O_{f,\alpha}(I) \approx \Psi_{f,\alpha}(I)\,.$$

(ii) *Let $\alpha \in (-\infty, \infty)$. Then for any $f \in L^2_{\text{loc}}(\boldsymbol{R}^n)$,*

$$\|f\|^2_{Q_\alpha(\boldsymbol{R}^n)} \approx \sup_I O_{f,\alpha}(I) \approx \sup_I \Psi_{f,\alpha}(I)\,,$$

where the supremum is taken over all cubes in $\boldsymbol{R}^n$ with dyadic sidelength.

Observe that if we replace $f(I)$ in $\Phi_f(I)$ with a constant c_I depending on the cube I, we obtain the following identity ((5.1) in [EJPX]):

$$|I|^{-1}\int_I |f(x)-c_I|^2 dx = \Phi_f(I) + |f(I)-c_I|^2 . \tag{2.2}$$

This implies that $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ if and only if there exist a finite constant $\kappa > 0$ and a constant c_I for every cube $I \subset \boldsymbol{R}^n$ such that

$$|I|^{-1}\int_I |f(x)-c_I|^2 dx \leq \kappa .$$

With the help of Lemma 2.1, we can easily obtain a $Q_\alpha(\boldsymbol{R}^n)$-version of the last assertion about $\mathrm{BMO}(\boldsymbol{R}^n)$.

THEOREM 2.2. *Let $\alpha \in (-\infty, \infty)$ and $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$. Then the following conditions are equivalent*:

(i) $f \in Q_\alpha(\boldsymbol{R}^n)$.

(ii) *There exist a finite constant $\kappa > 0$ and a sequence $\{c_J\}_{J\in\mathcal{D}_k(I)}$ for every cube $I \subset \boldsymbol{R}^n$ and integer $k \geq 0$ such that*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} |J|^{-1}\int_J |f(x)-c_J|^2 dx \leq \kappa .$$

(iii) *There exists a finite constant $\kappa > 0$ such that for every cube $I \subset \boldsymbol{R}^n$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} |J|^{-1}\int_0^{\infty} t\, m_{f,J}(t)\, dt \leq \kappa ,$$

where $m_{f,I}(t) = |\{x \in I; |f(x)-f(I)| > t\}|$.

PROOF. It suffices to show the implication (ii)⇒(i). If (ii) is true, then for the constant κ and sequence c_J, one has:

$$\Psi_{f,\alpha}(I) \leq \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} |J|^{-1}\int_J |f(x)-c_J|^2 dx \leq \kappa ,$$

which implies

$$\sup_I O_{f,\alpha}(I) \leq C \sup_I \Psi_{f,\alpha}(I) \leq C\kappa$$

and hence $f \in Q_\alpha(\boldsymbol{R}^n)$, by Lemma 2.1.

Denote by $\delta(\cdot,\cdot)$ the dyadic distance between two points in $\boldsymbol{R}^n$:

$$\delta(x, y) = \inf\{\ell(I); x, y \in I \in \mathcal{D}\} .$$

It is clear that

$$|x-y| \leq \sqrt{n}\,\delta(x, y) .$$

For a cube $I \in \mathcal{D}$ and a function $f \in L^2(I)$ let

$$O^{(d)}_{f,\alpha}(I) = (\ell(I))^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{(\delta(x, y))^{n+2\alpha}}\, dx\, dy$$

be the dyadic fractional mean oscillation (of f on I). The dyadic version $Q^{(d)}_\alpha(\boldsymbol{R}^n)$ of $Q_\alpha(\boldsymbol{R}^n)$ is defined as the class of all measurable functions f on $\boldsymbol{R}^n$ such that

$$\|f\|_{Q^{(d)}_\alpha(\boldsymbol{R}^n)} := \sup_{I \in \mathcal{D}} (O^{(d)}_{f,\alpha}(I))^{1/2} < \infty\,.$$

The following basic fact is Lemma 7.1 in [EJPX]:

LEMMA 2.3 (EJPX). *Let $\alpha \in (-\infty, \infty)$. Then, for any cube $I \in \mathcal{D}$ and $f \in L^2(I)$,*

$$\Psi_{f,\alpha}(I) \approx O^{(d)}_{f,\alpha}(I)\,.$$

This lemma gives immediately that $f \in Q^{(d)}_\alpha(\boldsymbol{R}^n)$ if and only if $\sup_{I\in\mathcal{D}} \Psi_{f,\alpha}(I) < \infty$ (see also [EJPX], Theorem 7.2). Moreover we have:

- $Q^{(d)}_\alpha(\boldsymbol{R}^n)$ is always a subclass of the dyadic BMO space

$$\mathrm{BMO}^{(d)}(\boldsymbol{R}^n) = \{f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n);\ \sup_{I\in\mathcal{D}} \Phi_f(I) < \infty\};$$

- $Q^{(d)}_\alpha(\boldsymbol{R}^n) = \mathrm{BMO}^{(d)}(\boldsymbol{R}^n)$ if $\alpha \in (-\infty, 0)$;
- $Q^{(d)}_\alpha(\boldsymbol{R}^n) = \boldsymbol{C}$ if $\alpha > n/2$.

For these see Theorem 7.3 in [EJPX].

Of particular interest is the following identity (see [EJPX], Theorem 7.9):

- $Q_\alpha(\boldsymbol{R}^n) = Q^{(d)}_\alpha(\boldsymbol{R}^n) \cap \mathrm{BMO}(\boldsymbol{R}^n), \quad \alpha \in (-\infty, 1/2)$.

In other words, $Q_\alpha(\boldsymbol{R}^n)$ can be characterized by means of $Q^{(d)}_\alpha(\boldsymbol{R}^n)$ and $\mathrm{BMO}(\boldsymbol{R}^n)$. Accordingly, in order to study $Q_\alpha(\boldsymbol{R}^n)$ it is enough to work with its dyadic counterpart, which is easily understood.

In what follows, we give a characterization of $Q^{(d)}_\alpha(\boldsymbol{R}^n)$ in terms of martingales, which is an analogue of a one-dimensional result of Janson ([Ja], Theorem 10). For each $l \in \boldsymbol{Z}$, let $\mathcal{F}_l$ be the σ-field generated by the partition $\mathcal{D}_l$. Then to each $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ we associate the sequence of functions $f_l = E(f\,|\mathcal{F}_l)$. In fact, f_l is the function that takes the constant value $f(I)$ on each dyadic cube $I \in \mathcal{D}_l$.

THEOREM 2.4. *Let $\alpha \in (-\infty, \infty)$ and $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$. Then the following conditions are equivalent*:

(i) $f \in Q^{(d)}_\alpha(\boldsymbol{R}^n)$.

(ii) *There exists a finite constant $\kappa > 0$ such that for each $l \in \boldsymbol{Z}$,*

$$\sum_{k=0}^{\infty} 2^{2\alpha k} E(|f - f_{l+k}|^2 \big| \mathcal{F}_l) \le \kappa \quad \text{a.s.}$$

(iii) *There exists a finite constant $\kappa > 0$ such that for each $l \in \boldsymbol{Z}$,*

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 2^{2\alpha j}\right) E(|f_{l+k+1} - f_{l+k}|^2 \big| \mathcal{F}_l) \leq \kappa \quad \text{a.s.}$$

PROOF. Let $l \in \boldsymbol{Z}$. If $I \in \mathcal{D}_l$ and $J \in \mathcal{D}_k(I) \subset \mathcal{D}_{l+k}$, then $|J| = 2^{-kn}|I|$, $f(J) = f_{l+k}$ on J, and hence

$$\begin{aligned}
\Psi_{f,\alpha}(I) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} |J|^{-1} \int_J |f(x) - f_{l+k}(x)|^2 dx \\
&= \sum_{k=0}^{\infty} 2^{2\alpha k} |I|^{-1} \int_I |f(x) - f_{l+k}(x)|^2 dx \qquad (2.3)\\
&= \sum_{k=0}^{\infty} 2^{2\alpha k} E(|f - f_{l+k}|^2 \big| \mathcal{F}_l)\,.
\end{aligned}$$

Thus (ii) is equivalent to $\sup_{I \in \mathcal{D}} \Psi_{f,\alpha}(I) < \infty$, which is equivalent to (i) by Lemma 2.3.

Note that for any nonnegative integer k one has, as in the one-dimensional case:

$$E(|f - f_{l+k}|^2 | \mathcal{F}_l) = \sum_{j=k}^{\infty} E(|f_{l+j+1} - f_{l+j}|^2 \big| \mathcal{F}_l)\,.$$

Inserting this into (2.3) and changing the order of summation, we get:

$$\sum_{k=0}^{\infty} 2^{2\alpha k} E(|f - f_{l+k}|^2 | \mathcal{F}_l) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} 2^{2\alpha k}\right) E(|f_{l+j+1} - f_{l+j}|^2 \big| \mathcal{F}_l)\,,$$

which implies the equivalence of (ii) and (iii).

Moreover, if $\alpha > 0$, $\sum_{k=0}^{j} 2^{2\alpha k} \approx 2^{2\alpha j}$ so (iii) is equivalent to

$$\sum_{j=0}^{\infty} 2^{2\alpha j} E(|f_{l+j+1} - f_{l+j}|^2 \big| \mathcal{F}_l) < \infty\,,$$

which is an exact extension of Theorem 10 in [Ja] to $\boldsymbol{R}^n$.

For the case $\alpha < 0$, recalling that $Q_\alpha^{(d)}(\boldsymbol{R}^n) = \mathrm{BMO}^{(d)}(\boldsymbol{R}^n)$, it follows from Theorem 2.4 that $f \in \mathrm{BMO}^{(d)}(\boldsymbol{R}^n)$ if and only if there is a finite constant $\kappa > 0$ such that

$$E(|f - f_j|^2 \big| \mathcal{F}_l) \leq \kappa \quad \text{a.s.}$$

for all $l \in \boldsymbol{Z}$ and all $j \geq l$. (See, for example, remark (b) in Section II.2 of [FeSt].)

Note that $Q_\alpha(\boldsymbol{R}^n)$ is translation invariant whereas $Q_\alpha^{(d)}(\boldsymbol{R}^n)$ is not. In fact, Theorem 7.8 in [EJPX] states that a function f is in $Q_\alpha(\boldsymbol{R}^n)$ if and only if all its translates belong to $Q_\alpha^{(d)}(\boldsymbol{R}^n)$. More precisely:

LEMMA 2.5 (EJPX). *Let $\alpha \in (-\infty, \infty)$ and let τ_t be the translation operator*

$$\tau_t f(x) = f(x - t)\,.$$

Then $f \in Q_\alpha(\boldsymbol{R}^n)$ if and only if $\tau_t f \in Q_\alpha^{(d)}(\boldsymbol{R}^n)$ for all $t \in \boldsymbol{R}^n$ with

$$\sup_{t\in\boldsymbol{R}^n} \|\tau_t f\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)} < \infty\,.$$

This suggests an extension of the general result of Garnett and Jones regarding BMO and BMO$^{(d)}$ ([GJ], Theorem, p. 352). In order to do so, we need two lemmas. The first one is a generalization of Lemma 5.6 in [EJPX] to the case of an arbitrary number of cubes.

LEMMA 2.6. *Let $\alpha \in (-\infty, 1/2)$. If a cube I is contained in the union of l cubes of the same size, namely $I \subset I^1 \cup \cdots \cup I^l$, with $|I^j| = |I|$ for $1 \le j \le l$, then*

$$\Phi_f(I) \le \sum_{j=1}^{l} \Phi_f(I^j) + \frac{2(l-1)}{l^2} \sum_{1\le i<j\le l} |f(I^i) - f(I^j)|^2 \tag{2.4}$$

and

$$\Psi_{f,\alpha}(I) \le C_l \left(\sum_{j=1}^{l} \Psi_{f,\alpha}(I^j) + \sum_{1\le i<j\le l} |f(I^i) - f(I^j)|^2 \right). \tag{2.5}$$

PROOF. The proof is similar to that of Lemma 5.6 in [EJPX]. First, using identity (2.2) with the constant $c_I = c := l^{-1}\sum_{i=1}^{l} f(I^i)$, we have

$$\begin{aligned}
\Phi_f(I) &\le |I|^{-1}\int_I |f(x)-c|^2dx \le \sum_{j=1}^{l} |I^j|^{-1}\int_{I^j} |f(x)-c|^2dx \\
&= \sum_{j=1}^{l} \left\{ \Phi_f(I^j) + |f(I^j) - c|^2 \right\} \\
&\le \sum_{j=1}^{l} \left\{ \Phi_f(I^j) + \left(l^{-1}\sum_{i=1}^{l} |f(I^j) - f(I^i)| \right)^2 \right\} \\
&\le \sum_{j=1}^{l} \left\{ \Phi_f(I^j) + \frac{(l-1)}{l^2}\sum_{i\ne j} |f(I^i) - f(I^j)|^2 \right\} \\
&= \sum_{j=1}^{l} \Phi_f(I^j) + \frac{2(l-1)}{l^2}\sum_{1\le i<j\le l} |f(I^i) - f(I^j)|^2\,.
\end{aligned}$$

For (2.5) we can, as is done in [EJPX], assume by homogeneity that $I = I^0 + \mathbf{x}$, with $I^0 = [0,1]^n$ and $\mathbf{x} = x^1\mathbf{e}^1 + \cdots + x^n\mathbf{e}^n$, where $x^i \in [0,1)$ and $\mathbf{e}^i$ is the unit vector in the ith coordinate direction. Then each I^j in the statement of the lemma can be assumed to be of the form $I^j = I^0 + \mathbf{v}^j$, where $\mathbf{v}^j$ is a vector whose coordinates are 0's and 1's. There are at most 2^n such choices, and any I^j which is not of this form would be superfluous and would just add to the right-hand-side of (2.5). Note that if $x^i = 0$, we need only consider those I^j's for which the vector $\mathbf{v}^j$ has a zero in the ith coordinate. In [EJPX] it was assumed $x^2 = \cdots = x^n = 0$ and there were only two cubes containing I (i.e. $l = 2$).

Now suppose k is a nonnegative integer. Following the notation in [EJPX], we denote by $\mathcal{D}_k^*$ the union $\bigcup_{j=1}^l \mathcal{D}_k(I^j)$ of the $l2^{nk}$ dyadic cubes of sidelength 2^{-k} contained in $I^1 \cup \cdots \cup I^l$. If $I_k \in \mathcal{D}_k(I)$, then

$$I_k = [0, 2^{-k}]^n + \mathbf{x} + t_k^1 \mathbf{e}^1 + \cdots t_k^n \mathbf{e}^n ,$$

where t_k^j is a number of the from $m2^{-k}$, m an integer, $0 \leq m < 2^k$. Now the cube $I_k^0 = [0, 2^{-k}]^n + t_k^1 \mathbf{e}^1 + \cdots t_k^n \mathbf{e}^n$ is in $\mathcal{D}_k(I^0)$, so $I_k = I_k^0 + \mathbf{x}$ is contained in l dyadic cubes $I_k^1 \cup \cdots \cup I_k^l$, each of which is in $\mathcal{D}_k^*$. Here l is the same number of cubes as in the statement of the lemma. In fact, if the point $\mathbf{x} + t_k^1 \mathbf{e}^1 + \cdots t_k^n \mathbf{e}^n$ belongs to the cube $I_k^1 \in \mathcal{D}_k^*$, then the other cubes are given by $I_k^j = I_k^1 + 2^{-k}\mathbf{v}^j$, where the vectors $\mathbf{v}^j$, $1 \leq j \leq l$ are as above (depending on the number of nonzero coordinates x^i of $\mathbf{x}$). Note that if x^i is an integer multiple of 2^{-k}, it is possible that I_k is contained in less than l cubes, but we can always use l cubes.

Applying (2.4) to I_k, we get

$$\begin{aligned} \Phi_f(I_k) &\leq \sum_{j=1}^{l} \Phi_f(I_k^j) + \frac{2(l-1)}{l^2} \sum_{1 \leq i < j \leq l} |f(I_k^i) - f(I_k^j)|^2 \\ &= \sum_{j=1}^{l} \Phi_f(I_k^j) + \frac{2(l-1)}{l^2} \sum_{1 \leq i < j \leq l} |f(I_k^1 + 2^{-k}\mathbf{v}^i) - f(I_k^1 + 2^{-k}\mathbf{v}^j)|^2 . \end{aligned}$$

Again following [EJPX], let us denote by $\mathcal{D}_k^0$ the set of $J \in \mathcal{D}_k^*$ such that $J + 2^{-k}\mathbf{v}^j \in \mathcal{D}_k^*$ for $j = 1, \ldots, l$. Since different I_k in $\mathcal{D}_k(I)$ corresponds to different I_k^1 in $\mathcal{D}_k^0$, if we sum over all $I_k \in \mathcal{D}_k(I)$ and over all $k \geq 0$, we get

$$\begin{aligned} \Psi_{f,\alpha}(I) &= \sum_{k=0}^{\infty} \sum_{I_k \in \mathcal{D}_k(I)} 2^{(2\alpha - n)k} \Phi_f(I_k) \\ &\leq l \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k^*} \Phi_f(J) \\ &\quad + \frac{2(l-1)}{l^2} \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k^0} \sum_{1 \leq i < j \leq l} |f(J + 2^{-k}\mathbf{v}^i) - f(J + 2^{-k}\mathbf{v}^j)|^2 \qquad (2.6) \\ &\leq l \sum_{j=1}^{l} \Psi_{f,\alpha}(I^j) \\ &\quad + \frac{2(l-1)}{l^2} \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k^0} \sum_{1 \leq i < j \leq l} |f(J + 2^{-k}\mathbf{v}^i) - f(J + 2^{-k}\mathbf{v}^j)|^2 . \end{aligned}$$

Continuing as in [EJPX], fix $k \geq 0$, a cube $J \in \mathcal{D}_k^0$, and vectors $\mathbf{v}^i$ and $\mathbf{v}^j$. For simplicity, write $J_0 = J + 2^{-k}\mathbf{v}^i$, $K_0 = J + 2^{-k}\mathbf{v}^j$. Then as in [EJPX] we have two sequences of dyadic cubes $J_0 \subset J_1 \subset \cdots \subset J_m$ and $K_0 \subset K_1 \subset \cdots \subset K_m$ with sidelengths $\ell(J_r) = 2^{-k+r}$ and

$J_m = K_m$ being the smallest dyadic cube containing J_0 and K_0. Thus we can repeat estimate (5.14) in [EJPX], using Cauchy-Schwarz and the fact that $|f(J_r) - f(J_{r-1})|^2 \le 2^n \Phi_f(J_r)$:

$$|f(J_0) - f(K_0)|^2 \le C \sum_{r=1}^{m} r^2 (\Phi_f(J_r) + \Phi_f(K_r)) \,. \tag{2.7}$$

Note that since $I^1 \cup \cdots \cup I^l \subset [0,2]^n$, we must have $m \le k+1$. Moreover, in the case $m = k+1$, J_k and K_k must be I^i and I^j for some i, j, $1 \le i < j \le l$. In this case we end up with the following analogue of estimate (5.15) in [EJPX]:

$$|f(J_0) - f(K_0)|^2 \le C \sum_{r=1}^{k} r^2 (\Phi_f(J_r) + \Phi_f(K_r)) + C|f(I^i) - f(I^j)|^2 \,. \tag{2.8}$$

Now we need to sum over all $J \in \mathcal{D}_k^0$ and all choices of $J_0 = J + 2^{-k}\mathbf{v}^i$ and $K_0 = J + 2^{-k}\mathbf{v}^j$, $1 \le i < j \le l$. As pointed out in [EJPX], the cubes J_r and K_r in (2.7) belong to $\mathcal{D}_{k-r}^*$, $1 \le r \le k$. Conversely, each cube $J' \in \mathcal{D}_{k-r}^*$ corresponds to a J_r or K_r only for those J_0 and K_0 which lie adjacent to the boundaries of its 2^n dyadic subcubes of sidelength 2^{-k+r-1} (otherwise both J_0 and K_0 would lie in a dyadic cube of sidelength 2^{-k+r-1}, thereby making $m \le r-1$). Thus there are at most $4n2^{(n-1)r}$ choices of J_0 or K_0 corresponding to $J' \in \mathcal{D}_{k-r}^*$. For the case $m = k+1$, in which we apply (2.8), each cube I^j, $1 \le j \le l$, can appear as J_k or K_k for at most $4n2^{(n-1)k}$ choices of J_0 or K_0. Thus we get

$$\begin{aligned}
&\sum_{J \in \mathcal{D}_k^0} \sum_{1 \le i < j \le l} |f(J + 2^{-k}\mathbf{v}^i) - f(J + 2^{-k}\mathbf{v}^j)|^2 \\
&\qquad \le C \sum_{r=1}^{k} \sum_{J' \in \mathcal{D}_{k-r}^*} r^2 2^{(n-1)r} \Phi_f(J') + C 2^{(n-1)k} \sum_{1 \le i < j \le l} |f(I^i) - f(I^j)|^2 \,.
\end{aligned}$$

Finally, summing over all $k \ge 0$ and letting $s = k - r$, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} 2^{(2\alpha - n)k} \sum_{J \in \mathcal{D}_k^0} \sum_{1 \le i < j \le l} |f(J + 2^{-k}\mathbf{v}^i) - f(J + 2^{-k}\mathbf{v}^j)|^2 \\
&\quad \le C \sum_{k=0}^{\infty} 2^{(2\alpha - n)k} \left\{ \sum_{r=1}^{k} \sum_{J' \in \mathcal{D}_{k-r}^*} r^2 2^{(n-1)r} \Phi_f(J') + C 2^{(n-1)k} \sum_{1 \le i < j \le l} |f(I^i) - f(I^j)|^2 \right\} \\
&\quad = C \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{J' \in \mathcal{D}_s^*} r^2 2^{(n-1)r + (2\alpha - n)(r+s)} \Phi_f(J') \\
&\qquad + C \sum_{k=0}^{\infty} 2^{(2\alpha - 1)k} \sum_{1 \le i < j \le l} |f(I^i) - f(I^j)|^2 \\
&\quad = C \sum_{r=0}^{\infty} r^2 2^{(2\alpha - 1)r} \sum_{1 \le j \le l} \sum_{s=0}^{\infty} \sum_{J' \in \mathcal{D}_s(I^j)} 2^{(2\alpha - n)s} \Phi_f(J')
\end{aligned}$$

$$+ C \sum_{k=0}^{\infty} 2^{(2\alpha-1)k} \sum_{1\le i<j\le l} |f(I^i) - f(I^j)|^2$$
$$\le C \sum_{1\le j\le l} \Psi_{f,\alpha}(I^j) + C \sum_{1\le i<j\le l} |f(I^i) - f(I^j)|^2 ,$$

since $\alpha < 1/2$ implies that $\sum_{k=0}^{\infty} 2^{(2\alpha-1)k}$ and $\sum_{r=0}^{\infty} r^2 2^{(2\alpha-1)r}$ are finite. Inserting this into (2.6), we get (2.5). Note that the constant C_l appearing in (2.5) can be replaced by a constant depending only on the dimension n and α, since, as explained above, we can assume $l \le 2^n$.

The second lemma required is an extension of Lemma 6 in [Ja] to $\boldsymbol{R}^n$.

LEMMA 2.7. *Suppose J is a fixed dyadic cube in $\boldsymbol{R}^n$, with sidelength 2^K for some $K \in \boldsymbol{Z}$. Let $I \subset \boldsymbol{R}^n$ be a cube (not necessarily dyadic) of sidelength 2^k, $k \in \boldsymbol{Z}$, and suppose $t \in \boldsymbol{R}^n$. Consider the intersection $(I+t)\cap J$ of the translated cube $I+t$ with the fixed cube J. Let I'_t be the smallest dyadic cube of sidelength at least $2^{\min\{k,K\}}$ which contains $(I+t)\cap J$, and let $\ell(I'_t) = 2^L$. Define $m_J(I,t)$ as follows:*

(i) *if $|(I+t)\cap J| = 0$, set $m_J(I,t) = 0$;*
(ii) *if $|(I+t)\setminus J| = 0$ (i.e. $(I+t)\subset J$), set $m_J(I,t) = L-k$;*
(iii) *if $|(I+t)\setminus J| > 0$ and $|(I+t)\cap J| > 0$, set $m_J(I,t) = \max\{0, K-k\}$.*

Then for every nonnegative integer M,

$$|\{t \in \boldsymbol{R}^n : m_J(I,t) > M\}| \le 2^{(K+1)n-M} , \tag{2.9}$$

and for $p \in (0,\infty)$,

$$\int_{\boldsymbol{R}^n} (m_J(I,t))^p dt \le 2^{(K+1)n+1} \sum_{l=1}^{\infty} l^p 2^{-l} \le C_{n,p}|J| < \infty . \tag{2.10}$$

PROOF. By definition, $m_J(I,t) \ge 0$ in all cases. Moreover, the cube $I'_t \subset J$ implies that $L \le K$ and therefore $m_J(I,t) \le \max\{0, K-k\}$ in all cases. If $k \ge K$, then $m_J(I,t)$ is identically 0 and there is nothing to prove. Thus we may assume $k < K$, and prove (2.9) for integers M with $0 \le M < K-k$.

Since (2.9) is translation invariant in I, we can fix $J = [0, 2^K]^n$, and assume $I = [0, 2^k]^n$. Observe that $m_J(I,t) > M$ is equivalent to the fact that $I+t$ intersects J (nontrivially) but every dyadic subcube $J' \subset J$ with sidelength $\ell(J') = 2^{k+M}$ does not contain $I+t$. Thus for each such dyadic subcube J' we get a set of vectors

$$S_{J'} = \{t \in J';\ I+t \text{ is not contained in } J'\}$$

with volume $|S_{J'}|$ equal to $2^{(k+M)n} - (2^{k+M} - 2^k)^n$. In addition, we have the set

$$S_0 = \{t \in \boldsymbol{R}^n;\ |(I+t)\cap J| > 0, |I+t\setminus J| > 0\}$$

with volume $|S_0|$ equal to $(2^K + 2^k)^n - 2^{Kn}$. Thus

$$\begin{aligned}
|\{t \in \boldsymbol{R}^n : m_J(I,t) > M\}| &= \sum_{J' \in \mathcal{D}_{K-k-M}(J)} |S_{J'}| + |S_0| \\
&= 2^{(K-k-M)n}(2^{(k+M)n} - (2^{k+M} - 2^k)^n) + (2^K + 2^k)^n - 2^{Kn} \\
&= 2^{Kn}[1 - (1 - 2^{-M})^n] + 2^{Kn}[(1 + 2^{k-K})^n - 1] \\
&\le 2^{Kn}[n2^{-M} + (2^n - 1)2^{k-K}] \\
&\le 2^{(K+1)n-M},
\end{aligned}$$

and (2.9) holds.

Now for $p > 0$ one has:

$$\begin{aligned}
\int_{\boldsymbol{R}^n} (m_J(I,t))^p dt &= \sum_{l=0}^{K-k} l^p |\{t \in \boldsymbol{R}^n : m_J(I,t) = l\}| \\
&\le \sum_{l=1}^{K-k} l^p |\{t \in \boldsymbol{R}^n : m_J(I,t) > l - 1\}| \\
&\le 2^{(K+1)n+1} \sum_{l=1}^{\infty} l^p 2^{-l},
\end{aligned}$$

proving (2.10).

With these two lemmas we have the following analogue of the result of Garnett and Jones (see also Theorem 5 in [Ja] for a one-dimensional Q_p version):

THEOREM 2.8. *Let $\alpha \in (-\infty, 1/2)$. Suppose F is a function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ such that for each $t \in \boldsymbol{R}^n$, $F(t,\cdot) \in Q_\alpha^{(d)}(\boldsymbol{R}^n)$ with support in a fixed dyadic cube, and $\int_{\boldsymbol{R}^n} F(t,x)dx = 0$. Moreover, assume $\|F(t,\cdot)\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)}$ is essentially bounded as a function of t. Then for every $N \in (0,\infty)$, the averaging function f_N belongs to $Q_\alpha(\boldsymbol{R}^n)$, where f_N is defined on $\boldsymbol{R}^n$ by*

$$f_N(x) = \frac{1}{(2N)^n} \int_{[-N,N]^n} F(t, x+t)dt .$$

PROOF. We proceed by analogy with Janson's proof in the case of the circle (see [Ja]). Set $f_t(x) = F(t,x)$ and $h_t(x) = F(t, x+t)$. Let J denote the fixed dyadic cube which contains the support of f_t for every t. Assume that I is a cube in $\boldsymbol{R}^n$ (not necessarily dyadic) of sidelength 2^k, $k \in \boldsymbol{Z}$. Consider the translate $I + t$ for some $t \in [-N,N]^n$. We want to estimate $\Psi_{h_t,\alpha}(I) = \Psi_{f_t,\alpha}(I+t)$ in terms of $\|f_t\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)}$. More specifically, we want to prove the estimate

$$(\Psi_{f_t,\alpha}(I+t))^{1/2} \le C(m_J(I,t) + 1)\|f_t\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)}, \tag{2.11}$$

where $m_J(I,t)$ is the integer defined in Lemma 2.7. There are several cases depending on the nature of the intersection $(I + t) \cap J$.

The trivial case is when the measure $|(I+t)\cap J|=0$, in which case Lemma 2.7 defines $m_J(I,t)$ to be zero, and on the other hand $f_t=0$ almost everywhere on $I+t$ gives $\Psi_{f_t,\alpha}(I+t)=0$.

The next case is that in which $|(I+t)\setminus J|=0$, or more simply (if we consider closed cubes) $I+t\subset J$. Then $I+t$ is contained in the union of at most 2^n adjacent dyadic cubes of equal sidelength, namely $I+t\subset I_1\cup\cdots\cup I_l$, with $\ell(I_j)=\ell(I)=2^k$ for $1\le j\le l$, I_j dyadic, and $l\le 2^n$. Moreover, if I_t' is the smallest dyadic cube containing $I+t$, then $I_1\cup\cdots\cup I_l\subset I_t'$.

By Lemma 2.6,

$$(\Psi_{f_t,\alpha}(I+t))^{1/2}\le C_l\left(\sum_{j=1}^{l}(\Psi_{f_t,\alpha}(I_j))^{1/2}+\sum_{1\le j<k\le l}|f_t(I_j)-f_t(I_k)|\right)$$
$$\le C\|f_t\|_{Q_\alpha^{(d)}(\mathbf{R}^n)}+C\sum_{1\le j<k\le l}|f_t(I_j)-f_t(I_k)|, \tag{2.12}$$

where the constants depend only on α and n (since $l\le 2^n$). For each cube I_j, consider the sequence of dyadic cubes $I_j=J_{j,0}\subset J_{j,1}\subset\cdots\subset J_{j,m}=I_t'$ with $\ell(J_{j,i+1})=2\ell(J_{j,i})$. Here $m=L-k$, where $\ell(I_t')=2^L$, and this is exactly the integer $m_J(I,t)$ defined by Lemma 2.7 in this case. Recall (see (5.4) in [EJPX]) that if $I\subset J$ then $|f(I)-f(J)|^2\le(|J|/|I|)\Phi_f(J)$. Thus

$$\begin{aligned}|f_t(I_j)-f_t(I_k)|&\le\sum_{i=1}^{m}|f_t(J_{j,i})-f_t(J_{j,i-1})|+\sum_{i=1}^{m}|f_t(J_{k,i})-f_t(J_{k,i-1})|\\&\le 2^{n/2}\sum_{i=1}^{m}(\Phi_{f_t}(J_{j,i}))^{1/2}+2^{n/2}\sum_{i=1}^{m}(\Phi_{f_t}(J_{k,i}))^{1/2}\\&\le 2^{n/2+1}m_J(I,t)\|f_t\|_{Q_\alpha^{(d)}(\mathbf{R}^n)}.\end{aligned}$$

Inserting this in (2.12), we get (2.11).

The remaining case is that in which $I+t$ intersects both J and its complement, i.e. $|(I+t)\setminus J|>0$ and $|(I+t)\cap J|>0$. Again write $I+t\subset I_1\cup\cdots\cup I_l$, with $\ell(I_j)=\ell(I)=2^k$ for $1\le j\le l$, I_j dyadic, and $l\le 2^n$, and use Lemma 2.6 to get inequality (2.12).

Now for each pair j,k, we have three cases. If both I_j and I_k have interiors which are disjoint from J, then $f_t(I_j)=f_t(I_k)=0$. If both I_j and I_k are contained in J, we proceed as above, using the fact that in this case $k\le K$, where $\ell(J)=2^K$, and $m_J(I,t)=K-k$, so if L is as above (i.e. $\ell(I_t')=2^L$), then $L-k\le m_J(I,t)$. Thus we again get

$$|f_t(I_j)-f_t(I_k)|\le 2^{n/2+1}m_J(I,t)\|f_t\|_{Q_\alpha^{(d)}(\mathbf{R}^n)}.$$

Otherwise, one of these two cubes, say I_k, must have disjoint interior with J, while the other, I_j, must either contain or be contained in J. Since f_t is supported in J, this gives $f_t(I_k)=0$ and $|f_t(I_j)-f_t(I_k)|=|f_t(I_j)|$.

If $I_j\supset J$, then $\int_{I_j}f_t(x)dx=\int_J f_t(x)dx=\int_{\mathbf{R}^n}f_t(x)dx=0$ so $f_t(I_j)=0$ and $|f_t(I_j)-f_t(I_k)|=0=m_J(I,t)$ (since $k\ge K$). Otherwise $I_j\subset J$, hence we let $J_{j,i}$ be the sequence of dyadic cubes with $I_j = J_{j,0} \subset J_{j,1} \subset \cdots \subset J_{j,m} = J$, where again in this case $m = m_J(I,t) = K - k$, and $\ell(J_{j,i+1}) = 2\ell(J_{j,i})$. Thus we can write

$$\begin{aligned} |f_t(I_j)| = |f_t(I_j) - f_t(J)| &\le \sum_{i=1}^{m} |f_t(J_{j,i}) - f_t(J_{j,i-1})| \le 2^{n/2} \sum_{i=1}^{m} (\Phi_{f_t}(J_{j,i}))^{1/2} \\ &\le 2^{n/2} m_J(I,t) \|f_t\|_{Q_\alpha^{(d)}(\boldsymbol{R})} . \end{aligned}$$

Inserting the estimates for both cases into (2.12), we again get (2.11).

Finally, in order to estimate $(\Psi_{f_N,\alpha}(I))^{1/2}$, we note that $(\Psi_{f,\alpha}(I))^{1/2}$ may be regarded as a weighted L^2 norm on $I \times I$ by writing (see [Ja], (10)):

$$\begin{aligned} \Psi_{f,\alpha}(I) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \Phi_f(J) \\ &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k-1} \sum_{J \in \mathcal{D}_k(I)} |J|^{-2} \int_J \int_J |f(x) - f(y)|^2 dx dy \\ &= \int_I \int_I |f(x) - f(y)|^2 \left(|I|^{-2} \sum_{k=0}^{\infty} \sum_{J \in \mathcal{D}_k(I)} 2^{(2\alpha+n)k-1} \chi_J(x) \chi_J(y) \right) dx dy . \end{aligned}$$

Applying Minkowski's integral inequality to

$$f_N = (2N)^{-n} \int_{[-N,N]^n} h_t dt ,$$

we get

$$\begin{aligned} (\Psi_{f_N,\alpha}(I))^{1/2} &\le (2N)^{-n} \int_{[-N,N]^n} (\Psi_{h_t,\alpha}(I))^{1/2} dt \\ &\le C(2N)^{-n} \int_{[-N,N]^n} (m_J(I,t) + 1) \|f_t\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)} dt \\ &\le C \sup_t \|F(t,\cdot)\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)} \left(1 + (2N)^{-n} \int_{t \in \boldsymbol{R}^n} m_J(I,t) dt \right) \\ &\le C \sup_t \|F(t,\cdot)\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)} (1 + (2N)^{-n} |J|) \end{aligned}$$

by (2.10) in Lemma 2.7. This shows that $\Psi_{f_N,\alpha}(I)$ is uniformly bounded (by a constant depending on the volume of J) when I is a cube of sidelength 2^k for any integer k. By Lemma 2.1, $f_N \in Q_\alpha(\boldsymbol{R}^n)$, with norm

$$\|f_N\|_{Q_\alpha(\boldsymbol{R}^n)} \le C_{N,J} \sup_{t \in \boldsymbol{R}^n} \|F(t,\cdot)\|_{Q_\alpha^{(d)}(\boldsymbol{R}^n)} .$$

3. Atomic decomposition for $Q_\alpha(\boldsymbol{R}^n)$. We follow the terminology of [FJ], Section 4, for the following:

DEFINITION 3.1. We call a function a_I a $(0,\infty)$-atom if there exists a cube $I \subset \boldsymbol{R}^n$ and integers $N_1 \ge 1$ and $N_2 \ge 0$, such that:

(a) supp $a_I \subset 3I$;
(b) $|D^\gamma a_I(x)| \le (\ell(I))^{-|\gamma|}$, for $D^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_n)^{\gamma_n}$ and $|\gamma| \le N_1$;
(c) $\int_{\boldsymbol{R}^n} x^\gamma a_I(x)dx = 0$, for $|\gamma| \le N_2$.

Uchiyama proves (cf. [U], Lemmas 3.1–3.4) that $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ if and only if f can be written as

$$f = \sum_{I \in \mathcal{D}} s_I a_I$$

with $\{a_I\}_{I\in\mathcal{D}}$ a sequence of $(0,\infty)$-atoms and $\{s_I\}_{I\in\mathcal{D}}$ a sequence of coefficients such that the measure $\mu = \sum_I |s_I|^2 |I| \delta_{(x_I, \ell(I))}$ satisfies the Carleson condition

$$\|\mu\|_c := \sup_{I\in\mathcal{D}} \mu(S(I))/|I| < \infty\,.$$

Here $\delta_{(x,t)}$ is used to denote the unit mass at the point (x,t) in the upper half-space $\boldsymbol{R}^{n+1}_+ = \{(x,t);\, x \in \boldsymbol{R}^n, t > 0\}$, and $S(I)$ denotes the "Carleson box" above the cube I, i.e.

$$S(I) = \{(y,t) \in \boldsymbol{R}^{n+1}_+;\, y \in I, 0 < t < \ell(I)\}.$$

This condition on the measure is equivalent (cf. [FJ], Section 4) to the following condition on the coefficients (which we will also call the Carleson condition):

$$\|\{s_I\}_{I\in\mathcal{D}}\|_{\mathcal{C}} := \left(\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subseteq I} |s_J|^2 |J| \right)^{1/2} < \infty\,.$$

In fact, in Uchiyama's result, condition (b) on the atoms is replaced by the Lipschitz condition $\|a_I\|_{\mathrm{Lip1}} \le \ell(I)^{-1}$, where we use $\|f\|_{\mathrm{Lip1}}$ to denote the Lip1-norm of $f \in C(\boldsymbol{R}^n)$, namely

$$\|f\|_{\mathrm{Lip1}} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|}\,,$$

and in condition (c) Uchiyama has $N_2 = 0$. In [FJ] the result is stated with more smoothness and cancellation conditions on the atoms.

We will extend this result to $Q_\alpha(\boldsymbol{R}^n)$ for $\alpha \in (0,1)$. There is a similar characterization of $Q_\alpha(\boldsymbol{R}^n)$ in terms of wavelets, given in [EJPX], Section 6, whose proof is based on that of Meyer [M] for the case of BMO. However, for the atomic decomposition we cannot assume orthogonality and we need to compensate with quasi-orthogonality, as in Uchiyama [U]. The idea of the quasi-orthogonal decomposition goes back to Chang and Fefferman [CF] (see also [St], Section IV.4.5).

DEFINITION 3.2. For $\alpha \in (-\infty,\infty)$, $I \in \mathcal{D}$ and $\mathbf{s} = \{s_K\}_{K\in\mathcal{D}}$ (a sequence associated to all the dyadic cubes in $\boldsymbol{R}^n$), let

$$U_{\mathbf{s},\alpha}(I) := \sum_{K\subseteq I} \left(\frac{\ell(K)}{\ell(I)}\right)^{n-2\alpha} |s_K|^2\,.$$

We say that $\{s_K\}_{K\in\mathcal{D}}$ is $\mathcal{C}_\alpha$-sequence provided that

$$\|\{s_K\}_{K\in\mathcal{D}}\|_{\mathcal{C}_\alpha} := \left(\sup_{I\in\mathcal{D}} U_{\mathbf{s},\alpha}(I)\right)^{1/2} < \infty\,.$$

For $\alpha = 0$, this is just the Carleson condition $\|\{s_K\}_{K\in\mathcal{D}}\|_{\mathcal{C}} < \infty$ above. Note that for $\alpha \geq 0$, a $\mathcal{C}_\alpha$-sequence is also a $\mathcal{C}_0$-sequence, since

$$U_{\mathbf{s},0}(I) \leq U_{\mathbf{s},\alpha}(I)\,.$$

Before we continue we need to recall a basic geometric fact about cubes in $\boldsymbol{R}^n$. If I, J are cubes, $\ell(I) = a\ell(J)$, and $bI \cap cJ \neq \emptyset$ for some positive numbers a, b, c, then

$$I \subset (a + ab + c)J\,.$$

In particular, if we restrict I to be a dyadic cube, then for a fixed J there are at most $(1 + b + c/a)^n$ such choices of I.

The next lemma extends Uchiyama's quasi-orthogonality result, Lemma 3.3 in [U] (see also [St], Lemma 4.5.1).

LEMMA 3.3. *Let $j \in \boldsymbol{N}$ and suppose $\{s_K\}_{K\in\mathcal{D}}$ is a $\mathcal{C}_\alpha$-sequence. For integers $l < m$, consider a collection $\mathcal{F}$ of dyadic cubes such that $2^l \leq \ell(K) \leq 2^m$ for every element K of $\mathcal{F}$. Define*

$$f(x) = \sum_{K\in\mathcal{F}} s_K a_K(x)\,, \quad x \in \boldsymbol{R}^n\,, \tag{3.1}$$

where the functions a_K satisfy:

(a′) $\operatorname{supp} a_K \subseteq 2^j K$;

(b′) $\|a_K\|_{\mathrm{Lip}1} \leq 2^{-j}(\ell(K))^{-1}$;

(c′) $\int_{\boldsymbol{R}^n} a_K(x)dx = 0$.

If $\alpha \in [0, 1)$, then there exists a constant C, independent of the choice of l, m and $\mathcal{F}$, such that

(i)

$$\|f\|_{L^2}^2 \leq C2^{2jn} \sum_{K\in\mathcal{F}} |s_K|^2 \ell(K)^n$$

and

(ii)

$$\int\!\!\int \frac{\left|\sum_{\{K\in\mathcal{F};\,x,y\in 2^jK\}} s_K[a_K(x) - a_K(y)]\right|^2}{|x-y|^{n+2\alpha}}\,dxdy \leq C2^{2j(n-\alpha)} \sum_{K\in\mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha}\,.$$

PROOF. Since $\alpha \geq 0$, $\{s_K\}$ is also $\mathcal{C}_0$-sequence and therefore part (i) is just Lemma 3.3 in [U]. We omit the proof since we will repeat the argument for the proof of part (ii).

Suppose $J, K \in \mathcal{F}$ with $\ell(J) \le \ell(K)$ and $2^j J \cap 2^j K \neq \emptyset$. Then

$$\begin{aligned} &\int_{2^j J \cap 2^j K} \int_{2^j J \cap 2^j K} \frac{|a_K(x) - a_K(y)||a_J(x) - a_J(y)|}{|x-y|^{n+2\alpha}}\, dx dy \\ &\quad \le 2^{-2j} \ell(K)^{-1} \ell(J)^{-1} \int_{2^j J} \int_{2^j J} |x-y|^{2-n-2\alpha}\, dx dy \\ &\quad \le C 2^{j(n-2\alpha)} \ell(K)^{-1} \ell(J)^{n+1-2\alpha}\,. \end{aligned} \tag{3.2}$$

Now for each cube $K \in \mathcal{F}$ and integer $k \ge 0$, let

$$\mathcal{G}_k(K) = \{J \in \mathcal{F};\, \ell(J) = 2^{-k}\ell(K) \text{ and } 2^j J \cap 2^j K \neq \emptyset\}\,.$$

Take Ω to be a bounded set in $\boldsymbol{R}^n$, so that the number of $K \in \mathcal{F}$ for which $2^j K \cap \Omega \neq \emptyset$ is finite. Thus if we integrate over $\Omega \times \Omega$, we can, after expanding the square inside the integral, interchange the order of summation and integration (since the sum is finite) and use (3.2) to get

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{\left|\sum_{\{K \in \mathcal{F}:\, x, y \in 2^j K\}} s_K [a_K(x) - a_K(y)]\right|^2}{|x-y|^{n+2\alpha}}\, dx dy \\ &\le \sum_{K \in \mathcal{F}} |s_K| \sum_{k=0}^{\infty} \sum_{J \in \mathcal{G}_k(K)} |s_J| \int_{2^j J \cap 2^j K} \int_{2^j J \cap 2^j K} \frac{|a_K(x) - a_K(y)||a_J(x) - a_J(y)|}{|x-y|^{n+2\alpha}}\, dx dy \\ &\le C \sum_{k=0}^{\infty} \sum_{K \in \mathcal{F}} |s_K| \sum_{J \in \mathcal{G}_k(K)} |s_J| 2^{j(n-2\alpha)} \ell(K)^{-1} \ell(J)^{n+1-2\alpha} \\ &\le C 2^{j(n-2\alpha)} \sum_{k=0}^{\infty} 2^{-k(n+1-2\alpha)} \left(\sum_{K \in \mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha} \right)^{1/2} \\ &\quad \times \left(\sum_{K \in \mathcal{F}} \left(\sum_{\mathcal{G}_k(K)} |s_J| \right)^2 \ell(K)^{n-2\alpha} \right)^{1/2} \\ &\le C 2^{j(n-2\alpha)} \sum_{k=0}^{\infty} 2^{-k(n+1-2\alpha)} \left(\sum_{K \in \mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha} \right)^{1/2} \\ &\quad \times \left(\sum_{K \in \mathcal{F}} \sum_{\mathcal{G}_k(K)} |s_J|^2 2^{(j+k)n} \ell(K)^{n-2\alpha} \right)^{1/2} \\ &\le C 2^{j(n-2\alpha)} \sum_{k=0}^{\infty} 2^{-k(n+1-2\alpha)} \left(\sum_{K \in \mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha} \right)^{1/2} \\ &\quad \times \left(\sum_{J \in \mathcal{F}} |s_J|^2 2^{2jn + k(2n-2\alpha)} \ell(J)^{n-2\alpha} \right)^{1/2} \\ &\le C 2^{2j(n-\alpha)} \sum_{k=0}^{\infty} 2^{-k(1-\alpha)} \sum_{K \in \mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha} \end{aligned}$$

$$\leq C2^{2j(n-\alpha)} \sum_{K\in\mathcal{F}} |s_K|^2 \ell(K)^{n-2\alpha} \quad (\text{since } \alpha < 1)\,.$$

Note that we have used the the basic geometric fact above (see remark before the statement of the lemma) to bound the cardinality of $\mathcal{G}_k(K)$ by a constant multiple of $2^{(j+k)n}$ and the cardinality of $\{K : J \in \mathcal{G}_k(K)\}$ (for a fixed J) by a constant multiple of 2^{jn}.

We have thus proved Part (ii) of the lemma for the double integral over any set $\Omega \times \Omega$, $\Omega \subset \boldsymbol{R}^n$ bounded, and therefore it holds for the integral over $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

Continuing as in [U], we give an analogue of Uchiyama's Lemma 3.4 in the case of $Q_\alpha(\boldsymbol{R}^n)$.

LEMMA 3.4. *Under the hypotheses of Lemma* 3.3, *if* $0 < \alpha < 1$, *we have*

$$\|f\|_{Q_\alpha(\boldsymbol{R}^n)} \leq C2^{jn} \|\{s_K\}\|_{\mathcal{C}_\alpha}\,,$$

where the constant C is independent of the choice of $\mathcal{F}$.

PROOF. Before we start, let us point out that one can remove all reference to $\mathcal{F}$ below by setting $s_K = 0$ for $K \notin \mathcal{F}$.

The proof is similar in notation and outline to that of [EJPX], Theorem 6.2 (see also [M], p. 154). By Lemma 2.1, in order to estimate $\|f\|_{Q_\alpha(\boldsymbol{R}^n)}$, it suffices to bound $O_{f,\alpha}(I)$ for arbitrary cubes I of dyadic sidelength. Fix such a cube I, and set

$$\mathcal{A}(I) = \{K \in \mathcal{D};\, 2^j K \cap I \neq \emptyset\}\,.$$

Note that condition (a$'$) implies $a_K = 0$ on I if $K \notin \mathcal{A}(I)$, so that in the sum (3.1) defining f, when $x \in I$, the only cubes K that appear are those with $K \in \mathcal{A}(I)$. Partition $\mathcal{A}(I)$ into

$$\mathcal{A}_1 = \mathcal{A}_1(I) = \{K \in \mathcal{A}(I);\, 2^j \ell(K) \leq \ell(I)\}\,,$$
$$\mathcal{A}_2 = \mathcal{A}_2(I) = \{K \in \mathcal{A}(I);\, \ell(I) < 2^j \ell(K)\}\,.$$

Then we have $f = f_1 + f_2$ on I, where

$$f_i = \sum_{K\in\mathcal{A}_i} s_K a_K\,, \quad i = 1, 2\,,$$

and

$$O_{f,\alpha}(I) \leq 2(O_{f_1,\alpha}(I) + O_{f_2,\alpha}(I))\,. \tag{3.3}$$

To take care of f_1, we again separate the sum into two parts:

$$\begin{aligned} O_{f_1,\alpha}(I) &= \ell(I)^{2\alpha-n} \int_I\int_I \frac{\left|\sum_{K\in\mathcal{A}_1} s_K[a_K(x) - a_K(y)]\right|^2}{|x-y|^{n+2\alpha}}\,dxdy \\ &\leq 2\ell(I)^{2\alpha-n} \int_I\int_I \frac{\left|\sum_{\{K\in\mathcal{A}_1;\, x,y\in 2^{j+1}\sqrt{n}K\}} s_K[a_K(x) - a_K(y)]\right|^2}{|x-y|^{n+2\alpha}}\,dxdy \\ &\quad + 4\ell(I)^{2\alpha-n} \int_I\int_I \frac{\left|\sum_{\{K\in\mathcal{A}_1;\, x\in 2^j K,\, y\notin 2^{j+1}\sqrt{n}K\}} s_K a_K(x)\right|^2}{|x-y|^{n+2\alpha}}\,dxdy \end{aligned}$$

$$:= \ell(I)^{2\alpha-n}(A+B)\,.$$

Take an integer m sufficiently large so that $2^m \geq 2\sqrt{n}$, and apply Lemma 3.3, Part (ii), with $2^{-m}a_K$ instead of a_K and $2^{j+m}K$ instead of 2^jK. Then, since $\alpha \geq 0$,

$$A \leq C_m 2^{2j(n-\alpha)} \sum_{K\in\mathcal{A}_1} |s_K|^2 \ell(K)^{n-2\alpha} \leq C 2^{2jn} \sum_{K\in\mathcal{A}_1} |s_K|^2 \ell(K)^{n-2\alpha}\,. \tag{3.4}$$

Noting that for $x \in 2^jK$ and $y \notin 2^{j+1}\sqrt{n}K$, $|x-y| \geq |x_K - y|/2$, we can apply Part (i) of Lemma 3.3 to the functions $a_K(x)$ and coefficients

$$\lambda_{K,y} = s_K |x_K - y|^{-n/2-\alpha} \chi_{\{K;\, y\notin 2^{j+1}\sqrt{n}K\}}\,,$$

for a fixed $y \in I$, to get

$$\begin{aligned} B &\leq C \int_I \int_I \Big| \sum_{\{K\in\mathcal{A}_1;\, x\in 2^jK,\, y\notin 2^{j+1}\sqrt{n}K\}} s_K a_K(x) |x_K - y|^{-n/2-\alpha} \Big|^2 \,dx dy \\ &= C \int_I \Big\| \sum_{K\in\mathcal{A}_1} \lambda_{K,y} a_K \Big\|^2_{L^2(I)} dy \\ &\leq C 2^{2jn} \int_I \sum_{K\in\mathcal{A}_1} |\lambda_{K,y}|^2 \ell(K)^n \,dy \\ &\leq C 2^{2jn} \sum_{K\in\mathcal{A}_1} |s_K|^2 \ell(K)^n \int_{I\setminus 2^{j+1}\sqrt{n}K} |x_K - y|^{-n-2\alpha}\,dy \\ &\leq C 2^{2jn} \sum_{K\in\mathcal{A}_1} |s_K|^2 \ell(K)^{n-2\alpha}\,. \end{aligned} \tag{3.5}$$

Thus (3.4) and (3.5) give

$$\begin{aligned} O_{f_1,\alpha}(I) &\leq C 2^{2jn} \ell(I)^{2\alpha-n} \sum_{K\in\mathcal{A}_1} |s_K|^2 \ell(K)^{n-2\alpha} \\ &\leq C 2^{2jn} \sum_{I'\in\mathcal{E}(I)} \sum_{K\subseteq I'} |s_K|^2 \left(\frac{\ell(K)}{\ell(I')}\right)^{n-2\alpha} \\ &\leq C 2^{2jn} \sup_{I'\in\mathcal{E}(I)} U_{\mathbf{s},\alpha}(I')\,, \end{aligned} \tag{3.6}$$

where we have used the notation $\mathcal{E}(I)$ to denote the collection of (at most 5^n) dyadic cubes I' with $\ell(I') = \ell(I)$ for which $I' \cap 3I \neq \emptyset$. Note that for $K \in \mathcal{A}_1$, we have $\ell(K) \leq 2^{-j}\ell(I)$ and $2^jK \cap I \neq \emptyset$, which implies $K \subset 3I$, hence $K \subseteq I'$ for a unique $I' \in \mathcal{E}(I)$.

Now for f_2, by the Lipschitz condition (b′) above, and using the fact that for every dyadic cube K, $|s_K|^2 \le U_{\mathbf{s},\alpha}(K)$, we have

$$
\begin{aligned}
O_{f_2,\alpha}(I) &= C\ell(I)^{2\alpha-n}\int_I\int_I \frac{\left|\sum_{K\in\mathcal{A}_2} s_K[a_K(x)-a_K(y)]\right|^2}{|x-y|^{n+2\alpha}}\,dxdy \\
&\le C\ell(I)^{2\alpha-n}\int_I\int_I\left(\sum_{K\in\mathcal{A}_2}|s_K|(2^j\ell(K))^{-1}\right)^2|x-y|^{2-n-2\alpha}\,dxdy \\
&\le C\ell(I)^{2\alpha-n}\left(\sup_{K\in\mathcal{A}_2}U_{\mathbf{s},\alpha}(K)\right)\left(\sum_{K\in\mathcal{A}_2}(2^j\ell(K))^{-1}\right)^2\ell(I)^{2+n-2\alpha} \\
&\le C\left(\sup_{K\in\mathcal{A}_2}U_{\mathbf{s},\alpha}(K)\right)\left(\sum_{K\in\mathcal{A}_2}\frac{\ell(I)}{2^j\ell(K)}\right)^2 \\
&\le C\left(\sup_{K\in\mathcal{A}_2}U_{\mathbf{s},\alpha}(K)\right)\left(\sum_{k=1}^{\infty}\sum_{K\in\mathcal{A}_2,2^j\ell(K)=2^k\ell(I)}2^{-k}\right)^2 \\
&\le C2^{2jn}\left(\sup_{K\in\mathcal{A}_2}U_{\mathbf{s},\alpha}(K)\right).
\end{aligned}
\tag{3.7}
$$

Here we have again used the geometric fact that for each $k \in \boldsymbol{N}$, there are at most $2^{(j+1)n}$ choices of $K \in \mathcal{A}_2$ with $2^j\ell(K) = 2^k\ell(I)$.

Therefore, by estimates (3.3), (3.6) and (3.7), we conclude

$$
O_{f,\alpha}(I) \le C2^{2nj}\left(\sup_{I'\in\mathcal{E}(I)}U_{\mathbf{s},\alpha}(I') + \sup_{K\in\mathcal{A}_2(I)}U_{\mathbf{s},\alpha}(K)\right) \le C2^{2nj}\|\{s_K\}\|^2_{\mathcal{C}_\alpha} \tag{3.8}
$$

for every cube I of dyadic sidelength. By Lemma 2.1, we have $f \in Q_\alpha(\boldsymbol{R}^n)$ with norm $\|f\|_{Q_\alpha(\boldsymbol{R}^n)} \le C2^{jn}\|\{s_K\}\|_{\mathcal{C}_\alpha}$. Note that nowhere in the proof did we use the number of elements of $\mathcal{F}$.

Before getting to the main theorem of this section, we need to review some results from [DX]. We first state a lemma which is a combination of Lemma 1.1 in [FJW] and Lemma 3.2 in [DX]. Here and below we will denote the Schwartz class of rapidly decreasing smooth functions on $\boldsymbol{R}^n$ by $\mathcal{S}$, and its dual, the space of tempered distributions, by $\mathcal{S}'$. For a function $\phi \in \mathcal{S}(\boldsymbol{R}^n)$, $\hat{\phi}$ will denote the Fourier transform of ϕ.

LEMMA 3.5. *Fix $N \in \boldsymbol{N}$. Then there exists a function $\phi : \boldsymbol{R}^n \to \boldsymbol{R}$ such that*

(1) $\operatorname{supp}\phi \subset \{x \in \boldsymbol{R}^n; |x| \le 1\}$;

(2) *ϕ is radial*;

(3) $\phi \in C^\infty(\boldsymbol{R}^n)$;

(4) $\int_{\boldsymbol{R}^n} x^\gamma\phi(x)dx = 0$ *if* $|\gamma| \le N$, $\gamma \in (\boldsymbol{N}\cup\{0\})^n$, $x^\gamma = x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n}$, $|\gamma| = \gamma_1+\gamma_2+\cdots+\gamma_n$;

(5) $\int_0^\infty(\hat{\phi}(t\xi))^2t^{-1}dt = 1$ *if* $\xi \in \boldsymbol{R}^n\setminus\{0\}$.

Moreover, if $\alpha \in (0,1)$, $f \in L^2_{\text{loc}}(\boldsymbol{R}^n)$, *and* $d\mu_{f,\phi,\alpha}(x,t) = |(f * \phi_t)(x)|^2 t^{-1-2\alpha}\,dt\,dx$, *then there is a constant* C, *independent of the choice of* f, *such that for any cubes* I *and* J *in* $\boldsymbol{R}^n$, *with center* $x_I = x_J$ *and with* $\ell(J) \geq 3\ell(I)$,

$$\mu_{f,\phi,\alpha}(S(I)) \leq C\ell(J)^{n-2\alpha} O_{f,\alpha}(J)\,.$$

Here again $S(I)$ *is the "Carleson box" over* I.

Next, we need to recall the duality result in [DX], identifying $Q_\alpha(\boldsymbol{R}^n)$ with the dual of the "Hardy-Hausdorff space" $HH^1_{-\alpha}(\boldsymbol{R}^n)$. This space can be characterized by the following atomic decomposition (Theorem 6.3 in [DX]):

THEOREM 3.6 (DX). *Let* $0 < \alpha < \min\{1, n/2\}$. *Define an* $HH^1_{-\alpha}$*-atom* a *to be a tempered distribution supported in a cube* I *and satisfying*:

(i)

$$|\langle a, \psi\rangle| \leq (O_{\psi,\alpha}(I))^{1/2}$$

for all $\psi \in \mathcal{S}$; *and*

(ii)

$$\langle a, \psi\rangle = 0$$

for any $\psi \in \mathcal{S}$ *which coincides in a neighborhood of* I *with a polynomial of degree* $\leq n/2+1$.

Then a tempered distribution f *on* $\boldsymbol{R}^n$ *belongs to* $HH^1_{-\alpha}$ *if and only if there is a sequence of* $HH^1_{-\alpha}$*-atoms* $\{a_j\}$, *and an* l^1 *sequence* $\{\lambda_j\}$, *such that* $f = \sum_j \lambda_j a_j$ *in the sense of distributions. Moreover,*

$$\|f\|_{HH^1_{-\alpha}(\boldsymbol{R}^n)} \approx \inf\left\{\sum_j |\lambda_j|;\quad f = \sum_j \lambda_j a_j\right\}.$$

As explained in [DX] (see Remark 2 after Lemma 6.2), an $HH^1_{-\alpha}$-atom a is actually a distribution in the homogeneous Sobolev space $\dot{L}^2_{-\alpha}(\boldsymbol{R}^n)$, and can thus be paired with a function ψ in the dual homogeneous Sobolev space $\dot{L}^2_\alpha(\boldsymbol{R}^n)$, namely a function satisfying

$$\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy < \infty\,.$$

In particular, we can take for ψ any Lipschitz function with compact support, such as a $(0,\infty)$-atom. Moreover, approximating ψ in $\dot{L}^2_\alpha(\boldsymbol{R}^n)$ by functions in $\mathcal{S}$, we see that condition (i) in Theorem 3.6 extends to the pairing of a with such ψ.

We are now in a position to prove the major result of this section.

THEOREM 3.7. *Let* $0 < \alpha < \min\{1, n/2\}$. *If* $\{a_I\}_{I\in\mathcal{D}}$ *is a sequence of* $(0,\infty)$*-atoms, and* $\{s_I\}_{I\in\mathcal{D}}$ *a* $\mathcal{C}_\alpha$*-sequence, then there exists a function* $f \in Q_\alpha(\boldsymbol{R}^n)$ *so that*

$$f = \sum_{I\in\mathcal{D}} s_I a_I = \lim_{k\to-\infty, m\to\infty} \sum_{I\in\mathcal{D}, 2^k\leq\ell(I)\leq 2^m} s_I a_I\,, \tag{3.9}$$

where the convergence is in $\mathcal{S}'(\boldsymbol{R}^n)$ *modulo constants and in the weak-$*$ topology in* $Q_\alpha(\boldsymbol{R}^n)$ *(viewed as the dual of* $HH^1_{-\alpha}(\boldsymbol{R}^n)$*).*

Conversely, if $f \in Q_\alpha(\boldsymbol{R}^n)$ *then there is a sequence* $\{a_I\}_{I\in\mathcal{D}}$ *of* $(0,\infty)$*-atoms, and a* $\mathcal{C}_\alpha$*-sequence* $\{s_I\}_{I\in\mathcal{D}}$ *such that* (3.9) *holds. Moreover,*

$$\|f\|_{Q_\alpha(\boldsymbol{R}^n)} \approx \|\{s_I\}\|_{\mathcal{C}_\alpha}.$$

PROOF. Let $\{s_I\}_{I\in\mathcal{D}}$ be a $\mathcal{C}_\alpha$-sequence, and $\{a_I\}_{I\in\mathcal{D}}$ a sequence of $(0,\infty)$-atoms. Note that $a_I/4$ satisfy conditions (a′)–(c′) of Lemma 3.3 with $j = 2$. Therefore if we denote by $\mathcal{F}_{k,m}$ the collection of cubes $K \in \mathcal{D}$ with $2^k < \ell(K) \le 2^m$, and set

$$f_{k,m} = \sum_{I\in\mathcal{F}_{k,m}} s_I a_I, \quad \text{for } k, m \in \boldsymbol{Z}, \quad k \le m\,,$$

then by Lemma 3.4,

$$\|f_{k,m}\|_{Q_\alpha(\boldsymbol{R}^n)} \le C\|\{s_I\}\|_{\mathcal{C}_\alpha}\,, \tag{3.10}$$

with a constant independent of k and m. From an analogous result for BMO (see [FJ], Theorem 4.1, or [St], Proposition IV.4.5), we know that as $k \to -\infty, m \to \infty$, $f_{k,m}$ converge in $\mathcal{S}'/\boldsymbol{C}$ and weak-$*$ in BMO (as the dual of H^1) to a function $f \in$ BMO. We want to show that $f \in Q_\alpha(\boldsymbol{R}^n)$ and in fact the convergence is weak-$*$ in $Q_\alpha(\boldsymbol{R}^n)$, viewed as the dual of $HH^1_{-\alpha}(\boldsymbol{R}^n)$.

First, for $k \le 0 \le m$, write

$$f_{k,m} = f_{k,0} + f_{0,m}.$$

Suppose g is an $HH^1_{-\alpha}$-atom, supported in a cube I. Let $\tilde{I}$ be the smallest concentric cube containing I with dyadic sidelength, say $\ell(\tilde{I}) = 2^r$.

Then for $0 < m < p$, we have, by condition (i) in Theorem 3.6,

$$|\langle f_{0,m} - f_{0,p}, g\rangle|^2 = |\langle f_{m,p}, g\rangle|^2 \le C\,O_{f_{m,p},\alpha}(I) \le C\,O_{f_{m,p},\alpha}(\tilde{I})\,.$$

If m is sufficiently large ($m \ge r-2$), we have $\ell(K) > \ell(\tilde{I})/4$ for all $K \in \mathcal{F}_{m,p}$, and hence we can repeat the calculations leading up to estimate (3.7) in the proof of Lemma 3.4 (with $j = 2$, $\mathcal{F} = \mathcal{F}_{m,p}$ and $\mathcal{A}_2 = \{K \in \mathcal{D};\, \ell(K) > \ell(\tilde{I})/4,\, 4K \cap \tilde{I} \ne \emptyset\}$) to get

$$\begin{aligned} O_{f_{m,p},\alpha}(\tilde{I}) &\le C\Big(\sup_{K\in\mathcal{A}_2} U_{\mathbf{s},\alpha}(K)\Big)\Bigg(\sum_{k=1}^{\infty}\ \sum_{K\in\mathcal{F}_{m,p}\,,\ell(K)=2^k\ell(\tilde{I})/4} 2^{-k}\Bigg)^2 \\ &\le C2^{4n}\|\{s_K\}\|^2_{\mathcal{C}_\alpha}\Bigg(\sum_{k>m+2-r} 2^{-k}\Bigg)^2. \end{aligned}$$

This shows $\langle f_{0,m} - f_{0,p}, g\rangle \to 0$ as $m, p \to \infty$.

To show $\langle f_{q,0} - f_{k,0}, g\rangle \to 0$ as $q, k \to -\infty$ we can use estimate (3.6), since for K in $\mathcal{F}_{q,k}$, $q < k \le r-2$, we have $\ell(K) \le \ell(\tilde{I})/4$. This means that

$$|\langle f_{q,0} - f_{k,0}\,, g\rangle|^2 \le C\,O_{f_{q,k}\,,\alpha}(\tilde{I})$$

$$\leq C 2^{4n} \sum_{I' \in \mathcal{E}(\tilde{I})} \sum_{K \in \mathcal{F}_{q,k}, K \subseteq I'} |s_K|^2 \left(\frac{\ell(K)}{\ell(I')}\right)^{n-2\alpha},$$

where

$$\mathcal{E}(\tilde{I}) = \{I' \in \mathcal{D}; \ell(I') = \ell(\tilde{I}), I' \cap 3\tilde{I} \neq \emptyset\}.$$

Now $\mathcal{E}(\tilde{I})$ has at most 5^n elements, and for each of the cubes $I' \in \mathcal{E}(\tilde{I})$, the finiteness of $U_{\mathbf{s},\alpha}(I')$ implies that

$$\sum_{K \in \mathcal{F}_{q,k}, K \subseteq I'} \left(\frac{\ell(K)}{\ell(I')}\right)^{n-2\alpha} |s_K|^2 \to 0 \quad \text{as } q, k \to -\infty,$$

giving $\langle f_{q,0} - f_{k,0}, g\rangle \to 0$ as $q, k \to -\infty$.

We have thus shown that $\lim_{k\to-\infty, m\to\infty} \langle f_{k,m}, g\rangle$ exists for any g which is an $HH^1_{-\alpha}$-atom, or a finite linear combination of such atoms. Since the finite linear combinations of $HH^1_{-\alpha}$-atoms form a dense subset of the predual $HH^1_{-\alpha}(\boldsymbol{R}^n)$, and by (3.10) the sequence $\{f_{k,m}\}$ is uniformly bounded in $Q_\alpha(\boldsymbol{R}^n)$, we conclude that $f_{k,m}$ converge weak-$*$ in $Q_\alpha(\boldsymbol{R}^n)$ to some function in $Q_\alpha(\boldsymbol{R}^n)$. This must be the same as the function f (the weak-$*$ limit in BMO), since $H^1(\boldsymbol{R}^n) \subset HH^1_{-\alpha}(\boldsymbol{R}^n)$ (see [DX]). Thus $f \in Q_\alpha(\boldsymbol{R}^n)$ with

$$\|f\|_{Q_\alpha(\boldsymbol{R}^n)} \leq C \|\{s_K\}\|_{\mathcal{C}_\alpha}. \tag{3.11}$$

Now let $f \in Q_\alpha(\boldsymbol{R}^n)$. For the atomic decomposition, we will follow the construction in the proof of Lemma 3.1 in [U], which in turn is based on [CF]. Let ϕ be as in Lemma 3.5. Then we can use Calderón's reproducing formula to obtain:

$$\begin{aligned} f(x) &= \int_0^\infty (\phi_t * \phi_t * f)(x) \frac{dt}{t} \\ &= \sum_{I \in \mathcal{D}} \int_{T(I)} \phi_t(x-y)(\phi_t * f)(y) \frac{dydt}{t} \\ &= \sum_{I \in \mathcal{D}} b_I(x), \end{aligned}$$

where $T(I)$ is the upper half of the "Carleson box", namely

$$T(I) = \{(y,t) \in \boldsymbol{R}^{n+1}_+; y \in I, \ell(I)/2 \leq t < \ell(I)\},$$

and the convergence is in the sense of distributions modulo constants (i.e. in $\mathcal{S}'(\boldsymbol{R}^n)/\boldsymbol{C}$—see, for example, [FJW], Appendix), or alternatively, in the weak-$*$ sense in BMO (see [St], Section IV.4.5.3).

From the support and cancellation conditions on ϕ, we can conclude that b_I is supported in $3I$ and $\int_{\boldsymbol{R}^n} x^\gamma b_I(x)dx = 0$ for $|\gamma| \leq N = N_2$. Moreover, as in the proof of Lemma 3.1 in [U] (for higher derivatives see also [St], Section IV.4.5.3), we can differentiate inside the integral to obtain

$$|D_x^\gamma b_I(x)| = \left| \int_{T(I)} (D_x^\gamma \phi_t(x-y))(\phi_t * f)(y) \frac{dydt}{t} \right|$$

$$\le \|D^\gamma \phi\|_{L^2}\left(\int_{\ell(I)/2}^{\ell(I)} t^{-n-2|\gamma|-1}dt\right)^{1/2}\left(\int_{T(I)} |(\phi_t * f)(y)|^2 \frac{dydt}{t}\right)^{1/2}$$
$$\le C_{\phi,\gamma}\ell(I)^{-n/2-|\gamma|}\left(\int_{T(I)} |(\phi_t * f)(y)|^2 \frac{dydt}{t}\right)^{1/2}.$$

Thus if we let $a_I = b_I/s_I$, where

$$s_I = C_{N_2}|I|^{-1/2}\left(\int_{T(I)} |(\phi_t * f)(y)|^2 t^{-1}dtdy\right)^{1/2},$$

and C_{N_2} is so chosen to be larger than $C_{\phi,\gamma}$ for all $|\gamma| \le N_2$, then a_I is a $(0,\infty)$-atom and $f = \sum_{I\in\mathcal{D}} s_I a_I$ in $\mathcal{S}'(\boldsymbol{R}^n)/\boldsymbol{C}$.

In order to verify that $\{s_I\}_{I\in\mathcal{D}}$ is a $\mathcal{C}_\alpha$-sequence, we apply Lemma 3.5 to obtain that for any $I \in \mathcal{D}$,

$$\begin{aligned} U_{\mathbf{s},\alpha}(I) &= C(\ell(I))^{2\alpha-n}\sum_{J\subseteq I}\ell(J)^{n-2\alpha}|J|^{-1}\int_{T(J)}|(\phi_t * f)(y)|^2 t^{-1}dtdy \\ &\le C(\ell(I))^{2\alpha-n}\sum_{J\subseteq I}\int_{T(J)}|(\phi_t * f)(y)|^2 t^{-1-2\alpha}dtdy \\ &= C(\ell(I))^{2\alpha-n}\int_{S(I)}|(\phi_t * f)(y)|^2 t^{-1-2\alpha}dtdy \\ &\le CO_{f,\alpha}(3I), \end{aligned}$$

so taking the supremum over I,

$$\|\{s_I\}\|_{\mathcal{C}_\alpha} \le C\|f\|_{Q_\alpha(\boldsymbol{R}^n)}. \tag{3.12}$$

Finally, note that by the first part of the theorem, $\sum s_I a_I$ converge in the weak-$*$ sense to a limit in $Q_\alpha(\boldsymbol{R}^n)$, and since we already have the weak-$*$ convergence in BMO to f, this limit must be f. By (3.11) and (3.12), $\|f\|_{Q_\alpha(\boldsymbol{R}^n)} \approx \|\{s_I\}\|_{\mathcal{C}_\alpha}$, as desired.

This completes the proof of the theorem.

Note that in proving the weak-$*$ convergence in $Q_\alpha(\boldsymbol{R}^n)$ of the sum of atoms $\sum s_I a_I$, we did not use the full force of Lemma 3.4, but rather only the case $j = 2$. In fact, using Lemma 3.4 and the following lemma of Uchiyama (Lemma 3.5 in [U]), we can prove a stronger result.

LEMMA 3.8 (Uchiyama). *Let $I \subset \boldsymbol{R}^n$ be a cube with center x_I, and suppose $b \in C^1(\boldsymbol{R}^n)$ satisfies*

(i) $\int_{\boldsymbol{R}^n} b(x)dx = 0$;
(ii) $|b(x)| \le (\ell(I))^{n+1}/(\ell(I) + |x - x_I|)^{n+1}$;
(iii) $\left|\partial_{x_i} b(x)\right| \le (\ell(I))^{n+1}/(\ell(I) + |x - x_I|)^{n+2}$ *for* $i = 1, \dots, n$.

Then there exists a sequence $\{a_j\}_{j=0}^{\infty}$ *of functions in* $C^1(\boldsymbol{R}^n)$ *such that*

$$b(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} a_j(x)\,,$$

and for each j *the function* a_j *satisfies conditions* (a′) – (c′) *of Lemma* 3.3 (*with respect to the cube* I).

We will call a function $b \in C^1(\boldsymbol{R}^n)$ satisfying conditions (i) – (iii) of Lemma 3.8 a $(0, \infty)$-molecule (see [FJ], Section 3). Note that every $(0, \infty)$-atom is also a molecule (up to a constant). Conversely, the lemma may be thought of as the decomposition of a molecule into atoms. We now state the extension of Theorem 3.7, namely the weak-$*$ convergence in $Q_\alpha(\boldsymbol{R}^n)$ of a sum of molecules. An analogous result for BMO is Theorem 4.1(b) in [FJ].

THEOREM 3.9. *Let* $0 < \alpha < \min\{1, n/2\}$. *If* $\{b_I\}_{I\in\mathcal{D}}$ *is a sequence of* $(0, \infty)$-*molecules, and* $\{s_I\}_{I\in\mathcal{D}}$ *a* $\mathcal{C}_\alpha$-*sequence, then there exists a function* $f \in Q_\alpha(\boldsymbol{R}^n)$ *so that*

$$f = \sum_{I\in\mathcal{D}} s_I b_I = \lim_{k\to-\infty, m\to\infty} \sum_{I\in\mathcal{D}, 2^k\le\ell(I)\le 2^m} s_I b_I\,,$$

where the convergence is in $\mathcal{S}'(\boldsymbol{R}^n)$ *modulo constants and in the weak-*$*$ *topology in* $Q_\alpha(\boldsymbol{R}^n)$ (*viewed as the dual of* $HH^1_{-\alpha}(\boldsymbol{R}^n)$).

PROOF. Let $\{s_I\}_{I\in\mathcal{D}}$ be a $\mathcal{C}_\alpha$-sequence, and $\{a_I\}_{I\in\mathcal{D}}$ a sequence of $(0, \infty)$-molecules. By Lemma 3.8,

$$b_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} a_{I,j}(x)\,,$$

where each $a_{I,j}$ possesses properties (a′) – (c′) of Lemma 3.3 with respect to I and j. Following the proof of Theorem 3.7, we denote by $\mathcal{F}_{k,m}$ the collection of cubes $K \in \mathcal{D}$ with $2^k < \ell(K) \le 2^m$, and set

$$f_{k,m,l}(x) = \sum_{I\in\mathcal{F}_{k,m}} s_I \sum_{j=0}^{l} 2^{-j(n+1)} a_{I,j}(x)\,, \quad \text{for } k, m, l \in \boldsymbol{Z}, \quad k \le m, l \ge 0\,.$$

With help of Lemma 3.4 we obtain that

$$\begin{aligned}
\|f_{k,m,l}\|_{Q_\alpha(\boldsymbol{R}^n)} &\le \left\| \sum_{I\in\mathcal{F}_{k,m}} s_I \sum_{j=0}^{l} 2^{-j(n+1)} a_{I,j} \right\|_{Q_\alpha(\boldsymbol{R}^n)} \\
&\le \sum_{j=0}^{l} 2^{-j(n+1)} \left\| \sum_{I\in\mathcal{F}_{k,m}} s_I a_{I,j} \right\|_{Q_\alpha(\boldsymbol{R}^n)} \\
&\le C \sum_{j=0}^{\infty} 2^{-j(n+1)} 2^{jn} \|\{s_I\}\|_{\mathcal{C}_\alpha} \\
&\le C \|\{s_I\}\|_{\mathcal{C}_\alpha}
\end{aligned} \tag{3.13}$$

with C independent of k, m and l. Similarly,

$$\|f_{k,m,l_1} - f_{k,m,l_2}\|_{Q_\alpha(\boldsymbol{R}^n)} \le C \sum_{j=l_1}^{l_2} 2^{-j(n+1)} 2^{jn} \|\{s_I\}\|_{\mathcal{C}_\alpha} \to 0 \quad \text{as } l_1, l_2 \to \infty.$$

Thus $\lim_{l\to\infty} f_{k,m,l}$ exists in $Q_\alpha(\boldsymbol{R}^n)$, and since

$$f_{k,m}(x) = \lim_{l\to\infty} f_{k,m,l}(x) = \sum_{I\in\mathcal{F}_{k,m}} s_I \sum_{j=0}^{\infty} 2^{-j(n+1)} a_{I,j}(x)$$

converges pointwise, an application of Fatou's lemma gives $f_{k,m} = \lim_{l\to\infty} f_{k,m,l}$ in $Q_\alpha(\boldsymbol{R}^n)$. Moreover, by (3.13),

$$\sup_{k,m} \|f_{k,m}\|_{Q_\alpha(\boldsymbol{R}^n)} \le C\|\{s_I\}\|_{\mathcal{C}_\alpha}.$$

As in the proof of Theorem 3.7, in order to show that $f_{k,m}$ converges weak-$*$ in $Q_\alpha(\boldsymbol{R}^n)$, it remains to show that $\lim_{k\to-\infty, m\to\infty}\langle f_{k,m}, g\rangle$ exists for g in a dense subset of the predual $HH^1_{-\alpha}(\boldsymbol{R}^n)$.

Again write $f_{k,m} = f_{k,0} + f_{0,m}$ for $k \le 0 \le m$, take g to be an $HH^1_{-\alpha}$-atom supported in a cube I, and let $\tilde{I}$ be the smallest concentric cube containing I with dyadic sidelength. Then by condition (i) in Theorem 3.6, Fatou's lemma, and Minkowski's inequality, we have, for $0 < m < p$,

$$\begin{aligned}
|\langle f_{0,m} - f_{0,p}, g\rangle| &\le C(O_{f_{0,m}-f_{0,p},\alpha}(I))^{1/2} \\
&\le C(O_{f_{0,m}-f_{0,p},\alpha}(\tilde{I}))^{1/2} \\
&\le C \liminf_{l\to\infty} (O_{f_{0,m,l}-f_{0,p,l},\alpha}(\tilde{I}))^{1/2} \\
&\le C \liminf_{l\to\infty} \sum_{j=0}^{l} 2^{-j(n+1)} (O_{h_j,\alpha}(\tilde{I}))^{1/2} \\
&= C \sum_{j=0}^{\infty} 2^{-j(n+1)} (O_{h_j,\alpha}(\tilde{I}))^{1/2},
\end{aligned} \tag{3.14}$$

where

$$h_j = \sum_{K\in\mathcal{F}_{m,p}} s_K a_{K,j}.$$

Imitating (3.7) in the proof of Lemma 3.4 (with $\mathcal{F} = \mathcal{F}_{m,p}$, $2^m \ge 2^{-j}\ell(\tilde{I})$), we have

$$\begin{aligned}
O_{h_j,\alpha}(\tilde{I}) &\le C\left(\sup_{K\in\mathcal{A}_2(\tilde{I})} U_{\mathbf{s},\alpha}(K)\right)\left(\sum_{k=1}^{\infty} \sum_{K\in\mathcal{F}_{m,p}\cap\mathcal{A}_2(\tilde{I}),2^j\ell(K)=2^k\ell(\tilde{I})} 2^{-k}\right)^2 \\
&\le C2^{2jn}\|\{s_K\}\|^2_{\mathcal{C}_\alpha}\left(\sum_{k>m+j-\log_2\ell(\tilde{I})} 2^{-k}\right)^2 \\
&= C2^{-2m+2j(n-1)}\|\{s_K\}\|^2_{\mathcal{C}_\alpha}\ell(\tilde{I})^2.
\end{aligned}$$

Thus

$$|\langle f_{0,m} - f_{0,p}, g\rangle| \leq C2^{-m}\|\{s_K\}\|_{\mathcal{C}_\alpha}\ell(\tilde{I})\sum_{j=0}^{\infty}2^{-2j} \to 0$$

as $m, p \to \infty$.

For the case of $\langle f_{q,0} - f_{k,0}, g\rangle$ as $k, q \to -\infty$, we repeat (3.14) with h_j now standing for $\sum_{K\in\mathcal{F}_{q,k}} s_K a_{K,j}$. Assuming $q < k \leq \log_2 \ell(\tilde{I}) - j$ so that $K \in \mathcal{F}_{q,k}$ implies $2^j\ell(K) \leq \ell(\tilde{I})$, another application of (3.6) in the proof of Lemma 3.4 gives

$$O_{h_j,\alpha}(\tilde{I}) \leq C2^{2jn}\sum_{I'\in\mathcal{E}(\tilde{I})}\sum_{K\in\mathcal{F}_{q,k},K\subseteq I'}|s_K|^2\left(\frac{\ell(K)}{\ell(I')}\right)^{n-2\alpha},$$

where we recall that

$$\mathcal{E}(\tilde{I}) = \{I' \in \mathcal{D}; \ell(I') = \ell(\tilde{I}), I' \cap 3\tilde{I} \neq \emptyset\}.$$

Noting again that $\mathcal{E}(\tilde{I})$ has at most 5^n elements, and for each of the cubes $I' \in \mathcal{E}'(\tilde{I})$, $\sum_{K\in\mathcal{F}_{q,k},K\subseteq I'}(\ell(K)/\ell(I'))^{n-2\alpha}|s_K|^2 \to 0$ as $q, k \to -\infty$ by the convergence of the series defining $U_{\mathbf{s},\alpha}(I')$, we have that for each $j \geq 0$, $2^{-jn}O_{h_j,\alpha}(\tilde{I})^{1/2} \to 0$ as $q, k \to -\infty$. Moreover, $2^{-jn}O_{h_j,\alpha}(\tilde{I})^{1/2}$ are bounded by a constant multiple of $\sup_{I\in\mathcal{D}} U_{\mathbf{s},\alpha}(I)^{1/2} = \|\{s_K\}_{K\in\mathcal{D}}\|_{\mathcal{C}_\alpha}$. This, together with (3.14), implies that $\langle f_{q,0} - f_{k,0}, g\rangle \to 0$ as $q, k \to -\infty$.

The proof of weak-$*$ convergence in $Q_\alpha(\boldsymbol{R}^n)$ is now complete. The convergence in $\mathcal{S}'/\boldsymbol{C}$ is proved in Remark 3.2 of [FJ].

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