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The dynamical stability of differentially rotating discs – II

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Summary. We extend our investigation of the dynamical stability of differentially rotating fluid tori of uniform entropy to the case in which there is a gradient of specific angular momentum. This case is much more complicated and in this paper we limit ourselves to an analytic approach. Our overall conclusions are that (i) the dynamical instabilities found to exist in constant specific angular momentum tori persist in this case; (ii) additional unrelated Kelvin–Helmholtz-like instabilities are introduced by allowing a gradient in specific angular momentum and (iii) the general unstable mode is a mixture of these two.

1 Introduction

In a previous paper (Papaloizou & Pringle 1984, hereafter Paper I) we studied the stability of a differentially rotating, perfect barotropic fluid with constant specific angular momentum, h, with particular reference to accretion tori. We used both analytic and numerical techniques, both of which were in agreement in showing that the tori are expected to be dynamically unstable to global modes with low azimuthal wavenumber m as well as to modes with high m. In this paper we extend the work of Paper I and consider tori in which the specific angular momentum is not constant. Because the problem is technically more complex than that considered in Paper I, a large amount of parameter space needs to be explored to obtain a basic idea of the stability properties. In this paper we content ourselves with an analytic approach. A numerical exploration will be presented subsequently.

Some idea of the complexity of the problem can be obtained from the studies of the effect of compressibility on plane parallel shear flows (Blumen, Drazin & Billings 1975; Turland 1976; Drazin & Davey 1977; Ray 1982). It is found that the effect of compressibility is to add a further mode of instability to the classical Kelvin–Helmholtz instability, and that the earlier concept of a critical Mach number across the shear above which stability set in (e.g. Blumen 1970) was erroneous. We note in passing (see also Turland 1976) that this finding is of relevance to the shear flow present around astrophysical jets and should lead to a reassessment of previous stability analyses in which the concept of a critical Mach number is used (e.g. Blandford & Pringle 1976). The new mode of instability is caused by the propagation of sound waves across the shear flow and it was this mode which was responsible for the instabilities found in Paper I. These ideas are drawn together in a powerful piece of work by Grinfeld (1984) who gives sufficient conditions for stability.

The structure of the paper is as follows. In Section 2 we set up the equations governing the structure of the equilibrium configuration. We consider the flow around a central gravitating point mass, although it should be noted that the general conclusions drawn do not depend intimately on this assumption. We consider in particular the case in which the torus takes the form of a thin ring. We derive the linearized perturbation equations (as in Paper I) but here also derive the equations governing the Lagrangian displacement, $\boldsymbol{\xi}$.

In Section 3 we present some general results. We present explicitly the Liapounov function found by Grinfeld (1984) for the two-dimensional rotational shear flow and derive from it sufficient conditions for stability. We also show for the fully three-dimensional case, but for a power-law rotation law that any unstable mode must corotate with the flow at some point.

In Section 4 we consider tori with nearly constant specific angular momentum, $h = h(\varpi)$, where (ϖ, ϕ, z) denote cylindrical polar coordinates, by use of perturbation theory, taking the h = constant case as the zeroth order state. We show that unstable modes found in the h = constant case can be extended continuously to cases for which $h' \equiv dh/d\varpi$ is small. We calculate explicitly the perturbation to the eigenvalue of the unstable fundamental mode of the isothermal thin ring discussed in Paper I.

In Section 5 we consider tori with small minor radii, and modes with low m for which the gradient of h cannot be treated as a perturbation. For small values of m there are modes which are almost independent of z. We show that for power-law rotation laws of the form $\Omega(\varpi) \propto \varpi^{-q}$, the modes are unstable for $q > \sqrt{3}$ and stable for $q < \sqrt{3}$. Instability in this case is driven by the classical Kelvin–Helmholtz mechanism and the modes do not have a sonic character. We also show that modes which are odd in z and have low m are stable. The analysis breaks down for high m where the modes regain their sonic character.

In Section 6 we consider high-m modes but for analytic tractability in the limit of a flat torus ($q \simeq 3/2$). We show that there exist sonic modes which are driven both by the classic instability and by the sonic mechanism. This illustrates the expectation that general unstable modes are likely to be driven by both mechanisms.

In Section 7 we present a discussion of the findings of this paper and summarize our conclusions in Section 8.

2 The basic equations

In this section we set out the equations governing the equilibrium configuration and derive the linear perturbation equations. Since these derivations closely follow Section 2 of Paper I we do not include all the details here. We have, however, extended the derivation of the perturbation equations to include expressions for the Lagrangian displacement which we require in Section 3.

2.1 THE EQUILIBRIUM

We consider the equilibrium configuration of a non-self-gravitating differentially rotating fluid. We use cylindrical polar coordinates (ϖ, ϕ, z) in an inertial frame and assume that the fluid rotates about the z-axis. We assume that the flow is under the influence of an external potential, ψ_p, which is that due to a gravitating point mass, M, situated at the origin; thus $\psi_p = -GM/(\varpi^2 + z^2)^{1/2}$. As in Paper I we adopt a polytropic equation of state, so that pressure, p, and density, ρ, are related by $p = A\rho^\Gamma$ where A is a constant. The polytropic index, n, is defined by $\Gamma = 1 + 1/n$. This implies that the angular velocity Ω, must be a function of ϖ alone.

In this paper we consider power-law rotation laws of the form

$$\Omega = \Omega_0(\varpi/\varpi_0)^{-q} \tag{2.1}$$

where q, Ω_0 and ϖ_0 are constants. As in Paper I we define a rotational potential $\psi_{\rm rot}$, where here

$$\psi_{\rm rot} = \Omega_0^2 \varpi_0^{2q}/[(2q-2)\varpi^{2q-2}]. \tag{2.2}$$

The equilibrium density configuration is then given by (see Paper I):

$$(n+1)p/\rho + \psi_p + \psi_{\rm rot} = C = \text{const}, \tag{2.3}$$

and we note that $p/\rho = A\rho^{1/n}$. Such a density configuration defines a torus centred on the origin. The density maximum of the torus lies in the $z = 0$ plane, and for convenience we define it to lie on the ring ($\varpi = \varpi_0$, $z = 0$).

In this paper we shall be interested in tori of small extent, that is, tori for which any internal dimension is very much less than ϖ_0. For these we may obtain an approximation to the structure by expanding $\psi_p + \psi_{\rm rot}$ about $\varpi = \varpi_0$, $z = 0$. Since we have a density maximum there, we have $\nabla(\psi_p + \psi_{\rm rot}) = 0$ there and the leading terms in the expansion are quadratic. Performing the expansion, equation (2.3) becomes (*cf.* Paper I, section 5.2)

$$A(n+1)\rho^{1/n} + \frac{GM}{2\varpi_0^3}[(2q-3)x^2 + z^2] = C' \tag{2.4}$$

where we define $x = \varpi - \varpi_0$ and find

$$C' = C + \Omega_0^2\varpi_0^2(2q-3)/(2q-2). \tag{2.5}$$

The surface of the torus ($\rho = 0$) is given by

$$(2q-3)x^2 + z^2 = \frac{2C'\varpi_0^3}{GM}. \tag{2.6}$$

In this approximation the density is given by

$$\rho = \rho_0\left[1 - \frac{x^2}{a^2} - \frac{z^2}{(2q-3)a^2}\right]^n, \tag{2.7}$$

where $\rho_0 = [C'/A(n+1)]^n$ is the maximum value of the density, and $a^2 = 2(n+1)\varpi_0^3 p_0/[GM\rho_0(2q-3)] \ll \varpi_0^2$.

In the case treated in Paper I (constant angular momentum, $q = 2$) the surfaces of constant density in the (ϖ, z) plane are concentric circles. For equilibria which are stable to axisymmetric perturbations (Rayleigh's criterion) we are interested in values of $q \leqslant 2$. In this case the constant density surfaces become similar concentric ellipses elongated in the ϖ-direction with ratios of major to minor axes of $1/(2q-3)^{1/2}$, or equivalently with eccentricity $[2(2-q)]^{1/2}$. As q tends towards the value of 3/2 from above the eccentricity of the ellipses tends to unity. This is because when q tends to 3/2, the rotation rate becomes Keplerian and radial pressure support tends to zero. The maximum and minimum radii of the torus occur on the midplane at $\varpi_\pm = \varpi_0 \pm a$.

2.2 THE PERTURBATION EQUATIONS

The velocity of the unperturbed toroidal equilibrium is $\mathbf{v}_0 = (0, \varpi\Omega, 0)$ and we write the velocity perturbation as $\mathbf{v}' = (v'_\varpi, v'_\phi, v'_z)$. The full velocity $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$ satisfies the fluid equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \psi, \tag{2.8}$$

where $\psi = \psi_{\rm p} + \psi_{\rm rot}$, and

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \mathbf{v}). \tag{2.9}$$

Since the unperturbed configuration is independent of time, t, and azimuth, ϕ, we assume the perturbed quantities to depend on t and ϕ in the form $\exp\,[i(m\phi + \sigma t)]$ where m is the azimuthal wavenumber, and σ is a (possibly) complex eigenfrequency. We write

$$\sigma = \sigma_{\rm R} + i\gamma \tag{2.10}$$

where $\sigma_{\rm R}$ and γ are real and for instability we have $\gamma < 0$.

The components of the Lagrangian displacement $\boldsymbol{\xi} = (\xi_\varpi, \xi_\phi, \xi_z)$ associated with the velocity perturbation are given by (Chandrasekhar 1961)

$$\xi_\varpi = v'_\varpi / i\bar{\sigma},$$

$$\xi_\phi = v'_\phi / i\bar{\sigma} - \varpi \cdot (d\Omega/d\varpi) \cdot (v'_\varpi/\bar{\sigma}^2),$$

and

$$\xi_z = v'_z / i\bar{\sigma}, \tag{2.11}$$

where $\bar{\sigma} \equiv \sigma + m\Omega$. The relationship between p' and $\mathbf{v}'$ which comes from linearizing equation (2.8) is derived in Paper I and is

$$i\bar{\sigma} v'_\varpi - 2\Omega v'_\phi = -\frac{\partial}{\partial \varpi}\left(\frac{p'}{\rho}\right),$$

$$i\bar{\sigma} v'_\phi + h' v'_\varpi/\varpi = -\frac{imp'}{\varpi\rho},$$

$$i\bar{\sigma} v'_z = -\frac{\partial}{\partial z}\left(\frac{p'}{\rho}\right), \tag{2.12}$$

where $h' \equiv d(\varpi^2\Omega)/d\varpi$.

Using the variable $W \equiv p'/(\rho\bar{\sigma})$, and equation (2.11) equations (2.12) can be written as

$$v'_\varpi - \xi_\phi h'/\varpi = i\partial W/\partial\varpi,$$

$$v'_\phi + \xi_\varpi h'/\varpi = -mW/\varpi,$$

$$v'_z = i\partial W/\partial z. \tag{2.13}$$

Now $\mathbf{v}'$ can be eliminated and we obtain the components of $\boldsymbol{\xi}$ in terms of W:

$$(\bar{\sigma}^2 - \kappa^2)\, \xi_\varpi = \partial(W\bar{\sigma})/\partial\varpi + 2m\Omega W/\varpi,$$

$$(\bar{\sigma}^2 - \kappa^2)\, \xi_\phi = im\bar{\sigma}W/\varpi + 2i\Omega(\partial W/\partial\varpi),$$

$$\bar{\sigma}^2 \xi_z = \partial(W\bar{\sigma})/\partial z, \tag{2.14}$$

where $\kappa^2 \equiv 2\Omega h'/\varpi$.

The perturbed continuity equation is

$$i\bar{\sigma}\rho' = -\operatorname{div}(\rho \mathbf{v}') \tag{2.15}$$

or, equivalently,

$$\rho' = -\operatorname{div}(\rho \boldsymbol{\xi}). \tag{2.16}$$

The perturbed equation of state for isentropic perturbations yields

$$p' = \left(\frac{\Gamma p}{\rho}\right)\rho'. \tag{2.17}$$

Using the definition of W, and equations (2.14), (2.16) and (2.17) we deduce

$$\frac{\bar{\sigma}^2\rho^2 W}{\Gamma p} = -\operatorname{div}(\rho\nabla W) + \frac{i}{\varpi}\frac{\partial}{\partial\varpi}(\rho h'\xi_\phi) + \frac{m\rho h'}{\varpi^2}\cdot\xi_\varpi, \tag{2.18}$$

and hence that

$$\frac{\bar{\sigma}^2\rho^2 W}{\Gamma p} = -\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi\bar{\sigma}^2}{D}\frac{\partial W}{\partial\varpi}\right) + \frac{\rho m^2}{\varpi^2}\cdot\frac{\bar{\sigma}^2}{D}\cdot W - \frac{\partial}{\partial z}\left(\rho\frac{\partial W}{\partial z}\right) - \frac{\bar{\sigma}mW}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho h'}{\varpi D}\right), \tag{2.19}$$

where $D \equiv \bar{\sigma}^2 - \kappa^2$.

This equation is the same as the equation (3.18) derived in Paper I.

3 Some general results

A comprehensive review of the stability of cylindrically symmetric (and plane parallel) shear flows for an incompressible, inviscid fluid is given by Drazin & Reid (1981). In this section we make use of the more recent results by Grinfeld (1984) to generalize the results to compressible media for two-dimensional flows. We also show that the result (due to Rayleigh) that any unstable mode has a corotation point within the flow is carried over to the full three-dimensional equilibria we consider here.

3.1 TWO-DIMENSIONAL FLOW

We consider here a flow, and perturbed flow, which are two-dimensional (i.e. independent of z). The perturbed fluid equations (Section 2.2) then simplify to

$$\frac{\partial v'_\varpi}{\partial t} + im\Omega v'_\varpi - 2\Omega v'_\phi = -\frac{\partial}{\partial\varpi}\left(\frac{p'}{\rho}\right), \tag{3.1}$$

$$\frac{\partial v'_\phi}{\partial t} + im\Omega v'_\phi + h' v'_\varpi/\varpi = -\frac{imp'}{\varpi\rho}, \tag{3.2}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho}\right) + im\Omega\frac{\rho'}{\rho} + \frac{1}{\rho}\operatorname{div}(\rho\mathbf{v}') = 0. \tag{3.3}$$

where now $\rho = \rho(\varpi)$ can be regarded as a surface density, and as before we assume $p' = (\Gamma p/\rho)\rho'$. For such a flow the unperturbed vorticity is $w_0 = h'/\varpi$, and the perturbed vorticity is

$$w' = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi v'_\phi) - \frac{im}{\varpi}v'_\varpi. \tag{3.4}$$

For ease of analysis we consider the flow between two fixed concentric cylinders, so that the boundary conditions are that $v'_\varpi = 0$. Following Grinfeld (1984) we consider the real functional

$$I = \int \rho\varpi d\varpi d\phi \left\{ \left| \mathbf{v}' + \frac{\rho'}{\rho}\varpi\Omega\hat{\mathbf{e}}_\phi \right|^2 + \left(\frac{\Gamma p}{\rho} - \varpi^2\Omega^2\right)\left|\frac{\rho'}{\rho}\right|^2 + \frac{\rho\varpi\Omega}{d(h'/\rho\varpi)/d\varpi}\left|\frac{\rho w' - (h'/\varpi)\rho'}{\rho^2}\right|^2 \right\}, \tag{3.5}$$

where $\hat{\mathbf{e}}_\phi$ is the unit vector in the ϕ-direction. It is straightforward to show that I is a constant of the motion. Since I is the sum of three terms each of which consists of a positive definite quantity multiplied by a coefficient, we can deduce sufficient conditions for stability, The conditions are that if in some frame the coefficients are positive throughout the flow, then no perturbation can become unbounded, since I is constant, and therefore the flow is stable. In this case the relevant frames to consider are those rotating with constant angular velocity Ω_c. In such a frame the equation (3.5) is changed only by replacing the explicitly occurring Ω by $\Omega - \Omega_c$. The general stability criterion is then:
If there is a value of Ω_c for which

$$\varpi^2(\Omega - \Omega_c)^2 < \Gamma p/\rho \tag{3.6}$$

and

$$(\Omega - \Omega_c)\frac{d}{d\varpi}\left(\frac{h'}{\rho\varpi}\right) > 0 \tag{3.7}$$

everywhere in the fluid, then the flow is stable.

Inequality (3.7) is impossible to satisfy only if $d(h'/\rho\varpi)/d\varpi$ is zero somewhere in the fluid. This is the generalization of Rayleigh's inflexion point criterion to rotating, compressible flows. Inequality (3.6) has no analogue for incompressible fluids. It is the same as the criterion found in Section 4 (equation 4.24) of Paper I, which was, however, derived for full three-dimensional perturbations.

3.2 COROTATION OF UNSTABLE MODES

Here we show that for any unstable mode, the real part of $\bar{\sigma}$ vanishes at some point in the unperturbed flow. The proof given holds only for power-law rotation laws, but is not restricted to toroidal equilibria and takes the z-dependence of the unperturbed equilibrium fully into account. We assume that at the fluid boundary either $\rho = 0$, or $W = 0$, or the normal derivative of W vanishes. This facilitates integration by parts without the introduction of surface terms.

We take equation (2.18), multiply both sides by W^*, where * denotes complex conjugate, and integrate over the volume occupied by the unperturbed fluid. After some integrations by parts we obtain

$$\int \frac{\bar{\sigma}^2\rho^2}{\Gamma p}|W|^2 d\tau = \int \rho|\nabla W|^2 d\tau + I_1, \tag{3.8}$$

where

$$I_1 = \int \frac{m\rho h'}{\varpi^2}\,\xi_\varpi W^* d\tau - i\int \frac{\rho\xi_\phi h'}{\varpi}\frac{\partial W^*}{\partial\varpi}\,d\tau. \tag{3.9}$$

Similarly, multiplying the parts of equation (2.14) by $\rho\xi_\varpi^*$, $\rho\xi_\phi^*$ and $\rho\xi_z^*$ respectively, integrating over the fluid volume and adding the results we obtain,

$$-\int \bar{\sigma}^2\rho\,|\boldsymbol{\xi}|^2 d\tau + \int \kappa^2\rho\,(|\xi_\varpi|^2 + |\xi_\phi|^2)\,d\tau = \int \bar{\sigma}W\,\mathrm{div}\,(\rho\boldsymbol{\xi}^*)\,d\tau + I_2, \tag{3.10}$$

where

$$I_2 = -\int \frac{2m\Omega}{\varpi}\,\xi_\varpi^*\,W\rho d\tau - 2i\int \Omega\xi_\phi^*\,\frac{\partial W}{\partial\varpi}\,\rho d\tau. \tag{3.11}$$

For a power-law rotation law $h'/(\varpi\Omega)$ is a constant and thus $I_2 = -2\Omega\varpi I_1^*/h'$. On the rhs of equation (3.10) we use equation (2.16) and the definition of W to note that $\mathrm{div}\,(\rho\boldsymbol{\xi}) = -\bar{\sigma}W\rho^2/(\Gamma p)$. Thus equation (3.10) can be rewritten as

$$-\int \bar{\sigma}^2\rho\,|\boldsymbol{\xi}|^2 d\tau + \int \kappa^2\rho\,(|\xi_\varpi|^2 + |\xi_\phi|^2)\,d\tau = -\int \frac{|\bar{\sigma}|^2|W|^2}{\Gamma p}\rho^2 d\tau - \frac{2\Omega\varpi}{h'}I_1^*. \tag{3.12}$$

We now eliminate I_1 between equations (3.8) and (3.12) to obtain

$$\frac{2\Omega\varpi}{h'}\int \frac{\bar{\sigma}^{*2}\rho^2|W|^2}{\Gamma p}\,d\tau - \int \bar{\sigma}^2\rho\,|\boldsymbol{\xi}|^2 d\tau = -\int \frac{|\bar{\sigma}|^2|W|^2}{\Gamma p}\rho^2 d\tau$$

$$-\int \kappa^2\rho\,(|\xi_\varpi|^2 + |\xi_\phi|^2)\,d\tau + \frac{2\Omega\varpi}{h'}\int \rho\,|\nabla W|^2 d\tau. \tag{3.13}$$

Recalling that $\sigma = \sigma_{\mathrm{R}} + i\gamma$, the imaginary part of (3.13) then gives

$$\gamma\int\left\{\frac{2\Omega\varpi\rho^2|W|^2}{h'\Gamma p} + \rho\,|\boldsymbol{\xi}|^2\right\}(\sigma_{\mathrm{R}} + m\Omega)\,d\tau = 0. \tag{3.14}$$

It follows at once that for an unstable mode ($\gamma \neq 0$), since the part of the integrand in (3.14) in curly brackets is positive definite, then $\sigma_{\mathrm{R}} + m\Omega$ must vanish at some point in the range of integration. Thus an unstable mode must have $\sigma_{\mathrm{R}} = -m\Omega$ at some point in the flow.

4 Tori with nearly constant specific angular momentum

In Paper I we discussed tori with constant specific angular momentum for which h = constant. We showed both analytically and numerically that instabilities existed in such tori. In Paper I we asserted that the instabilities must persist if the rotation law is changed slightly. Here we demonstrate the truth of that assertion.

4.1 PERTURBATION THEORY FOR UNSTABLE MODES

By inspection of equation (2.19) we see that we may in general expect a real continuum spectrum of singular modes for which $D = \bar{\sigma}^2 - \kappa^2 = 0$. However, for unstable modes (for which $\gamma \neq 0$), the quantity D is nowhere zero and we may apply standard perturbation theory.

When h = constant, equation (2.19) takes the form

$$L(\rho, W) = \frac{\bar{\sigma}^2 \rho^2 W}{\Gamma p} \tag{4.1}$$

where the elliptic operator L is defined by

$$L(\rho, W) \equiv -\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\rho\varpi\frac{\partial W}{\partial \varpi}\right) + \frac{\rho m^2 W}{\varpi^2} - \frac{\partial}{\partial z}\left(\rho\frac{\partial W}{\partial z}\right). \tag{4.2}$$

Consider a torus for which h = constant, and for which $\rho = \rho_1$, $p = p_1$ and $\Omega = \Omega_1$. We consider a particular unstable mode for which $W = W_1$, $\sigma = \sigma_1$ and $\bar{\sigma} = \bar{\sigma}_1$, with $\mathrm{Im}(\sigma_1) = \mathrm{Im}(\bar{\sigma}_1) \neq 0$.

We now consider the torus to be perturbed slightly so that the angular momentum is nearly, but not exactly, constant. We suppose the structure of the torus is changed slightly and write $\rho = \rho_1 + \delta\rho$, $p = p_1 + \delta p$ and $\Omega = \Omega_1 + \delta\Omega$. As a result of these changes we expect the mode to be perturbed and write $\sigma = \sigma_1 + \delta\sigma$ and $W = W_1 + \delta W$. We now calculate δW and $\delta\sigma$. To carry out the perturbation we note that $\kappa^2 = 2\Omega h'/\varpi$ is a small quantity, so that we may write

$$\frac{\bar{\sigma}^2}{D} = \frac{\bar{\sigma}^2}{\bar{\sigma}^2 - \kappa^2} \simeq 1 + \frac{\kappa^2}{\bar{\sigma}_1^2}. \tag{4.3}$$

In making this expansion we have used the fact that $|\bar{\sigma}_1|$ is bounded below for an unstable mode, so that the expansion is valid for small enough κ.

On substituting these expansions into equation (2.19) we obtain

$$\frac{\bar{\sigma}_1^2 \rho_1^2}{\Gamma p_1}\left(W_1 + \delta W + \left[\frac{2\delta\rho}{\rho_1} - \frac{\delta p}{p_1}\right] W_1\right) + \frac{\bar{\sigma}_1 \rho_1^2 W_1}{\Gamma p_1}(2\delta\sigma + 2m\delta\Omega)$$
$$= L(\rho_1, W_1 + \delta W) + L(\delta\rho, W_1) - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\rho_1\varpi\frac{\partial W_1}{\partial\varpi}\frac{\kappa^2}{\bar{\sigma}_1^2}\right) + \frac{m^2\rho_1}{\varpi_1^2}W_1\frac{\kappa^2}{\bar{\sigma}_1^2}$$
$$- \frac{\bar{\sigma}_1 m W}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho_1 h'}{\varpi\bar{\sigma}_1^2}\right). \tag{4.4}$$

To find $\delta\sigma$, we multiply this equation by W_1, and integrate over the volume of the torus. After doing some integrations by parts, and using the equation satisfied by W_1, we obtain

$$J_1 \cdot \delta\sigma = J_2 + J_3 + J_4 + J_5, \tag{4.5}$$

where

$$J_1 = 2\int \frac{\rho_1^2 \bar{\sigma}_1 W_1^2}{\Gamma p_1} d\tau,$$

$$J_2 = -\int \frac{\rho_1^2 \bar{\sigma}_1^2 W_1^2}{\Gamma p_1}\left(\frac{2m\delta\Omega}{\bar{\sigma}_1} + \frac{2\delta\rho}{\rho_1} - \frac{\delta p}{p_1}\right) d\tau,$$

$$J_3 = \int \delta\rho (\nabla W_1)^2 d\tau,$$

$$J_4 = \int \frac{\kappa^2 \rho_1}{\bar{\sigma}_1^2}\left[\frac{m^2 W_1^2}{\varpi^2} + \left(\frac{\partial W_1}{\partial\varpi}\right)^2\right] d\tau,$$

and

$$J_5 = -\int \frac{\bar{\sigma}_1 m W_1^2}{\varpi} \frac{\partial}{\partial \varpi}\left(\frac{\rho_1 h'}{\varpi \bar{\sigma}_1^2}\right) d\tau$$

$$= \int \frac{\rho_1 h'}{\varpi^2 \bar{\sigma}_1^2} \frac{\partial}{\partial \varpi}(\bar{\sigma}_1 m W_1^2)\, d\tau.$$

The last two terms can be combined and we write $J_4 + J_5 = J_6$, where

$$J_6 = \int \frac{\kappa^2 \rho_1}{\bar{\sigma}_1^2}\left(\frac{\partial W_1}{\partial \varpi}\right)\left(\frac{\partial W_1}{\partial \varpi} + \frac{\bar{\sigma}_1 m W_1}{\Omega_1 \varpi}\right) d\tau. \tag{4.6}$$

We remark that of the terms determining $\delta\sigma$, J_2 and J_3 come from changes in the background density, pressure and rotation law. However, the basic analysis given in Paper I still holds regardless of changes of this type and so these terms do not change the qualitative picture obtained from tori with constant h. Changes of physical significance arise from J_4 and J_5 (i.e. J_6) which depend on h'.

We now give rough order of magnitude estimates of the terms J_1 and J_6. To do so we note that in general for moderate values of m, W_1 changes on a scale length not less than ϖ, even when the torus is small. This follows from equation (4.1) for modes almost independent of z, such as those found in Paper I, if we remember that characteristically

$$\bar{\sigma}^2 \sim m^2 \left(\frac{d\Omega}{d\varpi}\right)^2_{\varpi=\varpi_0} \qquad a^2 \sim \frac{m^2 \Omega_0^2 a^2}{\varpi_0^2} \sim \frac{m^2 p_0}{\rho_0 \varpi_0^2} \tag{4.7}$$

where subscript zero denotes the value of the quantity at (ϖ_0, 0) and a is the minor radius (see Section 2). We also use the fact that for an unstable mode $|\bar{\sigma}_1| \geqslant |\gamma|$. Then for moderate values of m, we estimate

$$|J_6| \sim \int \frac{\kappa^2 \rho_1 |W_1|^2}{|\gamma|^2 \varpi^2} d\tau \tag{4.8}$$

and

$$|J_1| \sim \frac{|\gamma|\, \rho_0}{\Gamma p_0} \int \rho_1 |W_1|^2 d\tau. \tag{4.9}$$

These estimates are probably reasonable for $|\gamma| \sim a\Omega_0/\varpi_0$, when W_1 changes on a length scale comparable to ϖ. Note that if W_1 changes on a length scale long compared to ϖ, then $|J_6|$ is correspondingly reduced. When W_1 changes on a scale ϖ we have the estimate

$$|\delta\sigma| \sim \frac{\bar{\kappa}^2}{|\gamma|^3} \frac{\Gamma p_0}{\rho_0 \varpi_0^2} \sim \frac{\bar{\kappa}^2}{|\gamma|^3} \Omega_0^2 \left(\frac{a}{\varpi_0}\right)^2 \tag{4.10}$$

where $\bar{\kappa}^2$ is an appropriate average. As expected we see that if κ^2 is small everywhere, then $|\delta\sigma|$ is small, and the instabilities found for constant h also exist for tori with nearly constant h.

4.2 APPLICATION TO THE THIN ISOTHERMAL RING

Once a particular mode W_1 and corresponding eigenvalue σ_1 are known for a constant angular momentum configuration, the correction, $\delta\sigma$, to σ_1 caused by a slight deviation from the

$h' = 0$ rotation law can be calculated by evaluating the integrals involved in equation (4.5). It would be possible to do this for example for the modes determined numerically in Paper I. As an illustration of the procedure we calculate $\delta\sigma$ for the case of the thin isothermal ring (Paper I, section 5.2) for which the eigenmodes can be found analytically in the h = constant case. An extensive discussion of this case was given in Paper I, so we only briefly review the relevant points here and refer the reader to Paper I for further details.

The isothermal equation of state is $p = \rho c_s^2$ where the sound speed c_s is constant. In this case the density distribution corresponding to a thin ring is (*cf.* equation 2.7)

$$\rho = \rho_0 \exp \{-[(2q-3)x^2 + z^2]/2b^2\}$$

where

$$b^2 = c_s^2 \varpi_0^3/GM \ll \varpi_0^2. \tag{4.11}$$

For the unperturbed, constant-h torus, we have $q = 2$ and the density distribution is

$$\rho_1 = \rho_0 \exp[-(x^2+z^2)/2b^2]. \tag{4.12}$$

For the perturbation we consider a power-law rotation law with q slightly less than 2 and define $\epsilon = 2 - q$. We note that $\epsilon > 0$ and assume ϵ to be small. By comparison of (4.11) and (4.12) we find to lowest order in ϵ that

$$\delta\rho/\rho_1 = \epsilon(x^2/b^2), \tag{4.13}$$

and note that by isothermality $\delta\rho/\rho_1 = \delta p/p_1$. For a small torus we may expand Ω about the centre and retain only the first term; thus to first order in b/ϖ_0 we may write

$$\Omega = \Omega_0 - q\Omega_0 x/\varpi_0. \tag{4.14}$$

Hence we obtain

$$\delta\Omega/\Omega_1 = \epsilon x/\varpi_0. \tag{4.15}$$

The relevant eigenfunctions W_1 are given in Paper I. The unstable modes are independent of z, and for a fixed m, the fastest growing mode is the fundamental which is of the form

$$W_1 = \exp(-\tfrac{1}{2}a_1x^2 - a_2x),$$

where

$$a_1 = -\frac{1}{2b^2} + \left[\frac{1}{4b^4} - \frac{4m^2}{b^2\varpi_0^2}\right]^{1/2},$$

and

$$a_2 = \frac{2m\Omega_0(\sigma_1 + m\Omega_0)}{\varpi_0 c_s^2}\left[\frac{1}{4b^4} - \frac{4m^2}{b^2\varpi_0^2}\right]^{-1/2}. \tag{4.16}$$

The eigenvalue σ_1 is given by

$$(\sigma_1 + m\Omega_0)^2 = c_s^2\left(1 - \frac{16m^2b^2}{\varpi_0^2}\right)\left[\frac{m^2}{\varpi_0^2} - \frac{1}{2b^2} + \left(\frac{1}{4b^4} - \frac{4m^2}{b^2\varpi_0^2}\right)^{1/2}\right]. \tag{4.17}$$

For sufficiently small m, $\sigma_1 + m\Omega_0$ is purely imaginary and we now have all the relevant information to evaluate the integrals in equation (4.5), and hence to find $\delta\sigma$. The integrals are straightforward to evaluate, but somewhat tedious and we content ourselves here with making the further simplification of considering the case $m^2b^2/\varpi_0^2 \ll 1$. In Paper I we showed that the fastest growing mode occurred for $m^2b^2/\varpi_0^2 \cong 0.04$. Under this approximation we find

$$a_1 = -4m^2/\varpi_0^2, \tag{4.18}$$

and

$$(\sigma_1 + m\Omega_0)^2 = -3m^2b^2\Omega_0^2/\varpi_0^2. \tag{4.19}$$

Thus for the unstable mode,

$$\sigma_1 + m\Omega_0 = -i\sqrt{3}\, mb\Omega_0/\varpi_0, \tag{4.20}$$

and

$$a_2 = -4i\sqrt{3}m^2b/\varpi_0^2. \tag{4.21}$$

The main simplification in the evaluation of the integrals occurs because now W_1 is slowly varying in comparison to ρ_1, and so its variation may be neglected in evaluating J_1, J_2 and J_3. For these integrals we may take $W_1 = 1$. However, clearly the variation of W_1 may not be neglected in evaluating J_6 (equation 4.6), but in fact after differentiation we may set $W_1 = 1$ here also. Finally, because we are dealing with a small torus, the quantities such as ϖ, κ^2 and Ω may be replaced by their values at the density maximum; similarly we have $d\tau = 2\pi\varpi_0\, dx\, dz$, and since everything but ρ_1 is independent of z, the z-integration just gives rise to a constant factor throughout which we ignore. In this way we obtain

$$J_1 = -\frac{8\pi m\rho_0}{\Omega_0 b}\int_{-\infty}^{\infty}\left(\frac{x}{b}+\frac{i\sqrt{3}}{2}\right)\exp(-x^2/2b^2)\,dx, \tag{4.22}$$

$$J_2 = \frac{8\pi m^2\rho_0\epsilon}{\varpi_0}\int_{-\infty}^{\infty}\left(\frac{x}{b}+\frac{i\sqrt{3}}{2}\right)^2\left[-\frac{x^2}{b^2}+\frac{x/b}{(x/b+i\sqrt{3}/2)}\right]\exp(-x^2/2b^2)\,dx, \tag{4.23}$$

$$J_3 = \frac{2\pi m^2\rho_0\epsilon}{\varpi_0}\int_{-\infty}^{\infty}\frac{x^2}{b^2}\exp(-x^2/2b^2)\,dx, \tag{4.24}$$

and

$$J_6 = \frac{8\pi m^2\rho_0\epsilon}{\varpi_0}\int_{-\infty}^{\infty}\left[\frac{2(x/b+i\sqrt{3})^2}{(x/b+i\sqrt{3}/2)^2}-\frac{(x/b+i\sqrt{3})}{(x/b+i\sqrt{3}/2)}\right]\exp(-x^2/2b^2)\,dx. \tag{4.25}$$

These integrals are all real and simple to evaluate, and equations (4.5), (4.6) then yield

$$\delta\sigma = 1.61\, i\sqrt{3}m\Omega_0 b\epsilon/\varpi_0 \tag{4.26}$$

or equivalently

$$\sigma_1 + \delta\sigma = -m\Omega_0 - \frac{i\sqrt{3}mb\Omega_0}{\varpi_0}(1-1.61\epsilon), \tag{4.27}$$

where $\epsilon = 2 - q > 0$, and the approximations we have made require $\epsilon \ll m^2b^2/\varpi_0^2 \ll 1$. We note that this value of $\delta\sigma$ is smaller in modulus than the estimate given by equation (4.10) by a factor $\sim (b^2/\varpi_0^2)$ because in this case the scale on which W_1 changes in the isothermal ring is ϖ_0^2/b rather than ϖ_0. As expected the effect of departure from constancy of h is to reduce the growth rate of the instability slightly.

We remark that the numerical value (1.61) obtained here depends on precisely what perturbation is carried out: here we changed q but kept c_s and ρ_0 constant whereas another possibility would be to change q and keep c_s and the total torus mass constant (equivalent to a small change in ρ_0). Nevertheless the result that a slight change in the rotation law leads to a corresponding slight change in the growth rate of the mode can be expected to hold in general.

5 Stabilization of modes with low m in small tori

We have seen in Section 4 that the effect of a small positive gradient in specific angular momentum is to reduce the growth rate of an unstable mode. Indeed when there is a positive gradient of specific angular momentum there is a natural local frequency of oscillation, κ, which one would expect on physical grounds to impart stability. In this section we investigate conditions under which particular modes can be stabilized by this effect.

A convenient measure of the amount of shear in the torus is given by $\Delta\Omega \equiv \Omega_- - \Omega_+ > 0$ where Ω_+ and Ω_- are the minimum and maximum rotation rates in the torus, occurring at ϖ_+ and ϖ_- respectively. The dimensionless ratio $\Delta\Omega/\kappa$ then gives a measure of the strength of the driving of the shear instability compared to the stabilizing term. In particular, when $\Delta\Omega/\kappa$ is small in some sense one might expect any instability driven by shear to be inhibited or possibly removed. Such conditions are the opposite extreme to those pertaining to the case of almost constant specific angular momentum for which $\kappa \simeq 0$ and $\Delta\Omega/\kappa \gg 1$. We therefore consider tori in which $\Delta\Omega/\kappa$ is small. To do so we consider small tori for which the equilibrium density can be represented by equation (2.7). We have seen (Section 3.2) that for unstable modes, we must have $\sigma_R = -m\Omega$ at some point in the torus. Thus for weakly unstable modes (with small γ), $|\sigma + m\Omega|$ is also small throughout the torus. In fact

$$\frac{|\sigma+m\Omega|^2}{\kappa^2} = \frac{|\sigma_R+m\Omega|^2+\gamma^2}{2(2-q)\Omega^2} \leqslant \frac{m^2(\Delta\Omega)^2+\gamma^2}{2(2-q)\Omega_+^2}. \tag{5.1}$$

For a torus of small extent with $\varpi_\pm = \varpi_0 \pm a$, we may use equation (4.14) to write $\Delta\Omega = 2qa\Omega_0/\varpi_0$, and hence find that

$$\frac{|\sigma+m\Omega|^2}{\kappa^2} \lesssim \frac{4q^2m^2a^2\Omega_0^2/\varpi_0^2+\gamma^2}{2(2-q)\Omega_0^2}. \tag{5.2}$$

Thus if γ is small, $|\sigma+m\Omega|/\kappa$ is small provided that $m^2a^2/\varpi_0^2 \ll (2-q)/2q^2$. We note that the reverse of this strong inequality was required in Section 4.2. We now investigate the stability of modes for which this condition is satisfied. We note that for given q and m, the inequality can always be satisfied for a torus of sufficiently small a.

5.1 MODES WHICH ARE EVEN FUNCTIONS OF z

The equation governing linear stability is equation (2.19). When $|\sigma+m\Omega| \ll \kappa$, the quantity D may be replaced in that equation by $-\kappa^2$, and the equation becomes

$$\frac{\bar{\sigma}^2\rho^2 W}{\Gamma p} = -\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\rho\varpi\frac{\bar{\sigma}^2}{\kappa^2}\frac{\partial W}{\partial\varpi}\right) - \frac{m^2\rho}{\varpi^2}\cdot\frac{\bar{\sigma}^2}{\kappa^2}\cdot W - \frac{\partial}{\partial z}\left(\rho\frac{\partial W}{\partial z}\right) + \frac{m\bar{\sigma}W}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho h'}{\varpi\kappa^2}\right). \tag{5.3}$$

It is convenient to work in terms of $Q \equiv W\bar{\sigma} = p'/\rho$, for which the equation can be written

$$\bar{\sigma}^2\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi}{\kappa^2}\frac{\partial Q}{\partial\varpi}\right) - \frac{m^2\rho Q}{\kappa^2\varpi^2} - \frac{\rho^2 Q}{\Gamma p}\right] + \frac{\bar{\sigma}mQ}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{2\Omega\rho}{\kappa^2}\right) = \frac{\partial}{\partial z}\left(\rho\frac{\partial Q}{\partial z}\right). \tag{5.4}$$

For a small torus we write as before $x = \varpi - \varpi_0$, and neglect the variation of ϖ and of Ω (except where it occurs in $\bar{\sigma}$). We then have approximately

$$\bar{\sigma}^2 \left[\frac{1}{\kappa^2} \frac{\partial}{\partial x} \left(\rho \frac{\partial Q}{\partial x} \right) - \frac{m^2 \rho Q}{\kappa^2 \varpi_0^2} - \frac{\rho^2 Q}{\Gamma p} \right] + \frac{\bar{\sigma} m Q}{\varpi_0} \frac{\partial}{\partial x} \left(\frac{2 \Omega \rho}{\kappa^2} \right) = \frac{\partial}{\partial z} \left(\rho \frac{\partial Q}{\partial z} \right). \tag{5.5}$$

For the moment we have for convenience retained the $2\Omega/\kappa^2$ inside the differential of the final term on the lhs, although this is not necessary.

Equation (5.5), although simplified, is still a complicated partial differential equation. The solutions for Q can be classified according to whether they are even or odd functions of z. We consider here the even functions, and indeed find modes for which z-dependence plays only a minor role. We consider the odd functions in Section 5.2.

To investigate solutions of equation (5.5) we write

$$Q = \sum_{k=0}^{\infty} U_k(x) V_k(x, z), \tag{5.6}$$

where the functions $V_k(x, z)$ are specified orthogonal functions of z, for each value of x, with weight ρ^2/p. That is

$$\int V_k(x, z) V_j(x, z) \frac{\rho^2}{p} dz = N_k(x) \delta_{kj}, \tag{5.7}$$

where $N_k(x)$ is some suitable normalizing function of x, and the integral is taken along a line of constant z through the extent of the torus. It follows that we may choose $V_k(x, z)$ to satisfy the equation

$$\frac{\partial}{\partial z} \left(\rho \frac{\partial V_k}{\partial z} \right) + \lambda_k \frac{\rho^2}{p} V_k = 0 \tag{5.8}$$

where λ_k is an appropriate eigenvalue for the eigenfunction V_k. Using equation (2.7) for the density distribution and the polytropic equation of state, equation (5.8) is seen to be equivalent to

$$\frac{\partial}{\partial \xi} \left[(1 - \xi^2)^n \frac{\partial V_k}{\partial \xi} \right] + \lambda_k (1 - \xi^2)^{n-1} a^2 (2q - 3) \frac{\rho_0}{p_0} V_k = 0 \tag{5.9}$$

where ρ_0 and p_0 are the density and pressure in the centre of the torus, n is the adiabatic index, and

$$\xi = z / [(2q - 3)(a^2 - x^2)]^{1/2}. \tag{5.10}$$

We note that $x = \varpi - \varpi_0$ is to be regarded as constant as far as integrating (5.8) is concerned. If the torus extends to the zero-pressure surface, the boundary conditions to be used in conjunction with equation (5.9) are simply the regularity conditions at $\xi = \pm 1$. The eigenvalue problem so presented is a standard one (Whittaker & Watson 1927). It has polynomial solutions for V_k such that

$$V_k(x, z) = C_k^{n-1/2}(\xi); \qquad k = 0, 1, 2, \ldots, \tag{5.11}$$

where $C_k^{\nu}(\xi)$ denotes the Gegenbauer polynomial of degree k. The corresponding eigenvalues are

$$\lambda_k = \frac{k(k + 2n - 1)}{(2q - 3)} \cdot \frac{p_0}{\rho_0 a^2}. \tag{5.12}$$

We note that the eigenvalues are independent of x, although the eigenfunctions V_k are not.

We now substitute the expression for Q (equation 5.6) into equation (5.5), multiply both sides by $V_k(x, z)$, $k = 0, 1, 2, \ldots$ and integrate with respect to z. This gives us an infinite set of equations for the unknown functions $U_k(x)$, $k = 0, 1, 2, \ldots$ which take the form

$$\left(\frac{\bar{\sigma}^2}{\Gamma} - \lambda_k\right) N_k U_k = \bar{\sigma} I_{jk} U_j + \frac{\bar{\sigma}^2}{\kappa^2}\left\{\frac{\partial}{\partial x}(U_j J_{jk}) + K_{jk}\frac{\partial U_j}{\partial x} + \frac{\partial}{\partial x}\left(L_{jk}\frac{\partial U_j}{\partial x}\right) - \frac{m^2}{\varpi_0^2} U_j L_{jk}\right\}. \quad (5.13)$$

Here the repeated suffixes j are summed and

$$I_{jk} = -\frac{\bar{\sigma}}{\kappa^2}\int \rho \frac{\partial V_j}{\partial x}\frac{\partial V_k}{\partial x} dz + \frac{m}{\varpi_0}\int V_j V_k \frac{\partial}{\partial x}\left(\frac{2\Omega\rho}{\kappa^2}\right) dz,$$

$$J_{jk} = +\int \rho V_k \frac{\partial V_j}{\partial x} dz,$$

$$K_{jk} = -J_{kj},$$

and

$$L_{jk} = \int \rho V_j V_k \, dz. \quad (5.14)$$

We do not attempt to solve the set of equations (5.13) in full generality. However, for small tori with $|\bar{\sigma}/\kappa| \ll 1$ there are solutions for which all terms in the series for Q (equation 5.6) except the first one ($k = 0$) are small. These solutions are approximately separable in x and z and, because the eigenfunction V_0 is constant, are essentially independent of z. This solution is an even function of z, and the odd terms in the expression for Q may be taken as zero.

To obtain this approximate solution we consider equation (5.13) with $k = 0$, and assume that to zeroth order the only term to contribute from the sums on the rhs is the $j = 0$ term. Since V_0 is constant, and $\lambda_0 = 0$, we obtain

$$\frac{\bar{\sigma}^2}{\Gamma} U_0 \int \frac{\rho^2 dz}{p} = \frac{m\bar{\sigma} U_0}{\varpi_0}\frac{d}{dx}\left(\frac{2\Omega\Sigma}{\kappa^2}\right) + \frac{\bar{\sigma}^2}{\kappa^2}\left\{\frac{d}{dx}\left(\Sigma\frac{dU_0}{dx}\right) - \frac{m^2}{\varpi_0^2}\Sigma U_0\right\}, \quad (5.15)$$

where the surface density $\Sigma(x)$ is defined by

$$\Sigma(x) = \int \rho(x, z)\, dz. \quad (5.16)$$

By inspecting the equation (5.13) for $k > 0$, and again retaining only the $j = 0$ term in the sums we may estimate the ratio $|U_k|/|U_0|$ for $k > 0$. We assume that U_0 varies on the same scale as the torus, a, or slower and recall from inequality (5.2) that for small γ we have $|\bar{\sigma}|^2/\kappa^2 \lesssim m^2a^2/\varpi_0^2$. Hence we obtain the estimate

$$|U_k| \lesssim |U_0| \left(\frac{m^2a^2}{\varpi_0^2}\right)\cdot\left(\frac{N_0}{N_k}\right)^{1/2}. \quad (5.17)$$

This supports our contention that there is a mode which is dominated by the $k = 0$ term in the series.

To proceed with equation (5.15) we note that the definition of (equation 5.16) and the expression for the density distribution (equation 2.7) yield

$$\Sigma = K_n \rho_0 a(2q-3)^{1/2}\left(1 - \frac{x^2}{a^2}\right)^{n+1/2}, \quad (5.18)$$

where the constant K_n is defined by

$$K_n = 2 \int_0^1 (1-\xi^2)^n d\xi = \frac{2^{2n+1}[\Gamma(n+1)]^2}{\Gamma(2n+2)}. \tag{5.19}$$

Similarly we find

$$\int \frac{\rho^2 dz}{p} = \frac{(2n+1)(n+1)}{n(2q-3)a^2\Omega_0^2} \cdot \frac{\Sigma}{(1-x^2/a^2)}. \tag{5.20}$$

Using these, equation (5.15) becomes

$$\frac{d}{dx}\left(\Sigma \frac{dU_0}{dx}\right) - U_0 \left\{\frac{(2n+1)\kappa^2\Sigma}{(2q-3)\Omega_0^2 a^2(1-x^2/a^2)} + \frac{m^2\Sigma}{\varpi_0^2} - \frac{m\kappa^2}{\bar{\sigma}\varpi_0}\frac{d}{dx}\left(\frac{2\Sigma\Omega}{\kappa^2}\right)\right\} = 0. \tag{5.21}$$

This equation is similar in form to the z-component of the linearized vorticity equation for a two-dimensional disturbance in an axisymmetric incompressible flow and indeed is analogous to linearized equations for perturbations of two-dimensional plane-parallel incompressible shear flows (Drazin & Reid 1981). By using the standard procedures applied there it follows that a necessary condition for instability is that $d(2\Omega\Sigma/\kappa^2)/dx = 0$ somewhere in the flow (*cf.* Section 3.1). For the small torus approximation we are using this occurs at the density maximum $x = 0$. In this approximation we write $\bar{\sigma} = \hat{\sigma} - mq\Omega_0 x/\varpi_0$ where $\hat{\sigma} \equiv \sigma + m\Omega_0$, write $\Omega = \Omega_0$ and $\kappa^2 = 2(2-q)\Omega_0^2$. We may then rewrite equation (5.21) as

$$\frac{d}{dx}\left(\Sigma \frac{dU_0}{dx}\right) - \frac{U_0\Sigma(n+\tfrac{1}{2})}{a^2-x^2}\left\{\frac{4(2-q)}{2q-3} + \frac{2m^2(a^2-x^2)}{\varpi_0^2(2n+1)} + \frac{4x}{q(\lambda - x)}\right\} = 0 \tag{5.22}$$

where

$\lambda = \hat{\sigma}\varpi_0/(mq\Omega_0)$.

By analogy with the incompressible shear flow problem we search for non-singular solutions of equation (5.22) which correspond to marginally stable solutions with σ, or equivalently λ, real. Such solutions exist only when σ is chosen so that $\sigma + m\Omega = 0$ at the same value of λ at which $d(2\Omega\Sigma/\kappa^2)/dx$ vanishes. In the small torus approximation this means that $\lambda = 0$. We can also neglect the second term in curly brackets provided that

$$\frac{m^2a^2}{\varpi_0^2} \ll \frac{4(2n+1)(2-q)}{2q-3}.$$

The equation for the neutral mode then becomes

$$\frac{d}{dx}\left(\Sigma \frac{dU_0}{dx}\right) - \frac{U_0\Sigma(n+\tfrac{1}{2})}{a^2-x^2} \cdot \frac{4(3-q^2)}{q(2q-3)} = 0. \tag{5.23}$$

Using equation (5.18) we see that this equation has solutions regular at $x = \pm a$ only if $U_0(x) = C_k^n(x/a)$ where as before C_n^ν represents the Gegenbauer polynomial, and $k = 0, 1, 2, \ldots$. The solutions exist only for the particular relevant values, $q(k)$, of q which satisfy the equation

$$(1+2n)(q^2-3) = q(q-\tfrac{3}{2})k(k+2n) \tag{5.24}$$

and lie in the range $\tfrac{3}{2} < q \leq 2$.

The only value of k for which relevant values of $q(k)$ exist are $k=0$, $q(0)=\sqrt{3}$ and $k=1, q(1)=2$. However, for $q=2$, the approximations used here are invalid. We are interested therefore in the fundamental mode with $k=0$, for which $U_0=$ constant and $q(0)=\sqrt{3}$. It is straightforward to demonstrate, using perturbation theory that in the neighbourhood of this neutral mode stability occurs for $q<\sqrt{3}$ and instability for $q>\sqrt{3}$. Perturbation theory for this mode with $q \simeq \sqrt{3}$ yields the growth rate

$$-\gamma = -\operatorname{Im}(mq\Omega_0\lambda/\varpi_0)$$
$$= -\frac{mK_{n-1/2}}{\pi}\cdot\frac{q(3-q^2)}{2q-3}\cdot\frac{a\Omega_0}{\varpi_0} \qquad (5.25)$$

where K_n is defined in equation (5.19).

We see therefore that for small enough tori, the modes with low enough m that m^2a^2/ϖ_0^2 is appropriately small are stable provided that $q<\sqrt{3}$. We note that the approximations we have made in this section have ruled out the possibility of sonic-type modes and that the instabilities we have been considering here are driven by a similar mechanism to the shear flow instability. The sonic-type modes are regained even in small tori if we allow values of $m \gtrsim \varpi_0/a$, and these modes may still be unstable.

5.2 MODES WHICH ARE ODD FUNCTIONS OF Z

The modes considered in Section 5.1 were essentially independent of z. In this section we use the same approximation but consider modes for which Q is an odd function of z. Since in this approximation the modes correspond to the shear-driven modes in incompressible shear flows, we might expect the z-dependence to aid stabilization. Indeed we prove that all such odd modes are stable.

To do this we define a new variable $X = Q/\bar{\sigma}^{1/2}$ and rewrite equation (5.4) in the form

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi\bar{\sigma}}{\kappa^2}\frac{\partial X}{\partial\varpi}\right)+\frac{mX}{\varpi}\frac{\partial}{\partial\omega}\left(\frac{2\Omega\rho}{\kappa^2}\right)+\frac{X}{2\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi}{\kappa^2}m\frac{d\Omega}{d\varpi}\right)-\frac{X\rho m^2}{4\bar{\sigma}\kappa^2}\left(\frac{d\Omega}{d\varpi}\right)^2$$
$$=\frac{1}{\bar{\sigma}}\frac{\partial}{\partial z}\left(\rho\frac{\partial X}{\partial z}\right)+\bar{\sigma}\left(\frac{m^2\rho X}{\varpi^2\kappa^2}+\frac{\rho^2 X}{\Gamma p}\right). \qquad (5.26)$$

We multiply this equation by ϖX^* and integrate over the cross-section of the torus, to obtain

$$\int \bar{\sigma}\left\{\frac{\rho\varpi}{\kappa^2}\left|\frac{\partial X}{\partial\varpi}\right|^2+\left(\frac{m^2\rho}{\varpi\kappa^2}+\frac{\rho^2\varpi}{\Gamma p}\right)|X|^2\right\}d\varpi dz$$
$$=\int\frac{\rho\varpi}{\bar{\sigma}}\left\{\left|\frac{\partial X}{\partial z}\right|^2-\frac{m^2\Omega'^2}{4\kappa^2}|X|^2\right\}d\varpi\, dz$$
$$+\int |X|^2\frac{\partial}{\partial\varpi}\left[\frac{m\rho}{\kappa^2}(2\Omega+\tfrac{1}{2}\varpi\Omega')\right]d\varpi\, dz. \qquad (5.27)$$

We now take the imaginary part of this equation to obtain

$$\gamma\int\left\{\frac{\rho\varpi}{\kappa^2}\left|\frac{\partial X}{\partial\varpi}\right|^2+|X|^2\left(\frac{m^2\rho}{\varpi\kappa^2}+\frac{\rho^2\varpi}{\Gamma p}\right)+\frac{\rho\varpi}{|\bar{\sigma}|^2}\left[\left|\frac{\partial X}{\partial z}\right|^2-\frac{m^2\Omega'^2}{4\kappa^2}|X|^2\right]\right\}d\varpi\, dz=0. \qquad (5.28)$$

In this equation all the terms are of the same sign except for the last one. It follows therefore that if at each relevant value of ϖ,

$$\int \rho \left| \frac{\partial X}{\partial z} \right|^2 dz > \frac{m^2 \Omega'^2}{4\kappa^2} \int \rho \, |X|^2 \, dz, \tag{5.29}$$

we must have $\gamma = 0$, and there can be no unstable mode.

Consider now the minimum possible value of the ratio

$$R = \frac{\int \rho \, |\partial X/\partial z|^2 \, dz}{\int (\rho^2/p) |X|^2 \, dx}, \tag{5.30}$$

at each value of ϖ in the torus, subject to the constraint that X be an odd function of z. By Rayleigh's principle in Sturm–Liouville theory, the minimum value of R is attained when X satisfies equation (5.8) and is equal to the corresponding eigenvalue. The smallest eigenvalue corresponding to an odd eigenfunction is $\lambda_1 = 2np_0/[\rho_0 a^2 (2q-3)]$ and hence for all odd X (at each value of ϖ),

$$\int \rho \left| \frac{\partial X}{\partial z} \right|^2 dz \geqslant \frac{2np_0}{\rho_0 a^2 (2q-3)} \int \frac{\rho^2}{p} |X|^2 dz. \tag{5.31}$$

Further, since ρ/p has its minimum value at the centre of the torus we also have

$$\int \rho \left| \frac{\partial X}{\partial z} \right|^2 dz \geqslant \frac{2n}{(2q-3)a^2} \int \rho \, |X|^2 \, dz. \tag{5.32}$$

Using the inequalities (5.29) and (5.32) we see at once that if throughout the torus

$$\frac{m^2 a^2}{\varpi^2} < \frac{16n(2-q)}{q^2(2q-3)} \tag{5.33}$$

then no unstable mode can exist. This inequality is almost the same as that demanded by our approximations and we deduce that in small tori, the modes with low enough m that $m^2 a^2/\varpi_0^2$ is appropriately small and which are odd functions of z are stable.

6 Instability of modes with high m

In Section 5 we considered low-m modes in small tori. These modes did not have the character of sound waves and, when unstable, were driven by an analogous mechanism to that which drives the classic shear flow instability. For those modes we required $|\bar{\sigma}| \ll \kappa$. In the opposite limit $|\bar{\sigma}| \gg \kappa$, the modes acquire the character of sound waves and for large m we can ensure that $|\sigma + m\Omega| \gg \kappa$ throughout most of the flow except for a small region near corotation for weakly unstable modes. In this case we can replace D by $\bar{\sigma}^2$ in equation (2.19) and write

$$\frac{\bar{\sigma}^2 \rho^2 W}{\Gamma p} = -\frac{1}{\varpi} \frac{\partial}{\partial \varpi} \left(\rho \varpi \frac{\partial W}{\partial \varpi} \right) + \rho \frac{m^2 W}{\varpi^2} - \frac{\partial}{\partial z} \left(\rho \frac{\partial W}{\partial z} \right) - \bar{\sigma} \frac{mW}{\varpi} \frac{\partial}{\partial \varpi} \left(\frac{\rho h'}{\varpi \bar{\sigma}^2} \right). \tag{6.1}$$

In the limit of large m, bearing in mind that apart from in a small region near corotation we have $|\sigma + m\Omega| \propto m$, the last term in (6.1) is small compared to the other W term and can be neglected. We then have

$$\frac{\bar{\sigma}^2 \rho^2 W}{\Gamma p} = -\frac{1}{\varpi} \frac{\partial}{\partial \varpi} \left(\rho \varpi \frac{\partial W}{\partial \varpi} \right) + \rho \frac{m^2 W}{\varpi^2} - \frac{\partial}{\partial z} \left(\rho \frac{\partial W}{\partial z} \right). \tag{6.2}$$

However, equation (6.2) is the governing equation for the constant angular momentum case (Paper I) and is obtained from (equation 2.19) by setting $h'=0$. The mathematical properties of equation (6.2) derived in Paper I do not depend on the detailed form of Ω or ρ and lead us to expect instability. The modes have the character of sound waves, and their instability is connected to the condition (3.6) and not to the classical shear flow mechanism corresponding to condition (3.7). However, apart from this inherent tendency for sound waves to be unstable, the shear flow mechanism may operate near to corotation where equation (6.2) breaks down. In general we expect modes to exist which are unstable as a result of the combination of the two effects.

To investigate the stability of sonic modes with large m in full generality using equation (2.19) is a difficult problem. To obtain something tractable we seek conditions under which the modes are almost independent of z. The problem then becomes one in ordinary rather than partial differential equations. We start by returning to the basic equations (2.12) together with (2.15) and (2.17). From these we may derive the following pair of equations

$$\frac{\bar{\sigma}}{m}\frac{\partial}{\partial\varpi}(\rho\varpi v'_\phi)=iv'_\varpi\left[\rho\bar{\sigma}-\frac{h'^2}{m\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi}{h'}\right)+\frac{h'^2\bar{\sigma}\rho^2}{\Gamma p m^2}\right]-\frac{ih'}{m}\frac{\partial}{\partial z}(\rho v'_z)+v'_\phi\left[\frac{\bar{\sigma}\varpi}{m}\frac{\partial\rho}{\partial\varpi}-\frac{h'\bar{\sigma}^2\rho^2\varpi}{m^2\Gamma p}\right], \tag{6.3}$$

and

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi v'_\varpi)=\frac{\bar{\sigma}\rho^2 h'}{m\Gamma p}v'_\varpi-\frac{\partial}{\partial z}(\rho v'_z)-v'_\phi\frac{im\rho}{\varpi}\left[1-\frac{\bar{\sigma}^2\rho\varpi^2}{m^2\Gamma p}\right]. \tag{6.4}$$

We can integrate equations (6.3) and (6.4) with respect to z through the torus under the assumption that v'_ϖ and v'_ϕ are independent of z, to obtain a pair of first-order differential equations for v'_ϖ, v'_ϕ. If ρ and p were independent of z this would correspond to the case discussed in Section 3.1. However, if we attempt to justify this approach for a torus by following a similar procedure to that adopted in Section 5 we find that it can only be justified if $q\cong 3/2$ and the resulting modes do not vary on too short a length scale.

We let $\eta=v'_\phi$, $\zeta=iv'_\varpi/m$ and $\mu=\sigma/m$, so that $\bar{\sigma}=m/(\mu+\Omega)=m\bar{\mu}$, and obtain from equations (6.3) and (6.4):

$$\frac{d}{d\varpi}(\Sigma\varpi\eta)=A\eta+B\zeta, \tag{6.5}$$

and

$$\frac{d}{d\varpi}(\Sigma\varpi\zeta)=C\eta+D\zeta, \tag{6.6}$$

where

$$A=\varpi\frac{d\Sigma}{d\varpi}-h'\bar{\mu}\varpi\int\frac{\rho^2\,dz}{\Gamma p},$$

$$B=m^2\Sigma-\frac{h'^2}{\varpi\bar{\mu}}\frac{d}{d\varpi}\left(\frac{\Sigma\varpi}{h'}\right)+h'^2\int\frac{\rho^2\,dz}{\Gamma p},$$

$$C=\Sigma-\bar{\mu}^2\varpi^2\int\frac{\rho^2\,dz}{\Gamma p},$$

and

$$D = \bar{\mu} h' \varpi \int \frac{\rho^2 \, dz}{\Gamma p}.$$

We note that the coefficient B is singular at corotation where $\bar{\sigma} = 0$, if σ is real, and that the singular term has the factor $d(\Sigma\varpi/h')/d\varpi$. This quantity plays a crucial role in driving non-sonic shear instabilities (Section 3.1), and we might expect it to play a similar role here. However, equations (6.3) and (6.4) allow solutions with a sonic character, especially for large m.

By referring back to equation (6.2) with z-dependence neglected, it can be seen that for large m the solution is oscillatory in character in particular at the boundaries where ρ/p becomes infinite. As m increases the number of oscillations in the solution also increases, and as in Paper I we expect high-m modes to be trapped near to ϖ_+ and ϖ_-.

We can discuss the onset of instability by searching for marginally stable non-singular modes. As before these modes can only occur when μ is chosen so that $\bar{\mu} = 0$ at the point where $d(\Sigma\varpi/h')/d\varpi = 0$. Solutions of equations (6.5) and (6.6) which satisfy the appropriate regularity conditions at the boundaries will exist only for special values of other parameters. Here we adopt m as parameter, and assume it is so large that it can be treated as a continuous variable. Since the number of oscillations in the solution increases with m, we expect there to be critical values of m for which marginal modes exist. We suppose that such a value $m = m_0$ exists and let $\eta = \eta_0$, $\zeta = \zeta_0$, $\mu = \mu_0$, $A = A_0$, $B = B_0$, $C = C_0$ and $D = D_0$. We now seek unstable modes for values of m near to m_0. We write $m = m_0 + \delta m$ and $\mu = \mu_0 + \delta\mu$ with δm, $\delta\mu$ small. We then note that η and ζ satisfy the equations

$$\frac{d}{d\varpi}(\Sigma\varpi\eta) - (A_0\eta + B_0\zeta)$$
$$= [(m_0 + \delta m)^2 - m_0^2]\,\Sigma\zeta + \frac{h'^2}{\varpi}\frac{d}{d\varpi}\left(\frac{\Sigma\varpi}{h'}\right) \cdot \frac{\delta\mu \cdot \zeta}{(\mu_0 + \Omega)(\mu_0 + \delta\mu + \Omega)} - h'\varpi\,\delta\mu \cdot \int \frac{\rho^2\,dz}{\Gamma p} \cdot \eta \tag{6.7}$$

and

$$\frac{d}{d\varpi}(\Sigma\varpi\zeta) - (C_0\eta + D_0\zeta)$$
$$= -\varpi^2 \int \frac{\rho^2\,dz}{\Gamma p} \cdot (2\mu_0 + 2\Omega + \delta\mu) \cdot \delta\mu \cdot \eta + \delta\mu \cdot h'\varpi \cdot \int \frac{\rho^2\,dz}{\Gamma p} \cdot \zeta. \tag{6.8}$$

To find the behaviour of $\delta\mu$ to lowest order in perturbation theory we multiply equation (6.7) by $\varpi\zeta_0$, and equation (6.8) by $\varpi\eta_0$. We then subtract and integrate with respect to ϖ over the torus and replace ζ by ζ_0 and η by η_0. We find, eventually,

$$2m_0\,\delta m \int \Sigma\zeta_0^2\varpi\,d\varpi + \delta\mu \int \left\{\frac{d}{d\varpi}\left(\frac{\Sigma\varpi}{h'}\right) \cdot \frac{h'^2\zeta_0^2}{(\mu_0 + \Omega)(\mu_0 + \delta\mu + \Omega)} + 2\int \frac{\rho^2\,dz}{\Gamma p} \cdot [\varpi^3(\mu_0 + \Omega)\eta_0^2 - \zeta_0\eta_0 h'\varpi^2]\right\} d\varpi = 0. \tag{6.9}$$

From this equation we may find $\delta\mu$. To evaluate the integral containing $(\mu_0 + \delta\mu + \Omega)^{-1}$

we must assume that $\delta\mu$ has a small negative imaginary part and write

$$\frac{1}{\mu_0+\Omega+\delta\mu}=P\left(\frac{1}{\mu_0+\Omega}\right)+i\pi\delta\,(\mu_0+\Omega) \tag{6.10}$$

where P is the principal value and δ is Dirac's δ-function. Because of the choice of μ_0, all other terms are non-singular. We thus obtain:

$$\delta\mu\left\{P\int\frac{d}{d\varpi}\left(\frac{\Sigma\varpi}{h'}\right)\frac{h'^2\zeta_0^2}{(\mu_0+\Omega)^2}\,d\varpi+\iint\frac{\rho^2\varpi^2}{\Gamma p}\,[2\varpi(\Omega+\mu_0)\eta_0^2-2\zeta_0\eta_0h']\,d\varpi\,dz\right.$$
$$\left.+i\pi\left[\frac{d^2}{d\varpi^2}\left(\frac{\Sigma\varpi}{h'}\right)\frac{h'^2\zeta_0^2}{\Omega'^2}\right]_{\mathrm{c}}\right\}=-2m\,\delta m\int\Sigma\zeta_0^2\varpi\,d\varpi, \tag{6.11}$$

where the subscript c denotes evaluation at the corotation point where $\mu_0+\Omega=0$.

It is clear from equation (6.10) that we can ensure that $\delta\mu$ has a small negative imaginary part as initially assumed. An exception to this occurs when either $h'=0$ or $\zeta_0=0$ at corotation, and then in general $\delta\mu$ is real to this order. However, we know from Paper I that tori with $h'=0$ are unstable. Instability in such a case comes about when the coefficient of $\delta\mu$ in equation (6.11) is zero, in which case $(\delta\mu)^2\propto\delta m$ so that for at least one sign of δm instability results (Section 4.2 of Paper I). Similar results have been obtained in the case of compressible plane parallel shear flow (Blumen *et al.* 1975; Drazin & Davey 1977).

7 Discussion

In this paper we have extended our investigation of the stability of thick rotating discs or tori to cover more general rotation laws than that corresponding to constant specific angular momentum. We have adopted an analytic approach which has limited the complexity of the cases we discuss. More general problems require a numerical approach and we shall present the results of such an approach in a subsequent paper.

Some idea of the kind of instabilities present in compressible shear flow can be found in the work of Blumen *et al.* (1975) and Drazin & Davey (1977) who consider plane parallel flow and look for non-singular neutral modes. They show that if the velocity difference across the shear is subsonic then the usual inflexion point criterion is relevant (see also Drury 1979). However, for supersonic shears the unstable modes take on the character of sound waves. These results are confirmed by Grinfeld (1984) who gives an explicit Liapounov function for the linearized perturbation equations for the plane parallel compressible flow. The plane parallel case is however, simpler than the one we consider because of Squire's theorem which says, loosely, that the disturbances in that case may always be regarded as two-dimensional. In rotating flows no such simplification is possible.

However, the application of Grinfeld's Liapounov function approach to rotating flows is instructive (though strictly valid only for two-dimensional perturbations of two-dimensional flows) and we have given the relevant Liapounov function explicitly in Section 3.1. This shows that instability can occur only if *either* there are supersonic relative motions *or* the gradient of the ratio of vorticity to density changes sign. The tori considered in Paper I had zero vorticity and were unstable because they had essentially zero-pressure surfaces where the sound velocity was zero or small enabling violation of the first (supersonic) condition. There is no analogue of these instabilities in incompressible fluids. The second condition is the analogue of Rayleigh's necessary criterion for shear-flow instabilities in incompressible fluids and any instabilities driven by the behaviour of the gradient of $h'/(\varpi\Sigma)$ do have an incompressible precedent in the Kelvin–Helmholtz instability. Although Grinfeld's analysis cannot be used to prove instability our work in Paper I may be regarded as an illustration of the unstable sound waves that can occur when there are relative supersonic motions in the unperturbed flow.

In this paper we have considered rotation laws of the form $\Omega \propto \varpi^{-q}$ with $3/2 < q \leqslant 2$, and a finite barotropic fluid under the influence of a single gravitating point mass. For such flows which have a density maximum within them both of Grinfeld's conditions are violated and for general q we might expect any unstable modes to be an inseparable mixture of classical shear-flow mode and sound wave. In Section 3.2 we proved the general condition that any unstable mode must corotate at some point in the torus.

When the torus does not have constant specific angular momentum a natural restoring frequency, κ, where $\kappa^2 = 2\Omega h'/\varpi$, exists in the fluid. In Section 4 we demonstrated that when κ is small results may be obtained by applying perturbation theory to the unstable modes of sonic character found in the $h' = 0$ case. We have shown that there are no singularity problems in doing this and that the unstable eigenvalues found in the $h' = 0$ case are only slightly modified when κ is small. We have calculated the change in the eigenvalue explicitly for the fundamental unstable mode of the thin isothermal ring.

When the epicyclic frequency κ cannot be regarded as small we show in Section 5.1 that if $q \neq 2$ and the torus has small cross-section ($a \ll \varpi$) then modes with low m exist which are almost independent of z. We show that such modes are unstable if $q > \sqrt{3}$ and stable if $q < \sqrt{3}$. When unstable, these modes do not have a sonic character and are driven by a mechanism analagous to the one which drives the classical Kelvin–Helmholtz instability. For such tori we have shown that modes with low m, but with odd symmetry in z, are stable. This analysis breaks down for large m (roughly when $m \gtrsim \varpi_0/a$) when the modes again take on a sonic character. In Section 6 we have investigated mainly sonic modes with high m. To obtain a tractable problem we have taken the limit of v'_ϖ and v'_ϕ independent of z which is valid for the two-dimensional flow of Section 3.1 or for a highly flattened torus with $q \simeq 3/2$ (i.e. near Keplerian rotation). We show that the sonic modes can be unstable with the instability being driven by the Kelvin–Helmholtz mechanism in addition to the mechanism discussed in Paper I. This is an example of the mixture of sonic and Kelvin–Helmholtz mode which can be expected for the general rotation law.

8 Conclusion

The instabilities, shown to exist in Paper I in tori with $h' = 0$, persist in tori with non-constant specific angular momentum. Furthermore there is the complication that additional unrelated Kelvin–Helmoltz-like instabilities occur in tori for which $h' \neq 0$. The general unstable mode is a mixture of these two types. These results imply that models of quasars which invoke accretion tori, and models which imagine that centrifugal force can vacate a funnel up the rotation axis along which jets might originate (e.g. Lynden-Bell 1978; Rees *et al.* 1982) are not viable.

Because of the efficiency with which these instabilities can transfer angular momentum, they will be of fundamental importance in determining permissible rotation laws of differentially rotating flows in the astrophysical context. They imply that thick accretion discs must evolve dynamically rather than doing so on the slow 'viscous' time-scale envisaged for their thin counterparts. In the stellar context it will be important to determine the effect of a radial (possibly stabilizing) entropy gradient on the instabilities. For the thick disc-like flows seen in star-forming regions the (possibly destabilizing) effects of self-gravity will need to be taken into account.

Even the relatively simple flows considered in this paper are complicated to analyse, and the analytic approach adopted here has limited use to the treatment of particular cases. In a subsequent paper we shall adopt a numerical approach which will enable us to extend our investigation.

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