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CRIMINOLOGY

THE DYNAMICS OF A HOMEOSTATIC PUNISHMENT PROCESS*

ALFRED BLUMSTEIN,** JACQUELINE COHEN,† AND DANIEL NAGIN‡

I. INTRODUCTION

In his now classic analysis of crime, Durkheim argues that some level of crime is "an integral part of all healthy societies . . . provided that it attains and does not exceed a certain level for each social type."¹ He contends that crime is an unavoidable consequence of the very processes which contribute to the maintenance of social cohesion. As the set of standards and beliefs which define and bound a society are specified, some types of behavior will be prohibited and those engaging in these behaviors will be considered criminals. Furthermore, the public condemnation and punishment that follows a criminal act serves to articulate and reinforce the common set of norms and sentiments which ultimately guides the actions of the members of the society, thereby further enhancing social cohesion. Thus, while crime is a natural outgrowth of the processes generating social solidarity, it is the social response to crime that particularly serves to consolidate and reinforce that solidarity.

Blumstein and Cohen² have re-examined Durkheim's theory of a stable level of crime and pose an alternative position emphasizing the stability of punishment. Their argument is that the standards or thresholds that define punishable behavior are adjusted in response to overall shifts in the behavior of the members of a society so that a roughly constant proportion of the population is always undergoing punishment. Thus, if many more individuals engage in behavior defined as punishable, the demarcation between criminal

and non-criminal behavior would be adjusted to re-designate at least part of the previously criminal behavior as non-criminal, or the intensity or duration of punishment for those convicted would be reduced. A similar but opposite reassessment would occur when fewer people commit currently punishable acts. Their principal evidence in support of this hypothesis is the stability of imprisonment rates in the United States over the period 1930-1970 and in Norway over the period 1880-1964 (Figures 1a and 1b). Canadian imprisonment rates over the period 1880-1959 have been obtained subsequently, and these (Figure 1c) show the same stability behavior.

In this paper, the theoretical structure and the empirical basis of this earlier work is extended, and some processes that might generate the stable level of punishment are hypothesized. First, the time series of the imprisonment data for the United States, Norway and Canada are analyzed to provide an empirical description of the structure of the data. These results indicate a striking similarity in the data structures in the three countries studied. Different models of the crime and imprisonment process are then explored in an effort to characterize an underlying process that would generate the kinds of time series observed. A sensitivity analysis is then performed to identify how the different parameters of one such model contribute to national differences in observed levels of punishment.

II. THE BASIC HOMEOSTATIC HYPOTHESIS

First it is necessary to review the stability of punishment theory. Blumstein and Cohen³ posit a statistical density function $f_B(x)$, representing the distribution of behavior in a society. The basic concept of such a distribution is that there exists a range of behavior which may be viewed at one extreme as being compulsively moralistic and at the other as being severely criminally deviant with all shades in between (see Figure 2). It is then hypothesized that society establishes a boundary, B_0 , defining the limits of legitimate

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¹ E. DURKHEIM, *THE RULES OF THE SOCIOLOGICAL METHOD* 66-67 (S. Solovay & J. Mueller trans. 1964).

² Blumstein & Cohen, *A Theory of the Stability of Punishment*, 64 J. CRIM. L. & C. 198 (1973).

³ Blumstein & Cohen, *supra* note 2.

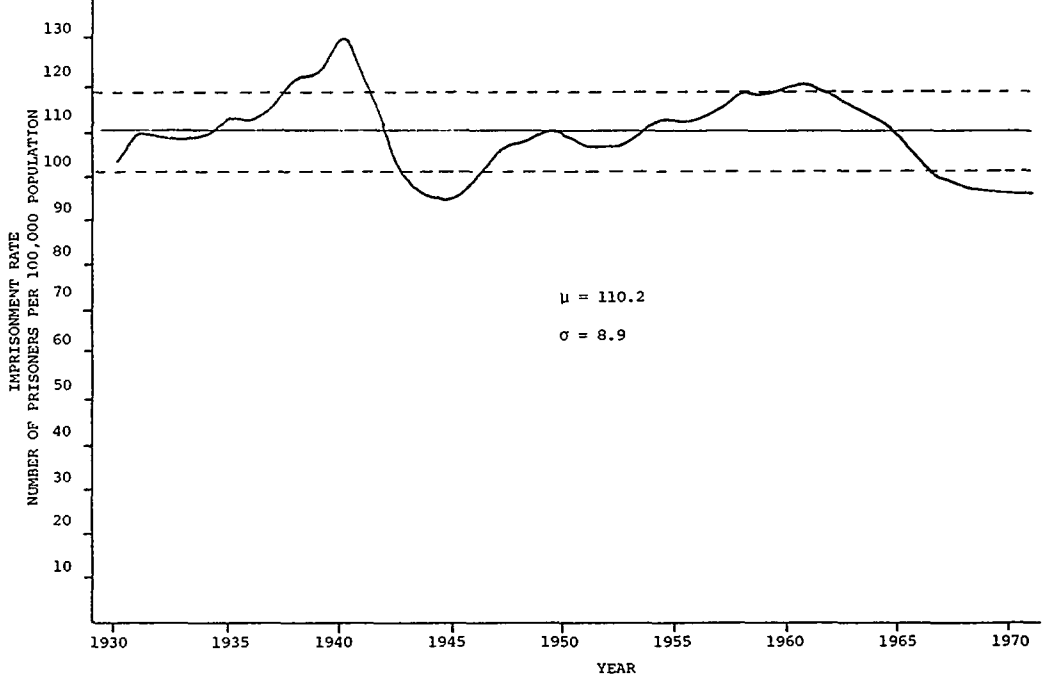


FIGURE 1a
Annual imprisonment rate in the United States: 1930-70⁴

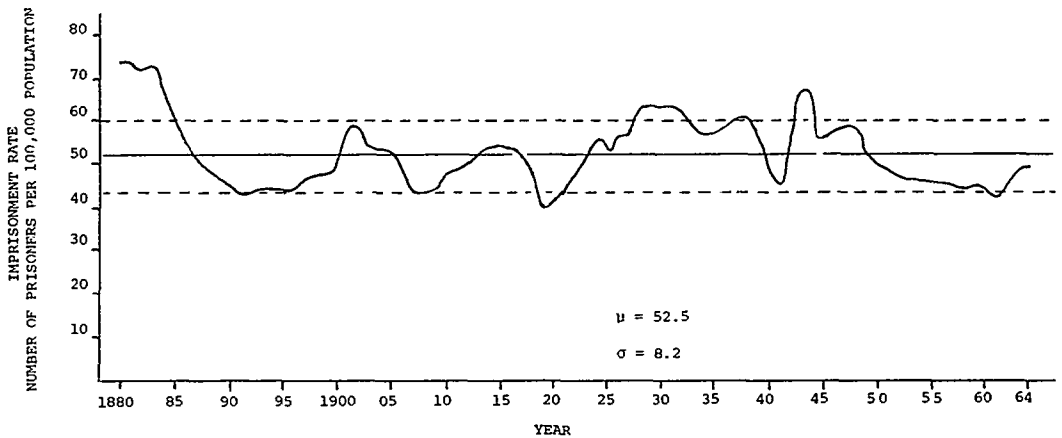


FIGURE 1b
Annual imprisonment rate in Norway: 1880-1964⁵

behavior. Individuals who engage in behavior $B > B_0$ are deemed punishable.

⁴U. S. BUREAU OF THE CENSUS, DEP'T OF COMMERCE, HISTORICAL STATISTICS OF THE UNITED STATES, COLONIAL TIMES TO 1962 AND REVISIONS 33 (1965); U. S. BUREAU OF THE CENSUS, DEP'T OF COMMERCE, STATISTICAL ABSTRACT OF THE UNITED STATES 1972, at 140-64 (93d ed. 1972); U. S. BUREAU OF THE CENSUS, DEP'T OF COMMERCE, STATISTICAL

A punishment probability function, $g(B)$, is introduced which reflects the probability that a person engaging in behavior beyond B_0 will be

ABSTRACT OF THE UNITED STATES 1970, at 138-61 (91st ed. 1970).

⁵ Letter from Nils Christie, Institute of Criminology and Criminal Law, University of Oslo, to A. Blumstein, 1970.

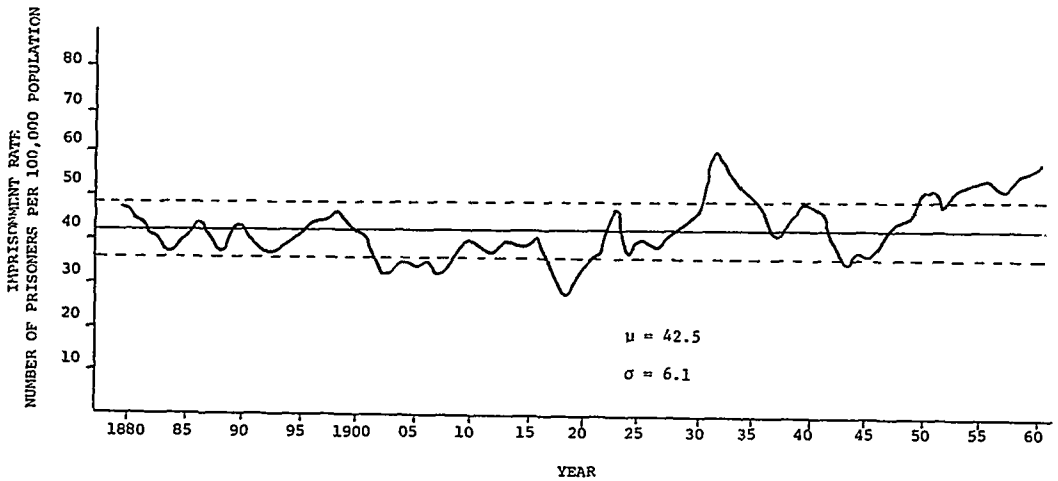


FIGURE 1c
Annual imprisonment rate in Canada: 1880-1959⁶

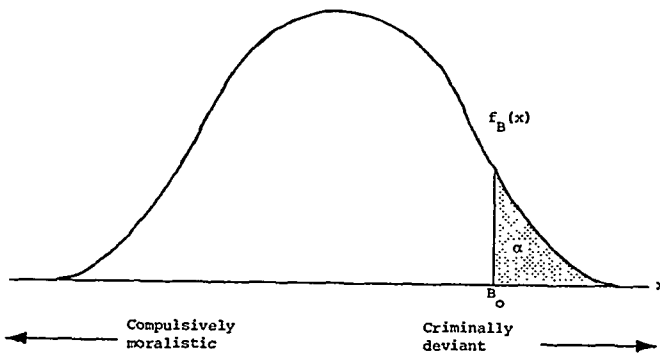


FIGURE 2
The behavior distribution

punished, and a punishment intensity function, $I(B)$, reflects the intensity of punishment applied to a punished individual at B . Thus, α , the aggregate amount of punishment delivered by society, is a function of the frequency of deviant behavior in that society and the expected punishment associated with deviant behavior.⁷

It is then hypothesized that α will be relatively stable over time in a given society, even though it may deviate somewhat for severely disruptive periods like wars or depressions. One means of maintaining the stable value of α in the face of changing behavior in the society is through redefi-

nition of the boundary, B_0 , between the criminal and the non-criminal. Under this homeostatic hypothesis, if behavior were to become less criminally deviant, that is, if $f_B(x)$ were to shift to the left, B_0 would be adjusted to $B_0' < B_0$, so that $\alpha(B_0) = \alpha'(B_0') = \alpha$.⁸

It is argued that the social forces accounting for stability include more than simple prison-cell capacity, or even the limited willingness of society to accept the economic burden of processing individuals through the criminal justice system, con-

⁸ In terms of the integral formulation, the hypothesis can be represented by:

$$\begin{aligned} \alpha'(B_0') &= \int_{B_0'}^{\infty} f'_B(x)g'(x)I'(x) dx \\ &= \int_{B_0}^{\infty} f_B(x)g(x)I(x) dx = \alpha \end{aligned}$$

⁶ HISTORICAL STATISTICS OF CANADA 634-59 (M. URQUHART & K. BUCKLY ed. 1965).

⁷ More precisely

$$\alpha = \int_{B_0}^{\infty} f_B(x)g(x)I(x) dx.$$

fining them and foregoing their productivity. Such an explanation does not account for the tendency of downward movements in imprisonment rates to reverse themselves and return to the mean. More fundamental considerations of social structure are probably at work. If too large a portion of the society is declared deviant, then the fundamental stability of the society may well be disrupted. Likewise, if too few are punished, the basic identifying values of the society will not be adequately articulated and reinforced, again leading to social instability. In the former case there will be pressures toward decriminalizing some behavior, while in the latter, there will be pressures for stricter law enforcement and perhaps more severe punishments.

III. TIME-SERIES ANALYSIS

Time-series analysis is often directed at a sequence of observations, such as those of Figure 1, in order to discover structures in the data, particularly relationships between an observation in period t and those in prior periods. In time-series analyses, two basic types of structures typically are explored: autoregression and moving averages. These can be studied either separately or in combination and, in many instances, can explain the systematic behavior of the time series.

In the autoregressive structure an observation at t is a weighted linear function of the observations from the preceding T periods, and the autoregression is said to be of order T . In the moving average process an observation at t is the result of stochastic variations about the mean.⁹ The stochastic variations in observations in successive time periods are related by an autoregressive type structure. Thus, the relationship between an observation at time t and prior observations occurs either through the serial correlation of stochastic deviations from the mean (moving average), or through serial correlation of the observations themselves (autoregression). While the difference between these two processes in terms of the behavior of the induced time series may not be obvious, their properties are very different. These differences permit the wide variety of time series which are encountered in practice to be estimated by making judicious use of autoregressive, moving average or mixed (autoregressive and moving average) processes.

⁹ See Appendix I for a more detailed description of autoregressive and moving average structures.

In order to gain further insight into the dynamics of the imprisonment process, time-series analysis was performed on the annual imprisonment rate data for the United States, Norway and Canada. Briefly, the analysis involves the following steps:

- 1) Using ordinary least squares, an autoregressive function of arbitrarily high order, say T , is estimated. If the autoregressive coefficient of the T^{th} subscript is statistically insignificant, an autoregressive relationship of order $T-1$ is estimated. This process is continued until a statistically significant autoregressive coefficient is found.
- 2) To determine if there is serial correlation of the stochastic component, ϵ_t (that is, a moving-average process), autoregressions again of arbitrarily high order t are run on the *deviations* of the actual data from those predicted by the estimated autoregression. If no significant autoregression coefficients are then found, there is strong evidence of no serial correlation in the stochastic component.

In the time-series analysis for each country, autoregression functions of order 4 ($T = 4$) were estimated and no significant coefficient ϕ_T was found until the second-order autoregression was estimated. When the stochastic components were checked for serial correlations, no significant autoregression relationships were found among the deviations. Figure 3 is a plot of the actual Canadian data against the values predicted by the estimated second-order autoregression for Canada. A visual inspection reveals both the high explanatory power of the regression and the seemingly random nature of the deviations.

Thus, one can reasonably conclude that the time series of the imprisonment rates for the United States, Norway and Canada each followed a second-order autoregressive process with no moving average component. If r_t is the imprisonment rate (prisoners/100,000 general population) in year t , we can adequately express r_t as a simple linear function of the imprisonment rates in the two immediately previous periods:

$$r_t = \delta + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \epsilon_t$$

where:

r_t = the daily average imprisonment rate in year t ,

δ, ϕ_1 = fixed parameters of the process, and

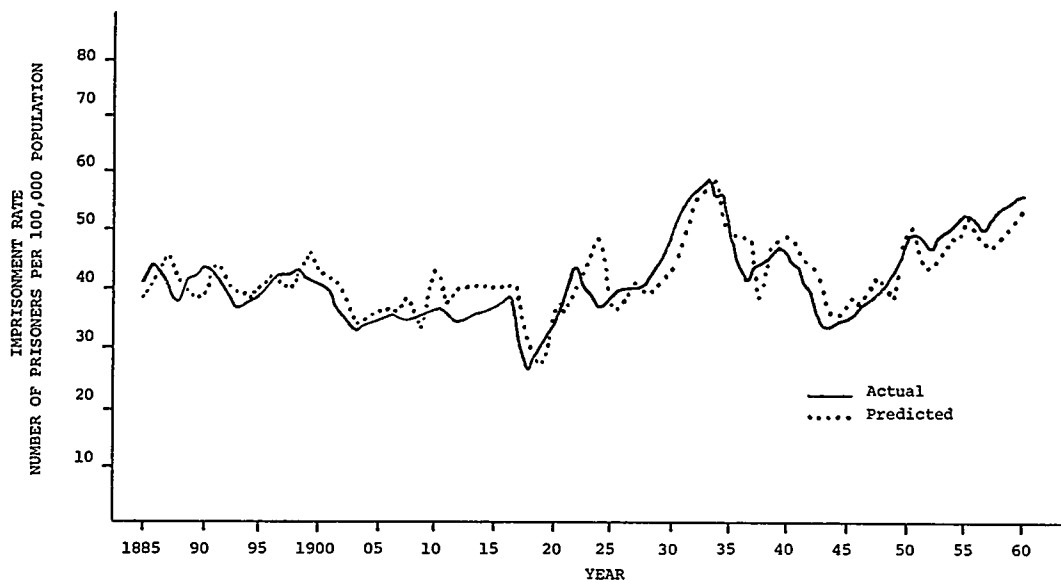


FIGURE 3
Actual vs. predicted imprisonment rate in Canada: 1885-1959

TABLE 1
ESTIMATED AUTOREGRESSION PARAMETERS FOR THE
ANNUAL IMPRISONMENT RATE¹⁰ (r_t) IN THE
UNITED STATES, NORWAY AND CANADA

$$r_t = \delta + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \epsilon_t$$

Parameter	USA	Norway	Canada
ϕ_1	1.42 (10.35)	1.17 (10.47)	1.25 (11.58)
ϕ_2	-.63 (-4.41)	-.35 (-3.13)	-.42 (-3.83)
δ	22.74 (2.76)	9.34 (3.15)	7.42 (3.04)
r^2	.84	.78	.79

ϵ_t = independent and identically distributed random variables with mean zero and variance σ^2 .

Table 1 presents the estimated autoregression parameters for each country. Given the wide range of possible structures for these data, the finding that the imprisonment rates in the three different countries follows a second-order autoregression strongly suggests that a similar mechanism

¹⁰The imprisonment rate is the average daily prison population per 100,000 general population. In the United States and Norway the rate base is 100,000 total population, while in Canada it is 100,000 population 16 years of age or older.

ism may be generating each, albeit with different driving parameters. It would be desirable to be able to identify a mechanism consistent with these empirical findings.

Processes following a second-order linear differential equation, not necessarily with constant coefficients,¹¹ generate second-order autoregressive functions. This connection is shown in Appendix II. Table 2 presents the parameters of the associated differential equation for each country as well as the characteristic time period (Π) of the cycles for each equation.¹²

Thus, a second-order differential equation is the mathematical characterization of a dynamic process that would generate the time series that were observed. Such an equation, however, is only an abstract representation that could describe any number of physical or social processes. One can posit a flow process in and out of prison that would

¹¹The general second-order differential equation with constant coefficients is: $r_t + cr_t + dr_t = F$, where r_t is the average daily imprisonment rate at t , and \dot{r}_t and \ddot{r}_t are respectively the first and second derivatives of r_t .

¹²A differential equation of the specified form results in cyclical behavior when $c^2 - 4d < 0$, and the period Π is obtained from:

$$\Pi = \frac{4\pi}{\sqrt{4d - c^2}}$$

TABLE 2

PARAMETERS FOR THE SECOND-ORDER DIFFERENTIAL EQUATION WHICH GENERATES THE ESTIMATED AUTOREGRESSIVE PROCESS FOR THE IMPRISONMENT RATE TIME-SERIES

$$\ddot{r}_t + c\dot{r}_t + dr_t = F$$

$$c = \frac{-\phi_1 - 2\phi_2}{\phi_2}$$

$$d = \frac{\phi_1 + \phi_2 - 1}{\phi_2}$$

$$F = \frac{-\delta}{\phi_2}$$

$$\Pi = \text{periodicity} = \sqrt{\frac{4\pi}{4d - c^2}}$$

Parameter	USA	Norway	Canada
c	.25	1.34	.98
d	.33	.51	.40
F	36.10	26.69	17.62
Π	11.2 yrs.	25.4 yrs.	15.7 yrs.

generate the differential equation consistent with the observed behavior of the time-series. With such a model the stability of imprisonment rates can be interpreted in terms of conceptually meaningful characteristics of a society; for example, the degree of punitiveness and the level of conformity. The first formulation is quite simple and requires only that the prison population remain stable through a simple balancing of receptions and releases. This formulation will be shown to be inconsistent with the observed behavior of the Canadian data. A second, more elaborate model which incorporates the homeostatic principles will be shown to be much more satisfactory and consistent with the Canadian data.

IV. EXPLORATION OF POSSIBLE EXPLANATORY MODELS

In this section, models of the social mechanism generating imprisonment rates are developed and their consistency with the observed stability and second-order autoregressive movement of the time series are explored. The models are developed by partitioning the total population of a society into three groups, one of which is the prison population. The flow rates of individuals among these groups is then examined. These simultaneous flows generate a system of simultaneous first-order differential equations. Such systems can be solved

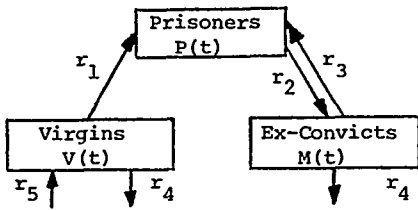
so that each population is defined solely as a function of its own derivatives (see Appendix IV). The result for any population group is in general a second-order differential equation, although in some systems, the second-order term vanishes, leaving only a first-order equation. We can judge the adequacy of each hypothesized structure by comparing the parameters of the differential equation for the imprisonment rate generated by the model with the same parameters derived from the autoregressive parameters estimated from the observed time series.

A. Prisoner, Ex-Convict, and Virgin Model

The first model to be examined partitions the total population T(t), into a prison population P(t), an ex-convict population M(t), and a population of individuals who have never been to prison (virgins) V(t). The possible flows in this structure are shown in Figure 4. Within this structure, the only mechanism for maintaining a stable imprisonment rate would be the balancing of releases from P(t) with receptions from V(t) and M(t).

When formalized, the relationship among the model flows can be used to derive a second-order differential equation for the imprisonment rate. The parameters (c, d and F) of this equation are functions of the various flow rates identified in Figure 4; their specific mathematical form is derived in Appendix III with their final form shown by equation (12) of that appendix. To assess the adequacy of this model, estimates of c and d generated by the model are compared with the estimates from the observed Canadian time series reported in Table 2. This comparison requires empirical estimates of the model's flow rates. The imprisonment rate of virgins, r₁, is exceedingly small. In Canada, for example, even if we were to assume that all receptions into prison in a year are of first-time offenders, r₁ would be no larger than .0004 and (r₂ + r₃) no larger than .7%. For the period 1880 to 1960 the exponential growth rate of the Canadian population was about 0.019 and r₁, the death rate, about .017. Therefore, using equation (12) in Appendix III, d is about 0.027, while c is about .79. In this model, c must be more than twenty-five times larger than d.

The values of c and d estimated from the Canadian autoregression parameters (Table 2) are .98 and .40, respectively. Thus, for Canada, Model I yields only a fair estimate of c and dramatically



- r_1 = imprisonment rate of virgins
- r_2 = release rate from prison
- r_3 = imprisonment rate of ex-convicts
- r_4 = death rate¹³
- r_5 = birth rate

FIGURE 4
Model I

underestimates d .¹⁴ The large underestimate of d will result in the model predicting nonoscillatory behavior in $r(t)$.¹⁵ This is, however, completely contrary to the strong cyclical behavior actually observed. It then appears that Model I, which considers only a steady-state balance of receptions and releases does not adequately explain the observed dynamics of the imprisonment rate. A more elaborate flow structure is required.

B. Prisoner, Criminal, Law-Abider Model

We now propose an alternative partitioning of the population into three subsets (Figure 5), now identified as "law-abiders," "criminals" and "prisoners," with the numbers in each group varying over time. In the context of the behavior distribution of Figure 2, the number of law-abiders at time t , $L(t)$, are those individuals whose behavior $B(t) < B_0(t)$. Likewise, the criminal population, $C(t)$, are those individuals with behavior $B(t) > B_0(t)$. The prison population, $P(t)$, are those individuals drawn from the criminal population who are confined in institutions at t .

¹³ For simplicity, differences between the death rate of ex-convicts and virgins, as well as the small death rate within prison, have been ignored.

¹⁴ When the predicted values of c and d are transformed into autoregressive form (equation (5) in Appendix II), the respective values of ϕ_1 and ϕ_2 are 1.54 and $-.55$. The predicted value of ϕ_1 , 1.54, is outside a 95% confidence interval of the value of 1.25 estimated from the actual data.

¹⁵ A necessary condition for oscillatory behavior is that $(c^2 - 4d) < 0$.

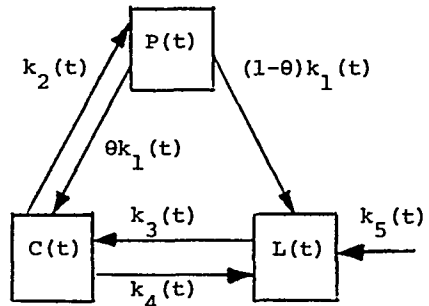


FIGURE 5
Model II—Stable imprisonment as a homeostatic process

The composition of populations changes continuously, as shown in the flow diagram of Figure 5. Some criminals are arrested, convicted and sent to prison at rate $k_2(t)$. Prisoners are regularly released from prison, with some returning to the criminal group [$\theta k_1(t)$] and others becoming law-abiders [$(1 - \theta)k_1(t)$]. There is also an important two-way flow between the criminal and law-abiding populations ($k_3(t)$ and $k_4(t)$). As $f_B(x)$, the behavior distribution in Figure 2, shifts to the right, for example, $C(t)$ increases and $L(t)$ decreases correspondingly. Similarly, a shift to the left, that is, to a population that is more law-abiding, results in a net flow from $C(t)$ to $L(t)$. These changes in the population composition would be reflected in changes in the normal flow rates, $k_i(t)$, among the population groups.

The possibility of flows between the criminal and law-abiding population is an important element of the model because these flows permit the incorporation of a central theme of the homeostatic notion, namely the redefinition of criminal behavior. Suppose, for example, that at time t_0 the system were in equilibrium and $P(t_0)/T(t_0)$ was the average long-term imprisonment rate. Now, suppose that at t_1 the behavior distribution $f_B(x)$ were to shift to the right, that is, the population were to become more criminal by current standards. This shift would be reflected in an increase in $k_3(t)$ to $k_3(t_1) > k_3(t_0)$. The increase in $k_3(t)$ would result in a net increase in the flow from $L(t)$ to $C(t)$. That increase would perturb the system from equilibrium and, holding all other $k_i(t)$ constant, would increase $P(t)/T(t)$ and $C(t)/T(t)$.

An increase in $P(t)/T(t)$, according to the homeostatic model, would set in motion the decriminalization of certain behavior by shifting the

demarcation between criminal and non-criminal behavior, B_0 . This shift would be reflected by readjustments in $k_3(t)$ and $k_4(t)$ such that $C(t)/T(t)$ and $L(t)/T(t)$ would return toward the equilibrium values.

Even when $f_B(x)$ and B_0 are stable, there is a regular flow between $C(t)$ and $L(t)$. A previously law-abiding college student begins dealing in drugs or a businessman finds that profits are substantially improved by criminal collusion with competitors. An occasional burglar gets married or gets a better job, and decides to cease his criminal activity. Thus, each population is continuously feeding the others.

One can formalize the description of these flows and again derive a second-order differential equation for the imprisonment rate. This is done in Appendix V under the preliminary assumption that the $k_i(t)$ are approximately constant over time.¹⁶ The parameters of this differential equation (c , d , and F') are functions of the flow rates for Figure 5. The adequacy of Model II is tested by determining the consistency of the model-generated equation with the observed dynamic behavior of an actual time series for imprisonment rates.

Toward this end, the model will be analyzed using rates associated with Canadian penitentiaries. The data were plotted in Figure 1. Visual inspection of the series indicates no obvious trend from 1880-1959. However, there does appear to be a marked change in the dynamic behavior after 1925. To reduce the time variation in the k 's (and, therefore, in c , d and F'), we restrict ourselves to the post-1925 series for the analysis.

To test the sufficiency of the derived differential equation (Appendix V), estimates of the k_i 's must be made to generate the theoretical values for c , d , and F' . This differential equation can be used to derive a theoretical autoregressive relationship by the approximation shown in Appendix II. An

¹⁶ This assumption of constant $k_i(t)$ disregards a central element of the stability of punishment theory, namely the changes in $k_3(t)$ and $k_4(t)$ that accompany the adjustment of the standards defining punishable behavior in response to shifts in objective behavior. The static nature of this representation results in serious limitation in the development and empirical analysis which follow. It does not, however, render it vacuous. If the model, even under the restriction of constant $k_i(t)$, can generate coefficients which are plausibly close to the actual values, then a rationale for exploring more complicated forms where the $k_i(t)$ vary will be established.

TABLE 3
THE RELEASE RATE ($k_1(t)$), AVERAGE DAILY PRISON POPULATION ($P(t)$) AND TOTAL POPULATION ($T(t)$) FOR CANADA: 1925-1960¹⁷

Year	$k_1(t)$	$P(t)$	$T(t)$ ¹⁸
1925	.37	2266	5,100,000
1930	.43	2868	6,700,000
1935	.55	3895	7,350,000
1940	.50	3736	7,850,000
1945	.46	3063	8,500,000
1950	.45	4380	9,400,000
1955	.52	5204	10,400,000
1960	.73	6141	11,500,000

empirical autoregression can then be run on the actual data to determine whether the parameters estimated from the data are comparable to those generated by the theoretical model.

The known values of the system characterized by Figure 5 are k_1 (the release rate), $T(t)$ and $P(t)$. Their values at five-year intervals from 1925 to 1960 are given in Table 3. The year 1940 was chosen to generate estimates for the model parameters. That year is about mid-way through the series, and its release rate k_1 and imprisonment rate/100,000 ($P/T \times 10^{-5}$) are the same as the means for the series.

The unknown values are: k_2 (the imprisonment rate of criminals); k_3 (the rate at which law-abiders become criminals); k_4 (the rate at which criminals become law-abiders); $(1 - \theta)$ (rehabilitation rate); and C (the size of the criminal population). Estimates for k_2 , k_3 , k_4 are made for equilibrium estimates of C/T of 1.5%, 1.0%, and 0.5%. Since individuals do not continuously behave in a criminal manner, a reasonable convention must be established to operationalize the idea of an individual belonging to the criminal population. A reasonable definition might categorize a person as a criminal in year t if he has committed an act for which he would have been imprisoned if caught and convicted.¹⁹ Then k_2 , the rate of

¹⁷ Prisoner statistics were obtained from unpublished statistics provided by the Office of Statistics, Secretariat of the Ministry of the Solicitor General, Government of Canada.

¹⁸ The total population includes only persons 16 years of age or older.

¹⁹ Note that this definition restricts the minimum time spent in the criminal population to 1 year.

imprisonment of the criminal population, is the ratio of prison receptions, a known value, to the estimate of the size of the criminal population.

The analysis is relatively insensitive to the value of θ , the portion of released prisoners returning directly to the criminal population. A plausible estimate is 0.33. Given our definition of membership in the criminal population, θ includes all those released prisoners who commit at least one crime within a year of their release. In a study of parole success, Gottfredson²⁰ reported that during a two year follow-up period, 38% of released prisoners returned to prison. In another study cited by Robison and Smith,²¹ 51% of released prisoners returned to prison during the three years immediately following their release. Since recidivism rates decline with each additional year following release and not all releasees who return to crime are apprehended, it is not unreasonable to assume that 33% of released prisoners return immediately to the criminal population.

The value of k_4 is calculated somewhat differently. If τ is the average time spent in C, then k_4 , the rate at which criminals leave C, is the reciprocal of τ . τ is assigned a value of 2 years for $C/T = 1.5\%$. For the other values of C/T , 1.0% and 0.5%, τ is taken to be successively larger. A smaller C is assumed to be associated with a larger τ to reflect a more "hard core" criminal population in C. Thus, for $C/T = 1.0\%$, we let $\tau = 3$ years and for $C/T = 0.5\%$, we let $\tau = 4$ years.

The remaining parameter to be estimated is k_3 . This parameter may be specified as the value which will maintain $C(t)$ at a constant level given the values of k_1 , k_2 , and k_4 . This is equivalent to assuming that the first derivative of $C(t)$ is zero.²²

The values of the k 's and the resulting differential equation and autoregression coefficients are given in Table 4 for the three assumed values of C/T . For comparison, the empirical second-order autoregression function estimated from the annual

Canadian imprisonment rate from 1925-1960 is as follows:

$$r_t = 1.23r_{t-1} - .43r_{t-2} + 9.17 \quad (1)$$

(8.26) (-2.89) (2.25)

where the values in parentheses are the t -values associated with each of the coefficients. A comparison of the parameter estimates of equation (1) with the corresponding autoregression parameters theoretically derived from the k_i in Table 4 show them to be roughly equivalent.²³ The coefficient of r_{t-1} , ϕ_1 , is overestimated by about 5% to 15%, whereas ϕ_2 is underestimated by about the same amount in each case. The relative direction of these differences is consistent with the high negative correlation ($-.82$) between the coefficients of r_{t-1} and r_{t-2} in the autoregression.

The value of the constant term is underestimated by as much as 60% in the theoretical estimates, $\delta' \times 10^5$. However, all of the estimates of $\delta' \times 10^5$ are within a 90% confidence interval of the regression value (2.57, 15.77).

Overall, despite the speculative, albeit plausible, nature of some of the parameter estimates, the model appears to do remarkably well in generating parameters consistent with those estimated from the actual data. The encouraging nature of these results indicates the potential merit of this approach to modeling the imprisonment process and justifies further work in this direction, especially efforts to examine the process without the restrictive assumption of constant flow rates. Furthermore, while acknowledging the tentative nature of Model II, one can cautiously begin to interpret the flow rates in the model in an effort to characterize those features of a society which contribute to its particular imprisonment rate.

V. IMPLICATIONS OF THE MODEL—A PARAMETRIC ANALYSIS

As a corollary to the hypothesis of the stability of crime, Durkheim also conjectured that the particular level of crime would vary among different "social types" and that it might be possible to

²⁰ D. GOTTFREDSON, *THE ROLE OF BASE EXPECTATIONS IN THE STUDY OF TREATMENTS* (1959).

²¹ Robison & Smith, *The Effectiveness of Correctional Programs*, 17 *CRIME AND DELINQUENCY* 67 (1971).

²² From the second equation in system (22) of Appendix V we have:

$$k_3 = \frac{-\theta k_1 P(t) + (k_2 + k_4)C(t)}{T(t) - P(t) - C(t)}$$

²³ The empirically estimated parameters in equation (1) are based on the imprisonment rate per 100,000 population, while the parameters in Table 4 are based on this rate per unit of population. Although the rates differ by a factor of 10^5 , the coefficients ϕ_i are unaffected and may be directly compared. However, the constant term δ in Table 4 must be multiplied by 10^5 when it is compared to the constant in equation (1).

TABLE 4
ESTIMATES OF FLOW PARAMETERS (k_i) FOR MODEL II AND THE ASSOCIATED COEFFICIENTS FOR THE
AUTOREGRESSION AND DIFFERENTIAL EQUATIONS GENERATED BY MODEL II USING ANNUAL
CANADIAN IMPRISONMENT RATES FROM 1925-1960

$$\ddot{f}(t) + c\dot{f}(t) + dr(t) = F'$$

$$r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \delta'$$

$C/T = 0.5\%$ $\tau = 4$	$C/T = 1.0\%$ $\tau = 3$	$C/T = 1.5\%$ $\tau = 2$
Flow Parameters:		
$k_1 = .50$	$k_1 = .50$	$k_1 = .50$
$k_2 = .046$	$k_2 = .023$	$k_2 = .015$
$k_3 = .0014$	$k_3 = .0035$	$k_3 = .0078$
$k_4 = .25$	$k_4 = .33$	$k_4 = .50$
$(1 - \theta) = .67$	$(1 - \theta) = .67$	$(1 - \theta) = .67$
Differential Equation Coefficients. ²⁴		
$c = .84$	$c = .89$	$c = 1.06$
$d = .16$	$d = .19$	$d = .28$
$F' = 6.4 \times 10^{-5}$	$F' = 8.1 \times 10^{-5}$	$F' = 11.7 \times 10^{-5}$
Autoregression Coefficients. ²⁵		
$\phi_1 = 1.42$	$\phi_1 = 1.39$	$\phi_1 = 1.31$
$\phi_2 = -.50$	$\phi_2 = -.48$	$\phi_2 = -.43$
$\delta' = 3.2 \times 10^{-5}$	$\delta' = 3.9 \times 10^{-5}$	$\delta' = 5.0 \times 10^{-5}$

C/T = average criminal population/total population.
 τ = mean stay in criminal population.

specify the level appropriate to each "social type."²⁶ Two of the authors have argued elsewhere²⁷ that Durkheim was not speaking of the level of actual criminal behavior that occurs,²⁸ but rather the level of punished criminal acts. Hence, it is the level of punishment meted out which remains stable, but varies in magnitude among different classes of societies.

A brief inspection of Figure 1 provides visual evidence for this corollary. While there is a stable process in each country with the annual imprisonment rate fluctuating around the mean, there are substantial differences among those means. The mean imprisonment rate for the United States is

2-3 times greater than the rate in either Norway or Canada.²⁹ In an effort to account for these differences, Model II will be interpreted in terms of some general societal characteristics. The ways in which these characteristics generate different imprisonment rates can then be examined within the framework identified by the model.

Two characteristics of societies important to the phenomena of crime and punishment are the

²⁴ These coefficients are estimated for (i) above using (20).

²⁵ These coefficients are estimated for (ii) above using (6) and the results for differential equation (i).

²⁶ E. DURKHEIM, *supra* note 1, at 66-67. A "social type" is simply a collection of similar societies. More formally, "social types" may be thought of as equivalent classes within the set of societies.

²⁷ Blumstein & Cohen, *supra* note 2, at 199.

²⁸ This would include any act that is a violation of some criminal statute.

²⁹ The definition of and institutional arrangements for prison populations vary considerably from country to country. The Canadian and U. S. data include only individuals in prisons and penitentiaries which are largely restricted to persons serving sentences of one year or more. In Norway, on the other hand, the typical sentence for the prison population rarely exceeds two months. Nevertheless, despite these differences, the selected prison statistics refer to the most severe penalty imposed in each country, aside from capital punishment. Our intention is to gain insight into the reasons for differences in the level of only the most severe form of punishment. From this perspective, then, the differences in definition allow cautious comparison of the rates while always keeping in mind the potential incompatibilities.

level of conformity within a society and the degree of punitiveness. The parameters k_1 and k_2 in Figure 5 reflect two aspects of the degree of punitiveness that are often cited, the severity and certainty of punishment. When other forms of punishment are ignored and only imprisonment is considered, the severity of punishment varies with the time actually served in prison. Since increases in the time served result in decreases in the release rate from prison, k_1 , the release rate, may be regarded as an inverse measure of the severity of punishment. The lower the value of k_1 , the more severe the punishment meted out. Alternatively, the flow rate of criminals to prison, k_2 , reflects the certainty of punishment for criminal behavior. The higher the value of k_2 , the more criminals are imprisoned.

Parameters k_3 and k_4 in Figure 5 are the flows between the law-abiding and criminal populations and together they reflect the overall level of conformity in a society. The magnitude of the flow from law-abiders to criminals, k_3 , provides some indication of the strength of the commitment to conformity within a society; the stronger the commitment, the smaller the outflow of law-abiders. The level of commitment to conformity in any society is probably a complex product of a number of different contributing factors, among them the successful internalization of the normative code, the deterrent effects associated with penalties and the heterogeneity of the society.

These factors affect the commitment to conformity differently and operate on very different dimensions of an individual's motivation. The more deeply rooted the norms and values of a society in the individual consciences of its members, the stronger will be their commitment to conformity. In this case the members conform out of a sense of duty or obligation. Deterrence, on the other hand, captures the extent to which individuals respond to the costs associated with the penalty structure. Effective deterrence will increase the strength of commitment to conformity.

Alternatively, greater heterogeneity in a society, be it cultural, ethnic, racial or religious, can weaken the overall commitment to conformity through the existence of competing normative systems which may be at odds with the official institutionalized standards. As the members of a society respond to the behavioral codes of different sub-cultures, there will be a larger variance in actual behavior and more chances of deviance.

While Model II does not permit distinguishing the contributions of these different factors, the effect of the resulting commitment to conformity can be examined through parameter k_3 .

Parameter k_4 is the flow from the criminal population to the law-abiding population. It reflects the endurance of the criminal role, or the extent to which individuals remain active criminals after committing a single crime. Thus, k_4 may be thought of as an inverse measure of the prevalence of hardcore criminality in a society. As k_4 gets smaller, fewer criminals return to the law-abiding population and the more enduring the criminal role.

The endurance of the criminal role is undoubtedly the result of a complicated process involving both the availability of opportunities to return to the law-abiding population and the existence of disincentives to remain a criminal. The opportunities to return are a function of the permanence of the stigma attached to being labeled a criminal and of institutionalized barriers which explicitly exclude former criminals from various aspects of a law-abiding life, for example, laws which bar known criminals from certain types of employment. The disincentives to remaining a criminal vary with the effectiveness of deterrents. The only deterrent explicitly identified in Model II is imprisonment. Nevertheless, a host of other unspecified deterrents, such as arrest and conviction, may also operate on the criminal population and be reflected in variations in the value of k_4 . In general, increases in both legitimate opportunities and criminal disincentives will be associated with decreases in the endurance of the criminal role and increases in k_4 .

Having identified each parameter in terms of punitiveness and conformity, the differential impact of these characteristics on the imprisonment rate and the level of criminality in a society can be explored. The flow process in Figure 5 can easily be translated into a Markov process in which the populations are the states of the process and the flow rates become the transition probabilities of moving from one state to another. Assuming the $k_i(t)$ are constant over time, the transition matrix for Model II is:

$$M = \begin{matrix} & \begin{matrix} P(t+1) & C(t+1) & L(t+1) \end{matrix} \\ \begin{matrix} P(t) \\ C(t) \\ L(t) \end{matrix} & \begin{bmatrix} 1 - k_1 & \theta k_1 & (1 - \theta)k_1 \\ k_2 & 1 - k_2 - k_4 & k_4 \\ 0 & k_3 & 1 - k_3 \end{bmatrix} \end{matrix}$$

TABLE 5

THE EQUILIBRIUM DISTRIBUTION AMONG PRISONERS (P), CRIMINALS (C) AND LAW-ABIDERS (L), ASSOCIATED WITH DIFFERENT VALUES OF THE PARAMETERS OF MODEL II

Parameter Values					Rates/100,000 Total Population			
k_1	k_2	k_3	k_4	θ	(P) Prisoners	(C) Criminals	(L) Law-Abiders	
I.								
.200					175.8	1406.1	98416.3	
.250					140.6	1406.5	98452.4	
.333	.025	.005	.333	.333	105.5	1407.0	98486.8	
.500					70.4	1407.5	98521.8	
1.000					35.2	1408.1	98556.6	
II.								
	.500	0.05	.333	.333	29.0	1449.0	98521.7	
					.010	70.4	1407.5	98521.8
					.025	134.3	1343.6	98521.8
					.050	192.8	1285.2	98522.0
					.075	246.3	1231.6	98521.8
	.100							
III.								
	.500	.025	.333	.333	14.2	284.9	99700.7	
					.001	42.5	849.5	99107.9
					.003	70.4	1407.5	98521.8
					.005	111.7	2232.4	97657.5
					.008	138.7	2774.2	97087.1
	.010							
IV.								
	.500	.025	.005	.333	112.6	2253.1	97633.7	
					.200	91.9	1838.8	98068.9
					.250	70.4	1407.5	98521.8
					.333	47.9	958.0	98993.9
					.500	24.5	489.3	99486.2
	1.000							
V.								
	.500	.025	.005	.333	.100	1384.9	98545.7	
					.250	1399.4	98530.4	
					.333	1407.5	98521.8	
					.500	1424.3	98504.2	
					.750	1450.1	98477.1	

Since this matrix is regular,³⁰ the equilibrium probability distribution among the three states can be obtained by raising the matrix to successive powers, M^n . As n becomes large, each row of M will approach the same equilibrium vector and any row of the matrix gives the equilibrium distribution.

This feature of matrix M permits the use of simulation techniques to examine the equilibrium distribution for different assigned values of the k_i and θ in M . By systematically changing the value of one parameter at a time, one can investigate the effect of that parameter alone on the equi-

librium distribution. Each parameter is assigned five values, while holding all other parameters constant. The entries in Table 5 are the equilibrium rates/100,000 total population for each of the three sub-populations of interest.³¹

Section I of Table 5 indicates the effects of varying the severity of punishment, $1/k_1$. As k_1 increases, punishments become less severe and the average imprisonment rate decreases sharply. In fact, as the average time served drops from 5 years to 1 year, the imprisonment rate also decreases five-fold. However, the proportion of criminals among the total population is virtually unaffected by changes in k_1 . This is largely due to P 's com-

³⁰ A transition matrix is regular if there is at least one path, perhaps multi-step, from each state to every other state.

³¹ The rates in Table 5 were found by multiplying the equilibrium probability of each state by 10^5 .

paratively small size with respect to both C and L. In fact, for all values of k_1 in the table, P is never even 0.2% of the total population and it represents at most only 12.5% of the criminal population.³² Thus, changes in k_1 , which affect the flow out of P, will have very little effect on the size of C. Any variations in the deterrent effect associated with changes in the release rate, k_1 , will be manifested in changes in k_3 and k_4 , the flows between criminals and law-abiders. Since these flows are held constant as k_1 varies, this effect cannot be detected in this analysis.

The variations in k_2 (section II, Table 5) reflect changes in the certainty of punishment. As k_2 increases, a higher proportion of criminals are imprisoned and the imprisonment rate increases. There is also some change in the relative size of the criminal population which decreases by 15% from 1449 to 1232 criminals/100,000 population as k_2 increases from .01 to .10. To the extent that the level of crime is a function of the number of criminals, the response of the criminal population to changes in k_1 and k_2 is consistent with the currently popular notion that it is the certainty of punishment and not its severity which has the greatest deterrent effect on crime.³³

Parameter k_3 is assumed to vary with the strength of the commitment to conformity in a society. The larger k_3 , the weaker that commitment and the more frequently law-abiders commit crimes. As section III of Table 5 reveals, increases in k_3 are accompanied by similar increases in both the relative size of the criminal population and the imprisonment rate.

The magnitude of parameter k_4 reflects the

³² These are not unreasonable bounds on the relative size of P. In the United States in 1970, for example, there were slightly less than 200,000 state and federal prisoners, or about 0.1% of the total population. BUREAU OF PRISONS, U. S. DEP'T OF JUSTICE, BULL. NO. 47, NATIONAL PRISONER STATISTICS: PRISONERS IN STATE AND FEDERAL INSTITUTIONS FOR ADULT FELONS: 1968, 1969, 1970 (April 1972).

During the same year there were 1,272,783 reported arrests for Index Crimes. FEDERAL BUREAU OF INVESTIGATION, U. S. DEP'T OF JUSTICE, UNIFORM CRIME REPORTS: 1970 (1970). Since the arrests of all police agencies are not contained in the reported figures and not all criminals are arrested, 2,500,000 is not an unreasonable estimate of the size of the criminal population. In this case the prisoner population is only 8 per cent of the criminal population.

³³ Wilson, *Lock 'em Up and Other Thoughts on Crime*, N. Y. Times, Mar. 9, 1975, §6 (Magazine), at 11, col. 1; *The Purpose of Prison*, NEWSWEEK, Feb. 10, 1975, at 36, col. 3 (quoting James Q. Wilson).

prevalence of "occasional" criminals as opposed to hard-core "careerists" in the criminal population. As k_4 increases, more criminals return to the law-abiding population, indicating criminality of a more transitory nature. It is thus no surprise that as k_4 increases (section IV, Table 5), both the relative size of the criminal population and the imprisonment rate decrease. In fact, a five-fold increase in k_4 from 0.2 to 1.0 is accompanied by a five-fold decrease in the rates of criminals and prisoners in the population.

The last section of Table 5 presents the effects of changes in θ , the recidivism rate of released prisoners. It is clear that the populations are virtually insensitive to changes in recidivism. Sizable increases in θ have very little effect on the size of the criminal and prison populations. As with parameter k_1 , the lack of effect on the criminal population is due to the extremely small size of P, which in section V of the table is less than 0.1% of the total population and represents only 5% of the criminal population. The variations in the number flowing from this small P to C that result from changes in θ will hardly be noticed in C. Furthermore, since θ determines the *distribution* of the flow out of P and not the magnitude of that flow, changes in θ have virtually no effect on the size of P.

With the exception of θ , changes in any one parameter of the model result in important differences in the imprisonment rate. The most striking consequence of the model, however, is the predominant effect of k_3 or k_4 alone on the criminal population. This has important policy implications for the control of crime. If Model II is an accurate representation of the flow process among law-abiders, criminals and prisoners, the results in Table 5 suggest that the activities of the criminal justice system, reflected in isolated changes in parameters k_1 , k_2 or θ , alone have very little impact on the size of the criminal population.

According to Model II, manipulations of only the time served in prison ($1/k_1$) or the various efforts in prisons to reduce recidivism (θ) will not affect the incidence of criminals. Furthermore, singly increasing the rate at which criminals go to prison (k_2) has only a marginal effect on the criminal population, while greatly expanding the prison population. According to Model II, although the imprisonment policies of a society are important in determining the imprisonment rate, taken one by one they are for the most part in-

consequential to the extent of criminality in a society.

The size of the criminal population is most responsive to the parameters reflecting the level of conformity, namely k_3 and k_4 . To the extent that conformity is a function of an effective socialization process and/or the homogeneity of a society, very little in the form of implementable policies can be done to reduce the proportion of criminals. However, to the extent that deterrence and opportunities for return to the law-abiders are operating, more reasonable attempts can be made to reduce criminality. Certainly, any efforts to remove barriers to a return to the law-abiding population which increase the value of k_4 will decrease the level of criminality. The more interesting policy implication, however, is the important role of deterrence in reducing crime. Inasmuch as effective general deterrence increases incentives to remain a law-abider (decreases k_3), while effective special deterrence increases incentives to leave the criminal population (increases k_4), the level of conformity increases and the proportion of criminals decreases. The exact mechanisms involved in optimizing these deterrence effects are then vital to efforts to reduce crime.

The results in Table 5 identify only the effects of "pure" changes in the parameters and as such they are necessarily artificial. Undoubtedly, several of the parameters will vary at the same time, and the actual population distributions will reflect the cumulative effect of these different parameters, as well as any interactive effects due to functional relationships among the parameters. Nevertheless, looking at the effects of each parameter alone does provide some opportunity for exploring the indirect implications of the model.

VI. SUMMARY

It has been conjectured that a homeostatic process operates within a society to maintain a stable level of punishment. This process is presumed to work through adaptive responses to changes in criminal behavior. In the short run these responses might involve changes in sentencing policies, such as an increase in the number of persons sentenced to prison or a decrease in the length of sentences imposed. In the long run, the limits of criminal behavior may actually be redefined through changes in law and/or in practice. The result is either the decriminalization of previously criminal acts or the addition of newly prohibited acts to the criminal code.

Evidence of the stability of punishment, in particular, has been presented. The national imprisonment rates in three countries were shown to be trendless time series, each generated by a second-order autoregressive process. Two models specifying the flow of individuals among different population groups were specified in an effort to identify the underlying dynamic process responsible for this stability.

Model I, which requires only a simple balancing of prison receptions and releases, was shown to be inadequate. For reasonable estimates of the parameter values of this process, it does not yield the observed cyclical behavior in imprisonment rates. A second model, which includes movements between the law-abiding and criminal populations, results in a better fit between the predicted and actual time series. Furthermore, Model II can be interpreted in terms of the levels of punitiveness and conformity in a society, thereby integrating the model into the existing body of work on deviance and social control.

The model, however, requires further development if its adequacy is to be fully explored. The major limitation in the development presented here is the assumption of constant flow rates among the populations. A central feature of the stability of punishment theory is adaptive behavior. In the context of our model, incorporation of adaptive behavior would require time-varying k 's. The incorporation of time-varying k_i 's into the model in a manner that is consistent with the theory would represent a major extension to our work. Also, the model does not explicitly incorporate deterrent effects. A further elaboration of the relationship of the flow rates to the deterrence process would further enhance the generality of the model by providing some synthesis of the stability of punishment with the notion of deterrence.

APPENDIX I

TIME SERIES ANALYSIS

The autoregressive structure is defined by

$$Y_t = \delta + \sum_{i=1}^T \phi_i Y_{t-i} + \epsilon_t \quad (1)$$

where

- Y_t is the observation in period t ,
- δ, ϕ_i are the fixed parameters of the generating process,
- ϵ_t 's are independent and identically dis-

tributed random variables with zero mean and variance σ^2 .

Equation (1) states that the observation at t (Y_t) is a weighted linear function of a constant and the observations of T prior periods, plus an independent stochastic error, ϵ_t . The time series analysis provides a means for estimating the number of prior periods, if any, for which the ϕ 's are significantly different from zero. The "order" of the autoregressive process is equal to largest subscript of the non-zero ϕ 's. For example, if $\phi_3 > 0$ and $\phi_i = 0$ for all $i > 3$, the process is called a "third-order" autoregression.

The autoregressive structure assumes the stochastic component, ϵ_t , to be independent of the stochastic components of prior observations. In time-series data, this is often not the case and the ϵ_t 's may be serially correlated over one or many periods.

A moving-average process is defined by:

$$Y_t = u + \epsilon_t \tag{2}$$

where, now:

$$\epsilon_t = \mu_t + \sum_{i=1}^T \gamma_i \mu_{t-i} \tag{3}$$

where:

u, γ_i are fixed parameters of the generating process,

μ_t are independent and identically distributed random variables with mean zero variance σ^2 .

The analyses provide a means for estimating u and the γ_i which are different from zero. As with autoregressive processes, the "order" of the moving average is defined by the maximum subscript of the γ_i 's which are different from zero.

APPENDIX II

THE DIFFERENTIAL EQUATION REPRESENTATION OF A SECOND-ORDER AUTOREGRESSIVE PROCESS

Processes following a second-order linear differential equation, not necessarily with constant coefficients, generate second-order autoregressive functions. By approximating the derivatives in the differential equation by difference equations, that is, if r_t is the imprisonment rate at time t , and its first two time derivatives are denoted by \dot{r}_t and \ddot{r}_t , then we approximate \dot{r}_t and \ddot{r}_t by:

$$\dot{r}_t = r_t - r_{t-1}$$

$$\ddot{r}_t = (r_t - r_{t-1}) - (r_{t-1} - r_{t-2})$$

The general second-order differential equation with constant coefficients is $\ddot{r}_t + cr_t + dr_t = F$, and, in the approximating difference equation, we have:

$$\begin{aligned} \ddot{r}_t + cr_t + dr_t &= (r_t - r_{t-1}) - (r_{t-1} - r_{t-2}) \\ &+ c(r_t - r_{t-1}) + dr_t = F. \end{aligned} \tag{4}$$

Equation (4) then leads to the second-order autoregressive function:

$$\begin{aligned} r_t &= \left[\frac{2 + c}{1 + c + d} \right] r_{t-1} \\ &+ \left[\frac{-1}{1 + c + d} \right] r_{t-2} + \frac{F}{1 + c + d} \end{aligned} \tag{5}$$

where ϕ_1, ϕ_2 , and δ are expressed in terms of c, d and F .

APPENDIX III

DERIVATION OF SECOND-ORDER DIFFERENTIAL EQUATION ASSOCIATED WITH MODEL I

The relationship among the flows of Model I may be formalized as follows:

$$\begin{aligned} \dot{P}(t) &= -r_2P(t) + r_3M(t) + r_1V(t) \\ \dot{M}(t) &= r_2P(t) - (r_3 + r_4)M(t) \\ \dot{V}(t) &= -(r_1 + r_4)V(t) + r_5T(t) \end{aligned} \tag{6}$$

where

$\dot{P}(t), \dot{M}(t), \dot{V}(t)$ = rate of change at t of the respective populations

r_1 = imprisonment rate of virgins

r_2 = release rate from prison

r_3 = imprisonment rate of ex-convicts

r_4 = death rate³⁴

r_5 = birth rate

Since the sum of $P(t), M(t)$ and $V(t)$ is the total population at time $t, T(t)$, then $V(t)$ may be replaced in the first equation of (6) by:

$$V(t) = T(t) - P(t) - M(t)$$

The dynamic behavior of $P(t)$ can now be expressed by a system of two flow equations where:

$$\begin{aligned} \dot{P}(t) &= -(r_2 + r_1)P(t) + (r_3 - r_1)M(t) + r_1T(t) \\ \dot{M}(t) &= r_2P(t) - (r_3 + r_4)M(t) \end{aligned} \tag{7}$$

³⁴ For the purpose of simplicity the differences between the death rate of ex-cons and of virgins and the small number of deaths of prisoners have been ignored.

or, in matrix form

$$\dot{Y} = AY + F$$

where

$$Y = \begin{bmatrix} P(t) \\ M(t) \end{bmatrix} \quad A = \begin{bmatrix} -(r_1 + r_2) & (r_3 - r_1) \\ r_2 & -(r_3 + r_4) \end{bmatrix}$$

$$F = \begin{bmatrix} r_1 T(t) \\ 0 \end{bmatrix}$$

Using the procedure outlined in Appendix IV, $P(t)$ may be translated to:

$$\ddot{P}(t) + a\dot{P}(t) + bP(t) = F_p \quad (8)$$

where

$$\begin{aligned} a &= (r_1 + r_2 + r_3 + r_4) \\ b &= (r_3 + r_4)(r_1 + r_2) - r_2(r_3 - r_1) \\ &= r_1(r_2 + r_3) + r_4(r_1 + r_2) \\ F_p &= -(r_1 + r_2)r_1T(t) + (r_1 + r_2 + r_3 + r_4) \\ &\quad r_1T(t) + r_1T(t) = (r_3 + r_4)r_1T(t) \\ &\quad + r_1T(t) \end{aligned}$$

Equation (8) is a differential equation describing the dynamic behavior of the total prison population, $P(t)$, whereas the autoregressions and their implied differential equations are expressed in terms of a *rate* of imprisonment per population. However, a translation between the two can be made; when $r(t)$ is the imprisonment rate per *unit* of population:

$$P(t) = r(t)T(t) \quad (9a)$$

then:

$$\dot{P}(t) = \dot{r}(t)T(t) + r(t)\dot{T}(t) \quad (9b)$$

$$\ddot{P}(t) = \ddot{r}(t)T(t) + 2\dot{r}(t)\dot{T}(t) + r(t)\ddot{T}(t) \quad (9c)$$

As a first estimate of $T(t)$, we assume that after accounting for "deaths," $T(t)$ grows exponentially, $T(t) = T_0e^{st}$. Then:

$$P(t) = T_0e^{st}r(t) \quad (10a)$$

$$\dot{P}(t) = T_0e^{st}(r(t) + gr(t)) \quad (10b)$$

$$\ddot{P}(t) = T_0e^{st}(\ddot{r}(t) + 2g\dot{r}(t) + g^2r(t)) \quad (10c)$$

We then substitute equations (10) into (8) and divide the equation by $T(t)$. Then:

$$\begin{aligned} [\ddot{r}(t) + 2g\dot{r}(t) + g^2r(t)] \\ + a[\dot{r}(t) + gr(t)] + br(t) = \frac{F_p}{T_0} e^{-st} \quad (11) \end{aligned}$$

Rearranging terms,

$$\ddot{r}(t) + cr(t) + dr(t) = F'^{35} \quad (12)$$

where

$$\begin{aligned} c &= a + 2g = r_1 + r_2 + r_3 + r_4 + 2g \\ d &= b + ag + g^2 = r_1(r_2 + r_3) + r_4(r_1 + r_2) \\ &\quad + ag + g^2 \\ F' &= r_3r_1 + r_4r_1 + r_1g \end{aligned}$$

APPENDIX IV

Suppose we have a system of simultaneous flows among three populations, $A(t)$, $B(t)$, $C(t)$, where:

$$\dot{A}(t) = a_{11}A(t) + a_{12}B(t) + a_{13}C(t) \quad (a)$$

$$\dot{B}(t) = a_{21}A(t) + a_{22}B(t) + a_{23}C(t) \quad (13) \quad (b)$$

$$\dot{C}(t) = a_{31}A(t) + a_{32}B(t) + a_{33}C(t) \quad (c)$$

such that

$$A(t) + B(t) + C(t) = T(t) \quad (14)$$

with:

$$T(t) = \text{total population at } t$$

a_{ij} may possibly be zero.

Since $C(t) = T(t) - A(t) - B(t)$, system (13) may be re-written as:

$$\begin{aligned} \dot{A}(t) &= (a_{11} - a_{13})A(t) + (a_{12} - a_{13})B(t) \\ &\quad + a_{13}T(t) \end{aligned}$$

$$\begin{aligned} \dot{B}(t) &= (a_{21} - a_{23})A(t) + (a_{22} - a_{23})B(t) \\ &\quad + a_{23}T(t) \end{aligned} \quad (15)$$

or in matrix notation:

$$\dot{Y} = AY + F \quad (16)$$

where:

$$Y = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix} \quad \dot{Y} = \begin{bmatrix} \dot{A}(t) \\ \dot{B}(t) \end{bmatrix} \quad F = \begin{bmatrix} a_{13}T(t) \\ a_{23}T(t) \end{bmatrix}$$

$$A = \begin{bmatrix} (a_{11} - a_{13}) & (a_{12} - a_{13}) \\ (a_{21} - a_{23}) & (a_{22} - a_{23}) \end{bmatrix}$$

³⁵ It should be noted that equation (12) is based on the imprisonment rate per *unit* of population, while the estimated differential equations in Table 2 are based on the rate per 100,000 population. Although the rates differ by a factor of 10^5 , the coefficients c and d are unaffected and may be directly compared. The constant term F' , however, must be multiplied by 10^5 when it is compared to the constant term F in Table 2.

Taking the derivative of (16), we get:

$$\dot{Y} = AY + \dot{F} \tag{17}$$

Substituting (16) for \dot{Y}

$$\dot{Y} = A^2Y + AF + \dot{F} \tag{18}$$

Let a and b be the coefficients of the quadratic equation resulting from taking the determinant of $[A - \lambda I]$:

$$\begin{vmatrix} (a_{11} - \lambda) - \lambda & (a_{12} - a_{12}) \\ (a_{21} - a_{23}) & (a_{22} - a_{23}) - \lambda \end{vmatrix} = \begin{vmatrix} c_{11} - \lambda & c_{12} \\ c_{21} & c_{22} - \lambda \end{vmatrix} = 0 \tag{19}$$

or

$$(c_{11} - \lambda)(c_{22} - \lambda) - c_{21}c_{12} = 0$$

$$\lambda^2 - (c_{11} + c_{22})\lambda + (c_{11}c_{22} - c_{21}c_{12}) = 0^{36} \tag{20}$$

Thus,

$$a = -(c_{11} + c_{22})$$

$$b = c_{11}c_{22} - c_{21}c_{12}$$

Adding the sum $(a\dot{Y} + bY)$ to both sides of (18)

$$\begin{aligned} \dot{Y} + a\dot{Y} + bY &= (A^2Y + a\dot{Y} + bY) + AF + \dot{F} \\ &= (A^2Y + aAY + bY) + aF \\ &\quad + AF + \dot{F} \\ &= [A^2 + aA + bI]Y + aF \\ &\quad + AF + \dot{F} \\ &= aF + AF + \dot{F} \end{aligned} \tag{21}$$

since $A^2 + aA + bI = 0$ and (21) are no longer simultaneous.

APPENDIX V

DERIVATION OF THE SECOND-ORDER DIFFERENTIAL EQUATION ASSOCIATED WITH MODEL II

The relationship among the flows of Model II can be formalized as follows:

$$\begin{aligned} \dot{P}(t) &= -k_1(t)P(t) + k_2(t)C(t) \\ \dot{C}(t) &= \theta k_1(t)P(t) - k_2(t)C(t) - k_4(t)C(t) \\ &\quad + k_3(t)L(t) \\ \dot{L}(t) &= (1 - \theta)k_1(t)P(t) + k_4(t)C(t) \\ &\quad - k_3(t)L(t) + k_5(t)T(t) \end{aligned} \tag{22}$$

where

$\dot{P}(t), \dot{C}(t), \dot{L}(t)$ = rate of change at t of the respective populations (i.e., their first derivatives)

$k_1(t)$ = release rate from prison at t

$k_2(t)$ = imprisonment rate of the criminal population at t

$k_3(t)$ = rate at which law-abiders become criminals at t

$k_4(t)$ = rate at which criminals become law-abiders at t

$k_5(t)$ = net population growth rate at t

θ = portion of the persons released from prison who return to criminal activity

Since the sum of $P(t), C(t)$ and $L(t)$ is the total population at t, $T(t)$, we can replace $L(t)$ by

$$L(t) = T(t) - C(t) - P(t)$$

and the dynamic behavior of $P(t)$ can be expressed by the two flow equations:

$$\dot{P}(t) = -k_1(t)P(t) + k_2(t)C(t) \tag{23}$$

$$\begin{aligned} \dot{C}(t) &= [\theta k_1(t) - k_3(t)]P(t) - [k_2(t) + k_3(t) \\ &\quad + k_4(t)]C(t) + k_3(t)T(t) \end{aligned}$$

In matrix form:

$$\dot{Y} = AY + F$$

where:

$$\dot{Y} = \begin{bmatrix} \dot{P}(t) \\ \dot{C}(t) \end{bmatrix} \quad Y = \begin{bmatrix} P(t) \\ C(t) \end{bmatrix} \quad F = \begin{bmatrix} 0 \\ k_3(t)T(t) \end{bmatrix}$$

$$A = \begin{bmatrix} -k_1(t) & k_2(t) \\ (\theta k_1(t) - k_3(t)) & -(k_2(t) + k_3(t) + k_4(t)) \end{bmatrix}$$

The equations in (23) are a first-order system of simultaneous differential equations like those examined in the discussion of Model I, but here the coefficients are not necessarily constant. In the case of constant coefficients each population was defined solely in terms of its own derivatives, for instance:

$$\ddot{P}(t) + a\dot{P}(t) + bP(t) = F_P$$

and a, b and F_P were determined from the matrix A (Appendix IV). A similar solution in terms of its own derivatives also exists for each population when the coefficients are not constant, namely:

$$\dot{Y} + a(t)\dot{Y} + b(t)Y = F(t) \tag{24}$$

However, now the time-varying coefficients, $a(t), b(t)$ and $F(t)$, are in general complicated and, in this case, elusive functions of the $k_i(t)$. Neverthe-

³⁶The values of λ which satisfy equation (20) are the eigenvalues of A.

less, as a point of departure we can explore the dynamic character of this model by assuming the $k_i(t)$ are approximately constant.

Under the assumption of constant k_i , the differential equation governing the behavior of $P(t)$, the prison population, is:

$$\ddot{P}(t) + a\dot{P}(t) + bP(t) = F_p \quad (25)$$

where:

$$a = k_1 + k_2 + k_3 + k_4$$

$$b = k_1[(1 - \theta)k_2 + k_3 + k_4] + k_2k_3$$

$$F_p = k_2k_3I(t)$$

We can change (25) into a differential equation describing the behavior of the *rate* of imprisonment per unit of population, $r(t)$ using the procedure outlined in (8) through (12) in Appendix III to yield:

$$\ddot{r} + cr + dr = F' \quad (26)$$

where

$$c = a + 2g$$

$$d = b + ag + g^2$$

$$F' = k_2k_3.$$