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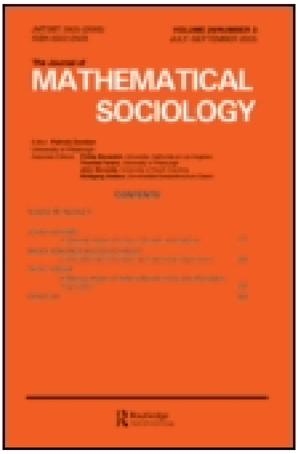

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THE DYNAMICS OF CLOSENESS AND BETWEENNESS

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Although both betweenness and closeness centrality are claimed to be important for the effectiveness of someone's network position, it has not been comprehensively studied which networks emerge if actors strive to optimize their centrality in the network in terms of betweenness and closeness. We study each of these centrality measures separately, but we also analyze what happens if actors value betweenness and closeness simultaneously. Network dynamics differ considerably in a scenario with either betweenness or closeness incentives compared with a scenario in which closeness and betweenness incentives are combined. There are not only more stable networks if actors' betweenness and closeness are combined, but also these stable networks are less stylized.

Keywords: betweenness centrality, closeness centrality, network dynamics, network formation, social networks

1. INTRODUCTION

Freeman (1979) already realized that to describe the centrality of actors in a network, one network measure is not sufficient. Even when only considering the communication activities in a network, it makes sense to distinguish different measures. More specifically, he distinguished degree centrality for the extent to which an actor is active in the process of communication in the network, betweenness centrality as the extent to which an actor is essential to channel information between other actors, and closeness centrality as a measure of independence of an actor to receive information quickly from any position in the network. Subsequently, it was realized that central positions in the network can provide benefits for the actors in these positions, but that depending on the application different types of central

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positions are beneficial. For example, Burt (1992) showed that controlling information exchange between other actors can be beneficial for promotion within a firm. Although Burt operationalized control of information using his formalization of structural holes, betweenness centrality can also be considered as a measure for such a control of information. In other circumstances, closeness can be more beneficial. For example, if information needs to travel along shorter paths, the information is more reliable compared with information that has to travel along long paths. Therefore, if fast and accurate access to information is necessary and especially if network channels are noisy, closeness centrality will be beneficial.

Given that it is meanwhile acknowledged that certain network positions provide benefits for actors, it is also plausible that actors consciously choose their relations to optimize their network positions in an incentive-guided manner (see Flap, 2003). In the last 15 years, the developments on modeling incentive-based network formation have been tremendous in sociology and in economics. The two textbooks by Goyal (2007) and Jackson (2008) are examples for the development in economics. The special issue of *Social Networks* (see Snijders & Doreian, 2010) provides a broad overview of recent developments in sociology. We will not review this literature here again but connect to the most related studies below.

1.1. Motivation

We add to the theoretical research on network dynamics in three ways. First, although Freeman (1979) has been arguing already in the late 1970s what the beneficial features are of different central positions in networks, it has not been studied in detail how the classic centrality measures can drive network formation. This exercise complements the theory of centrality that originally measures the effect of network positions on individual opportunities (see, e.g., Wasserman & Faust, 1994), but not the effect of individual behavior on network structure. If network dynamics show that advantageous network positions are *not* stable in a dynamic context, effects of advantageous network positions on individual opportunities might be smaller than expected. The reason is that the advantages of the network positions can only be exploited for a short time.

Second, the centrality indices are based on network statistics that are applicable in many different contexts—from ancient marriages (Padgett & Ansell, 1993) to research and development collaborations (Walker, Kogut, & Shan, 1997). Correspondingly, centrality-oriented linking behavior covers basic types of linking behavior, beyond a single application. One type of linking behavior is oriented toward access to resources via short paths (closeness). Another type of linking behavior is oriented toward becoming a mediator or broker by lying between others (betweenness). By using closeness and betweenness centrality as operationalizations for these two types of incentives, we deviate from most of the current literature on strategic network formation. This enables us to compare, for example, network dynamics based on betweenness with network dynamics based on structural holes (Buskens & Van de Rijt, 2008). Studying resemblances and discrepancies not only serves to assess the robustness of previous results but might also be informative for the theoretical distinction between different measures.

Third, after comparing the dynamics of closeness and betweenness, we analyze the combination of closeness and betweenness incentives to determine network formation. For contexts in which different types of incentives are salient simultaneously, it would be misleading to analyze them separately because combining closeness and betweenness incentives leads to qualitatively different network dynamics compared to looking at each of the incentives separately. So far, there is hardly any research on the interplay between different types of incentives to predict network formation processes, although it is likely that multiple incentives are important simultaneously.[1] For example, considering the Medici's position in the marriage network in Renaissance Florence, Padgett and Ansell (1993) illustrate that for the trading abilities betweenness played a major role, but for actors with low betweenness, it was important to be at least close to the other actors.

1.2. Relation to the Literature

In this article, we will introduce a model in which actors strive for closeness and betweenness (centrality), while links are costly. Several models on strategic network formation resemble this "centrality model." The dynamics of intermediation rents in terms of structural holes is studied by Buskens and Van de Rijt (2008) and Goyal and Vega-Redondo (2008), as well as by Willer (2007) and Kleinberg, Suri, Tardos, and Wexler (2008).[2] Each of those models uses a different operationalization of structural holes. None of the models uses betweenness centrality, although Burt not only proposes some new measures for brokerage but also employs betweenness (Burt, 2002). Hummon (2000) and Doreian (2006) study the dynamics of the "connections model," originally introduced by Jackson and Wolinsky (1996). Covering incentives for short paths, the connections model is closely related to closeness incentives (as examined in a comparison of these two models in Buechel, 2008). Moreover, Fabrikant, Luthra, Maneva, Papadamitriou, and Shenker (2003) introduce a model where actors' utility is decreasing with their average path length (the network statistic on which closeness is based on). This model is adapted to bilateral link formation and further studied by Corbo and Parkes (2005). Finally, Holme and Ghoshal (2006, 2009) study a model (within a different framework, though) where actors optimize their closeness, while links are costly.

What has not been done in the literature is *to contrast* and *to combine* the dynamics of "closeness-type" incentives to (with) the dynamics of "betweenness-type" incentives. We explicitly investigate how the dynamics of closeness differ from the dynamics of betweenness and examine what happens if both centrality incentives matter simultaneously. To study which networks emerge for different incentives, we use three complementary tools. First, we derive general propositions on properties of stable networks using analytic tools. Second, we enumerate all stable networks with eight or less actors and different relative weights for closeness and betweenness. Finally, we simulate a dynamic process to estimate the likelihood of different stable networks in the theoretical conditions that we consider.

[1]A first step in this direction is made by Sato (1997), who analyzes a combination of brokerage and closure for small networks.

[2]The model of Kleinberg et al. (2008) is less comparable because it studies the unilateral (not the bilateral) formation of network links.

As we will show, emerging networks for pure closeness incentives are rarely star networks, but frequently star-like networks in the sense that they are sparse and connected. Also, the dynamics of pure betweenness lead to a special class of networks (complete bipartite networks) in most of the settings. For a combination of closeness and betweenness incentives we observe that there are more and qualitatively different stable networks.

The next section introduces the model and the methods. Section 3 contrasts the closeness dynamics with betweenness dynamics. Section 4 examines the interaction of closeness and betweenness dynamics, and Section 5 concludes.

2. MODEL AND METHODS

2.1. Basic Definitions

2.1.1. Networks and Features of Networks. We consider a finite set of actors N with typical elements i or j and size $n \geq 3$. The bilateral relationships among these actors are modeled as an undirected (and dichotomous) network. Let $\mathcal{G}$ be the set of all those networks and G a typical element. With $ij \in G$ we denote the presence of the link between actors i and j in G. Let $G \cup ij$ be the network obtained when the link between actors i and j is added to network G, while $G \backslash ij$ denotes the network when the link between actors i and j is removed from network G.

A path between two actors i and j is a sequence of distinct actors $i_1 i_2 i_3 \ldots i_k$ such that $i_1 = i$, $i_k = j$, and $i_l i_{l+1} \in g\ \forall l \in \{1, \ldots, k-1\}$. The distance $d_G(i, j)$, or simply $d(i, j)$ between two actors is the length of their shortest path(s), where the length is the number of links in the sequence.[3] Neighbors have distance 1; neighbors of neighbors that are not directly connected are at distance 2; and pairs that cannot reach each other via any number of other actors are defined to have distance M, a number larger than any possible actual distance in a network. We work with the conventions $M = n$ and $d(i, i) = 0\ \forall i \in N$. Let the diameter of a network be the maximal distance between two connected actors in the network. A network is called connected if there exists a path between any two actors in the network. A set of connected actors is called a component if there is no path to actors outside of this set. A link is called a bridge if its deletion increases the number of components in a network.

Let us define some network architectures that will be used throughout the text. In the complete network K_n every possible link is present, while in the empty network, the complement of the complete network, $\overline{K}_n$ no link is present. A network is a tree if all links are bridges and the network is connected. If all links are bridges and the network consists of multiple components, the network is called a forest. A network is complete bipartite (K_{n_1,n_2}) if it can be partitioned into two (nonempty) groups of actors (of sizes n_1 and n_2) such that no link is present within a group and all links are present across groups. A special case is the balanced complete bipartite network K_{n_1,n_1} with $n = 2n_1$. When referring to complete bipartite networks (CB), we assume that there are at least two actors in each group. If one group consists of one actor, this is called a star network $K_{1,n-1}$ (in which one actor is linked to every other

[3]Below, we introduce notation for properties of actors in a network G omitting G if there is no confusion about which network is considered.

actor, while there are no other links). A circle of size $k(\geq 3)$ is a sequence of k distinct actors $i_1 i_2 \ldots i_k$ such that $i_l i_{l+1} \in G \; \forall l \in \{1,\ldots, k\}$, where $i_{k+1} := i_1$. A circle network C_n is a network with no links besides a circle of size $k = n$. Eliminating one link of a circle network leads to a line network P_n.

2.1.2. Degree, Closeness, and Betweenness. The degree $d_G(i)$ or simply $d(i)$ of an actor i is the number of links actor i has in network G. An isolate is an actor with $d(i) = 0$ and a pendant is an actor with $d(i) = 1$ (the link to this latter actor is called a loose end). The average degree of a network is defined as $d(G) := \frac{1}{n} \sum_{i \in N} d(i)$. Network density is defined as $D(G) := \frac{d(G)}{n-1}$, the average degree as a proportion of the maximal possible average degree. Degree can be considered as a measure of centrality (Freeman, 1979). But besides the beneficial aspects of many links, there are also costs (time, effort) involved. We assume that the costs of maintaining relationships are the same for any link independent of the number of links an actor has and exceed those benefits that are restricted to direct contacts.[4] This means that maintaining links is costly.[5]

The idea of closeness reaches back to the origins of social network analysis. An actor is considered as "central" in a social network if his distance to other actors is small (Sabidussi, 1966). Freeman (1979) uses the inverse average distance (of an actor to all actors in the network) to formalize closeness $\left(\frac{n-1}{\sum_{j \in N} d(i,j)}\right)$. As argued in Buechel (2008), it is equally reasonable to operationalize closeness as the reverse average distance $\left(-\frac{\sum_{j \in N} d(i,j)}{n-1}\right)$. The advantage of the latter definition is that any change in closeness is proportional to a change in average distances (as also argued in Valente & Foreman, 1998). Usually closeness is not defined for actors that are not connected via any number of others. We extend closeness to all undirected networks using M as the distance of not connected pairs. In this article, we use the normalized version of the reverse average distance. Closeness of actor i in network G is in that case equal to

$$C_C(i) = \frac{M}{M-1} - \frac{\sum_{j \in N} d(i,j)}{(M-1)(n-1)}. \tag{1}$$

$C_C(i) = 0$ for isolates, while $C_C(i) = 1$ for an actor who is directly connected to all others in the network. As examined in Buechel (2008), this choice of operationalization (as opposed to Freeman's definition) affects the results but does not fundamentally change them. The reason for this consistency is that in our analysis of closeness the results are strongly driven by the ordering of closeness centralities in different network positions and the results are to a lesser extent driven by the precise differences between the centralities.

[4]Without this assumption every actor wants to be directly linked to every other actor, independently of any other benefits. Alternatively, one could assume increasing marginal costs of links, but we do not need this more complex cost structure to develop our main point.

[5]We only consider costs for link maintenance and do not take into account specific costs for creating or deleting links.

Betweenness was introduced by Anthonisse (1971) and Freeman (1979) and used in many studies thereafter (e.g., Song, Nerur, & Teng, 2007). The betweenness of an actor i is proportional to the number of pairs j and k for whom i lies on the shortest path (also called "geodesic"). If j and k have more than one geodesic, the fraction of shortest paths going through i is used. Formally,

$$C_B(i) = \frac{2}{(n-1)(n-2)} \sum_{j<k(j \neq i, k \neq i)} \frac{g_{jk}^i}{g_{jk}}, \tag{2}$$

where g_{jk} is the number of geodesics between j and k, and g_{jk}^i indicates the number of shortest paths between j and k that go through i; the fraction $\frac{g_{jk}^i}{g_{jk}}$ is replaced by zero, when $g_{jk} = 0$. The constant before the fraction normalizes betweenness to be between 0 (an actor is on no shortest path between two other actors) and 1 (the center in a star network).

2.2. Utility Function and Actor Behavior

We assume that closeness and betweenness are the benefits derived from the network structure, while direct links are costly. Let $c > 0$ be the costs of one link and $\lambda \in [0, 1]$ the relative importance (weight) of betweenness versus closeness benefits. In this "centrality model" we represent the behavior for any actor i by the following utility function:

$$u_i(G) = (1-\lambda)C_C(i) + \lambda C_B(i) - cd(i). \tag{3}$$

We analyze the model for all possible parameter combinations, as they represent different contexts including high costs and low costs for maintaining links as well as pure closeness incentives ($\lambda = 0$), pure betweenness ($\lambda = 1$), and both closeness and betweenness being important ($0 < \lambda < 1$).[6]

In this formulation the utility function is linear in closeness, betweenness, and degree. This assumption implies that the effect of a change in one centrality measure is independent of the level the three centrality measures have. For example, the costs of a link are independent of the number of links an actor already has and independent of his closeness and betweenness. Such a formulation is a convenient choice, but clearly restricts generality.[7]

In our model actors have homogeneous preferences. It is an interesting question to ask how networks evolve when actors differ in their preferences (see, e.g., Galeotti, Goyal, & Kamphorst, 2006). But since applications of our model are very different in nature, we put emphasis on the different contexts that influence

[6]Instead of setting the slopes (λ and $1-\lambda$) in relation to each other, we could also have defined them independently. Both notations allow us to represent exactly the same behavior and there is no difference when examining stability (and efficiency). The relative notation can be advantageous for comparative statics because c then measures the costs in comparison to one unit of benefit.

[7]Goyal and Joshi (2006), De Jaegher and Kamphorst (2008), and Buechel (2008) study models where the assumption is partially relaxed (by allowing for increasing and decreasing marginal returns).

everybody's choice, not on the difference between actors (as also argued in Burger & Buskens, 2009).

Finally, we assume that actors in our model decide about links myopically. This means that actors consider the consequences of their actions on the current network structure, but do not anticipate the potential reactions of others (cf. Jackson & Wolinsky, 1996).

2.3. Methods to Study Emergence

To study which networks are likely to emerge for different incentives under the assumptions specified above, we employ three complementary methods: formal derivations, enumeration, and simulation. We introduce in this subsection each method while we sometimes provide a basic result as an illustration. Sections 3 and 4 summarize the results for different parameters using each method.

2.3.1. Formal Derivations. To find the networks that are likely to emerge, the first step is to exclude all those networks in which individual actors have incentives and possibilities to change the network. Jackson and Wolinsky (1996) proposed such a stability condition that takes into account that, typically for social networks, the establishment of a relationship needs the agreement of both actors involved, while the dissolution can be done unilaterally. Accordingly, a network G is (pairwise) stable if

(i) $\forall\, ij \in G, \quad u_i(G) \geq u_i(G \backslash ij)$ and $u_j(G) \geq u_j(G \backslash ij)$ and

(ii) $\forall\, ij \notin G, \quad u_i(G \cup ij) > u_i(G) \Rightarrow u_j(G \cup ij) < u_j(G)$.

To establish stability one typically needs the maximal incentive (change in benefits) of any actor to sever a link and the maximal incentive (change in benefits) of any two actors to add a link and compare them to linking costs c. Because benefits are based only on closeness and betweenness, the crucial aspects for a focal actor i are the change in distances $\sum_{j \in N} d(i,j)$ ($\hat{=}$ nonnormalized *closeness*) and the change in the number of shortest paths he is on $\sum_{j<k(j \neq i, k \neq i)} \frac{g^i_{jk}}{g_{jk}}$ ($\hat{=}$ nonnormalized *betweenness*), what we call his "brokerage." Plugging in these changes into the utility function above yields the following condition: *if a new link for some actor i in some network G means a decrease in distances of X and an increase in brokerage of Y, this actor is willing to form the link only if*

$$c \leq \frac{(1-\lambda)[X]}{(M-1)(n-1)} + \frac{\lambda 2[Y]}{(n-1)(n-2)}. \tag{4}$$

Although deriving the changes in distances and brokerage for a given situation might be tedious, it is a straightforward task.

Formal derivations are first of all used to establish the existence of stable networks and to characterize a significant boundary of the parameters. In particular, the following proposition specifies for which costs the complete network is uniquely stable. Clearly, for costs smaller than the derived threshold, no interesting results can be expected.

TABLE 1 Number of Stable Networks for a Network of Size n

Network size n	5	6	7	8
Stable networks for $\lambda = 0$ (Closeness)	6	12	21	45
Stable networks for $\lambda = 0.5$	9	20	45	117
Stable networks for $\lambda = 1$ (Betweenness)	4	9	18	37
Total number of nonisomorphic networks	34	156	1,044	12,346
Fraction (of stable networks for $\lambda = 0.5$)	26%	13%	4.3%	0.95%

Proposition 1. *In the centrality model there exists at least one stable network for any parameters* $(\lambda, c) \in [0, 1] \times \mathbb{R}_+$. *Moreover, if* $c < \frac{1-\lambda}{(n-1)(M-1)}$, *the complete network* K_n *is uniquely stable.*

The proof of all propositions can be found in the Appendix. The existence of a stable network is assured by three simple networks—the complete network, the empty network, and a star network—of which at least one is stable. The complete network is stable for $c \le \frac{1-\lambda}{(n-1)(M-1)}$. When this condition holds with equality, actors are indifferent between keeping and removing a link with the minimal possible benefits—that is a link that only serves to reducing the distance to one other actor by the amount of one, while it does not provide any brokerage. Above the upper bound for the stability of the complete network there is (possibly) a multitude of stable networks. We explore those networks further in Sections 3 and 4. For sufficiently high c, the empty network is uniquely stable, since no link benefit can justify its costs. The proof of existence is completed by showing that the star network is stable when the complete and empty network are not stable.

It is straightforward to analyze for which parameter settings a certain network is stable and not stable (we do this exercise in Subsection 4.1 for some prominent networks). Moreover, the formal derivations are used to characterize the stable networks by properties they must or must not satisfy.

2.3.2. Enumeration. For small n, one can check the stability for any possible network structure using brute force computer power. We apply this for $n \le 8$. To provide an overview of the stable networks that exist, we checked in which range of costs c each network is stable for a fixed $\lambda \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ and using $M = n$. We call networks "stable for λ" if there exists a cost range with positive support in which the network is stable. By excluding those networks that are only stable for an infinitely small cost range, that is, one single cost value, we do not expect to lose reasonable candidates for the emerging networks, because the networks would lose their stability due to the smallest perturbations in the cost c.[8] Table 1 shows how many different stable networks exist fixing $\lambda \in \{0, 0.5, 1\}$ and combining all cost levels.[9]

[8]We cannot make the same robustness check for perturbations of λ. To run the enumeration, we either have to fix λ and search for ranges of c for a given network, or fix c and search for a ranges of λ. We chose to fix λ, because there are some canonical candidates of λ to be analyzed, i.e., $\lambda = 0$ and $\lambda = 1$, while this is not true for c.

[9]With the described procedure, we find all stable networks, except those that are stable for some λ that we did not consider, while they are not stable for all λ's we did consider. This number of networks is likely to be small because most of the stable networks we find are stable for multiple values of λ.

While the total number of nonisomorphic networks explodes with size n, the number of stable networks increases much more gradually. So our stability notion, despite being a minimal requirement, can already exclude many networks from being part of a prediction even without fixing the cost parameter c.

While the enumeration provides a full picture of the candidates for emerging networks, it does not reveal which networks are most likely the endpoint of a dynamic process in which, for example, in any period two actors randomly meet. Each meeting is a possibility for those actors to change their relationship. We use simulations to elaborate on the expected structural features of emerging networks for different parameter values in such a dynamic process.[10]

2.3.3. Simulation. The third method to investigate the emergence of stable networks is a simulation of myopic improvement dynamics. To run a simulation, one has to fix both behavioral parameters, weight λ and costs c (as well as the basic settings, n and M). Then, the simulation takes the following steps:

1. Start with some network.
2. Pick a pair of actors $\{i, j\}$ at random (every pair with equal probability).
3. If ij does not exist, form the link ij if both i and j improve their utility (at least one strictly); if the link ij exists, sever the link if either i or j improves strictly by severing it; keep the current status of the link in all other cases.
4. Go back to step 2 and repeat the steps for the actual new situation until no pair of actors wants to change anymore.

The procedure starts with one given network and follows a sequence of deviations toward some stable network (cf. Doreian, 2006; Willer, 2007). Similar simulations can be found in Hummon (2000) or Buskens and Van de Rijt (2008).

It is the nature of such a simulation that one has to choose a few parameter settings out of a continuum of possibilities. As in the enumeration, we fixed $M = n$ in any simulation. As parameter setting we chose the weights $\lambda = 0$, 0.1, 0.5, 0.9, 1 and four cost levels ($c =$ *very low*, *low*, *medium*, *high*) which are illustrated by the dots in Figure 1. The weights include models where closeness and betweenness incentives are analyzed separately as well as models in which they are combined with different weights—one balanced model $\lambda = 0.5$, and cases that check for nonlinearities in the dynamic process when going from a model with only closeness or betweenness to a combined model. The cost levels are defined according to analytical considerations as follows (in increasing order): *very low* $:= \frac{1}{2^3 n} - \varepsilon$, *low* $:= \frac{1}{2^2 n} - \varepsilon$, *med* $:= \frac{1}{2^1 n} - \varepsilon$, and *high* $:= \frac{1}{2^0 n} - \varepsilon$, where $\varepsilon = 0.001$, so, e.g., for $n = 5$ costs levels are $c = 0.024$, 0.049, 0.099, and 0.199. The subtraction of $\epsilon = 0.001$ only serves to avoid potential

[10]The formal derivations and the enumeration are based on the notion of pairwise stability, which is conceptually not very restrictive. However, we do not work with stronger notions of stability for three reasons: (a) As the enumeration shows, only a small subset of all networks is pairwise stable. (b) We let the enumeration also check for unilateral stability (Buskens & Van de Rijt, 2008), which is a stronger stability concept than pairwise stability, but it turns out that this refinement does not heavily decrease the number of equilibrium networks in our model. (c) The simulation partially serves as an equilibrium selection device and provides itself an indication for the more or less important stable networks.

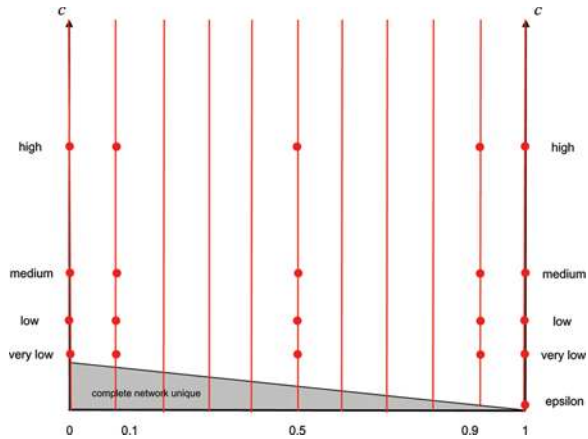

FIGURE 1 Setting of the parameters for the simulation and the enumeration method (color figure available online).

situations in which actors are indifferent between having or not having a particular link. For $\lambda = 1$ there was an additional run for "epsilon costs" $c = \epsilon = 0.001$, that is, a cost level sufficiently small such that any increase in betweenness benefits would justify its costs (for not too high n). By starting twice (or three times) with each configuration, there are $2(4 \cdot 5 + 1) = 42$ (respectively 63) runs per starting network.

Figure 1 illustrates the three methods in the parameter space with weight λ on the horizontal axis and cost c on the vertical axis. The dots stand for the 21 settings of the simulation. The enumeration "collects" all stable networks along the vertical lines. By formal derivations we find thresholds, for example, for the uniqueness of the complete network, which can be represented by regions in the parameter space.

As starting networks for the simulation we took all nonisomorphic networks for network size $n = 3, \ldots, 8$ and a sample stratified by density of around 2,500 networks for network sizes 14 and 20. To give a specific example: For $n = 14$ we used a sample of 2,432 starting networks. Each of them was used for 42 runs. On average it took 137 (median is 56) iterations to reach a stable network.

The purpose of the simulation is two-fold. First, for small network sizes, for which we know all stable networks from enumeration, we use the simulation to attach probabilities of emergence. The second purpose of the simulation is to run computational experiments. Starting with the same network structures but using different utility parameters provides important insight how changes in the utility of actors affect the emerging network structure. In the following we employ all three methods presented here (formal derivations, enumeration, and simulation) to answer specific questions about the consequences of closeness and betweenness incentives on the network dynamics.[11]

[11]In this article, we selected the most important results to illustrate the difference between various centrality models. For example, we only present the enumeration and simulation results for $n = 8$. Some additional results that corroborate the main message of the current article can be found in Buechel (2009). Further results can also be requested from the authors.

We argue that each of the three methods has its significant strengths and weaknesses such that omitting one of them would not lead to a sufficient examination of our model. Clearly, without formal derivations, enumeration and simulation are black boxes leading to data which can be described but not generalized. Omitting the enumeration, we do not get a full picture of the candidates for stable networks. This is an issue because the dynamics of the simulation are not only driven by the utility function—the point of interest—but also by the process of link formation.[12] That is, the rule that a pair of actors is drawn to revise their relationship might induce different network structures than, for example, the rule that a single actor is drawn who can change the relationship that is most valuable for him. Finally, without the simulation, we assess a dynamic question (which networks emerge when ...) by only static methods. Moreover, we would not have had numerical examples for $n \geq 10$ such that we might miss important features of emerging networks.

3. CLOSENESS VERSUS BETWEENNESS INCENTIVES

This section first describes the dynamics of closeness incentives and then turns to betweenness dynamics.

3.1. Dynamics of Closeness

For pure closeness incentives ($\lambda = 0$), actors face a trade-off between short distances and linking costs. This is equivalent to a linear version of the model introduced in Buechel (2008) and almost equivalent to the model proposed by Fabrikant et al. (2003), where the benefit function is also linearly decreasing with the sum of distances.[13] In the original formulation of the Fabrikant model, analyzed by Corbo and Parkes (2005) for bilateral network formation, the distance of not connected actors is set to $M = \infty$ while this assumption is relaxed by Brandes, Hoefer, and Nick (2008). Corbo and Parkes (2005) identify some classes of stable networks and also mention the difficulty in finding all stable networks.

Moreover, actors striving for short paths is similar to the utility function of the connections model, discussed in Jackson and Wolinsky (1996) and Hummon (2000), where the value of each connected actor decreases with his distance.[14] The star network is the predominantly discussed stable network of the connections model, but different other stable networks were found (see Hummon, 2000). In Buechel (2008) it is shown that the set of stable networks of the (symmetric) connections model almost coincides with the set of stable networks in the model with linear closeness benefits (which is the centrality model for $\lambda = 0$).

However, it has not been characterized what the stable networks look like. So the question remains whether the star or star-like networks are a typical

[12]We thank Ulrik Brandes for pointing out this issue.

[13]For $\lambda = 0$ the benefits are just an affine linear transformation of benefits in the Fabrikant model, while for the costs c anyhow any possible value is considered. As a consequence, the two models lead to the same sets of stable and efficient networks. However, they do not lead to the same absolute values of utility, for example, when computing ratios of the values of different networks, as Corbo and Parkes (2005) do.

[14]Consistently, Borgatti and Everett (2006) list the benefits of the connections model among the "closeness-like" centrality indices.

outcome for closeness-type incentives and which other networks can occur. The star belongs to the family of the tree networks. Among the trees, the star is the network with the minimal sum of distances. Therefore, star-like networks can be described as connected, sparse with short distances. Below we analyze to which extent the stable and emerging networks for closeness incentives ($\lambda = 0$) satisfy these three properties.

3.1.1. Formal Derivations. For $c < \frac{1}{(n-1)}$ all stable networks must be connected because the marginal benefit of linking to an actor in a different component is at least $\frac{M-1}{(n-1)(M-1)} = \frac{1}{n-1}$ (see Buechel, 2008, Prop. 5). This threshold is slightly above $c = high$ (so we obtain connected networks for all cost levels of the simulation when $\lambda = 0$).

Concerning distances, one can find an upper bound for the diameter in a stable network.[15] The following proposition is based on the minimal benefit two actors—who are separated by a given distance—gain from linking.

Proposition 2. *In the centrality model with $\lambda = 0$, the following holds: The diameter of a stable network is smaller or equal to p, with $p = max\{\sqrt{4c(n-1)(M-1)+1}, 1\}$.*

Let us study the implications of this result in a numerical example: for $c = low\left(= \frac{1}{2^2 n} - \epsilon \approx \frac{1}{2^2 n}\right)$, the boundary is $p = \sqrt{\frac{(n-1)(M-1)}{n} + 1}$. That means that the maximal distance between connected actors that can emerge in a simulation with $M = n = 8$ is two and in a simulation with $M = n = 14$ this is three. For $c = medium$, the maximal possible distance is three for size 8 and five for size 14.

The sparsity of stable networks also can be shown analytically. For each level of c, we find an upper bound for the average degree $d(G)$. We cannot exclude a high degree actor directly because a star-like position leads to high benefits which can compensate for the costs. But there is a link between the existence of small circles and the average degree that we can use to get the following result:

Proposition 3. *In the centrality model with $\lambda = 0$, the following holds: If $c > \frac{9n}{16(n-1)(M-1)}$, $d(G) < \frac{1}{2}n + \frac{1}{2}$ for any stable network G and if $c > \frac{4n}{5(n-1)(M-1)}$, $d(G) < \sqrt{n}$ for any stable network G.*

The first part of the statement does not drastically restrict the candidates for emerging networks. It restricts the density $D(G)$ of the stable networks not to be higher than around 60%. The second part applies for higher costs, for example, $c = high$. It restricts the stable networks of size 8 to have less than 11 links, networks of size 14 to have less than 25 links.[16]

[15] The corresponding result is already stated in Fabrikant et al. (2003).

[16] The result on average degree and the result on the diameter get stronger for bigger sizes of the networks. For $n = M = 100$ and $c = low$ the diameter is not larger than 9; and $c = high$ restricts the density to be less than around 10%.

TABLE 2 Properties of Stable Networks for Pure Closeness Incentives $\lambda = 0$ (Enum. $n = 8$)

	All networks	Stable networks
Number of networks	12,346	45
Number of trees	253	19
Number of connected networks	11,117	43
Mean number of links	14	9.09
Mean of AV'DIS	1.779	2.149
Mean of AV'DIS'C	1.563	1.982

3.1.2. Enumeration. While the formal derivations provide upper bounds, the enumeration reveals to what extent the set of all stable networks for closeness incentives satisfies the three properties of interest (sparsity, connectedness and short distances). Table 2 shows the enumeration results. The first column describes the properties of all nonisomorphic networks and serves as a benchmark; the second column contains all stable networks for closeness incentives.

The first two rows of the table show that out of the 12,346 nonisomorphic networks, only 253 (that is 0.2%) networks are trees. While for the set of stable networks for pure closeness $\lambda = 0$ there are 19 trees, which makes 42%.[17] To interpret the rows in the middle, note that a tree of size 8 is connected with exactly 7 links. The table shows that, indeed, most stable networks are connected and sparse with an average of 9 links per network. The last two rows assess the distances. The average distance, AV'DIS, measures the distance between any pair of actors in a network (using $M = n = 8$ for not connected pairs); AV'DIS'C only considers connected pairs. The set of stable networks exhibits relatively high distances. While a star has an average distance of 1.75, in the set of stable networks there are many with higher distances. In fact, only three of the 45 stable networks exhibit a lower average distance than an arbitrarily chosen network. This is the only aspect of star-like networks that is not clearly matched in the set of stable networks. The stable networks are sparse and connected, but do not exhibit short average distances compared to an arbitrary network. Of course, one needs to realize that denser networks tend to have shorter distances.

The enumeration results (above) do not differentiate by the level of c. Analytically, it is easy to show that trees can only be stable in the range $c \in \left[\frac{1}{(M-1)(n-1)}, \frac{1}{(n-1)}\right]$ (for $\lambda = 0$).[18] Above this range there are only few stable networks. For example, using again the enumeration for $n = 8$, there are three networks that are stable for higher costs. Those are the empty network, the circle network and a network consisting of a circle of size 7 plus one isolate. Let us now analyze which networks emerge within the cost range of trees.

3.1.3. Simulation. We ran a simulation with three settings of c where trees are stable, starting with any possible network for $n = 8$. Table 3 shows the frequency

[17]For other weights ($\lambda = 0.1, 0.2, \ldots, 1$), the fraction of trees is not above 22%.

[18]Below that cost range the complete network is uniquely stable as shown in Proposition 1; above this range no network with loose ends can be stable as will be shown in Remark 2.

TABLE 3 Fraction of Trees Emerging for Closeness Incentives (Sim. $n = 8$)

	Low cost	Medium cost	High cost
Stable networks	12	10	20
Trees emerging	1.0%	11.4%	90.7%
Star emerging	1.0%	0.6%	0.1%
Number of links	12.29	8.58	7.09
Average distance	1.56	1.90	2.34

with which a tree and specifically the star network emerges as well as the number of links and average distance of the emerging networks. The average distance equals the average distance between connected actors since all the emerging networks must be connected for $c < \frac{1}{n-1}$.

Table 3 shows that the star network itself is not a good prediction for the dynamics of closeness.[19] Trees are the dominant structure for high levels of c. The fourth row shows the average number of links for the emerging networks. The emerging networks are sparse, but become denser when c is reduced.

By drawing all of the frequently emerging networks in this simulation, we made the following observations: For $c = medium$, the dominant architecture consists of loose ends and some links forming a circle of size 4 or 5 (but not smaller). For $c = low$, we find more of these circles in the dominant architecture, but there are typically no loose ends.

Summarizing, the expectation that the dynamics of closeness lead to star-like networks is partially confirmed by the three methods. Stable networks be sparse and must not contain long distances (formal derivations). Virtually all stable networks are connected and sparse (enumeration). The star network rarely emerges, although for high costs, the typical emerging networks are trees (simulation).

3.2. Dynamics of Betweenness

For pure betweenness ($\lambda = 1$), every actor is striving for brokerage opportunities. This is similar to three models that are based on Burt's idea of structural holes. Buskens and Van de Rijt (2008) find that complete bipartite networks are the most likely outcome of network dynamics. Willer (2007) finds the circle network as the most likely to emerge, but since he only considers networks up to size $n = 4$, the circle network cannot be distinguished from a balanced complete bipartite network. In the model of Goyal and Vega-Redondo (2008), actors not only seek brokerage opportunity but also derive benefits from the size of their component and try to avoid being mediated by others. With a strong notion of stability, they find the star network as most likely outcome. Since in our model for $\lambda = 1$ actors only optimize their brokerage benefits, we expect that the dynamics most closely resemble the results of Buskens and Van de Rijt (2008).

[19]This result is consistent with the argument of Watts (2001) analyzing the dynamics of the connections model. In particular, she shows that for a dynamic process like the one we consider here (in the simulation), the probability that the star network is reached, converges to zero for n going to infinity.

TABLE 4 Properties of Stable Networks for Pure Betweenness Incentives $\lambda = 1$ (Enum. $n = 8$)

	All networks	Stable networks
Number of networks	12,346	37
Number of connected networks	11,117	19
Mean number of links (for connected subset)	14 (14.41)	12.7 (16.16)
Mean of AV'DIS'C	1.563	1.466
Fraction of networks with diameter of 2	38.6%	86.5%
Mean fraction of 3-circles	13.3%	9.0%
Number of networks without any 3-circle	3.3%	54%

Bipartite networks are characterized by not containing any circle of odd length. Since this precludes 3-circles, bipartite networks cannot be extremely dense. However, complete bipartite networks are quite dense and contain only distances of length 1 and 2.

3.2.1. Formal Derivations. Similar to the case of closeness incentives, one can formally restrict the distances of stable networks by considering what two (distant) actors gain from linking.

Proposition 4. *In the centrality model with $\lambda = 1$, the following holds: (i) Any network with a diameter of size $p(\geq 4)$ or larger is not stable if $c < \frac{(\lfloor \frac{p}{2} \rfloor - 1)\lfloor \frac{p}{2} \rfloor}{(n-1)(n-2)}$. Moreover, (ii) for sufficiently low c, any network with a diameter of three or larger is not stable.*

Proposition 4 (ii) shows that, in line with the expectation of complete bipartite networks, only distances of 1 and 2 occur between connected actors in stable networks. If the stable networks are complete bipartite, they are also connected and furthermore a pair of actors at distance 2 is not directly linked since 3-circles are precluded in bipartite networks. However, formal results on the (non)existence of circles in stable networks (as well as results on the average degree of stable networks) are more challenging for betweenness incentives. To establish which other stable networks exist for small network size, we turn to the enumeration.

3.2.2. Enumeration. Table 4 shows to which extent the stable networks for $\lambda = 1$ satisfy the expected properties of being connected, not containing a 3-circle, and having a diameter of 2.

First, it is notable that almost 50% of the stable networks are disconnected.[20] This fact has to be considered when interpreting the other statistics. The mean density for the stable networks is lower than in an arbitrary network, but this can be explained by the overrepresentation of disconnected networks. The last rows of the table show that a considerable number of stable networks satisfy the requirement of not containing a circle of length 3 (and that stable networks have fewer 3-circles on average). The middle rows of the table show that the distances (between connected actors) of the stable networks are short and, indeed, there are few networks with distances larger than 2.

Those results give a first suggestion that the stable networks resemble complete bipartite networks. Let us now check how many of the emerging networks really are

[20]We will provide an explanation for this observation in Subsection 4.4.

TABLE 5 Fraction of Complete Bipartite Networks (CBs) Emerging (Sim. $n = 8$)

	Epsilon costs	Very low costs	Low cost
Stable networks	19	9	4
All CBs with or without isolates	40.4%	78.3%	61.1%
CBs (2:6, 3:5, 4:4) without isolates	29.0%	38.7%	0.9%
Balanced CBs (4:4, 3:3, 2:2) with or without isolates	13.4%	37.6%	61.1%
Balanced CB (4:4) without isolates	12.5%	25.4%	0.9%

complete bipartite when starting with different values of c. In our model, it can be shown that any complete bipartite network can only be stable for $c \leq \frac{2-\frac{4}{n}}{(n-1)(n-2)}$. Nonetheless, as the enumeration reveals, there are not many stable networks above this range: only 9 out of 37 stable networks for $n = 8$. Of the other 28 stable networks, many resemble complete bipartite networks. Some of them do not belong to this class in a strict sense, for example, a network with two isolates and a (4:2)-complete-bipartite component.

3.2.3. Simulation. We ran the simulation for three settings of c, where complete bipartite networks might be stable. Table 5 presents the frequency of emergence for different sets of complete bipartite networks (with at least two actors in each group). It is notable that for $c = low$ the empty network emerges in 20.8% of the cases and for costs higher than depicted ($c = medium$ and $c = high$) the empty network emerges in 99.9% (resp. 94.0%) of the simulation runs, while also the circle network is stable. As the table shows, the class of complete bipartite networks is, indeed, the dominant structure. Moreover, it can be observed that for small costs c, rather the connected ones emerge; for higher costs c, rather the ones with the same group size emerge. The balanced complete bipartite network (which has groups of the same size and is connected), is the most frequently emerging network.

The expectation that the dynamics of betweenness lead to complete bipartite networks is confirmed. The stable networks exhibit similar properties; that is, they are dense, have short distances, and frequently do not contain 3-circles (formal derivation and enumeration). The typical emerging structure, besides the empty network, is a complete bipartite component with possibly some isolates (simulation).

Having characterized the emerging networks for pure closeness incentives and for pure betweenness incentives, the next question is how those results carry over to a scenario with combined incentives.

4. INTERACTION OF CLOSENESS AND BETWEENNESS INCENTIVES

In this section we let the relative importance of closeness and betweenness vary, i.e., $0 < \lambda < 1$.

4.1. Formal Derivations

Let us first have a look at some prominent networks.

- In the empty network $\overline{K_n}$, adding a link only increases closeness for the actors involved, while their betweenness remains zero. Therefore, the empty network is stable if this marginal benefit derived from a change in closeness centrality is smaller than the linking costs.

- Similarly, in the complete network K_n, removing a link decreases the closeness of the actor involved, while his betweenness remains zero.
- Since any link is a bridge in the star network $K_{1,n-1}$, the dissolution of a link leads to a substantial reduction of closeness for both actors involved. The addition of a link does not increase the betweenness and hardly increases the closeness of the two (peripheral) actors.
- In the circle network C_n, an additional link across the circle provides a significant amount of both closeness and betweenness benefits. Removing a link also reduces both betweenness as well as closeness for both actors involved. Rather than dissolving a link, two actors are willing to form an additional one, across the circle. Therefore, the circle network can be expected to be stable only for relatively high linking costs.
- In the balanced complete bipartite network K_{n_1,n_1}, all actors have some betweenness as well as high closeness, because distance is at most 2 and each pair that is at distance 2 is mediated by all the actors in the other group. Adding a link is not very beneficial in terms of closeness and betweenness, while the loss of removing a link is a bit larger. Consequently, if links are rather cheap, but not too cheap these networks can be stable.

These observations lead to Proposition 5 presenting the parameter combinations for which the five prominent network structures are stable.

Proposition 5. *In the centrality model the following holds:*

1. *The complete network K_n is stable if and only if $c \leq \frac{1-\lambda}{(n-1)(M-1)}$.*
2. *The empty network $\overline{K_n}$ is stable if and only if $c \geq \frac{1-\lambda}{n-1}$.*
3. *A star network $K_{1,n-1}$ is stable if and only if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$.*
4. *Let n be a multiple of 4. Then a circle network C_n is stable if and only if $\frac{(1-\lambda)[\frac{1}{8}n^2-\frac{1}{2}n+1]}{(M-1)(n-1)} + \frac{2\lambda[\frac{1}{8}n^2-\frac{3}{4}n+1]}{(n-1)(n-2)} \leq c \leq \frac{(1-\lambda)[\frac{1}{4}n^2-\frac{1}{2}n]}{(M-1)(n-1)} + \frac{2\lambda[\frac{1}{8}n^2-\frac{1}{2}n+\frac{1}{2}]}{(n-1)(n-2)}$.*
5. *The balanced complete bipartite network K_{n_1,n_1} (for even n) is stable if and only if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq \frac{2(1-\lambda)}{(n-1)(M-1)} + \frac{2\lambda[1-\frac{2}{n}]}{(n-1)(n-2)}$.*

Figure 2 illustrates Proposition 5 depicting the parameter space with weight λ on the horizontal axis and cost c on the vertical axis. It indicates the "regions" of the parameter space where the complete network, the star network, the balanced complete bipartite network and the circle network are stable.[21] The empty network is stable above the dotted line.

[21]Results look different for small network size and slightly different for networks with an odd number of actors.

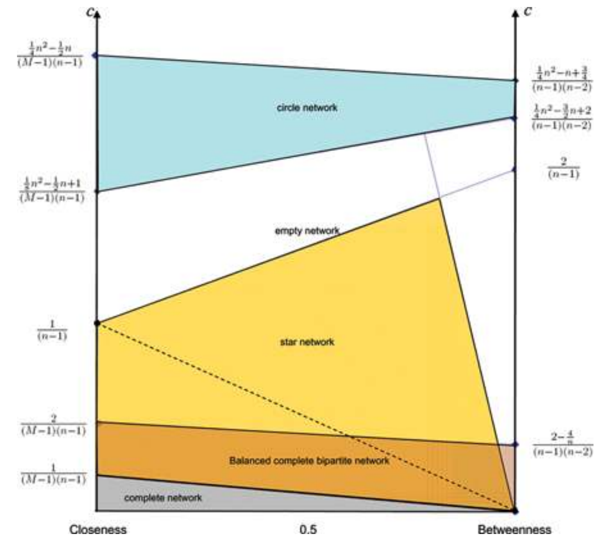

FIGURE 2 "Parameter map" with stability for some prominent networks (color figure available online).

The complete network is stable for $\lambda < 1$ if costs c are low enough. The empty network is trivially stable for $\lambda = 1$ and stable for $\lambda < 1$ if costs are not too low.[22] The lower bound for the star network and the balanced complete bipartite network coincides with the upper bound for the complete network. The upper bound for the star network is first increasing and then decreasing in λ. We will return to this point in Subsection 4.4. For $\lambda = 1$, the star network is not stable. Figure 2 indicates that the balanced complete bipartite network and the circle network, both can be stable for any weight λ. While the circle networks are high cost phenomena, the complete bipartite networks are low cost phenomena.

The result shows that for the five prominent networks the conditions for stability (lower or upper bounds of c) are linear in λ (respectively, piecewise linear in λ for the star network). While three of them can be stable for any λ, the star and the complete network are only stable if $\lambda < 1$. We use the enumeration to examine how many other networks are stable for certain λ's.

4.2. Enumeration

By enumeration we can compare all stable networks for different incentives. Figure 3 depicts the number of stable networks for different λ's (for $n = 8$). The networks are shaded by the range of λ for which they are stable. The first observation is that there are more stable networks for each level of mixed incentives than for pure incentives ($\lambda \in \{0, 1\}$). All 45 networks that are stable for closeness incentives ($\lambda = 0$) are also stable for some other λ. Eight of them are stable for any λ (e.g., the empty network); 15 are stable for any λ, but pure betweenness (like the star or the complete

[22]The condition for stability of the complete network of Proposition 5.1 coincides with the condition for its uniqueness of Proposition 1 (setting the inequality strict). This is not true for the empty network: the threshold for uniqueness is larger (in terms of costs) than the threshold for stability.

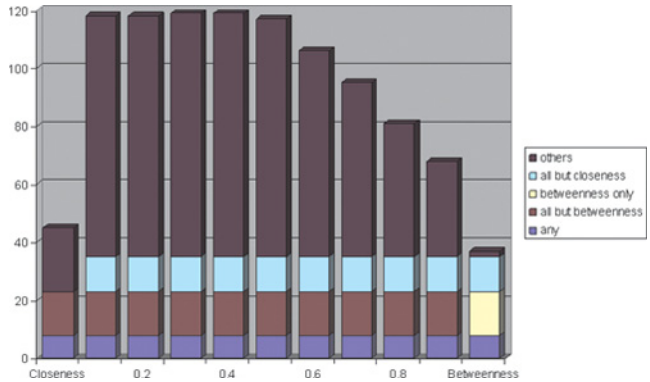

FIGURE 3 Number of stable networks by λ (enum. $n = 8$; color figure available online).

network). For pure betweenness ($\lambda = 1$), there are 37 stable networks. Fifteen of them are not stable for any other λ (we used). These networks consist of disconnected components in which actors want to connect the components as soon as a small amount of closeness incentives is introduced. Only three networks of the other categories are found stable for only one weight. The other stable networks for betweenness are typically also stable for any other λ (e.g., the balanced complete bipartite network or the empty network). Thus, there is strong indication that the stable networks across certain λ's do not differ heavily, except for the case of pure incentives. Since most of the stable networks are neither stable for pure closeness nor for pure betweenness but for mixed incentives, many candidates of emerging networks are not covered by pure incentives.

Measuring certain properties of the set of stable networks further indicates that pure incentives are special cases. As an example, Figure 4 shows the boxplots of the density (indicating the mean and the quartiles) for the different sets of stable networks. (A mean density of 0.4 means that in a particular set of networks [on

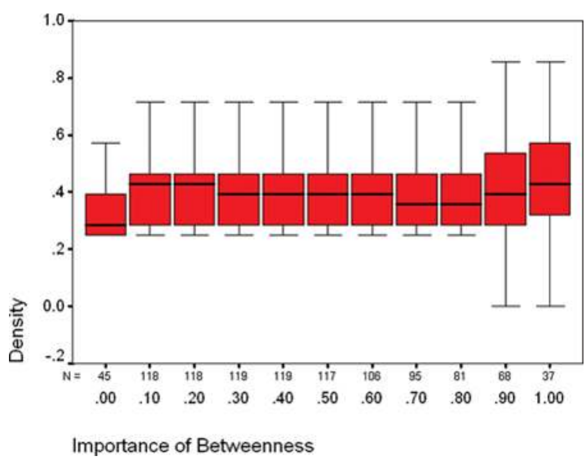

FIGURE 4 Distribution of density in stable networks (enum. $n = 8$; color figure available online).

average] 40% of all possible links are present.) While the stable networks for different mixed incentives have a similar distribution of density, the density of the stable networks for pure closeness ($\lambda = 0$) and for pure betweenness incentives ($\lambda = 1$) differs. Stable networks for betweenness incentives are denser on average, while variance is larger.

We can summarize that introducing a bit of betweenness (closeness) incentives into a pure closeness (betweenness) model heavily increases the number of stable networks and that stable networks for mixed incentives do not have the same properties as the stable networks for pure incentives. In that sense, the enumeration reveals that, although the weighting of benefits in our model is smooth (the benefits are a linear combination of closeness and betweenness), the results exhibit jumps. Before explaining why such phenomena occur, let us have a look at the simulation results.

4.3. Simulation

A necessary condition for a network to emerge in a dynamic process (like the simulation we use) is stability. So the stable networks found in the enumeration (for a certain λ) are now the candidates for emerging networks in the simulation (for this λ and different settings of c).

Table 6 presents the simulation results for $n = 8$. CONNECTED stands for the fraction of connected networks; LINKS for the number of links (which is proportional to the density); DEG'VAR stands for the variance of degree; AV'DIS measures the average distance (between all pairs) in a network; AV'DIS'C stands for

TABLE 6 Properties of Emerging Networks (Sim. with $n = 8$)

WEIGHT	COSTS	CONNECTED	LINKS	DEG'VAR	AV'DIS	AV'DIS'C
All networks		90%	14.0	0.42	1.78	1.56
0 (Closeness)	Very low	100%	28.0	0.00	1.00	1.00
	Low	100%	12.3	0.75	1.56	1.56
	Medium	100%	8.6	1.04	1.90	1.90
	High	100%	7.1	1.05	2.34	2.34
0.1	Very low	100%	28.0	0.00	1.00	1.00
	Low	100%	12.3	0.76	1.56	1.56
	Medium	100%	8.7	1.04	1.88	1.88
	High	100%	7.2	1.01	2.30	2.30
0.5	Very low	100%	17.4	0.66	1.38	1.38
	Low	100%	11.5	1.43	1.61	1.61
	Medium	100%	8.4	1.10	1.93	1.93
	High	100%	7.2	0.97	2.30	2.30
0.9	Very low	100%	15.1	0.65	1.47	1.47
	Low	100%	12.4	1.41	1.64	1.64
	Medium	100%	8.1	1.52	2.04	2.02
	High	0%	0.4	0.03	7.76	2.00
1 (Betweenness)	Epsilon	83%	17.2	1.03	1.69	1.36
	Very low	45%	13.2	1.17	2.65	1.41
	Low	1%	6.5	1.35	5.39	1.42
	Medium	0%	0.0	0.00	7.99	1.71
	High	0%	0.4	0.03	7.73	2.00

TABLE 7 Connectedness of Stable Networks (Enum. $n = 8$)

	0	0.1	0.3	0.5	0.7	0.9	1
Number of connected networks	43	116	118	116	94	65	19
Fraction of connected networks	96%	98%	99%	99%	99%	96%	51%

the average distance between connected actors. Since the simulation starts with any possible network for $n = 8$, the first row stands for the properties of the starting networks, while the values of the emerging networks can be interpreted as estimations for an arbitrary starting network.

Throughout any weight λ, there are some clear-cut relations between the costs of linking c and the properties of the emerging networks. The higher the costs, the lower the density and the higher the average distances. Higher costs also increase the probability that an emerging network is disconnected.[23]

The effects of different incentives, however, are not trivial. Changing the setting from $\lambda = 0$ to $\lambda = 0.1$ increases the candidates for emerging networks from 45 to 118 (as found by enumeration), but the properties of the emerging networks (e.g., LINKS) do not seem to be heavily affected. For the change from $\lambda = 1$ to $\lambda = 0.9$ there is a more drastic effect, for example, for the property CONNECTED: The emerging networks for pure betweenness incentives are frequently disconnected. Besides connectedness, none of the properties is influenced by the weight λ in one specific direction.

4.4. Interplay of Closeness and Betweenness Incentives: Connectedness

In this section we have observed that the interaction between closeness and betweenness incentives leads to nontrivial dynamics. The enumeration reveals that most of the candidates for emerging networks are not found for pure incentives but for mixed incentives. Moreover, the emerging networks for $\lambda = 1$ (pure betweenness) substantially differ from the emerging networks in the other settings. To understand why such phenomena occur when mixing different incentives, let us analyze the interplay of closeness and betweenness incentives focusing on one structural feature: connectedness. We first explain why many of the stable and emerging networks for $\lambda = 1$ are not connected and then show why many of the stable networks for $\lambda < 1$ are not stable for $\lambda = 1$.

Table 7 shows the number and fraction of stable networks that are connected. One can observe that most of the stable networks are connected, except for $\lambda = 1$. Among the stable networks for $\lambda = 1$, 18 are disconnected; for $\lambda = 0.9$ this reduces to three (enumeration for $n = 8$). For many mixed incentives $\lambda \in \{0.3, \ldots, 0.8\}$ only the empty network is not connected and stable. Thus, there seems to be an inverse "u-shaped" relation between the weight λ and the number of connected networks: For mixed incentives, networks are more often connected than for pure incentives.

[23]We did not run simulations for very high costs, where the empty network is expected to emerge in most cases.

To form a connected network, the addition of bridges (links connecting two components) is necessary. An actor who forms a link to the other component can increase his closeness and his betweenness substantially depending on the sizes of the two components. Consider two actors i and j in different components of size $l+1$ (respectively $r+1$). It can be shown that in the centrality model actor i gains from a link to j by at least $(1-\lambda)\frac{(r+1)(M-\frac{1}{2}r-1))}{(M-1)(n-1)}+\lambda\frac{2l(r+1)}{(n-1)(n-2)}$.[24] The minimal threshold is attained when the r-component forms a line, because then the marginal closeness for actor i is minimal. Since there is a strong incentive to establish bridges, c must be very high in order to avoid that two actors in different components form a link. This is expressed in Remark 1.

Remark 1. *In the centrality model the following holds: If* $c<(1-\lambda)\frac{2M-3}{(M-1)(n-1)}+\lambda\frac{4}{(n-1)(n-2)}$, *the stable networks contain at most one nontrivial component.*

From the remark we can conclude that if a stable network is not connected, it usually consists of isolates (singleton components) in addition to one larger component. In fact, the enumeration does not yield any stable network with multiple non-trivial components. Why we would rather observe isolates in the stable networks for $\lambda=1$ and for $\lambda=0$ but not for mixed incentives is explained below.

4.4.1. Integration of Isolates. Consider a network G with an isolated actor i and an actor j who is already part of a larger group. Then, when closeness only matters ($\lambda=0$), actor i has a strong interest in the link ij, as this link is his first connection to the network (without ij, $C_C(i)=0$). Actor j's interest is restricted: creating ij means being directly connected to i, but does not have an impact on any other distance (there is no indirect benefit). So for high enough linking costs c, i is willing to link with j, but j rejects this offer. When betweenness only matters ($\lambda=1$), actor j has a high interest in the link ij, because it provides a substantial amount of betweenness. On the other hand, i is not interested in this link as his betweenness is zero with or without ij. So the link will not be formed. Finally, when both incentives matter ($\lambda\in[\epsilon, 1-\epsilon]$), the link can be formed because both actors do have a rather high interest in this link, but for different reasons: i wants to have access to the community (closeness incentives); j enjoys mediating i with all his connections (betweenness incentives).

This example suggests that networks with isolates are rather not stable for mixed incentives because two actors will add a link, while this is not necessarily true for $\lambda=1$ and $\lambda=0$. In fact, we have observed that several networks with isolates are stable for $\lambda=1$ but not for $\lambda<1$—this is because introducing closeness benefits ($\lambda<1$) would justify also for the isolated actor to add a link.

While the integration of isolates sheds some light on the puzzle why we observe that many of the stable networks for $\lambda=1$ are not stable for $\lambda<1$, it does not directly explain why we observe that many of the stable networks for $\lambda<1$ are not stable for $\lambda=1$.

[24]Interestingly, adding actors (increasing the l and r) has an additive effect on the change in closeness, but a multiplicative effect on the change in betweenness of i and j.

4.4.2. Stability of Loose Ends. The integration of an isolate is the complementary action of cutting a link to a pendant (actor at a loose end). Since cutting a link can be done unilaterally, the stability of a network with pendants is based on the minimum of two marginal benefits as shown in Remark 2.

Remark 2. *In the centrality model the following holds: If* $c > min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$, *no network with pendants (actors of degree one) is stable.*

As can be seen in Figure 2, the upper bound for the star network is exactly the boundary for loose ends. The costs for which no network with pendants can be stable is piecewise linear in λ, increasing first and then decreasing. The argument is the same as before (cf. the integration of isolates). For pure closeness incentives the neighbor of the pendant has limited interest in the link. This interest increases with the introduction of betweenness benefits. When λ approaches one, the pendant's interest in the link diminishes because the weight of closeness is decreasing and his marginal betweenness is zero.

For $\lambda = 1$, this threshold is zero such that no network with pendants can be stable (since we always assume in our model that $c > 0$). This excludes, among other networks, all trees from being stable. Therefore, several stable networks for $\lambda < 1$ are not stable for $\lambda = 1$. Moreover, it need not always be pendants who render many networks unstable. Generally, for $\lambda = 1$ many networks fail to be stable because actors do not have any incentive to keep a link, because the link is not a shortest path between any two other actors.[25] Introducing a bit of closeness benefits can justify keeping these relationships.

Analyzing the interplay of closeness and betweenness incentives provides an example of why network dynamics of multiple incentives are more complex than the dynamics of each type of incentives separately. Moreover, it shows another point. Although all actors do have the same preferences, that is, the same utility function, the formation of links can be driven by very different motives, based on different network positions.

5. CONCLUDING REMARKS

The innovations of this article are three-fold. First, although both betweenness and closeness centrality are cornerstones of social network analysis, it has hardly been explicitly studied which networks will emerge if actors follow incentives for these two positional advantages. We formulate such a model and derive the stable networks for each of the incentives separately. By also including costs for the number of links, we have covered degree centrality, the third centrality measure from the classic article by Freeman (1979). The characterization of the emerging networks can be illustrated in Figures 5 and 6 depicting two of the most frequently emerging networks. Typically, the dynamics of closeness lead to sparse networks, which are connected (trees). In the depicted network, the closeness centrality of actor 8 is high, while the centrality of all other actors is moderate. The dynamics of betweenness

[25]In 75% of all nonisomorphic networks ($n = 8$), some actors are willing to sever a link even for the smallest costs $c = \epsilon$ for pure betweenness $\lambda = 1$.

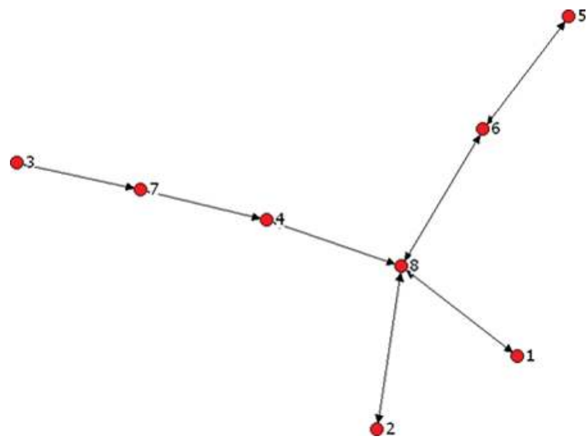

FIGURE 5 Very frequently emerging network for closeness incentives (sim. $n = 8$; color figure available online).

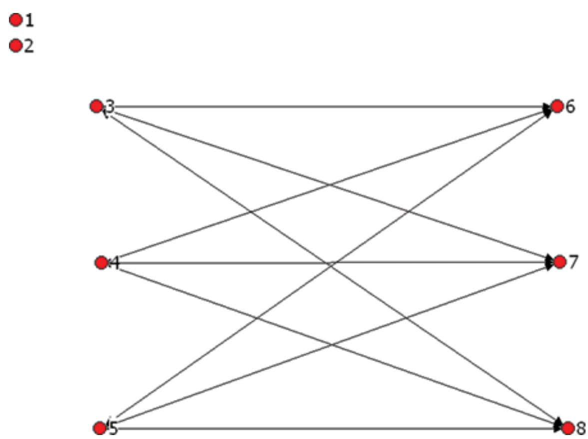

FIGURE 6 Very frequently emerging network for betweenness incentives (sim. $n = 8$; color figure available online).

typically lead to networks with isolates and a dense component which is bipartite (there are two groups without intragroup links). In such a network betweenness centrality is zero for the isolates and positive (but not high) for all actors in the component.[26] The distribution of the (betweenness) benefits depends on the group sizes in the bipartite component. In the depicted network, groups are of equal size with the implication that betweenness benefits are evenly distributed.

Second, we discuss the relation between our findings and earlier findings on related models. For closeness, there are some closely related exercises (Fabrikant et al., 2003; Corbo & Parkes, 2005) and the results of our model are comparable

[26]Indeed, the network depicted in Figure 5 has much higher average betweenness than the network depicted in Figure 6.

to these earlier results although none of these earlier studies explored the emerging stable networks using a combination of formal derivation, enumeration, and computer simulation. The earlier results only provided partial characterizations of stable networks, while we were able to summarize the complete set of stable networks at least for small network size. Our results as well relate to studies on the connections model (Jackson & Wolinsky, 1996) because also in that model actors' incentives are based on short distances. Although the stable network predominantly discussed in the literature for the connections model is a star network, we find that with closeness incentives the star can be stable but is not one of the frequently emerging networks. The star differs from the typically emerging networks in our simulations by its short distances and its extreme centralization (benefits in the star network are higher but less evenly distributed compared to our findings).

Incentives for betweenness are related to models based on structural holes because in both types of models actors strive to be between other actors. Our results strongly resemble results also found by Buskens and Van de Rijt (2008) for the dynamics of structural holes, namely, that the frequently emerging networks are complete bipartite networks and, in particular, the network with equal group sizes emerges frequently. This is less than self-evident from the outset because structural holes (as defined by Burt, 1992, and used by Buskens & Van de Rijt, 2008) are only about mediation over short distances while betweenness also values mediation over long distances. In our study, as well as in Buskens and Van de Rijt, star networks are not stable or are stable only for very small networks in contrast to results of Goyal and Vega-Redondo (2008) who find the star as the main stable network. The reason why Goyal and Vega-Redondo do not find complete bipartite networks as stable networks is most likely that in complete bipartite networks (except for stars) it is the case that everybody is mediating everybody.

Third, and maybe most importantly, there has hardly been any theoretical work that studies the interplay between different types of incentives to predict network formation processes. When combining incentives for closeness and betweenness, we find results that are not straightforward extensions of considering them separately. For a combination of closeness and betweenness incentives, many networks emerge that do neither emerge under closeness nor under betweenness incentives. We provide an explanation of this phenomenon based on the observation that two actors, despite similar preferences, can have quite different motivations of action. In the particular example of an isolated actor linking with a well-connected actor, the motivation of the isolated actor is access to the group (derived from closeness centrality), the motivation of the well-connected actor is his mediating position (derived from betweenness centrality) between the group and the isolated actor.

This last result shows that for understanding the emergence of real world networks, it can be crucial to consider multiple important network characteristics simultaneously. So far, most theories on network dynamics have studied a single type of incentive, which mostly resulted in very stylized networks such as stars or complete bipartite networks. While one suggestion to obtain more realistic networks is to assume that actors have heterogeneous preferences in networks, our study shows that with multiple incentives, this assumption is not necessary. The path dependency of the network formation process leads to different incentives being salient depending on the network position. This thus provides an alternative for obtaining networks with less stylized or more heterogeneous network positions to be stable even without starting with actors having heterogeneous preferences.

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APPENDIX: PROOFS OF THE PROPOSITIONS

A.1. Proofs of Section 2

Proposition 1. *In the centrality model there exists at least one stable network for any parameters $(\lambda, c) \in [0, 1] \times \mathbb{R}_+$. Moreover, if $c < \frac{1-\lambda}{(n-1)(M-1)}$, the complete network K_n is uniquely stable.*

Proof of Proposition 1. The two statements are independent.

- Existence follows almost directly from Proposition 5 of Section 4. For $\lambda = 1$, the empty network $\overline{K_n}$ is stable at any cost c, because a first link does not provide any betweennness. For $\lambda < 1$, the empty network $\overline{K_n}$ is stable if $c \geq \frac{1-\lambda}{n-1} =: \inf(\overline{K_n})$, the complete network K_n is stable if $c \leq \frac{1-\lambda}{(n-1)(M-1)} =: \sup(K_n)$ and the star network $K_{1,n-1}$ is stable if $\inf(K_{1,n-1}) \leq c \leq \min\{\sup_1(K_{1,n-1}), \sup_2(K_{1,n-1})\}$, with $\inf(K_{1,n-1}) := \frac{1-\lambda}{(n-1)(M-1)}$, $\sup_1(K_{1,n-1}) := \frac{1+\lambda}{n-1}$, and $\sup_2(K_{1,n-1}) := (1 - \lambda)$

$[\frac{M}{M-1} - \frac{2n-3}{(M-1)(n-1)}]$. It remains to be shown that if $\overline{K_n}$ and K_n are not stable, $K_{1,n-1}$ is stable. This follows directly from $\inf(K_{1,n-1}) = \sup(K_n)$, $\sup_1(K_{1,n-1}) \geq \inf(\overline{K_n})$ as $\frac{1+\lambda}{n-1} \geq \frac{1-\lambda}{n-1}$, and $\sup_2(K_{1,n-1}) \geq \inf(\overline{K_n})$ (by definition $n \geq 3$ and $M \geq n-1$, which implies $\frac{M(n-1)-2n+3}{M-1} \geq 1$).

- (a) The complete network is stable if no actor wants to cut a link. For $c \leq \frac{1-\lambda}{(n-1)(M-1)}$ this is true because the change in distances of cutting a link is 1, while the change in brokerage is 0. (b) Take any network $G \in \mathcal{G}\backslash\{K_n\}$. $\exists(i, j)$: $d(i, j) > 1$. By connecting their closeness increases by at least $\frac{1}{(n-1)(M-1)}$. So for $c < \frac{1-\lambda}{(n-1)(M-1)}$ the network will be unstable as i and j strictly improve by forming the link. □

A.2. Proofs of Section 3

Proposition 2. *In the centrality model with $\lambda = 0$, the following holds: The diameter of a stable network is smaller or equal to p, with $p = max\{\sqrt{4c(n-1)(M-1)+1}, 1\}$.*

Proof of Proposition 2. We show that in a network with a diameter of $d > p$, there exists a pair of actors who can increase their utility by forming a link. Take any network G with a diameter of $d > p \geq 1$.[27] Let i and j be two actors at maximal distance ($d(i, j) = d$) and consider one shortest path between them. By forming the link ij, actor i does not only decrease his distance to j, but also to some actors on this shortest path. Let $\Delta(d)$ stand for the change in distances stemming from that path. It is easy to derive that

$$\Delta(d) = \begin{cases} 2+4+6+\ldots+d-3+d-1 = \frac{1}{4}d^2 - \frac{1}{4}, \text{for odd } d \\ 1+3+5+\ldots+d-3+d-1 = \frac{1}{4}d^2, \text{for even } d. \end{cases} \quad (5)$$

This implies that from network G to network $G \cup ij$ the closeness of actor i changes by at least $\frac{d^2-1}{4(M-1)(n-1)}$. This also holds for j. It remains to be shown that the marginal costs c are lower than this marginal benefit. $p = \sqrt{4c(n-1)(M-1)+1}$ implies that $c = \frac{p^2-1}{4(M-1)(n-1)}$. Since $p < d$, the marginal costs are smaller than the marginal benefit. □

Proposition 3. *In the centrality model with $\lambda = 0$, the following holds: If $c > \frac{9n}{16(n-1)(M-1)}$, $d(G) < \frac{1}{2}n + \frac{1}{2}$ for any stable network G and if $c > \frac{4n}{5(n-1)(M-1)}$, $d(G) < \sqrt{n}$ for any stable network G.*

The proof of Proposition 3 relies on the following lemma:

Lemma 1. *In the centrality model with $\lambda = 0$, the following holds for $q \in \mathbb{N}$ with $q \geq 3$: If $c > \frac{n[q-2+\frac{1}{4}(q-3)^2]}{q(M-1)(n-1)}$, then any network with a circle of size q or smaller is not stable.*[28]

[27]Recall that the diameter is not M by definition.

[28]By definition of circles, we do not consider "circles" of size 2, 1, or 0.

Proof of Lemma 1. Consider a circle C_q of size q, with $3 \leq q \leq n$. Let N_q denote the set of q actors in the circle and let N_{-q} be the $n-q$ other actors. For a network that contains this circle $G \supseteq C_q$, and a link $ij \in C_q$, let $\psi(i, j, G)$ be the change in closeness for actor i when link ij is removed from network G, i.e. $\psi(i, j, G) := C_C(i)(G) - C_C(i)(G \backslash ij)$. Note that $\psi(i, j, G) \neq \psi(j, i, G)$ in general. Clearly, G is only stable if $\psi(i, j, G) \geq c$ for any ordered pair (i, j) such that ij in C_q.

Let n_1 be the number of actors in N_{-q} who are connected with N_q (in network $G \supseteq C_q$). And assume for the moment that q is odd.

- Suppose $n_1 = 0$. Then $\psi(i,j,G) = \frac{(q-2)+\frac{1}{4}(q-3)^2}{(M-1)(n-1)} =: \overline{\psi}$ for any pair (i, j) such that $ij \in C_q$. The derivation is very similar to that of the proof about the diameter in Proposition 2.
- Suppose $0 < n_1 \leq q$. Then $\min\limits_{(i,j):ij \in C_q} \psi(i,j,G) \leq 2\overline{\psi}$. If $n_1 = q$, it is possible to increase the change in closeness for any actor by a factor 2; that is, there is a network with $\psi(i,j,G) = 2\overline{\psi}$ for any (i, j) such that $ij \in C_q$. To see this let each actor in N_q be linked with exactly one of the n_1 actors and there are no additional links. In this particular network, the n_1 additional actors are equally allocated around the circle. If there is a pair (i, j) in a network such that $\psi(i,j,G) > 2\overline{\psi}$, there must be another pair (i', j') with $\psi(i',j',G) < 2\overline{\psi}$ because in that case the additional actors are concentrated on a particular side of the circle. If $n_1 < q$, the increase might even be smaller than $2\overline{\psi}$ for all actors on the circle.
- Similarly, suppose $q < n_1 \leq 2q$. Then $\min\limits_{(i,j):ij \in C_q} \psi(i,j,G) \leq 3\overline{\psi}$, where the equality can be obtained if $n_1 = 2q$ and a component of two actors is attached to each actor on the circle without links between these attached components (and without any other links between actors in N_q).
- More generally, suppose $n_1 \leq zq$ for some natural number z. Then $\min\limits_{(i,j):ij \in C_q} \psi(i,j,G) \leq (z+1)\overline{\psi}$, where the equality can be obtained if $n_1 = zq$ and a component of z actors is attached to each actor on the circle without additional links.
- Finally, suppose $n_1 = n - q$. Then $\min\limits_{(i,j):ij \in C_q} \psi(i,j,G) \leq (\frac{n-q}{q} + 1)\overline{\psi} = \frac{n}{q}\overline{\psi}$. If $\frac{n}{q}$ is a natural number, the condition can hold tightly. (One can construct such a network by arranging the actors in N_{-q} into q components of size $\frac{n}{q} + 1$ and adding exactly one link from each actor in C_q to exactly one group.) If $\frac{n}{q}$ is not a natural number, the inequality still holds true because it implicitly assumes that change in closeness ($\psi(i, j, G)$) can be distributed equally, which is then not possible.

We have so far established that for any $G \supseteq C_q$ with q odd we have

$$\min_{(i,j):ij \in C_q} \psi(i,j,G) \leq \frac{n}{q}\overline{\psi} = \frac{n[q-2+\frac{1}{4}(q-3)^2]}{q(M-1)(n-1)}. \tag{6}$$

If q is even, the derivation is analogous and leads to a threshold that is slightly smaller than $\overline{\psi}$. Thus, the statement above also holds in this case. If $c > \frac{n}{q}\overline{\psi}$, any network containing a circle of size q is not stable (since there is an actor i with a neighbor j such that $c > \psi(i, j, G)$). Moreover, the expression $\overline{\psi}$ is increasing in q. Thus, for any network with a circle smaller than q, this is also true. □

Proof of Proposition 3. To prove the result, we employ Lemma 1 and combine it with the following theorem (Th. 1.3.4 in Diestel, 2005): Let $d(G) = \frac{1}{n}\sum_{i \in N} d(i)$ be the average degree and $q(G)$ the size of the smallest circle in G, which is defined to be large if there are no circles. Let $\delta \in \mathbb{R}$ and $\rho \in \mathbb{N}$. Then,

$$\text{if } [A]\ d(G) \geq \delta (\geq 2) \quad \text{and } [B]\ q(G) \geq \rho, \text{ then } [C]\ n \geq n_0,$$

$$\text{with } n_0 = \begin{cases} 1 + \delta \sum_{k=0,\cdots,\frac{\rho-3}{2}} (\delta - 1)^k \text{ for } \rho \text{ odd} \\ 2 \sum_{k=0,\cdots,\frac{\rho}{2}-1} (\delta - 1)^k \text{ for } \rho \text{ even.} \end{cases}$$

We now transform the logical structure of *[A]* and *[B]* implies *[C]* into *not[C]* and *[B]* implies *not[A]*.

To get *[B]*, we fix a certain ρ, here $\rho = 4, 5$, and use the proposition that for $c > \frac{n[\rho - 2 + \frac{1}{4}(\rho-3)^2]}{\rho(M-1)(n-1)}$ there are no circles of size ρ or smaller in stable networks (see Lemma 1 above). To get *not[C]*, we choose δ such that $n_0 = n+1$ (this is possible as n_0 is a function of δ and ρ). These conditions together imply *not[A]*, that means that $d(G) < \delta$.

We now use this procedure for different values of ρ: Let $\rho = 4$. Then *[B]* reduces to $c > \frac{9n}{16(M-1)(n-1)}$. *not[C]* is achieved by choosing $\delta = \frac{1}{2}n + \frac{1}{2}$ because it implies that $n = n_0 - 1 = -1 + 2\sum_{k=0,1}(\delta - 1)^k = 2\delta - 1$. *not[A]* means that $d(G) < \delta$. So we get the result: If $c > \frac{9n}{16(M-1)(n-1)}$, $d(G) < \frac{1}{2}n + \frac{1}{2}$.

Let $\rho = 5$. Then *[B]* reduces to $c > \frac{4n}{5(M-1)(n-1)}$. *not[C]* is achieved by choosing $\delta = \sqrt{n}$ because it implies that $n = n_0 - 1 = \delta \sum_{k=0,1}(\delta - 1)^k = \delta^2$. *not[A]* means that $d(G) < \delta$. So we get the result: If $c > \frac{4n}{5(M-1)(n-1)}$, $d(G) < \sqrt{n}$. □

Proposition 4. *In the centrality model with $\lambda = 1$, the following holds: (i) Any network with a diameter of size $p(\geq 4)$ or larger is not stable if $c < \frac{(\lfloor \frac{p}{2} \rfloor - 1)\lfloor \frac{p}{2} \rfloor}{(n-1)(n-2)}$. Moreover, (ii) for sufficiently low c, any network with a diameter of three or larger is not stable.*

Proof of Proposition 4. For part (i) we show that in a network with a diameter d, two (connected) actors at maximal distance increase their benefits at least with $\frac{(\lfloor \frac{d}{2} \rfloor - 1)\lfloor \frac{d}{2} \rfloor}{(n-1)(n-2)}$ by establishing a link between them. This implies for c below that level that the network is unstable. If a network has a larger diameter than d, it also contains two actors at distance d.

Take any network G with a diameter of $d \geq 4$. Let i and j be two actors at maximal distance $d(i, j) = d$ and consider one of their shortest paths. Consider two actors i' and j' on that geodesic such that $d(i, i') + d(j, j') < \frac{d-1}{2}$ ($i', j' \neq i$, but $j' = j$ is allowed). It holds that i is not on any shortest path between i' and j'. It must also hold that $d(i', j') = d - d(i, i') - d(j, j')$. The distance cannot be shorter because this would imply that there exists a shorter path for i and j to connect. The distance cannot be longer since there is this path on the geodesic.

Establishing ij adds a new path from i' to j' (that uses i). This path is of length $d(i, i') + d(j, j') + 1 := p(i', j')$. It is shorter than their former shortest path, as straightforward transformations show:

$$d(i,i') + d(j,j') < \frac{d-1}{2} \Leftrightarrow d(i,i') + d(j,j') + 1 < d - d(i,i') - d(j,j') \tag{7}$$

$$\Leftrightarrow p(i',j') < d(i',j'). \tag{8}$$

Thus, $\frac{g^{i}_{i'j'}(G \cup ij)}{g_{i'j'}(G \cup ij)} - \frac{g^{i}_{i'j'}(G)}{g_{i'j'}(G)} = \frac{1}{1} - 0 = 1$. In other words, actor i increases his brokerage since he is on all shortest paths between i' and j' now, which he was not before. In order to compute the minimal change in brokerage, one can compute the number of pairs whose distance shortens in dependence of d. The straightforward derivation yields the following (where $\chi(d)$ is the number of pairs whose distance shortens coinciding with the change in brokerage and $\lfloor x \rfloor$ stands for x rounded to the next lower integer):

$$\chi(d) \geq 1 + 2 + 3 + 4 + \ldots + \lfloor \frac{d}{2} \rfloor - 1 = \frac{1}{2}\left(\lfloor \frac{d}{2} \rfloor - 1\right)\lfloor \frac{d}{2} \rfloor \tag{9}$$

This implies that from network G to network $G \cup ij$ the betweenness of actor i increases by at least $\frac{(\lfloor \frac{d}{2} \rfloor - 1)\lfloor \frac{d}{2} \rfloor}{(n-1)(n-2)}$. Since $\lambda = 1$, the marginal benefits are at least as high as the marginal costs c. The argument holds for both actors i and j such that G is not stable.

For part (ii)[29] assume that for $G \in \mathcal{G} \; \exists \; i, j$: $2 < d(i, j) < M$. Let $N_G(i)$: $= \{k \in N$: $ik \in G\}$, be the set of neighbors of i in network G and similarly $N_G(j)$. By the existence of a path longer than 2 (between i and j), we know that $N_G(i) \neq \emptyset$ and $N_G(j) \neq \emptyset$. As this path is a geodesic, we know that $\exists \; k$: $k \in N_G(i)$ and $k \notin N_G(j)$; and $\exists \; l$: $l \in N_G(j)$ and $l \notin N_G(i)$. In fact, $N_G(i) \cap N_G(j) = \emptyset$ which implies that $d(k, j) \geq 2$. Let $G' := G \cup ij$, be the network when we add the link ij. Then the path kij is a geodesic between k and j in G'. This generates some betweenness value for i. The same holds for j. As the marginal costs c are lower than any marginal benefit, we conclude $u_i(G) < u_i(G')$ and $u_j(G) < u_j(G')$, which contradicts stability. □

A.3. Proofs of Section 4

Proposition 5. *In the centrality model the following holds:*

1. *The complete network K_n is stable if and only if* $c \leq \frac{1-\lambda}{(n-1)(M-1)}$.
2. *The empty network $\overline{K_n}$ is stable if and only if* $c \geq \frac{1-\lambda}{n-1}$.
3. *A star network $K_{1,n-1}$ is stable if and only if* $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$.

[29]Part (i) does not count the marginal benefits i derives from pairs (i' and j') for which the establishment of ij means an additional shortest path. Those pairs also increase the marginal benefit of i, but the amount depends on the number of shortest paths. For part (ii) we also consider such pairs.

4. *Let n be a multiple of 4. Then a circle network C_n is stable if and only if* $\frac{(1-\lambda)[\frac{1}{8}n^2-\frac{1}{2}n+1]}{(M-1)(n-1)}+\frac{2\lambda[\frac{1}{8}n^2-\frac{3}{4}n+1]}{(n-1)(n-2)}\leq c\leq\frac{(1-\lambda)[\frac{1}{4}n^2-\frac{1}{2}n]}{(M-1)(n-1)}+\frac{2\lambda[\frac{1}{8}n^2-\frac{1}{2}n+\frac{1}{2}]}{(n-1)(n-2)}$.
5. *The balanced complete bipartite network K_{n_1,n_1} (for even n) is stable if and only if* $\frac{1-\lambda}{(n-1)(M-1)}\leq c\leq\frac{2(1-\lambda)}{(n-1)(M-1)}+\frac{2\lambda[1-\frac{2}{n}]}{(n-1)(n-2)}$.

Proof of Proposition 5. The results of Proposition 5 present lower and/or upper bounds of costs where a network is claimed to be stable. For conciseness, we denote with $\inf(G)$ the claimed lower bound of a network G and analogously the claimed upper bound with $\sup(G)$.

1. The complete network K_n can only be altered by deletion of a link. Any actor deleting any link increases his distances by 1 and does not change his brokerage. Therefore, no actor will sever a link for $c\leq\sup(K_n)$ and every actor wants to sever a link for higher costs.
2. The empty network $\overline{K_n}$ can only be altered by the addition of links. Any actor adding a link decreases his distances by $M-1$, while his brokerage remains zero. Thus, no actor will do that for $c\geq\inf(\overline{K_n})$ and any pair of actors is willing to add a link for $c<\inf(\overline{K_n})$.
3. In a star network $K_{1,n-1}$ only peripheral actors can add links. Any actor adding a link reduces his distances by 1 and does not change his brokerage. This leads to the $\inf(K_{1,n-1})$. The central actor severing a link increases his distances by $M-1$ and decreases his brokerage by $n-2$. A peripheral actor cutting a link increases his distances by $M-1+(n-2)(M-2)$ and does not change his brokerage. Plugging into the utility function yields that no actor wants to sever a link for $c\leq\min\{\sup_1(K_{1,n-1});\ \sup_2(K_{1,n-1})\}$, while some actor is willing to sever a link for higher costs.
4. Any actor severing any link increases his distances from the circle to the line network. For n even this is a change in distances of $\frac{1}{4}n^2-\frac{1}{2}n$ and a change in brokerage from $\frac{1}{8}n^2-\frac{1}{2}n+\frac{1}{2}$ to zero, yielding the upper bound. Two actors forming a link benefit the further away they are. For n a multiple of four, two actors on opposite sides (i.e. they have two shortest paths) can form a link building a network with two circles of odd length. Their change in distances can be derived as $\frac{1}{8}n^2-\frac{1}{2}n+1$, while their brokerage changes by $\frac{1}{8}n^2-\frac{3}{4}n+1$.[30]
5. In complete bipartite networks, additional links are only possible within a group. Since everybody is already indirectly linked, any actor adding a link reduces his distances by 1 without changing his brokerage. This yields the $\inf(K_{n_1,n_1})$. Since both groups consist of at least two actors ($n\geq 3$ and even), cutting one link ij only affects the distance between i and j. Their distance changes by 2. The brokerage for an actor i changes by $(\frac{n}{2}-1)(\frac{n}{2})^{-1}=1-\frac{2}{n}$, because he was on one of $\frac{n}{2}$ shortest paths between j and each of the $\frac{n}{2}-1$ actors in the other group. An actor is indifferent about cutting a link if $c=\frac{2(1-\lambda)}{(n-1)(M-1)}+\frac{2\lambda[1-\frac{2}{n}]}{(n-1)(n-2)}=\sup(K_{n_1,n_1})$. Therefore, for $c<\inf(K_{n_1,n_1})$, two actors form a link; for $c>\sup(K_{n_1,n_1})$, an actor will

[30]In the same way slightly different inequalities can be derived for other network sizes.

sever a link, and no actor can improve by changing a link for $\inf(K_{n_1,n_1}) \leq c \leq \sup(K_{n_1,n_1})$. □

Remark 1. *In the centrality model the following holds: If* $c < (1-\lambda)\frac{2M-3}{(M-1)(n-1)} + \lambda\frac{4}{(n-1)(n-2)}$, *the stable networks contain at most one nontrivial component.*

Proof of Remark 1. Consider a network with two components of size 2 and an actor i in one of them. By linking to the other component, actor i's distances decrease by $2M-1-2$, while his brokerage increases by $1\cdot 2$. This yields the minimal change in benefits for any link between two nontrivial components, since components larger than 2 imply stronger improvements. □

Remark 2. *In the centrality model the following holds: If* $c > min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$, *no network with pendants (actors of degree one) is stable.*

Proof of Remark 2. Take any network G with a pendant i and his neighbor j. We show that the condition implies that one of the actors wants to sever link ij.

1. Actor i does not reduce brokerage by severing this link. Removing the link increases his distances at least by $M-1$ (when actor j is also a pendant) and at most by $M-1+(n-2)(M-2)$ (when actor j is directly linked to all other actors). Therefore, actor i will not keep the link if $c > \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}$.
2. For actor j, severing the link increases his distances by $M-1$ and hence decreases his closeness by $\frac{1}{n-1}$. Moreover, he was on the shortest path between i and any other actor in this component. The more actors in this component, the higher the incentive to keep this link. The maximum brokerage of $n-2$ is attained for a connected network. Therefore, actor j certainly wants to sever the link for $c > \frac{1-\lambda}{n-1} + \frac{2\lambda(n-2)}{(n-1)(n-2)}$ rendering the network unstable. □